A complete description of bi-dimensional anisotropic strain-gradient elasticity

Nicolas Auffray, Justin Dirrenberger, G Rosi

A complete description of bi-dimensional anisotropic strain-gradient elasticity

N. Auffray[a,*], J. Dirrenberger[b], G. Rosi[c]

[a] *Université Paris-Est, Laboratoire Modélisation et Simulation Multi Echelle, MSME UMR 8208 CNRS, 5 bd Descartes, 77454 Marne-la-Vallée, France*
[b] *PIMM, Arts et Métiers-ParisTech/CNAM/CNRS UMR 8006, 151 bd de l'Hôpital, 75013 Paris, France*
[c] *Université Paris-Est, Laboratoire Modélisation et Simulation Multi Echelle, MSME UMR 8208 CNRS, 61 av du Général de Gaulle, 94010 Créteil Cedex, France*

Abstract

In the present paper spaces of fifth-order tensors involved in bidimensional strain gradient elasticity are studied. As a result complete sets of matrices representing these tensors in each one of their anisotropic system are provided. This paper completes and ends some previous studies on the subject providing a complete description of the anisotropic bidimensional strain gradient elasticity. It is proved that this behavior is divided into 14 non equivalent anisotropic classes, 8 of them being isotropic for classical elasticity. The classification and matrix representations of the *acoustical gyrotropic* tensor are also provided, these results may find interesting applications to the study of waves propagation in dispersive micro-structured-media.

Keywords: Strain gradient elasticity, Anisotropy, Higher-order tensors, Chirality, Acoustical activity

1. Introduction

Understanding and modeling wave propagation in periodic lattices is a problem of prime importance for the design of metamaterials. The shape of the elementary cell and its point group determines the elementary vibration modes of the lattice (Dresselhaus et al., 2008), hence the number and nature of acoustical and optical branches of the dispersion curves. The structure of branches are important since it determines the band gaps (Kittel, 2007), which are responsible for some "macroscopic" non standard effects. Another specific feature of wave propagation in periodic lattices is a strong directionality at high frequencies, which cannot always be described by a classical continuous formulation.

At the present time, the study of wave propagation in periodic lattices relies on FEM computation on a unit cell. Results of such simulations can be found in numerous references (Phani et al., 2006; Spadoni et al., 2009; Liu et al., 2011). The determination of a continuous substitution medium that would replace the explicit micro-structure might be valuable, especially regarding optimization purposes (Jensen and Sigmund, 2004). Since the loading wavelengths are a few times the scale of the unit cell, wave propagation through the medium is dispersive, i.e. the phase velocity of a wave depends on its frequency. And, as well-known, classical linear elasticity is not dispersive (Royer and Dieulesaint, 2000). This raises the question of how to model dispersivity in a continuous fashion. Such a formulation is appealing for studying wave transmission and reflection across material discontinuities, which could find natural applications both in biomechanics (Rosi et al., 2014) and nondestructive damage evaluation (dell'Isola et al., 2011).

*Corresponding author
Email address: Nicolas.auffray@univ-mlv.fr (N. Auffray)

This question began to be investigated in the field of condensed matter physics during the 1960s (Toupin, 1962; Portigal and Burstein, 1968). The aim was to circumvent the uses of cumbersome models of lattice dynamics in the modeling of dispersive behaviors. Physical motivations were two-fold:

Acoustical activity which concerns the rotation of the plane of polarization of a transverse wave through its propagation, was observed in some crystals. This effect, which can be encoded by a fifth-order *gyrotropic* tensor (Bhagwat et al., 1986), couples strain and strain-gradient effects.

Ballistic phonon imaging is an high-energy imaging technique used to investigate the anisotropic features of crystals. Using heat pulses of very high frequency (0.1-1 THz), for very low room temperature (Wolfe, 2005), the heat propagation is no more diffusive but ballistic and described by the elastic properties of the crystal lattice. To study departure from classical elasticity that occurs at high-frequency, DiVincenzo (1986) proposed a continuum extension that involved not only a fifth-order elasticity tensor but also a sixth-order one.

Nowadays problematics in metamaterials studies are very similar[1]. Since effective description using classical elasticity is not sufficient, one can use generalized continuum theories which are known to be dispersive. There are two ways to extend classical continuum mechanics (Toupin, 1962; Mindlin, 1964, 1965; Erigen, 1967; Mindlin and Eshel, 1968):

Higher-order continua: with this option the number of degrees of freedom is extended. These theories can model optical branches. The Cosserat elasticity in which local rotations are added as degrees of freedom belongs to this family (Cosserat and Cosserat, 1909). This enhancement can be extended further to obtain the micromorphic elasticity (Green and Rivlin, 1964; Mindlin, 1964; Germain, 1973).

Higher-grade continua: the other option is to keep the same degrees of freedom but to add higher-order gradients of the displacement field into the energy density. Within this framework no optical branch is added to the acoustical ones. Mindlin first strain-gradient elasticity (SGE) (Mindlin, 1964; Mindlin and Eshel, 1968), and second strain gradient elasticity (Mindlin, 1965) belongs to this family. Higher-grade continua can be conceived as low frequency, long wave-length approximations of higher-order continua (Mindlin, 1964).

This situation is sketched in Table 1.

In order to mimic the general approach followed in physics which are based on Taylor expansion of a non-local constitutive operator, the higher-grade path will be followed. In the present paper, Mindlin strain-gradient elasticity (SGE) will be considered. This model can be seen as a phenomenological approximation of the expansion used by Portigal and Burstein (1968) or DiVincenzo (1986)[2].

modeling anisotropic wave propagation in this framework requires knowing the matrix representations of

- **higher-order inertia tensors;**
- **higher-order elasticity tensors;**

[1]It is worth noting an important difference between these two approaches: for condensed matter physics, the number of degrees of freedom of the microproblem is finite, whereas this number is infinite for metamaterials.

[2]DiVicenzo's perturbative approach and the Mindlin strain-gradient phenomenological continuum agree on both the fourth- and fifth-order tensors, but differ for the sixth-order one. To obtain a strict agreement on this tensor, Mindlin second strain-gradient elasticity should be used (Mindlin, 1965). Since the nature of the correct continuum extension remains unclear today, attention will be concentrated in the present paper on the simplest consistent extension.

$$
\begin{tikzcd}
 & \textit{Rotation}: & \textit{Stretch}: \\
 & \text{Cosserat} \arrow[r] \arrow[d, "\simeq"] & \text{Micromorphic} \arrow[d, "\simeq"] \\
\text{Classical continua} \arrow[ur, "Higher-order"] \arrow[r, "Higher-grade"'] & \text{Koiter} \arrow[r] & \text{Strain-Gradient}
\end{tikzcd}
$$

Table 1: Basic extensions of a classical continuum. From the left to the right, *rotation* then *strech* are added to the kinematics. For higher-order continua these *extensions* are independent DOF, for higher-grade continua they are controlled by higher-order gradients of the displacement field.

in each anisotropic system. The first point has rarely been addressed in the literature, and at the present time little is known, except in some specific situations (Wang and Sun, 2002; Bacigalupo and Gambarotta, 2014). Despite its interest, this subject will not be considered in the present paper in which attention will be focused on higher-order elasticity tensors. If higher-order inertia tensors are specific to dynamics, higher-order elasticity tensors are also involved in statics. Hence, our results may find applications both for static and dynamics.

Concerning higher-order elasticity tensors, if matrix representations are known for the sixth-order elasticity tensor (Auffray et al., 2009a, 2013), the fifth-order tensor involved in this model has not been studied; it is the purpose of the present contribution to provide a complete set of anisotropic matrix representation for this fifth-order coupling tensor. For the sake of simplicity, our investigation will be restricted to a bidimensional physical space. As a consequence:

- **the description of the static anisotropic 2D strain-gradient elasticity model is now complete. 2D static strain-gradient elasticity possesses 14 different types of anisotropy, 8 of them being isotropic for classical elasticity;**

- **the complete set of gyrotropic tensors responsible for the so-called *acoustical activity*, that is the rotation of the plane of polarization of a transverse wave through its propagation[3], is also obtained.**

The paper is organized as follows. First, the constitutive law of SGE is recalled and results regarding symmetry classes are recapitulated. The main results obtained for the coupling elasticity (CE) tensor and the acoustic gyrotropic (AG) tensor are given in Section 3, where explicit matrix representations for all the symmetry classes are provided. In Section 4 results concerning tensors of SGE are summarized, and the complete classification of SGE law is given. The different kinds of coupling which may occur are detailed. It will be shown that, in 2D, fifth-order coupling elasticity plays a limited role in the modeling of chiral sensitivity. Finally, in Section 5 a few concluding remarks are drawn.

2. Strain-gradient elasticity

In this section, the basic equations of strain gradient elasticity are presented. The constitutive relations are considered first, followed by the equation of motion. These different relations

[3]If the nature of this effect is rather clear in a 3D space, its interpretation in 2D remains unclear.

involve the classical fourth-order elasticity tensor supplemented by a fifth-order coupling tensor and a sixth-order tensor. It should be noted that the motion equation only involves a particular combination of components of the fifth-order tensor. This leads to the definition of the fifth-order *acoustical gyrotropic* tensor. As will be shown hereafter, these two fifth-order tensors behave differently with respect to material symmetries.

2.1. Constitutive equations

In the strain-gradient theory of linear elasticity (Mindlin, 1964; Mindlin and Eshel, 1968), the constitutive law gives the symmetric Cauchy stress tensor σ and the hyperstress tensor τ in terms of:

- **the infinitesimal strain tensor:** ε
- **the strain-gradient tensor:** $\eta = \varepsilon \otimes \underline{\nabla}$ **which, using index notation, gives** $\eta_{ijk} = \varepsilon_{ij,k}$ **the comma denoting a derivation.**

through the two linear relations:

$$\begin{cases} \sigma_{ij} = C_{ijlm}\varepsilon_{lm} + M_{ijlmn}\eta_{lmn}, \\ \tau_{ijk} = M_{lmijk}\varepsilon_{lm} + A_{ijklmn}\eta_{lmn}. \end{cases} \tag{1}$$

Above,

- $\mathbf{C}$ is the classical fourth-order elastic tensor;
- $\mathbf{M}$ is the fifth-order coupling elastic (CE) tensor;
- $\mathbf{A}$ is the sixth-order second-order elastic (SOE) tensor.

These tensors satisfy the following index permutation symmetry:

$$C_{\underline{(ij)(lm)}} \quad ; \quad M_{(ij)(kl)m} \quad ; \quad A_{\underline{(ij)k}\ \underline{(lm)n}}$$

where the notation (..) stands for the minor symmetries, whereas $\underline{..}$ stands for the major one. In the case where the microstructure of a material exhibits centro-symmetry, the fifth-order elastic stiffness tensor $\mathbf{M}$ vanishes. It is worth noting that in even dimension the inversion is a proper transformation. As a consequence, and contrary to the 3D case, the vanishing of an odd-order tensor is not related to chirality. In 2D, odd-order tensors are null for even-order rotational invariant media (Auffray et al., 2009b). Hence, as it will be shown, the fifth-order coupling tensor exists both for chiral and achiral media.

2.2. Dynamics

As the foreseen applications concern dispersive elastodynamics, the associated equation and construction of the acoustic gyrotropic tensor are presented here. This topic will be considered more in depth in a forthcoming paper (Rosi and Auffray, 2015). **In the absence of body double-forces, the motion equation of a strain-gradient media subjected to body forces f_i reads:**

$$s_{ij,j} + f_i = \rho\ddot{u}_i - \kappa_{jik}\ddot{u}_{j,k} - \kappa_{jkil}\ddot{u}_{j,kl} \tag{2}$$

where s_{ij} is the *effective* second-order symmetric stress tensor, κ_{jik} is a third order micro-inertia tensor and κ_{jkil} a fourth order micro-inertia tensor (Mindlin and Eshel, 1968; Mindlin, 1964; Ben-Amoz, 1976; Askes and Aifantis, 2011; Bacigalupo and Gambarotta, 2014) . It is important to remark that the third order micro-inertia tensor κ_{jik} is vanishing for centrosymmertic materials (see e.g. Bacigalupo and Gambarotta

(**2014**)).
This tensor is defined as follows:

$$s_{ij} = \sigma_{ij} - \tau_{ijk,k}.$$

Using the general constitutive law (1), the effective second-order tensor takes the form

$$s_{ij} = C_{ijlm}\varepsilon_{lm} + (M_{ijklm} - M_{klijm})\varepsilon_{lm,k} - A_{ijklmn}\varepsilon_{lm,nk}$$

which can be rewritten in the following way

$$s_{ij} = C_{ijlm}\varepsilon_{lm} + M^{\sharp}_{ijklm}\varepsilon_{lm,k} - A_{ijklmn}\varepsilon_{lm,nk} \tag{3}$$

with the *dynamic* fifth-order tensor $\mathbf{M}^{\sharp}$ defined as

$$M^{\sharp}_{ijklm} = M_{ijklm} - M_{klijm}$$

This tensor possesses the following index symmetries:

$$M^{\sharp}_{\underline{(ij)}\,\overline{(kl)}m} \tag{4}$$

where the notation $\underline{..}\,\overline{..}$ indicates antisymmetry with respect to block permutation (Boutin, 1996; Triantafyllidis and Bardenhagen, 1996). In physics this tensor is known as the acoustical gyrotropic tensor and is responsible for the-called acoustical activity (Portigal and Burstein, 1968; Srinivasan, 1988).

2.3. Synthesis

Until now, $\mathbb{C}$ and $\mathbb{A}$, the vector spaces of $\mathbf{C}$ and $\mathbf{A}$, have been investigated, both in a 2D and 3D euclidean spaces (Mehrabadi and Cowin, 1990; Forte and Vianello, 1996; Auffray et al., 2009a, 2013). Also, the answers to the following three questions have been provided:
(a) How many symmetry classes and which symmetry classes do $\mathbb{C}$ and $\mathbb{A}$ have?
(b) For every given symmetry class, how many independent material parameters do $\mathbb{C}$ and $\mathbb{A}$ have?
(c) For each given symmetry class, what is the explicit matrix form of $\mathbf{C}$ and $\mathbf{A}$ relative to an orthonormal basis?

In 2D, for $\mathbb{C}$, He and Zheng (1996) demonstrated that the space of classical fourth-order tensors is divided in 4 classes. This result was also obtained by a different mean by Vianello (1997). For $\mathbb{A}$ the question was solved in 2D by Auffray et al. (2009a), the space of sixth-order tensors is more complex since it is divided in 8 classes. For the 3D case, the number of symmetry classes increases since $\mathbb{C}$ is now divided into 8 classes (Forte and Vianello, 1996), and $\mathbb{A}$ into 17 classes (Olive and Auffray, 2013; Auffray et al., 2013). At the present time, these questions remain open for the fifth-order tensor spaces $\mathbb{M}$ and $\mathbb{M}^{\sharp}$, both in 2D and 3D. **Some theoretical results are available concerning the 3D case (Olive and Auffray, 2014; Auffray, 2014), but without explicit construction.** In order to have a complete SGE theory to model dispersive media, answering the aforementioned three questions for $\mathbb{M}$ and $\mathbb{M}^{\sharp}$ is important. In the following this study will be conducted for $\mathbb{M}$ and results for $\mathbb{M}^{\sharp}$ will then be deduced.

2.4. Symmetry classes

Let $\mathbf{Q}$ be an element of the 2D orthogonal group[4] $\mathrm{O}(2)$. $\mathbf{M}$ is said to be invariant under the action of $\mathbf{Q}$ if

$$Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}M_{opqrs} = M_{ijklm}. \tag{5}$$

[4]The orthogonal group in 2D is defined as $\mathrm{O}(2) = \{\mathbf{Q} \in \mathrm{GL}(2) | \mathbf{Q}^T = \mathbf{Q}^{-1}\}$, in which $\mathrm{GL}(2)$ denotes the set of invertible transformations acting on $\mathbb{R}^2$.

The symmetry group of $\mathbf{M}$ is defined as the subgroup $\mathrm{G}_{\mathbf{M}}$ of $\mathrm{O}(2)$ constituted of all the orthogonal tensors leaving $\mathbf{M}$ invariant:

$$\mathrm{G}_{\mathbf{M}} = \{\mathbf{Q} \in \mathrm{O}(2) \,|Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}M_{opqrs} = M_{ijklm}\}. \tag{6}$$

As proposed by Forte and Vianello (1996) it is meaningful to consider two tensors $\mathbf{M}$ and $\mathbf{N}$ as exhibiting symmetry of the same kind if their symmetry groups are conjugate in the sense that

$$\text{there exists a } \mathbf{Q} \in \mathrm{O}(2) \text{ such that } \mathrm{G}_{\mathbf{N}} = \mathbf{Q}\mathrm{G}_{\mathbf{M}}\mathbf{Q}^T. \tag{7}$$

Thus, the symmetry classes associated to $\mathbf{M}$ can be naturally defined as the set $[\mathrm{G}_{\mathbf{M}}]$ of all the subgroups of $\mathrm{O}(2)$ conjugate to $\mathrm{G}_{\mathbf{M}}$:

$$[\mathrm{G}_{\mathbf{M}}] = \{\mathrm{G} \subseteq \mathrm{O}(2) \,\big|\mathrm{G} = \mathbf{Q}\mathrm{G}_{\mathbf{M}}\mathbf{Q}^T, \mathbf{Q} \in \mathrm{O}(2)\}. \tag{8}$$

In other words, the symmetry class to which $\mathbf{M}$ belongs corresponds to its symmetry group modulo its orientation, i.e. $\mathrm{O}(2)$. Furthermore, it is known (Zheng and Boehler, 1994) that in a bidimensional space, the symmetry class of a tensor is conjugate to a closed subgroup of $\mathrm{O}(2)$. The collection of these subgroups are known and are elements of the following set (Armstrong, 1983):

$$\{\mathrm{Id}, \mathrm{Z}_2^\pi, \mathrm{Z}_k, \mathrm{D}_k, \mathrm{SO}(2), \mathrm{O}(2)\}_{k\in\mathbb{N}_{>1}}$$

in which the following groups are involved:

- Id, the identity group;
- Z_k, the cyclic group[5] with k elements generated by $\mathbf{R}(2\pi/k)$, a rotation angle $2\pi/k$;
- $\mathrm{SO}(2)$, the infinitesimal rotation group, the cyclic limit group for $k \to \infty$;
- Z_2^π, where π denotes a mirror transformation through the y axis;
- D_k, the dihedral group with $2k$ elements generated by $\mathbf{R}(2\pi/k)$ and π;
- $\mathrm{O}(2)$, the infinitesimal orthogonal, the dihedral limit group for $k \to \infty$;

In the following a group will be said mirror-invariant, $\mathcal{M}$, if it contains the reflection-operation, π, and centro-invariant, $\mathcal{I}$, if it contains the inversion-operation $i = \mathbf{R}(\pi)$. **In 2D, and in contrary to 3D, the inversion implies the presence of an even-order rotation; hence the inversion is, in this case, a proper transformation. As a consequence, in 2D, chirality is not equivalent to non-centro symmetry, but to the lack of mirror symmetry only.** Hence the set of closed subgroups of $\mathrm{O}(2)$ can be divided in four subsets according to whether groups are mirror-invariant ($\mathcal{M}$) or not ($\overline{\mathcal{M}}$), centro-invariant ($\mathcal{I}$) or not ($\overline{\mathcal{I}}$). The following table contains the different cases:

	$\mathcal{I}$	$\overline{\mathcal{I}}$
$\mathcal{M}$	D_{2k},$\mathrm{O}(2)$	Z_2^π, D_{2k+1}
$\overline{\mathcal{M}}$	Z_{2k}, $\mathrm{SO}(2)$	Z_{2k+1}

Table 2: Classification of O(2) subsets according to their mirror- and centro-invariance

As will be seen, these four sets describe the different couplings that may, or may not, exist in the complete SGE model.

[5]It has to be noted that Z_2^π and Z_2 are isomorphic as group but not conjugate.

In a side paper (Auffray et al., 2015), it is proved that, in 2D, the vector space $\mathbb{M}$ is divided into 6 symmetry classes: one isotropic and five anisotropic. These results are summarized in Table 3. Some comments concerning this classification have to be made:

• In order to be complete, and even if it reduces to the null tensor, the isotropic symmetry class $[\mathrm{O}(2)]$ has been included in the classification;

• A tensor which is Z_5-invariant has its symmetry group conjugate to a D_5-invariant one. As a consequence the pentachiral class $[\mathrm{Z}_5]$ is empty.

Name	Oblique	Rectangular	Trichiral	Trigonal	Pentachiral	Pentagonal	Isotropic
$\mathrm{G}_{\mathcal{M}}$	Id	Z_2^{π}	Z_3	D_3	Z_5	D_5	$\mathrm{O}(2)$
$\#_{\mathrm{indep}}(\mathbf{M})$	18 (17)	9	6 (5)	3	2 (1)	1	0
$[\mathrm{G}_{\mathbf{M}}]$	[Id]	$[\mathrm{Z}_2^{\pi}]$	$[\mathrm{Z}_3]$	$[\mathrm{D}_3]$	$[\mathrm{D}_5]$	$[\mathrm{D}_5]$	$[\mathrm{O}(2)]$

Table 3: The names, the sets of subgroups $[\mathrm{G}_{\mathbf{M}}]$ and the numbers of independent components $\#_{\mathrm{indep}}(\mathbf{M})$ for the 6 symmetry classes of $\mathbb{M}$. The in-parenthesis number indicates the minimal number of components of the matrix in an appropriate basis.

The symmetry classes of the vector space $\mathbb{M}^{\sharp}$ are very different, since the classes $[\mathrm{Z}_3]$ and $[\mathrm{D}_5]$ are now empty. Results for the space of gyrotropic tensors[6] are summarized in Table 4.

Name	Oblique	Rectangular	Trichiral	Trigonal	Isotropic
$\mathrm{G}_{\mathcal{M}}$	Id	Z_2^{π}	Z_3	D_3	$\mathrm{O}(2)$
$\#_{\mathrm{indep}}(\mathbf{M}^{\sharp})$	6 (5)	3	2 (1)	1	0
$[\mathrm{G}_{\mathbf{M}^{\sharp}}]$	[Id]	$[\mathrm{Z}_2^{\pi}]$	$[\mathrm{D}_3]$	$[\mathrm{D}_3]$	$[\mathrm{O}(2)]$

Table 4: The names, the sets of subgroups $[\mathrm{G}_{\mathbf{M}^{\sharp}}]$ and the numbers of independent components $\#_{\mathrm{indep}}(\mathbf{M}^{\sharp})$ for the 4 symmetry classes of $\mathbb{M}^{\sharp}$. The in-parenthesis number indicates the minimal number of components of the matrix in an appropriate basis.

3. Matrix representations of the coupling elasticity tensor

The goal of the present section is to determine, for each symmetry class, the *explicit matrix form* of $\mathbb{M}$ and $\mathbb{M}^{\sharp}$ relative to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. **To that aim we follow a strategy introduced for classical elasticity by Mehrabadi and Cowin (1990) and extended to strain-gradient elasticity in Auffray et al. (2009a) and Auffray et al. (2013). This approach is summarized hereafter.**

3.1. Orthonormal basis and matrix component ordering

Let be defined the following spaces:

$$\begin{aligned}
\mathbb{T}_{(ij)} &= \{\mathrm{T} \in \mathbb{T}_{ij} | \mathrm{T} = \sum_{i,j=1}^{2} T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,\ T_{ij} = T_{ji}\} \\
\mathbb{T}_{(ij)k} &= \{\mathrm{T} \in \mathbb{T}_{ijk} | \mathrm{T} = \sum_{i,j,k=1}^{2} T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k,\ T_{ijk} = T_{jik}\}
\end{aligned}$$

which are, in 2D, respectively, 3- and 6-dimensional vector spaces. Therefore

[6] It can be noted that the following results are the same as for the space of piezoelectric tensors (Auffray et al., 2015; Vannucci, 2007).

- the first-order elasticity tensor $\mathbf{C}$ is a self-adjoint endomorphism of $\mathbb{T}_{(ij)}$;
- the coupling elasticity tensor $\mathbf{M}$ is a linear application from $\mathbb{T}_{(ij)k}$ to $\mathbb{T}_{(ij)}$;
- the second-order elasticity tensor $\mathbf{A}$ is a self-adjoint endomorphism of $\mathbb{T}_{(ij)k}$.

In order to express the Cauchy-stress tensor σ, the strain tensor ε, the strain-gradient tensor η and the hyperstress tensor τ as 3- and 6-dimensional vectors and write $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{A}$ as, respectively: a 3×3, 3×6 and 6×6 matrices, we introduce the following orthonormal basis vectors:

$$\begin{aligned}\widetilde{\mathbf{e}}_I &= \left(\frac{1-\delta_{ij}}{\sqrt{2}}+\frac{\delta_{ij}}{2}\right)(\mathbf{e}_i\otimes\mathbf{e}_j+\mathbf{e}_j\otimes\mathbf{e}_i), \quad 1\le I\le 3\\ \widehat{\mathbf{e}}_\alpha &= \left(\frac{1-\delta_{ij}}{\sqrt{2}}+\frac{\delta_{ij}}{2}\right)(\mathbf{e}_i\otimes\mathbf{e}_j+\mathbf{e}_j\otimes\mathbf{e}_i)\otimes\mathbf{e}_k, \quad 1\le \alpha\le 6\end{aligned}$$

where the summation convention for a repeated subscript does not apply. Then, the aforementioned tensors can be expressed as:

$$\widetilde{\varepsilon}=\sum_{I=1}^{3}\widetilde{\varepsilon}_I\widetilde{\mathbf{e}}_I,\quad \widetilde{\sigma}=\sum_{I=1}^{3}\widetilde{\sigma}_I\widetilde{\mathbf{e}}_I,\quad \widehat{\eta}=\sum_{\alpha=1}^{6}\widehat{\eta}_\alpha\widehat{\mathbf{e}}_\alpha,\quad \widehat{\tau}=\sum_{\alpha=1}^{6}\widehat{\tau}_\alpha\widehat{\mathbf{e}}_\alpha \tag{9}$$

$$\widetilde{\mathbf{C}}=\sum_{I,J=1,1}^{3,3}\widetilde{C}_{IJ}\widetilde{\mathbf{e}}_I\otimes\widetilde{e}_J \quad \overline{\mathbf{M}}=\sum_{I,\alpha=1,1}^{3,6}\overline{M}_{I\alpha}\widetilde{\mathbf{e}}_I\otimes\widehat{\mathbf{e}}_\alpha,\quad \widehat{\mathbf{A}}=\sum_{\alpha,\beta=1,1}^{6,6}\widehat{A}_{\alpha\beta}\widehat{\mathbf{e}}_\alpha\otimes\widehat{\mathbf{e}}_\beta, \tag{10}$$

so that the relations in (1) can be written in the matrix form

$$\begin{cases}\widetilde{\sigma}_I=\widetilde{C}_{IJ}\widetilde{\varepsilon}_J+\overline{M}_{I\alpha}\widehat{\eta}_\alpha\\ \widehat{\tau}_\alpha=\overline{M}_{\alpha J}\widetilde{\varepsilon}_J+\widehat{A}_{\alpha\beta}\widehat{\eta}_\beta\end{cases} \tag{11}$$

The relationship between the matrix components $\widetilde{\varepsilon}_I$ and ε_{ij}, and between $\widehat{\eta}_\alpha$ and η_{ijk} are

$$\widetilde{\varepsilon}_I=\begin{cases}\varepsilon_{ij} & \text{if } i=j,\\ \sqrt{2}\varepsilon_{ij} & \text{if } i\neq j;\end{cases}\qquad \widehat{\eta}_\alpha=\begin{cases}\eta_{ijk} & \text{if } i=j,\\ \sqrt{2}\eta_{ijk} & \text{if } i\neq j;\end{cases} \tag{12}$$

and, obviously, the same relations between $\widetilde{\sigma}_I$ and σ_{ij} and $\widehat{\tau}_\alpha$ and τ_{ijk} hold. For the constitutive tensors we have the following correspondences:

$$\widetilde{C}_{IJ}=\begin{cases}C_{ijkl} & \text{if } i=j \text{ and } k=l,\\ \sqrt{2}C_{ijkl} & \text{if } i\neq j \text{ and } k=l \text{ or } i=j \text{ and } k\neq l,\\ 2C_{ijkl} & \text{if } i\neq j \text{ and } k\neq l.\end{cases} \tag{13}$$

$$\overline{M}_{I\alpha}=\begin{cases}M_{ijklm} & \text{if } i=j \text{ and } k=l,\\ \sqrt{2}M_{ijklm} & \text{if } i\neq j \text{ and } k=l \text{ or } i=j \text{ and } k\neq l,\\ 2M_{ijklm} & \text{if } i\neq j \text{ and } k\neq l.\end{cases} \tag{14}$$

$$\widehat{A}_{\alpha\beta}=\begin{cases}A_{ijklmn} & \text{if } i=j \text{ and } l=m,\\ \sqrt{2}A_{ijklmn} & \text{if } i\neq j \text{ and } l=m \text{ or } i=j \text{ and } l\neq m,\\ 2A_{ijklmn} & \text{if } i\neq j \text{ and } l\neq m.\end{cases} \tag{15}$$

It remains to choose appropriate two-to-one and three-to-one subscript correspondences between ij and I, on one hand, and ijk and α, on the other hand. For the classical variables the standard two-to-one subscript correspondence is used, i.e:

I	1	2	3
ij	11	22	12

Table 5: The two-to-one subscript correspondence for 2D strain/stress tensors

The three-to-one subscript correspondence for strain-gradient/hyperstress tensor, specified in Table 5, is chosen in order to make the 6th-order tensor $\mathbf{A}$ block-diagonal for dihedral classes (c.f. Appendix B)[7].

α	1	2	3	
ijk	111	221	122	Privileged direction: 1
α	4	5	6	
ijk	222	112	121	Privileged direction: 2

Table 6: The three-to-one subscript correspondence for 2D strain-gradient/hyperstress tensors

The matrix representations of first- and second-order elasticity tensors have already been investigated. Their different forms are recalled in Appendix A and Appendix B. Hence, in the remaining subsection, attention will be devoted to CE tensors.

3.2. Transformation matrix

Using the introduced orthogonal bases and the subscript correspondences, the action of a rotation tensor $\mathbf{Q} \in \mathrm{O}(2)$ on $\mathbf{M}$ can be represented using two different matrices: a 3×3 matrix $\widetilde{\mathbf{Q}}$, and a 6×6 matrix $\widehat{\mathbf{Q}}$ in a way such that

$$Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}M_{opqrs} = \widetilde{Q}_{IJ}\overline{M}_{J\alpha}\widehat{Q}_{\alpha\beta} \tag{16}$$

where

$$\widetilde{Q}_{IJ} = \frac{1}{2}(Q_{io}Q_{jp} + Q_{ip}Q_{jo}) \quad ; \quad \widehat{Q}_{\alpha\beta} = \frac{1}{2}(Q_{io}Q_{jp} + Q_{ip}Q_{jo})Q_{kq} \tag{17}$$

with I and J being associated to ij and op, and α and β being associated to ijk and opq respectively. Thus, formula (5) expressing the invariance of $\mathbf{M}$ under the action of $\mathbf{Q}$ is equivalent to

$$\widetilde{\mathbf{Q}}\overline{\mathbf{M}}\widehat{\mathbf{Q}}^T = \overline{\mathbf{M}} \tag{18}$$

where $\widetilde{\mathbf{Q}}$ stands for the 3×3 matrix of components $\widetilde{Q}_{IJ}$, $\widehat{\mathbf{Q}}$ stands for the 6×6 matrix of components $\widehat{Q}_{\alpha\beta}$ and $\overline{\mathbf{M}}$ the 3×6 matrix of components $\overline{M}_{J\alpha}$.

3.3. Matrix representations of $\overline{\mathbf{M}}$ *and* $\overline{\mathbf{M}}^\sharp$ *for all symmetry classes*

We are now ready to give the explicit expressions of $\overline{\mathbf{M}}$ and $\overline{\mathbf{M}}^\sharp$ for each of the 5 anisotropic classes. Matrices will first be given in a brut form, and in a second time split into sub-matrices so as to make appear elementary building blocks. The order adopted to specify the expressions of $\overline{\mathbf{M}}$ for the symmetry classes $[\mathrm{Z}_k]$ and $[\mathrm{D}_k]$ is $k = 1, 3, 5$.

3.3.1. Symmetry class characterized by [Id]

Constitutive tensor. In this most general case, illustrated by Figure 1, the material in question is *fully anisotropic* and the CE matrix $\overline{\mathbf{M}}$ comprises 18 independent components. The explicit

[7]Further comments on the reason of such a choice can be found in Auffray et al. (2013).

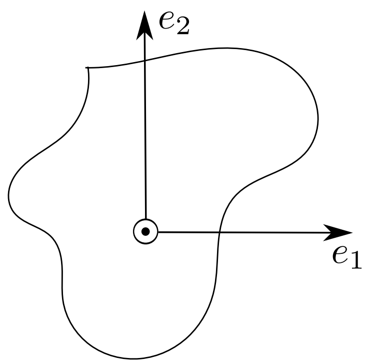


Figure 1: Oblique system (Id-invariance): the material is completely asymmetric.

expression of $\overline{\mathbf{M}}$ as a full 3×6 matrix is:

$$\overline{\mathbf{M}}_{\mathrm{Id}} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \end{pmatrix}$$

This matrix can be decomposed into sub-matrices that constitute elementary building blocks. To that aim we first define the nm-dimensional space $\mathcal{M}(n,m)$ composed of $n \times m$ matrices. Then, we can write $\overline{\mathbf{M}}$ in the following way

$$\overline{\mathbf{M}}_{\mathrm{Id}} = \begin{pmatrix} A^{(6)} & B^{(6)} \\ C^{(3)} & D^{(3)} \end{pmatrix}$$

where the form and number of independent components of each involved sub-matrix are specified by

- $A^{(6)},\ B^{(6)} \in \mathcal{M}(2,3)$;
- $C^{(3)},\ D^{(3)} \in \mathcal{M}(1,3)$;

For example, $A^{(6)}$ is an element of $\mathcal{M}(2,3)$ and contains 6 independent components while $C^{(3)}$ belongs to $\mathcal{M}(1,3)$ and comprises 3 independent components. **It should be noted that it is possible to find rotations that increases by one the number of zeros in $\overline{\mathbf{M}}_{\mathrm{Id}}$. For those particular angles, the matrix representation of the rotated tensor involves 17 components. If the associated physical is clear in classical elasticity Norris (1989), its counterpart, if any, for strain-gradient elasticity is unclear.**

Gyrotropic tensor. In this situation, the AG matrix $\overline{\mathbf{M}}^{\sharp}$ includes 6 independent components

$$\overline{\mathbf{M}}^{\sharp}_{\mathrm{Id}} = \begin{pmatrix} 0 & m^{\sharp}_{12} & m^{\sharp}_{13} & m^{\sharp}_{14} & 0 & m^{\sharp}_{16} \\ -m^{\sharp}_{12} & 0 & m^{\sharp}_{23} & 0 & -m^{\sharp}_{14} & m^{\sharp}_{26} \\ -m^{\sharp}_{16} & -m^{\sharp}_{26} & 0 & -m^{\sharp}_{23} & -m^{\sharp}_{13} & 0 \end{pmatrix}$$

These components are related to those of $\overline{\mathbf{M}}_{\mathrm{Id}}$ through the relations:

$$m^{\sharp}_{12} = m_{12} - m_{21} \quad ; \quad m^{\sharp}_{13} = m_{13} - m_{35} \quad ; \quad m^{\sharp}_{23} = m_{23} - m_{34}$$

$$m^{\sharp}_{16} = m_{16} - m_{31} \quad ; \quad m^{\sharp}_{14} = m_{14} - m_{25} \quad ; \quad \mathbf{m}^{\sharp}_{\mathbf{26}} = \mathbf{m}_{\mathbf{26}} - \mathbf{m}_{\mathbf{32}}$$

It should be noted that it is possible to find rotations that make **one more** zero appears in the previous matrix $\overline{\mathbf{M}}^{\sharp}_{\mathrm{Id}}$. Therefore, in these specific bases the matrix $\overline{\mathbf{M}}^{\sharp}_{\mathrm{Id}}$ is defined by 5 components.

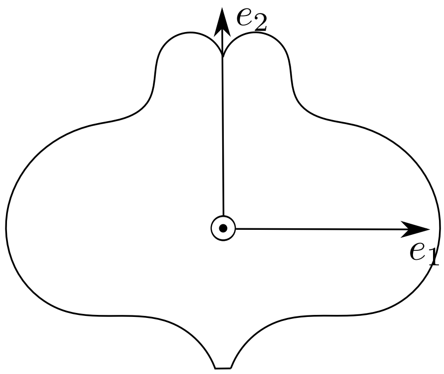


Figure 2: Rectangular system (Z_2^π-invariance): the material exhibits a symmetry plane.

3.3.2. Symmetry class characterized by $[Z_2^\pi]$

Constitutive tensor. The materials having the symmetry classes $[Z_2^\pi]$, shown in Figure 2 is referred to as being *rectangular*. The CE matrix $\overline{\mathbf{M}}$ contains 9 independent components. Using the three-to-one subscript correspondence given in Table 5, the associated matrix has the following brut form:

$$\overline{\mathbf{M}}_{Z_2^\pi} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & 0 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{34} & m_{35} & m_{36} \end{pmatrix}$$

Gyrotropic tensor. In this situation the AG matrix $\overline{\mathbf{M}}^\sharp$ includes 3 independent components

$$\overline{\mathbf{M}}^\sharp_{Z_2^\pi} = \begin{pmatrix} 0 & m^\sharp_{12} & m^\sharp_{13} & 0 & 0 & 0 \\ -m^\sharp_{12} & 0 & m^\sharp_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & -m^\sharp_{23} & -m^\sharp_{13} & 0 \end{pmatrix}$$

These components are related to those of $\overline{\mathbf{M}}_{Z_2^\pi}$ through the relations:

$$m^\sharp_{12} = m_{12} - m_{21} \quad ; \quad m^\sharp_{13} = m_{13} - m_{35} \quad ; \quad m^\sharp_{23} = m_{23} - m_{34}$$

3.3.3. Symmetry classes $[Z_3]$ *and* $[D_3]$

The materials having the symmetry classes $[Z_3]$ and $[D_3]$, as shown in Figures 3 and 4, are referred to as *trichiral* and *trigonal*, respectively.

Constitutive tensors. The CE matrix $\overline{\mathbf{M}}$ contains, respectively, 6 or 3 independent components. Using the three-to-one subscript correspondence given in Table 5, the CE matrices exhibiting the Z_3-symmetry and D_3-symmetry have the following brut forms:

$$\overline{\mathbf{M}}_{Z_3} =$$

$$\begin{pmatrix} m_{11} & -m_{11}-\frac{\sqrt{2}}{2}(m_{34}+m_{35}) & -\sqrt{2}m_{11}-\frac{1}{2}(3m_{34}-m_{35}) & m_{24}+\sqrt{2}m_{31} & -m_{24}-\frac{\sqrt{2}}{2}(m_{31}-m_{32}) & -\sqrt{2}m_{24}-\frac{1}{2}(m_{31}+m_{32}) \\ m_{11}+\sqrt{2}m_{34} & -m_{11}-\frac{\sqrt{2}}{2}(m_{34}-m_{35}) & -\sqrt{2}m_{11}-\frac{1}{2}(m_{34}+m_{35}) & m_{24} & -m_{24}-\frac{\sqrt{2}}{2}(m_{31}+m_{32}) & -\sqrt{2}m_{24}-\frac{1}{2}(3m_{31}-m_{32}) \\ m_{31} & m_{32} & \frac{\sqrt{2}}{2}(m_{31}-m_{32}) & m_{34} & m_{35} & \frac{\sqrt{2}}{2}(m_{34}-m_{35}) \end{pmatrix}$$

$$\overline{\mathbf{M}}_{D_3} = \begin{pmatrix} m_{11} & -m_{11}-\frac{\sqrt{2}}{2}(m_{34}+m_{35}) & -\sqrt{2}m_{11}-\frac{1}{2}(3m_{34}-m_{35}) & 0 & 0 & 0 \\ m_{11}+\sqrt{2}m_{34} & -m_{11}-\frac{\sqrt{2}}{2}(m_{34}-m_{35}) & -\sqrt{2}m_{11}-\frac{1}{2}(m_{34}+m_{35}) & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{34} & m_{35} & \frac{\sqrt{2}}{2}(m_{34}-m_{35}) \end{pmatrix}$$

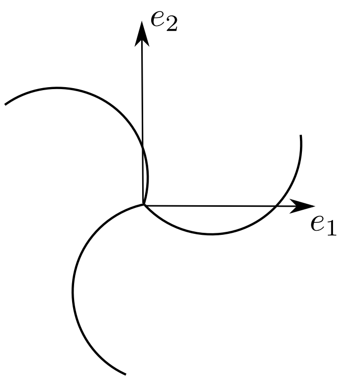


Figure 3: Trichiral system (Z_3-invariance): the material is $\frac{2\pi}{3}$-invariant.

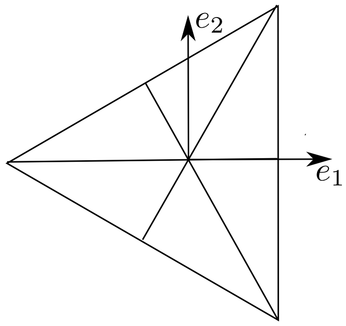


Figure 4: Trigonal system (D_3-invariance): the material is $\frac{2\pi}{3}$-invariant and exhibits 3 symmetry planes.

And, using block matrix notations,:

$$\overline{\mathbf{M}}_{\mathrm{Z}_3} = \begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(2)} & D^{(2)} \end{pmatrix} + \begin{pmatrix} f(D^{(2)}) & f(C^{(2)}) \\ 0 & 0 \end{pmatrix} \; ; \; \overline{\mathbf{M}}_{\mathrm{D}_3} = \begin{pmatrix} A^{(1)} & 0 \\ 0 & D^{(2)} \end{pmatrix} + \begin{pmatrix} f(D^{(2)}) & 0 \\ 0 & 0 \end{pmatrix}$$

First, the expressions of $A^{(1)}$ and $B^{(1)}$ with 1 independent component are specified by

$$A^{(1)} = \begin{pmatrix} a_{11} & -a_{11} & -\sqrt{2}a_{11} \\ a_{11} & -a_{11} & -\sqrt{2}a_{11} \end{pmatrix} \quad ; \quad B^{(1)} = \begin{pmatrix} b_{11} & -b_{11} & -\sqrt{2}b_{11} \\ b_{11} & -b_{11} & -\sqrt{2}b_{11} \end{pmatrix}$$

for the remaining independent components:

$$C^{(2)} = \begin{pmatrix} c_{11} & c_{12} & \frac{\sqrt{2}}{2}(c_{11} - c_{12}) \end{pmatrix} \quad ; \quad D^{(2)} = \begin{pmatrix} d_{11} & d_{12} & \frac{\sqrt{2}}{2}(d_{11} - d_{12}) \end{pmatrix}$$

and for the dependent ones:

$$f(C^{(2)}) = \begin{pmatrix} \sqrt{2}c_{11} & -\frac{\sqrt{2}}{2}(c_{11} - c_{12}) & -\frac{1}{2}(c_{11} + c_{12}) \\ 0 & -\frac{\sqrt{2}}{2}(c_{11} + c_{12}) & -\frac{1}{2}(3c_{11} - c_{12}) \end{pmatrix} \quad ;$$

$$f(D^{(2)}) = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2}(d_{11} + d_{12}) & -\frac{1}{2}(3d_{11} - d_{12}) \\ \sqrt{2}d_{11} & -\frac{\sqrt{2}}{2}(d_{11} - d_{12}) & -\frac{1}{2}(d_{11} + d_{12}) \end{pmatrix}$$

It should be noted that it is possible to find rotations that reduce the number of coefficients in $\overline{\mathbf{M}}_{\mathrm{Z}_3}$. For example, under a rotation of angle θ solution of:

$$\tan 3\theta = \frac{c_{11} - c_{12}}{d_{11} - d_{12}} \tag{19}$$

the former matrix is transformed into a new one:

$$\overline{\mathbf{M}}^{\star}_{\mathrm{Z}_3} = \begin{pmatrix} A^{(1)} & B^{\star(1)} \\ C^{\star(1)} & D^{(2)} \end{pmatrix} + \begin{pmatrix} f(D^{(2)}) & f(C^{\star(2)}) \\ 0 & 0 \end{pmatrix}$$

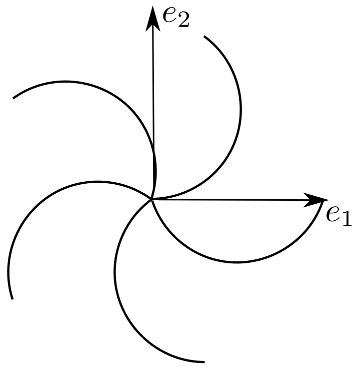


Figure 5: Pentachiral system (Z_5-invariance): the material is $\frac{2\pi}{5}$-invariant.

with

$$C^{\star(1)} = \begin{pmatrix} c^{\star}_{11} & c^{\star}_{11} & 0 \end{pmatrix} \quad ; \quad f(C^{\star(1)}) = \begin{pmatrix} \sqrt{2}c^{\star}_{11} & 0 & -c^{\star}_{11} \\ 0 & -\sqrt{2}c^{\star}_{11} & -c^{\star}_{11} \end{pmatrix}$$

In this specific basis the number of components needed to define $\overline{\mathbf{M}}_{\mathrm{Z}_3}$ is decreased by one. But it should be observed that after being rotated the resulting matrix is still different from $\overline{\mathbf{M}}_{\mathrm{D}_3}$. Therefore the two symmetry classes are distinct.

Gyrotropic tensor. The AG matrices $\overline{\mathbf{M}}^{\sharp}$ have the following shapes:

$$\overline{\mathbf{M}}^{\sharp}_{\mathrm{Z}_3} = \begin{pmatrix} 0 & m^{\sharp}_{12} & \frac{\sqrt{2}}{2}m^{\sharp}_{12} & m^{\sharp}_{14} & 0 & -\frac{\sqrt{2}}{2}m^{\sharp}_{14} \\ -m^{\sharp}_{12} & 0 & \frac{\sqrt{2}}{2}m^{\sharp}_{12} & 0 & -m^{\sharp}_{14} & -\frac{\sqrt{2}}{2}m^{\sharp}_{14} \\ \frac{\sqrt{2}}{2}m^{\sharp}_{14} & \frac{\sqrt{2}}{2}m^{\sharp}_{14} & 0 & -\frac{\sqrt{2}}{2}m^{\sharp}_{12} & -\frac{\sqrt{2}}{2}m^{\sharp}_{12} & 0 \end{pmatrix}$$

$$\overline{\mathbf{M}}^{\sharp}_{\mathrm{D}_3} = \begin{pmatrix} 0 & m^{\sharp}_{12} & \frac{\sqrt{2}}{2}m^{\sharp}_{12} & 0 & 0 & 0 \\ -m^{\sharp}_{12} & 0 & \frac{\sqrt{2}}{2}m^{\sharp}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2}m^{\sharp}_{12} & -\frac{\sqrt{2}}{2}m^{\sharp}_{12} & 0 \end{pmatrix}$$

These components are related to those of $\overline{M}_{\mathrm{Z}^{\pi}_3}$ through the relations:

$$m^{\sharp}_{12} = -2m_{11} - \frac{\sqrt{2}}{2}(3m_{34} + m_{35}) \quad ; \quad m^{\sharp}_{14} = 2m_{24} + \frac{\sqrt{2}}{2}(3m_{31} + m_{32})$$

It should be noted that it is possible to find rotations that transform the $\overline{\mathbf{M}}^{\sharp}_{\mathrm{Z}_3}$ into $\overline{\mathbf{M}}^{\sharp}_{\mathrm{D}_3}$. Therefore if for $\mathbf{M}$ the classes $[\mathrm{Z}_3]$ and $[\mathrm{D}_3]$ are distinct, this is no longer the case for $\mathbf{M}^{\sharp}$. It means that if the constitutive tensor $\mathbf{M}$ is chiro-sensitive, the gyrotropic tensor $\mathbf{M}^{\sharp}$ is not.

3.3.4. Symmetry classes $[\mathrm{Z}_5]$ *and* $[\mathrm{D}_5]$.

Constitutive tensors. Whether a material is Z_5-invariant or D_5-invariant, the number of independent parameters in the matrix representation is 2 or 1. But as it will be shown, there exists only one symmetry class, the pentagonal one $[\mathrm{D}_5]$. The CE matrices $\overline{\mathbf{M}}_{\mathrm{Z}_5}$ and $\overline{\mathbf{M}}_{\mathrm{D}_5}$ for pentachiral and pentagonal material systems (see Figures 5 and 6) are given respectively by

$$\overline{\mathbf{M}}_{\mathrm{Z}_5} = \begin{pmatrix} m_{11} & -m_{11} & -\sqrt{2}m_{11} & -m_{24} & m_{24} & \sqrt{2}m_{24} \\ -m_{11} & m_{11} & \sqrt{2}m_{11} & m_{24} & -m_{24} & -\sqrt{2}m_{24} \\ \sqrt{2}m_{24} & -\sqrt{2}m_{24} & -2m_{24} & \sqrt{2}m_{11} & -\sqrt{2}m_{11} & -2m_{11} \end{pmatrix}$$

$$\overline{\mathbf{M}}_{\mathrm{D}_5} = \begin{pmatrix} m_{11} & -m_{11} & -\sqrt{2}m_{11} & 0 & 0 & 0 \\ -m_{11} & m_{11} & \sqrt{2}m_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}m_{11} & -\sqrt{2}m_{11} & -2m_{11} \end{pmatrix}$$

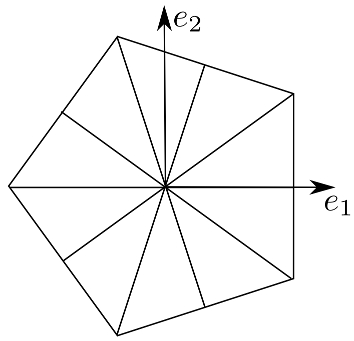


Figure 6: Pentagonal system (D_5-invariance): the material is $\frac{2\pi}{5}$-invariant and possesses 5 mirror lines.

And, using the block matrix notation:

$$\overline{\mathbf{M}}_{\mathrm{Z}_5} = \begin{pmatrix} A^{(1)} & B^{(1)} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ f(B^{(1)}) & f(A^{(1)}) \end{pmatrix} \quad ; \quad \overline{\mathbf{M}}_{\mathrm{D}_5} = \begin{pmatrix} A^{(1)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & f(A^{(1)}) \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} a_{11} & -a_{11} & -\sqrt{2}a_{11} \\ -a_{11} & a_{11} & \sqrt{2}a_{11} \end{pmatrix} \; ; \; B^{(1)} = \begin{pmatrix} -b_{11} & b_{11} & \sqrt{2}b_{11} \\ b_{11} & -b_{11} & -\sqrt{2}b_{11} \end{pmatrix}$$

$$f(A^{(1)}) = \begin{pmatrix} \sqrt{2}a_{11} & -\sqrt{2}a_{11} & -2a_{11} \end{pmatrix} \; ; \; f(B^{(1)}) = \begin{pmatrix} \sqrt{2}b_{11} & -\sqrt{2}b_{11} & -2b_{11} \end{pmatrix}$$

It is important to note that it is possible to find a rotation that reduces the number of coefficients of $\overline{\mathbf{M}}_{\mathrm{Z}_5}$. Under a rotation of angle θ solution of:

$$\tan 5\theta = \frac{-a_{11}}{b_{11}} \tag{20}$$

the former matrix is transformed into a new one:

$$\overline{\mathbf{M}}^{\star}_{\mathrm{Z}_5} = \begin{pmatrix} A^{\star(1)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & f(A^{\star(1)}) \end{pmatrix}$$

with $A^{\star(1)}$ having the same form as $A^{(1)}$. Therefore, after this rotation, $\overline{\mathbf{M}}^{\star}_{\mathrm{Z}_5} = \overline{\mathbf{M}}_{\mathrm{D}_5}$. Hence, as announced in Section 2.4, the symmetry class $[\mathrm{Z}_5]$ is empty.

Gyrotropic tensor. For these material symmetries, the gyrotropic tensor vanishes.

4. Complete 2D strain-gradient anisotropic systems

By combining the results of the previous section with previously published results (summarized in the appendices), the shapes of complete strain-gradient elasticity can be given for all the symmetry classes in 2D. To that aim, let us define the following space:

$$\mathbb{S}\mathrm{gr} = \{\mathcal{L} = (C_{ijkl}, M_{ijklm}, A_{ijklmn}) \in \mathbb{C} \times \mathbb{M} \times \mathbb{A}\}$$

which is the *complete space* of SGE. The symmetry group of $\mathcal{L}$ is defined as:

$$\mathrm{G}_{\mathcal{L}} = \mathrm{G}_{\mathbf{A}} \cap \mathrm{G}_{\mathbf{M}} \cap \mathrm{G}_{\mathbf{C}}$$

and, as for a single tensor, we can define the symmetry class of a linear law as:

$$[\mathrm{G}_{\mathcal{L}}] = \{\mathrm{G} \subseteq \mathrm{O}(2) | \mathrm{G} = \mathbf{Q}\mathrm{G}_{\mathcal{L}}\mathbf{Q}^T, \mathbf{Q} \in \mathrm{O}(2)\} \tag{21}$$

As the union of the symmetry classes for each tensor space of the constitutive law covers all the O(2)-subgroups allowed by the Hermann theorem (Auffray, 2008), there is no need to conduct a specific study to determinate the set of symmetry classes of $\mathbb{S}$gr. The number and type of symmetry classes are known for each tensor space of Mindlin strain-gradient elasticity.

Before stating the classification, let us recap some results:

• *Classical elasticity*: the classification has been done by He and Zheng (1996). Their results are synthesized in the following table:

Name	Digonal	Orthotropic	Tetragonal	Isotropic
$[G_{\mathbf{C}}]$	$[Z_2]$	$[D_2]$	$[D_4]$	$[O(2)]$
$\#_{\text{indep}}(\mathbf{C})$	6 (5)	4	3	2

Table 7: The names, the sets of subgroups $[G_{\mathbf{C}}]$ and the numbers of independent components $\#_{\text{indep}}(\mathbf{C})$ for the 4 symmetry classes of $\mathbb{C}$. The in-parenthesis number indicates the minimal number of components of the matrix in an appropriate basis.

• *Second-order elasticity*: the classification has been done by Auffray et al. (2009a), and are synthesized in the following table:

Name	Digonal	Orthotropic	Tetrachiral	Tetragonal
$[G_{\mathbf{A}}]$	$[Z_2]$	$[D_2]$	$[Z_4]$	$[D_4]$
$\#_{\text{indep}}(\mathbf{A})$	21 (20)	12	9 (8)	6
Name	Hexachiral	Hexagonal	Hemitropic	Isotropic
$[G_{\mathbf{A}}]$	$[Z_6]$	$[D_6]$	$[SO(2)]$	$[O(2)]$
$\#_{\text{indep}}(\mathbf{A})$	7 (6)	5	5	4

Table 8: The names, the sets of subgroups $[G_{\mathbf{A}}]$ and the numbers of independent components $\#_{\text{indep}}(\mathbf{A})$ for the 8 symmetry classes of $\mathbb{A}$. The in-parenthesis number indicates the minimal number of components of the matrix in an appropriate basis.

As a result we obtain 14 non equivalent symmetry classes, which are reported together with their number of independent components in the following table:

Name	Oblique	Rectangular	Digonal	Orthotropic	Trichiral	Trigonal	Tetrachiral	Tetragonal
$[G_{\mathcal{L}}]$	$[Id]$	$[Z_2^{\pi}]$	$[Z_2]$	$[D_2]$	$[Z_3]$	$[D_3]$	$[Z_4]$	$[D_4]$
$\#_{\text{indep}}(\mathcal{L})$	45 (44)	27	36 (35)	16	15 (14)	10	13 (12)	9
Name	Pentachiral	Pentagonal	Hexachiral	Hexagonal	Hemitropic	Isotropic		
$[G_{\mathcal{L}}]$	$[Z_5]$	$[D_5]$	$[Z_6]$	$[D_6]$	$[SO(2)]$	$[O(2)]$		
$\#_{\text{indep}}(\mathcal{L})$	9 (8)	7	9 (8)	7	7	6		

Table 9: The names, the sets of subgroups $[G_{\mathcal{L}}]$ and the numbers of independent components $\#_{\text{indep}}(\mathcal{L})$ for the 14 symmetry classes of $\mathcal{L}$. The in-parenthesis number indicates the minimal number of components of the law in an appropriate basis.

As a result, in each symmetry class, the constitutive law has the following synthetic form:

$$\mathcal{L}_{\mathrm{Id}} = \begin{pmatrix} \mathbf{C}_{\mathrm{Z}_2} & \mathbf{M}_{\mathrm{Id}} \\ \mathbf{M}^T_{\mathrm{Id}} & \mathbf{A}_{\mathrm{Z}_2} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_2^\pi} = \begin{pmatrix} \mathbf{C}_{\mathrm{Z}_2} & \mathbf{M}_{\mathrm{Z}_2^\pi} \\ \mathbf{M}^T_{\mathrm{Z}_2^\pi} & \mathbf{A}_{\mathrm{Z}_2} \end{pmatrix} \tag{22}$$

$$\mathcal{L}_{\mathrm{Z}_2} = \begin{pmatrix} \mathbf{C}_{\mathrm{Z}_2} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_2} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_2} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_2} & 0 \\ 0 & \mathbf{A}_{\mathrm{D}_2} \end{pmatrix} \tag{23}$$

$$\mathcal{L}_{\mathrm{Z}_3} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)} & \mathbf{M}_{\mathrm{Z}_3} \\ \mathbf{M}^T_{\mathrm{Z}_3} & \mathbf{A}_{\mathrm{Z}_6} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_3} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)} & \mathbf{M}_{\mathrm{D}_3} \\ \mathbf{M}^T_{\mathrm{D}_3} & \mathbf{A}_{\mathrm{D}_6} \end{pmatrix} \tag{24}$$

$$\mathcal{L}_{\mathrm{Z}_4} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_4} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_4} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4} & 0 \\ 0 & \mathbf{A}_{\mathrm{D}_4} \end{pmatrix} \tag{25}$$

$$\mathcal{L}_{\mathrm{Z}_5} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)} & \mathbf{M}_{\mathrm{D}_5} \\ \mathbf{M}^T_{\mathrm{D}_5} & \mathbf{A}_{\mathrm{SO}(2)} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_5} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)} & \mathbf{M}_{\mathrm{D}_5} \\ \mathbf{M}^T_{\mathrm{D}_5} & \mathbf{A}_{\mathrm{O}(2)} \end{pmatrix} \tag{26}$$

$$\mathcal{L}_{\mathrm{Z}_6} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_6} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_6} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)} & 0 \\ 0 & \mathbf{A}_{\mathrm{D}_6} \end{pmatrix} \tag{27}$$

$$\mathcal{L}_{\mathrm{SO}(2)} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)} & 0 \\ 0 & \mathbf{A}_{\mathrm{SO}(2)} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{O}(2)} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)} & 0 \\ 0 & \mathbf{A}_{\mathrm{O}(2)} \end{pmatrix} \tag{28}$$

In can be observed that among those 14 different classes, 8 of them are isotropic for classical elasticity. These classes will be refered to as *Cauchy-isotropic*:

$$\mathcal{I}_{\mathrm{so}} = \{[\mathrm{Z}_3], [\mathrm{D}_3], [\mathrm{Z}_5], [\mathrm{D}_5], [\mathrm{Z}_6], [\mathrm{D}_6], [\mathrm{SO}(2)], [\mathrm{O}(2)]\}$$

and among them the following 4 ones are chiro-sensitive

$$\mathcal{C}_{\mathrm{ir}} = \{[\mathrm{Z}_3], [\mathrm{Z}_5], [\mathrm{Z}_6], [\mathrm{SO}(2)]\}$$

in which only $[\mathrm{Z}_3], [\mathrm{Z}_6]$ are compatible with the crystallographic restriction. These different *Cauchy-isotropic* classes differ by the nature and the kind of second-order anisotropic coupling. Let us detail now the different kind of couplings that can be produced.

4.1. $\mathcal{S}$- and $\mathcal{O}$-type coupling

Auffray et al. (2009a) has pointed out that the sixth-order tensor $\mathbf{A}$ encodes some kind of chiral behavior. In Section 3, another type of coupling encoded by the fifth-order tensor $\mathbf{M}$ has been identified. These 2 couplings are distinct:

- **The fifth-order tensor couples first order and second order terms. This coupling of *order* ($\mathcal{O}$-type) is due to the lack of the central symmetry ($\overline{\mathcal{I}}$) that occurs for symmetry classes $[\mathrm{Z}_{2k+1}]$ and $[\mathrm{D}_{2k+1}]$. For these classes the stress and hyperstress equations are coupled. In other terms:**

  $$\frac{\partial \sigma}{\partial \eta} \neq 0 \quad \textbf{and} \quad \frac{\partial \tau}{\partial \varepsilon} \neq 0$$

  This coupling may exist both for chiral and achiral symmetry classes.

- Chiral coupling phenomena described by the sixth-order tensor are of *spatial type* ($\mathcal{S}$-type). This mechanism occurs for the symmetry classes $[\mathrm{Z}_k]$. In such cases a chiral coupling is created between the spatial directions and solely concerns second-order effects. Let consider $\mathcal{L}_{\mathrm{SO}(2)}$ which is the simplest example of this situation. For this class $\mathbf{M}$ is null, therefore

  $$\frac{\partial \sigma}{\partial \eta} = 0 \quad \text{and} \quad \frac{\partial \tau}{\partial \varepsilon} = 0$$

This means that first- and second-order elasticity are not coupled. Let us now consider the linear relation between τ and η in this particular case:

$$\begin{pmatrix} \tau_{111} \\ \tau_{221} \\ \sqrt{2}\tau_{122} \\ \tau_{222} \\ \tau_{112} \\ \sqrt{2}\tau_{121} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}}-a_{23} & 0 & a_{15} & -\frac{a_{15}}{\sqrt{2}} \\ & a_{22} & a_{23} & -a_{15} & 0 & -\frac{a_{15}}{\sqrt{2}} \\ & & \frac{a_{11}+a_{22}}{2}-a_{12} & \frac{a_{15}}{\sqrt{2}} & \frac{a_{15}}{\sqrt{2}} & 0 \\ & & & a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}}-a_{23} \\ & & & & a_{22} & a_{23} \\ & & & & & \frac{a_{11}+a_{22}}{2}-a_{12} \end{pmatrix} \begin{pmatrix} \eta_{111} \\ \eta_{221} \\ \sqrt{2}\eta_{122} \\ \eta_{222} \\ \eta_{112} \\ \sqrt{2}\eta_{121} \end{pmatrix}$$

The spatial coupling is encoded by the following antisymmetric matrix:

$$\begin{pmatrix} 0 & a_{15} & -\frac{a_{15}}{\sqrt{2}} \\ -a_{15} & 0 & -\frac{a_{15}}{\sqrt{2}} \\ \frac{a_{15}}{\sqrt{2}} & \frac{a_{15}}{\sqrt{2}} & 0 \end{pmatrix}$$

which disappears in the symmetry class $[\mathrm{O}(2)]$. This effect, which is present for all $[\mathrm{Z}_{2k}]$ symmetry classes, is a consequence of the absence mirror symmetries (Auffray et al., 2013).

Therefore the symmetry classes of SGE, can be split in 4 sets:

1. $\mathcal{SO}$: Constitutive laws belonging to this set present both spatial- and order-coupling. This set contains $[\mathrm{Z}_{2k+1}]$, and hence corresponds to $\overline{\mathcal{IM}}$ subgroup of $\mathrm{O}(2)$;
2. $\mathcal{S}$: Constitutive laws belonging to this set present spatial-coupling. This set contains $[\mathrm{Z}_{2k}]$ and $[\mathrm{SO}(2)]$, and hence corresponds to $\mathcal{I}\overline{\mathcal{M}}$ subgroup of $\mathrm{O}(2)$;
3. $\mathcal{O}$: Constitutive laws belonging to this set present order-coupling. This set contains $[\mathrm{D}_{2k+1}]$, and hence corresponds to $\overline{\mathcal{I}}\mathcal{M}$ subgroup of $\mathrm{O}(2)$;
4. $\mathcal{A}$: Constitutive laws belonging to this set are uncoupled. This set contains $[\mathrm{D}_{2k}]$ and $[\mathrm{O}(2)]$, and hence corresponds to $\mathcal{IM}$ subgroup of $\mathrm{O}(2)$.

This structure is summed-up in the following diagram:

$$\begin{tikzcd} \mathcal{SO} \arrow[r, "\sigma"] \arrow[d, "i"] & \mathcal{S} \arrow[d, "i"] \\ \mathcal{O} \arrow[r, "\sigma"] & \mathcal{A} \end{tikzcd}$$

in which σ denotes reflection and i the inversion.
Hence in 2D the chiral coupling is encoded by the second-order elasticity tensor, while non-centro-symetric coupling is encoded by the fifth-order tensor.

4.2. Discussion

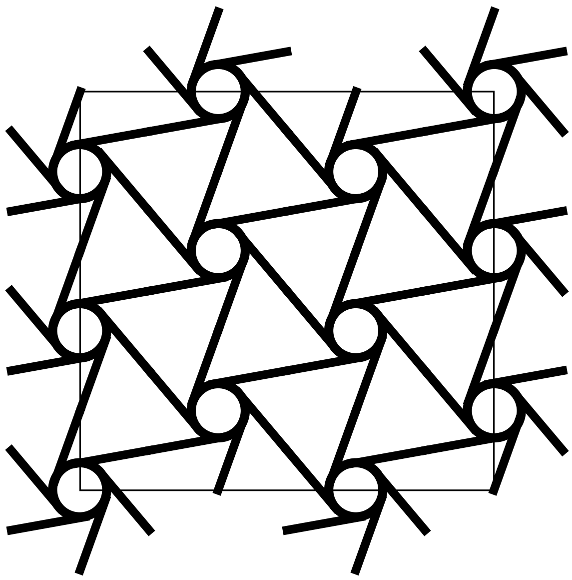
It is interesting to note that, in the literature devoted to chiral lattices (Spadoni et al., 2009; Liu et al., 2012; Spadoni and Ruzzene, 2012) and auxetic materials (Prall and Lakes, 1997; Dirrenberger et al., 2011), attention has only been focused on geometries that induce $\mathcal{S}$-type chiral coupling[8]. The main difficulty to explore $\mathcal{O}$-type chirality in 2D is that the associated rotational groups are not compatible with translational symmetry:

- if a material is both $[\mathrm{Z}_3]$ and invariant by translation, it is automatically $[\mathrm{Z}_6]$-invariant[9];

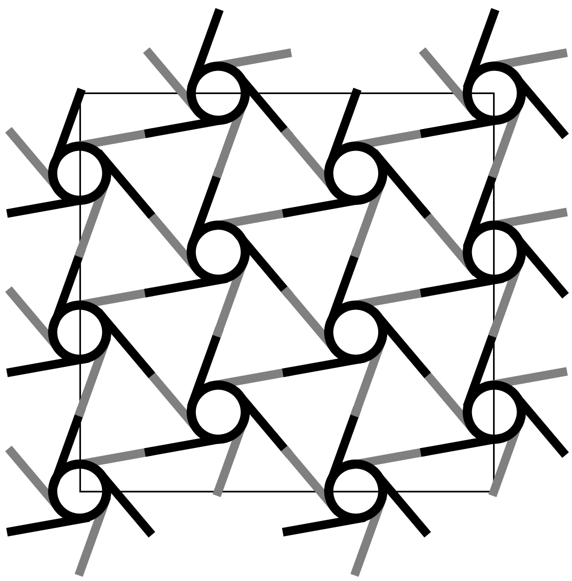
[8]Whereas in 3D, as the microstructure is generally neglected, attention is devoted to hemitropic SGE, i.e. to $\mathcal{O}$-type chiral coupling induced by the fifth-order tensor (Papanicolopulos, 2011).
[9]It is worth being noted that this symmetry may be broken down by using different materials in the tiling.

- a $[Z_5]$-invariance is not compatible with any translational invariance. It worth noting that this kind of rotational invariance can be found in quasi-crystallographic tilings such as the Penrose tilings.

Periodic tiling only present $\mathcal{S}$-type coupling. A well-known example is the hexachiral structure (Figure 7(a)). This tiling was proposed by Lakes (1991) and studied in Prall and Lakes (1997). Since then this material has bee studied by numerous authors (Spadoni et al., 2009; Liu et al., 2012; Dirrenberger et al., 2011, 2012, 2013; Bacigalupo and Gambarotta, 2014). The symmetry class of this pattern is $[Z_6]$, and hence, in the framework of SGE, there is no coupling between first- and second-order elasticity. This observation is in agreement with the one made by Spadoni and Ruzzene (2012) in the context of micropolar elasticity: first- and second-order elasticity are not coupled for hexachiral structures.



(a) Mono-material: hexachiral



(b) Bi-material: trichiral,

Figure 7: Mono- and bi- material patterns, which belong to the $[Z_6]$ and $[Z_3]$ respectively

However, if we consider a bi-material hexachiral pattern, i.e with ligaments made of a different material as in figure 7(b), this coupling between first- and second-order

elasticity is necessary. Indeed, this new material will now belong to the class $[\mathrm{Z}_3]$, for which the tensor M is not vanishing.
The framework presented in this paper is then particularly useful when considering chiral or periodic bi-material composites, whose higher order properties cannot be taken into account when using other formulations, e.g. Cauchy or Micropolar continua.

5. Conclusions

This paper completes some previous publications (Auffray et al., 2009a, 2010) on the description of anisotropic bidimensional strain-gradient elastic behavior. Two spaces of fifth-order tensors have been studied:

1. the space of coupling elasticity tensors involved in the constitutive law;
2. the space of gyroptropic tensors, responsible of the so-called *acoustic activity*.

In both cases, a complete set of anisotropic matrices has been provided. As a consequence, the anisotropic description of bidimensional strain-gradient elasticity is now complete. This behavior is divided into 14 non-equivalent anisotropic classes, 8 of them being isotropic for classical elasticity. We believe that those results will be useful for the continuous description of architectured material, and especially for the modeling of non-classical waves propagation.

Acknowledgements

The authors would like to gratefully acknowledge the *Fédération Francilienne de Mécanique* for financial support through its starting grant program. G. Rosi wishes to thank the *Faculté de Sciences et Technologie* of the *Université Paris-Est Créteil Val de Marne* for the financial support.

Appendix A. Matrix representations for C

$$C_{\mathrm{Z}_{(2)}} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ & c_{22} & c_{23} \\ & & c_{33} \end{pmatrix}, \quad C_{\mathrm{D}_{(2)}} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ & c_{22} & 0 \\ & & c_{33} \end{pmatrix} \tag{A.1}$$

$$C_{\mathrm{Z}_{(4)}} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ & c_{11} & -c_{13} \\ & & c_{33} \end{pmatrix}, \quad C_{\mathrm{D}_{(4)}} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ & c_{11} & 0 \\ & & c_{33} \end{pmatrix} \tag{A.2}$$

$$C_{\mathrm{O}(2)} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ & c_{11} & 0 \\ & & c_{11} - c_{12} \end{pmatrix} \tag{A.3}$$

Appendix B. Matrix representations for A

$$A_{\mathrm{Z}_{(2)}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{pmatrix}, \quad A_{\mathrm{D}_{(2)}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{44} & a_{45} & a_{46} \\ & & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{pmatrix} \tag{B.1}$$

$$A_{\mathrm{Z}_{(4)}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & a_{15} & a_{16} \\ & a_{22} & a_{23} & -a_{15} & 0 & a_{26} \\ & & a_{33} & -a_{16} & -a_{26} & 0 \\ & & & a_{11} & a_{12} & a_{13} \\ & & & & a_{22} & a_{23} \\ & & & & & b_{55} \end{pmatrix}, \quad A_{\mathrm{D}_{(4)}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{11} & a_{12} & a_{13} \\ & & & & a_{22} & a_{23} \\ & & & & & b_{55} \end{pmatrix} \tag{B.2}$$

$$A_{\mathrm{Z}_6} = \begin{pmatrix} a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}}-a_{23} & 0 & a_{15} & -\frac{a_{15}}{\sqrt{2}} \\ & a_{22} & a_{23} & -a_{15} & 0 & -\frac{a_{15}}{\sqrt{2}} \\ & & \frac{a_{11}+a_{22}}{2}-a_{12} & \frac{a_{15}}{\sqrt{2}} & \frac{a_{15}}{\sqrt{2}} & 0 \\ & & & a_{44} & a_{11}-a_{44}+a_{12} & \frac{3a_{11}-a_{22}}{\sqrt{2}}-a_{23}-\sqrt{2}a_{44} \\ & & & & a_{22}+a_{44}-a_{11} & \sqrt{2}(a_{44}-a_{11})+a_{23} \\ & & & & & \frac{-3a_{11}+a_{22}}{2}-a_{12}+2a_{44} \end{pmatrix} \tag{B.3}$$

$$A_{\mathrm{D}_6} = \begin{pmatrix} a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}}-a_{23} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & \frac{a_{11}+a_{22}}{2}-a_{12} & 0 & 0 & 0 \\ & & & a_{44} & a_{11}-a_{44}+a_{12} & \frac{3a_{11}-a_{22}}{\sqrt{2}}-a_{23}-\sqrt{2}a_{44} \\ & & & & a_{22}+a_{44}-a_{11} & \sqrt{2}(a_{44}-a_{11})+a_{23} \\ & & & & & \frac{-3a_{11}+a_{22}}{2}-a_{12}+2a_{44} \end{pmatrix} \tag{B.4}$$

$$A_{\mathrm{SO}(2)} = \begin{pmatrix} a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}}-a_{23} & 0 & a_{15} & -\frac{a_{15}}{\sqrt{2}} \\ & a_{22} & a_{23} & -a_{15} & 0 & -\frac{a_{15}}{\sqrt{2}} \\ & & \frac{a_{11}+a_{22}}{2}-a_{12} & \frac{a_{15}}{\sqrt{2}} & \frac{a_{15}}{\sqrt{2}} & 0 \\ & & & a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}}-a_{23} \\ & & & & a_{22} & a_{23} \\ & & & & & \frac{a_{11}+a_{22}}{2}-a_{12} \end{pmatrix} \tag{B.5}$$

$$A_{\mathrm{O}(2)} = \begin{pmatrix} a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}}-a_{23} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & \frac{a_{11}+a_{22}}{2}-a_{12} & 0 & 0 & 0 \\ & & & a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}}-a_{23} \\ & & & & a_{22} & a_{23} \\ & & & & & \frac{a_{11}+a_{22}}{2}-a_{12} \end{pmatrix} \tag{B.6}$$

References

Armstrong M.A. (1983). *Basic topology.* Springer.

Askes H. and Aifantis E. C. (2011). *Gradient elasticity in statics and dynamics: an overview of formulations, length scaleidentification procedures, finite element implementations and new results.* Int. J. Solids Struct., vol. 48 n° 13, pp 1962–1990.

Auffray N. (2008). *Démonstration du théorème d'Hermann à partir de la méthode Forte-Vianello.* CR Mécanique, vol. 336 n° 5, pp 458–463.

Auffray N. (2014). *Analytical expressions for odd-order anisotropic tensor dimension.* CR Mécanique, vol. 342 n° 5, pp 284–291.

Auffray N., Bouchet R., and Bréchet Y. (2009a). *Derivation of anisotropic matrix for bi-dimensional strain-gradient elasticity behavior.* Int. J. Solids Struct., vol. 46 n° 2, pp 440–454.

Auffray N., Bouchet R., and Bréchet Y. (2009b). *Mechanics of Microstructured Solids*, vol. 46, chapter Class-Jump Phenomenon for Physical Symmetries in Bi-dimensional Space, pp 1–11. Springer Berlin Heidelberg.

Auffray N., Bouchet R., and Bréchet Y. (2010). *Strain gradient elastic homogenization of bidimensional cellular media.* Int. J. Solids Struct., vol. 47 n° 13, pp 1698–1710.

Auffray N., Kolev B., and Olive M. (2015). *Handbook of bidimensional tensors. Part I : Decomposition and symmetry classes.* to be submitted, vol. .

Auffray N., Le Quang H., and He Q.C. (2013). *Matrix representations for 3D strain-gradient elasticity.* J. Mech. Phys. Solids, vol. 61 n° 5, pp 1202–1223.

Bacigalupo A. and Gambarotta L. (2014). *Second-gradient homogenized model for wave propagation in heterogeneous periodic media.* Int. J. Solids Struct., vol. 51 n° 5, pp 1052–1065.

Ben-Amoz M (1976). *A dynamic theory for composite materials.* Z. Angew. Math. Phys., vol. 27 n° 1, pp 83–99.

Bhagwat K. V., Wadhawan V. K., and Subramanian R. (1986). *A new fourth-rank tensor for describing the acoustical activity of crystals.* J. Phys. C, vol. 19 n° 3, pp 345.

Boutin C. (1996). *Microstructural effects in elastic composites.* Int. J. Solids Struct., vol. 33 n° 7, pp 1023–1051.

Cosserat E. and Cosserat F. (1909). *Théorie des corps déformables.* Hermann.

dell'Isola F., Placidi L., and Madeo A. (2011). *Linear plane wave propagation and normal transmission and reflection at discontinuity surfaces in second gradient 3D continua.* Z. Angew. Math. Mech., vol. 92 n° 1, pp 52–71.

Dirrenberger J., Forest S., and Jeulin D. (2012). *Elastoplasticity of auxetic materials.* Comp. Mater. Sci., vol. 64, pp 57–61.

Dirrenberger J., Forest S., and Jeulin D. (2013). *Effective elastic properties of auxetic microstructures: anisotropy and structural applications.* Int. J. Mech. Mater. Des., vol. 9 n° 1, pp 21–33.

Dirrenberger J., Forest S., Jeulin D., and Colin C. (2011). *Homogenization of periodic auxetic materials.* Procedia Eng., vol. 10, pp 1847–1852.

DiVincenzo D.P. (1986). *Dispersive corrections to continuum elastic theory in cubic crystals.* Phys. Rev. B, vol. 34 n° 8, pp 5450.

Dresselhaus M., Dresselhaus G., and Jorio A. (2008). *Group Theory, Application to the Physics of Condensed Matter.* Springer-Verlag.

Erigen A. C. (1967). *Theory of micropolar elasticity.* In : Fracture, vol. 2., ed. Leibowitz H., pp 621–629. Academic Press, New York.

Forte S. and Vianello M. (1996). *Symmetry classes for elasticity tensors.* J. Elasticity, vol. 43 n° 2, pp 81–108.

Germain P (1973). *The method of virtual power in continuum mechanics. Part 2: Microstructure.* SIAM J. Appl. Math., vol. 25 n° 3, pp 556–575.

Green A.E. and Rivlin R.S. (1964). *Multipolar continuum mechanics.* Arch. Ration. Mech. An., vol. 17 n° 2, pp 113–147.

He Q. C. and Zheng Q. S. (1996). *On the symmetries of 2D elastic and hyperelastic tensors.* J. Elasticity, vol. 43 n° 3, pp 203–225.

Jensen J. S. and Sigmund O. (2004). *Systematic design of photonic crystal structures using topology optimization: Low-loss waveguide bends.* Appl. Phys. Lett., vol. 84 n° 12, pp 2022–2024.

Kittel (2007). *Introduction to solid state physics.* John Wiley & Sons.

Lakes R. (1991). *Deformation mechanisms in negative Poisson's ratio materials: structural aspects.* J. Mater. Sci., vol. 26 n° 9, pp 2287–2292.

Liu X.N., Hu G.K., Sun C.T., and Huang G.L. (2011). *Wave propagation characterization and design of two-dimensional elastic chiral metacomposite.* J. Sound Vib., vol. 11 n° 330, pp 2536–2553.

Liu X.N., Huang G.L., and Hu G.K. (2012). *Chiral effect in plane isotropic micropolar elasticity and its application to chiral lattices.* J. Mech. Phys. Solids, vol. 60 n° 11, pp 1907–1921.

Mehrabadi M.M. and Cowin S.C. (1990). *Eigentensors of linear anisotropic elastic materials.* Quart. J. Mech. Appl. Math., vol. 43 n° 1, pp 15–41.

Mindlin R.D. (1964). *Micro-structure in linear elasticity.* Arch. Ration. Mech. An., vol. 16 n° 1.

Mindlin R.D. (1965). *Second gradient of strain and surface-tension in linear elasticity.* Int. J. Solids Struct., vol. 1 n° 4, pp 417–438.

Mindlin R.D. and Eshel N.N. (1968). *On first strain-gradient theories in linear elasticity.* Int. J. Solids Struct., vol. 4 n° 1, pp 109–124.

Norris A. N. (1989). *On the acoustic determination of the elastic moduli of anisotropic solids and acoustic conditions for the existence of symmetry planes.* Q. J. Mech. Appl. Math., vol. 42, pp 413–426.

Olive M. and Auffray N. (2013). *Symmetry classes for even-order tensors.* Math. Mech. Comp. Systems, vol. 1, pp 177–210.

Olive M. and Auffray N. (2014). *Symmetry classes for odd-order tensors.* Z. Angew. Math. Mech., vol. 94 n° 5, pp 421–447.

Papanicolopulos S.-A. (2011). *Chirality in isotropic linear gradient elasticity.* Int. J. Solids Struct., vol. 48 n° 5, pp 745–752.

Phani A. S., Woodhouse J., and Fleck N. A. (2006). *Wave propagation in two-dimensional periodic lattices.* J. Acoust. Soc. Am., vol. 119 n° 4, pp 1995–2005.

Portigal D. L. and Burstein E. (1968). *Acoustical activity and other first-order spatial dispersion effects in crystals.* Phys. Rev., vol. 170 n° 3, pp 673.

Prall D. and Lakes R.S. (1997). *Properties of a chiral honeycomb with a poisson's ratio of -1.* Int. J. Mech. Sci., vol. 39 n° 3, pp 305–314.

Rosi G. and Auffray N. (2015). *Wave propagation within strain-gradient framework.* To be submitted, vol. .

Rosi G., Nguyen V.-H., and Naili S. (2014). *Reflection of acoustic wave at the interface of a fluid-loaded dipolargradient elastic half-space.* Mech. Res. Commun., vol. 56, pp 98–103.

Royer D. and Dieulesaint E. (2000). *Elastic Waves in Solids II.* Springer.

Spadoni A. and Ruzzene M. (2012). *Elasto-static micropolar behavior of a chiral auxetic lattice.* J. Mech. Phys. Solids, vol. 60 n° 1, pp 156–171.

Spadoni A., Ruzzene M., Gonella S., and Scarpa F. (2009). *Phononic properties of hexagonal chiral lattices.* Wave Motion, vol. 46 n° 7, pp 435–450.

Srinivasan T. P. (1988). *A description of acoustical activity using irreducible tensors.* J. Phys. C, vol. 21, pp 4207–4219.

Toupin R. A. (1962). *Elastic materials with couple-stresses.* Arch. Rat. Mech. An., vol. 11 n° 1, pp 385–414.

Triantafyllidis N. and Bardenhagen S. (1996). *Elastic materials with couple-stresses.* J. Mech. Phys. Solids, vol. 44 n° 11, pp 1891–1928.

Vannucci P. (2007). *The polar analysis of a third order piezoelectricity-like plane tensor.* Int. J. Solids Struct., vol. 44 n° 24, pp 7803–7815.

Vianello M. (1997). *An integrity basis for plane elasticity tensors.* Arch. Mech., vol. 49, pp 197–208.

Wang Z. P. and Sun C. T. (2002). *Modeling micro-inertia in heterogeneous materials under dynamic loading.* Wave Motion, vol. 36, pp 473–485.

Wolfe J. P. (2005). *Imaging phonons: acoustic wave propagation in solids.* Cambridge University Press.

Zheng Q.-S. and Boehler J. P. (1994). *The description, classification, and reality of material and physical symmetries.* Acta Mech., vol. 102 n° 1-4, pp 73–89.