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André Kabakouala
L.M.P.T., U.F.R Sciences et Techniques, Université de Tours, Parc Grandmont, 37200 Tours, France.

Andre.Kabakouala@lmpt.univ-tours.fr

Abstract

In this paper, we present a new argument (see Lemma 3.4) that allows us to simplify the proof of stability of peakons established in Lin and Liu (2009) (Theorem 1.1).

1 Introduction

In this paper, we consider the Degasperis-Procesi equation (DP)

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

with $u(0) = u_0 \in L^2(\mathbb{R})$ and $(1-\partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$.

The DP equation is completely integrable (see [3]) and has been proved to be physically relevant for water waves (see [1]). It possesses, among others, the following conservation laws

$$E(u) = \int_{\mathbb{R}} yv = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right), \quad F(u) = \int_{\mathbb{R}} u^3 = \int_{\mathbb{R}} \left(-v_{xx}^3 + 12vv_{xx}^2 - 48v^2 v_{xx} + 64v^3\right), \tag{1.2}$$

where $y = (1-\partial_x^2)u$ and $v = (4-\partial_x^2)^{-1}u$. One can notice that the conservation law $E(\cdot)$ is equivalent to $\|\cdot\|_{L^2(\mathbb{R})}^2$. Indeed, using integration by parts (we assume that $u(\pm\infty) = v(\pm\infty) = v_x(\pm\infty) = 0$), it holds

$$\|u\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u^2 = \int_{\mathbb{R}} (4v - v_{xx})^2 = \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right) \sim E(u). \tag{1.3}$$

In the sequel we will denote

$$\|u\|_{\mathcal{H}} = \sqrt{E(u)}. \tag{1.4}$$

Applying $(1-\partial_x^2)^{-1}(\cdot)$ to (1.1), we obtain

$$u_t + \frac{1}{2}\partial_x u^2 + \frac{3}{2}(1-\partial_x^2)^{-1}\partial_x u^2 = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{1.5}$$

In this form, the DP equation admits explicit solitary waves called *peakons* (see [3]) that are defined by

$$u(t,x) = \varphi_c(x-ct) = c\varphi(x-ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}^*, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{1.6}$$

Our goal is to simplify the proof given in [7] of the stability of a single peakon for the DP equation. Recall that the proof of the stability for the Camassa-Holm equation (CH) in [2] follows from two integral relations between two conservation laws of CH, $\max_{\mathbb{R}} u$ and functions related to u. In [7] the proof is more complicated, since all the local maxima and minima of $v = (4-\partial_x^2)^{-1}u$ are involved in the relations. In this paper we present a simplification of this proof, where only the maximum of v is involved in the relations. Our proof is thus closer to the proof for CH in [2]. The main idea is the following: since u is L^2-close to the peakon $\varphi_c(\cdot - \xi)$, for some $\xi \in \mathbb{R}$, and $(1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, it is easy to check that u is actually C^0-close to the peakon, and thus v is C^2-close to the *smooth-peakon*:

$$\rho_c(x-\xi) = (4-\partial_x^2)^{-1}\varphi_c(x-\xi) = \frac{c}{3}e^{-|x-\xi|} - \frac{c}{6}e^{-2|x-\xi|}, \quad x \in \mathbb{R}. \tag{1.7}$$

First, since ρ_c, ρ_c' and ρ_c'' are very small with respect to the amplitude c outside of the interval $\Theta_0 = [-6.7, 6.7]$, we can restrict ourself to study v on $\Theta_\xi = [\xi - 6.7, \xi + 6.7]$. Now we observe that ρ_c'' has strictly negative values in the interval $\mathcal{V}_0 = [-\ln\sqrt{2}, \ln\sqrt{2}]$, with ρ_c' strictly positive on $[-6.7, -\ln\sqrt{2}]$ and ρ_c' strictly negative on $[\ln\sqrt{2}, 6.7]$. This forces v_x to change sign only one time on Θ_ξ, and thus v has only one local extremum (which is a maximum) on Θ_ξ. This fact will considerably simplify the proof of the stability.

2 Preliminaries

In this section, we briefly recall the global well-posedness results for the Cauchy problem of the DP equation (see [5] and [8]), and its consequences. For I a finite or infinite time interval of $\mathbb{R}_+$, we denote by $\mathcal{X}(I)$ the function space [1]

$$\mathcal{X}(I) = \left\{u \in C\left(I; H^1(\mathbb{R})\right) \cap L^\infty\left(I; W^{1,1}(\mathbb{R})\right),\ u_x \in L^\infty\left(I; BV(\mathbb{R})\right)\right\}. \tag{2.1}$$

Theorem 2.1 (Global Weak Solution; See [5] and [8])**.** *Assume that $u_0 \in L^2(\mathbb{R})$ with $y_0 = (1-\partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$. Then the DP equation has a unique global weak solution $u \in \mathcal{X}(\mathbb{R}_+)$ such that*

$$y(t,\cdot) = (1-\partial_x^2)u(t,\cdot) \in \mathcal{M}^+(\mathbb{R}), \quad \forall t \in \mathbb{R}_+ \tag{2.2}$$

and

$$|u_x(t,x)| \le u(t,x), \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.3}$$

Moreover $E(\cdot)$ and $F(\cdot)$ are conserved by the flow.

Remark 2.1 (Control of L^∞ Norm by L^2 Norm)**.** (2.3) and the well-known Sobolev embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ lead to

$$\|u\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}}\|u\|_{H^1(\mathbb{R})} \le \|u\|_{L^2(\mathbb{R})}. \tag{2.4}$$

3 Stability of peakons

In this section, we present our simplification of the proof of stability of peakons for the DP equation.

Theorem 3.1 (Stability of Peakons)**.** *Let $u \in \mathcal{X}([0,T[)$, with $0 < T \le +\infty$, be a solution of the DP equation and φ_c be the peakon defined in (1.6), traveling to the right at the speed $c > 0$. There exist $C > 0$ and $\varepsilon_0 > 0$ only depending on the speed c, such that if*

$$y_0 = (1-\partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \tag{3.1}$$

[1] $W^{1,1}(\mathbb{R})$ is the space of $L^1(\mathbb{R})$ functions with derivatives in $L^1(\mathbb{R})$ and $BV(\mathbb{R})$ is the space of function with bounded variation.

and

$$\|u_0 - \varphi_c\|_{\mathcal{H}} \le \varepsilon^2, \quad \text{with} \ \ 0 < \varepsilon < \varepsilon_0, \tag{3.2}$$

then

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{\mathcal{H}} \le C\sqrt{\varepsilon}, \quad \forall t \in [0, T[, \tag{3.3}$$

where $\xi(t) \in \mathbb{R}$ is the only point where the function $v(t,\cdot) = (4 - \partial_x^2)^{-1} u(t,\cdot)$ attains its maximum.

We first recall that $E(u) \sim E(\varphi_c)$ and $F(u) \sim F(\varphi_c)$ in $\mathbb{R}$, if $u \sim \varphi_c$ in $L^2(\mathbb{R})$, with $y \in \mathcal{M}^+(\mathbb{R})$ (see for instance [7] or [6]).

Lemma 3.1 (Control of Distances Between Energies; See [6])**.** *Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. If $\|u - \varphi_c\|_{\mathcal{H}} \le \varepsilon^2$, then*

$$|E(u) - E(\varphi_c)| \le O(\varepsilon^2) \tag{3.4}$$

and

$$|F(u) - F(\varphi_c)| \le O(\varepsilon^2), \tag{3.5}$$

where $O(\cdot)$ only depends on the speed c.

To prove Theorem 3.1, by the conservation of $E(\cdot)$, $F(\cdot)$ and the continuity of the map $t \mapsto u(t)$ from $[0, T[$ to $\mathcal{H}$ (since $\mathcal{H} \simeq L^2$), it suffices to prove that for any function $u \in L^2(\mathbb{R})$ satisfying $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, (3.4) and (3.5), if

$$\inf_{z \in \mathbb{R}} \|u - \varphi_c(\cdot - z)\|_{\mathcal{H}} \le \varepsilon^{1/4}, \tag{3.6}$$

then

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \le C\sqrt{\varepsilon}, \tag{3.7}$$

where $\xi \in \mathbb{R}$ is the only point of maximum of v.

Let us present some important properties of smooth-peakons, defined in (1.7), which will play a crucial role in the proof of Theorem 3.1. The smooth-peakon ρ_c belongs to $H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$ (by the Sobolev embedding) since φ_c belongs to $H^1(\mathbb{R})$ (defined in (1.6)). It is a positive even function, which admits a single maximum $c/6$ at point 0, and decays at infinity to 0 (see Fig. 1a). Its derivative ρ_c' belongs to $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$, it is an odd function, which vanishes only at the origin and has negative values on $[0, +\infty[$. It admits a a single minimum $-c/12$ at point ln2 and tends at infinity to 0 (see Fig. 1b). Its second derivative ρ_c'' belongs to $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$, it is an even function, which vanishes at $\pm\ln 2$, takes positive values on $]-\infty, -\ln 2[\cup]\ln 2, +\infty[$ and negative values on $[-\ln 2, \ln 2]$. It admits a single minimum $-c/3$ at point 0 and two maxima $c/24$ at points $\pm\ln 4$, and decays at infinity to 0 (see Fig. 1c).

Next, we will need the following estimates.

Lemma 3.2 (C^0, C^1 and C^2 Approximations)**.** *Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. If $\|u - \varphi_c\|_{\mathcal{H}} \le \varepsilon^{1/4}$, then*

$$\|u - \varphi_c\|_{C^0(\mathbb{R})} + \|v - \rho_c\|_{C^2(\mathbb{R})} \le O(\varepsilon^{1/8}) \tag{3.8}$$

and

$$\|v - \rho_c\|_{C^1(\mathbb{R})} \le O(\varepsilon^{1/4}). \tag{3.9}$$

Proof. Let us begin with the second estimate. From the definition of $E(\cdot)$ and $\mathcal{H}$ (see respectively (1.2) and (1.4)), one can see that $\|u\|_{\mathcal{H}}$ is equivalent to $\|v\|_{H^2(\mathbb{R})}$, since $\|v\|_{H^2(\mathbb{R})} \le \|u\|_{\mathcal{H}} \le 5\|v\|_{H^2(\mathbb{R})}$. Then, assumption u is $\mathcal{H}$-close to φ_c implies that v is H^2-close to ρ_c. Now, using the Sobolev embedding of $H^2(\mathbb{R})$ into $C^1(\mathbb{R})$, we deduce (3.9).

For the first estimate, note that the assumption $y = (1 - \partial_x^2)u \ge 0$ implies that $u = (1 - \partial_x^2)^{-1} y \ge 0$ and satisfies $|u_x| \le u$ on $\mathbb{R}$ (see (2.3)). Then, applying triangular inequality, and using that $|\varphi_c'| = \varphi_c$ on

$\mathbb{R}$ and (2.4), we have

$$\begin{aligned}\|u-\varphi_c\|_{H^1(\mathbb{R})} &\le \|u\|_{H^1(\mathbb{R})}+\|\varphi_c\|_{H^1(\mathbb{R})}\\ &\le 2\|u\|_{L^2(\mathbb{R})}+2\|\varphi_c\|_{L^2(\mathbb{R})}\\ &\le 2\|u-\varphi_c\|_{L^2(\mathbb{R})}+4\|\varphi_c\|_{L^2(\mathbb{R})}\\ &\le O(\varepsilon^{1/4})+O(1),\end{aligned}$$

where $\|\varphi_c\|_{L^2(\mathbb{R})}=c$. Therefore, applying the Gagliardo-Nirenberg inequality and using that $\|u-\varphi_c\|_{\mathcal{H}}\le \varepsilon^{1/4}$ (with $\mathcal{H}\simeq L^2$), we obtain

$$\begin{aligned}\|u-\varphi_c\|_{C^0(\mathbb{R})} &\le \|u-\varphi_c\|^{1/2}_{L^2(\mathbb{R})}\|u-\varphi_c\|^{1/2}_{H^1(\mathbb{R})}\\ &\le O(\varepsilon^{1/8})\Big(O(\varepsilon^{1/8})+O(1)\Big)\\ &\le O(\varepsilon^{1/8})\ .\end{aligned}$$

Finally to estimate the second term of the left-hand side of (3.8), we first notice that the continuity of $(4-\partial_x^2)^{-1}(\cdot)$ from $H^s(\mathbb{R})$ to $H^{s+2}(\mathbb{R})$ and the above estimates ensure that $\|v-\rho_c\|_{H^3}=O(1)$ and $\|v-\rho_c\|_{H^2}=O(\varepsilon^{1/4})$. These last estimates combined with the Gagliardo-Nirenberg inequality yield the result as above. □

The following lemma specifies the distance to minimize for stability.

Lemma 3.3 (Quadratic Identity; See [7]). *For any $u\in L^2(\mathbb{R})$ and $\xi\in\mathbb{R}$, it holds*

$$E(u)-E(\varphi_c)=\|u-\varphi_c(\cdot-\xi)\|^2_{\mathcal{H}}+4c\left(v(\xi)-\frac{c}{6}\right), \tag{3.10}$$

where $v=(4-\partial_x^2)^{-1}u$ and $\rho_c(0)=c/6$.

Proof. We follow the idea of Constantin and Strauss with the CH equation (see [2], Lemma 1). We compute

$$\begin{aligned}E(u-\varphi_c(\cdot-\xi)) &= E(u)+E(\varphi_c)-2\left\langle(1-\partial_x^2)\varphi_c(\cdot-\xi),(4-\partial_x^2)^{-1}u\right\rangle_{H^{-1},H^1}\\ &= E(u)+E(\varphi_c)-2\left\langle(1-\partial_x^2)\varphi_c(\cdot-\xi),v\right\rangle_{H^{-1},H^1},\end{aligned} \tag{3.11}$$

where $\langle\cdot,\cdot\rangle_{H^{-1},H^1}$ denotes the duality bracket $H^{-1}(\mathbb{R})$, $H^1(\mathbb{R})$. Now, using the definition of $\varphi_c'(\cdot-\xi)$ and integration by parts, we have

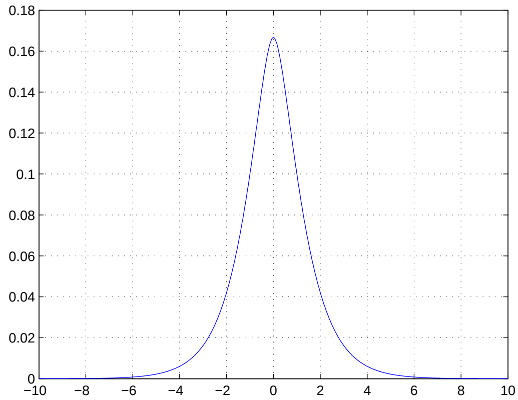
$$\begin{aligned}\left\langle(1-\partial_x^2)\varphi_c(\cdot-\xi),v\right\rangle_{H^{-1},H^1} &= \int_{\mathbb{R}} v\varphi_c(\cdot-\xi)+\int_{\mathbb{R}} v_x\varphi_c'(\cdot-\xi)\\ &= \int_{\mathbb{R}} v\varphi_c(\cdot-\xi)+\int_{-\infty}^{\xi} v_x\varphi_c(\cdot-\xi)-\int_{\xi}^{+\infty} v_x\varphi_c(\cdot-\xi)\\ &= 2cv(\xi).\end{aligned} \tag{3.12}$$

Recalling that the energy of peakons is given by

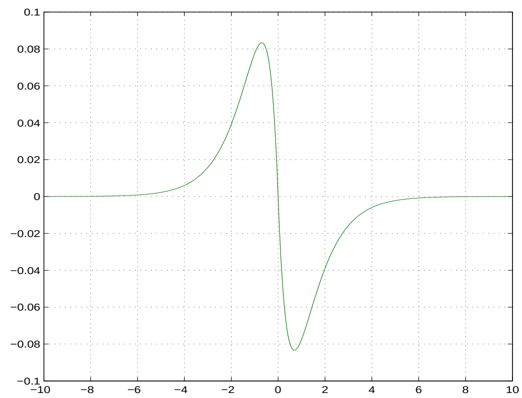
$$\begin{aligned}E(\varphi_c)=\left\langle(1-\partial_x^2)\varphi_c,(4-\partial_x^2)^{-1}\varphi_c\right\rangle_{H^{-1},H^1} &= \int_{\mathbb{R}}\rho_c\varphi_c+\int_{\mathbb{R}}\rho_c'\varphi_c'\\ &= \int_{\mathbb{R}}\rho_c\varphi_c+\int_{-\infty}^{0}\rho_c'\varphi_c-\int_0^{+\infty}\rho_c'\varphi_c=2c\rho_c(0)=\frac{c^2}{3}\ ,\end{aligned} \tag{3.13}$$

we obtain the lemma. □

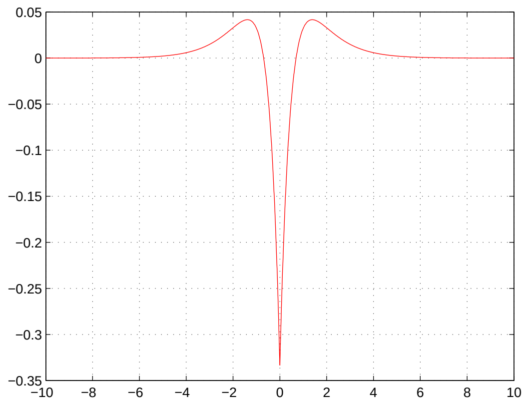
Now we will study carefully the local extrema of $v=(4-\partial_x^2)^{-1}u$. Let $u\in L^2(\mathbb{R})$ with $y=(1-\partial_x^2)u\in\mathcal{M}^+(\mathbb{R})$, and assume that (3.6) holds for some $z\in\mathbb{R}$. We consider the interval in which the mass of



(a) $\rho(x) = \frac{1}{3}e^{-|x|} - \frac{1}{6}e^{-2|x|}$ profile.



(b) $\rho'(x) = \left(\frac{1}{3}e^{-|x|} - \frac{1}{3}e^{-2|x|}\right)_{x<0} + \left(\frac{1}{3}e^{-2|x|} - \frac{1}{3}e^{-|x|}\right)_{x>0}$ profile.



(c) $\rho''(x) = \frac{1}{3}e^{-|x|} - \frac{2}{3}e^{-2|x|}$ profile

Figure 1: Variation of the smooth-peakon with the amplitude 1/6 at initial time.

smooth-peakons is concentrated, and the interval in which the mass of second derivative of smooth-peakons is strictly negative. In the sequel of this paper, the notation $\alpha \simeq \beta$ means that $0.9 \times \beta \leq \alpha \leq 1.1 \times \beta$. We set, for any $z \in \mathbb{R}$,

$$\Theta_z = [z - 6.7, z + 6.7], \quad \text{where} \;\; 6.7 \simeq \ln\left(\frac{20}{20 - \sqrt{399}}\right), \tag{3.14}$$

and

$$\mathcal{V}_z = \left[z - \ln\sqrt{2}, z + \ln\sqrt{2}\right]. \tag{3.15}$$

One can clearly see that $\mathcal{V}_0$ is a subset of Θ_0 (since $20/(20 - \sqrt{399}) > \sqrt{2}$). We chose the values ± 6.7 such that $\rho_c(\pm 6.7) \simeq c/2400 \simeq 4.1 \times 10^{-4}c$ as in [6]. Also, we have $\rho_c'(-6.7) = -\rho_c'(6.7) \simeq 4.1 \times 10^{-4}c$ and $\rho_c''(\pm 6.7) \simeq 4.1 \times 10^{-4}c$. Then ρ_c, ρ_c' and ρ_c'' are very small with respect to the amplitude c on $\mathbb{R} \setminus \Theta_0$.

We claim the following result.

Lemma 3.4 (Uniqueness of the Local Maximum)**.** *Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies* (3.6) *for some $z \in \mathbb{R}$. There exists $\varepsilon_0 > 0$ only depending on the speed c, such that if $0 < \varepsilon < \varepsilon_0$,* *then the function $v = (4 - \partial_x^2)^{-1}u$ admits a unique local extremum on Θ_z. This extremum is a maximum, and it holds*

$$v(x) \le \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus \Theta_z, \tag{3.16}$$

$$u(x) \le \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus \Theta_z. \tag{3.17}$$

Proof. The key is to study the impact of the assumption $y \in \mathcal{M}^+(\mathbb{R})$ on v. First, let us show that $|v_x| \le 2v$ on $\mathbb{R}$. We recall that from the assumption $y \ge 0$, we have $u \ge 0$ and $v \ge 0$ on $\mathbb{R}$. According to the definition of v, we have for all $x \in \mathbb{R}$,

$$v(x) = \frac{e^{-2x}}{4} \int_{-\infty}^{x} e^{2x'} u(x')dx' + \frac{e^{2x}}{4} \int_{x}^{+\infty} e^{-2x'} u(x')dx'$$

and

$$v_x(x) = -\frac{e^{-2x}}{2} \int_{-\infty}^{x} e^{2x'} u(x')dx' + \frac{e^{2x}}{2} \int_{x}^{+\infty} e^{-2x'} u(x')dx',$$

which yields

$$|v_x(x)| \le 2v(x), \quad \forall x \in \mathbb{R}. \tag{3.18}$$

Second, let us show that $u \le 6v$ on $\mathbb{R}$. Using the Fourier transform, one can check that

$$\begin{aligned}(1 - \partial_x^2)^{-1}(4 - \partial_x^2)^{-1}(\cdot) &= \mathcal{F}^{-1}\left[\frac{1}{3(1+\omega^2)} - \frac{1}{3(4+\omega^2)}\right](\cdot) \\ &= \frac{1}{3}(1 - \partial_x^2)^{-1}(\cdot) - \frac{1}{3}(4 - \partial_x^2)^{-1}(\cdot),\end{aligned} \tag{3.19}$$

and one can rewrite v as

$$v = (4 - \partial_x^2)^{-1}(1 - \partial_x^2)^{-1}y = \frac{1}{3}(1 - \partial_x^2)^{-1}y - \frac{1}{3}(4 - \partial_x^2)^{-1}y. \tag{3.20}$$

Then for all $x \in \mathbb{R}$,

$$\begin{aligned}u(x) - 6v(x) &= -(1 - \partial_x^2)^{-1}y(x) + 2(4 - \partial_x^2)^{-1}y(x) \\ &= -\frac{1}{2}\int_{\mathbb{R}} e^{-|x-x'|}y(x')dx' + \frac{1}{2}\int_{\mathbb{R}} e^{-2|x-x'|}y(x')dx' \\ &\le 0,\end{aligned} \tag{3.21}$$

since $e^{-2|\cdot|} \le e^{-|\cdot|}$ on $\mathbb{R}$.

We are now ready to prove the uniqueness of local maxima in Θ_z. Let us first study the sign of v_{xx} on $\mathcal{V}_z$. One can easy check that for all $x \in \mathcal{V}_0$,

$$\rho_c''(x) \le \frac{\sqrt{2}-2}{6}c. \tag{3.22}$$

Then, combining (3.8) and (3.22), taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, we have for all $x \in \mathcal{V}_z$,

$$v_{xx}(x) \le \frac{\sqrt{2}-2}{6}c + O(\varepsilon^{1/4}) \le \frac{\sqrt{2}-2}{600}c < 0,$$

which implies that v_x is strictly decreasing on $\mathcal{V}_z$. Let us study the sign of v_x on $\Theta_z \setminus \mathcal{V}_z$. One can easily check that

$$\rho_c'\left(-\ln\sqrt{2}\right) = \frac{\sqrt{2}-1}{6}c \quad \text{and} \quad \rho_c'\left(\ln\sqrt{2}\right) = -\frac{\sqrt{2}-1}{6}c, \tag{3.23}$$

and that $\rho_c'(x) \geq 10^{-4}c$ for all $x \in [-6.7, -\ln\sqrt{2}]$. Then using (3.9) and taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, we have $v_x(x) \geq 4 \times 10^{-5}c > 0$ for all $x \in [z - 6.7, z - \ln\sqrt{2}]$. Proceeding in the same way, we obtain $v_x(x) \leq -4 \times 10^{-5}c < 0$ for all $x \in [z + \ln\sqrt{2}, z + 6.7]$. Since v_x is strictly decreasing on $\mathcal{V}_z$ and changes sign, v_x vanishes once on $\mathcal{V}_z$ and thus on Θ_z. Hence, v admits a single local extremum on Θ_z, which is a maximum since $v_{xx} < 0$ on $\mathcal{V}_z$.

Now, using that ρ_c is increasing on $\mathbb{R}^-$, (3.9) and taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, it holds for all $x \in]-\infty, z - 6.7[$,

$$v(x) = \rho_c(x - z) + O(\varepsilon^{1/4}) \leq \frac{c}{2400} + O(\varepsilon^{1/4}) \leq \frac{c}{300}.$$

Proceeding in the same way for $x \in]z + 6.7, +\infty[$, we obtain (3.16).

Combining (3.8), (3.21) and proceeding as for the estimate (3.16), we get (3.17). Note that $\varphi_c(\pm 6.7) \simeq 1.2 \times 10^{-3}c$. This completes the proof of the lemma. $\square$

Under the assumptions of Lemma 3.4, v has got a unique point of global maximum on $\mathbb{R}$. In the sequel of this section, we will denote by ξ this point of global maximum and we set $M = v(\xi) = \max_{x\in\mathbb{R}} v(x)$. The next two lemmas can be directly deduced from the similar lemmas established in [7] (see also [6]).

Lemma 3.5 (Connection Between $E(\cdot)$ and M^2; See [7])**.** *Let $u \in L^2(\mathbb{R})$ and $v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R})$. Define the function g by*

$$g(x) = \begin{cases} 2v + v_{xx} - 3v_x, & x < \xi, \\ 2v + v_{xx} + 3v_x, & x > \xi. \end{cases} \tag{3.24}$$

Then it holds

$$\int_{\mathbb{R}} g^2(x)dx = E(u) - 12M^2. \tag{3.25}$$

Lemma 3.6 (Connection Between $F(\cdot)$ and M^3; See [7])**.** *Let $u \in L^2(\mathbb{R})$ and $v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R})$. Define the function h by*

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi, \\ -v_{xx} + 6v_x + 16v, & x > \xi. \end{cases} \tag{3.26}$$

Then it holds

$$\int_{\mathbb{R}} h(x)g^2(x)dx = F(u) - 144M^3. \tag{3.27}$$

Sketch of proof. The proof of Lemmas 3.5-3.6 follows by direct computation, using integration by parts, with $v_x(\xi) = 0$ and $v(\pm\infty) = v_x(\pm\infty) = v_{xx}(\pm\infty) = 0$. See [7] (also [6]) to undersand the technique. $\square$

We can now connect the conservation laws.

Lemma 3.7 (Connection Between $E(\cdot)$ and $F(\cdot)$)**.** *Let $u \in L^2(\mathbb{R})$, with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies* (3.6) *for some $z \in \mathbb{R}$. There exists $\varepsilon_0 > 0$ only depending on the speed c, such that if $0 < \varepsilon < \varepsilon_0$, then it holds*

$$M^3 - \frac{1}{4}E(u)M + \frac{1}{72}F(u) \leq 0. \tag{3.28}$$

Proof. The key is to show that $h \leq 18M$ on $\mathbb{R}$. Note that by (3.9) we know that $18M \geq c/4$ and that Lemma 3.4 ensures that $\xi \in \Theta_z$ for ε_0 small enough. Let us set $\lambda = z - 6.7$, $\mu = z + 6.7$, and rewrite the function h as

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \lambda, \\ u - 6v_x + 12v, & \lambda < x < \xi, \\ u + 6v_x + 12v, & \xi < x < \mu, \\ -v_{xx} + 6v_x + 16v, & x > \mu\ . \end{cases}$$

If $x \in \mathbb{R} \setminus \Theta_z$, using that $v_{xx} = 4v - u$, (3.16) and (3.17), it holds

$$h \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c}{9} \leq 18M.$$

If $\lambda < x < \xi$, then $v_x \geq 0$, and using that $u \leq 6v$ on $\mathbb{R}$, we have

$$h = u - 6v_x + 12v \leq 18v.$$

If $\xi < x < \mu$, then $v_x \leq 0$, and similarly using that $u \leq 6v$ on $\mathbb{R}$, we get

$$h = u + 6v_x + 12v \leq 18v.$$

Therefore, it holds

$$h(x) \leq 18 \max_{x \in \mathbb{R}} v(x) = 18M, \quad \forall x \in \mathbb{R}. \tag{3.29}$$

Now, combining (3.25), (3.27) and (3.29), we get

$$F(u) - 144M^3 = \int_{\mathbb{R}} h(x) g^2(x) dx \leq \|h\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} g^2(x) dx \leq 18M(E(u) - 12M^2),$$

and we obtain the lemma. □

Proof of Theorem 3.1. We argue as El Dika and Molinet in [4]. As noticed after the statement of the theorem, it suffices to prove (3.7) assuming that $u \in L^2(\mathbb{R})$ satisfies (3.1), (3.2) and (3.6). We recall that $M = v(\xi) = \max_{x \in \mathbb{R}} v(x)$ and we set $\delta = c/6 - M$. We first remark that if $\delta \leq 0$, combining (3.4) and (3.10), it holds

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq |E(u_0) - E(\varphi_c)|^{1/2} \leq O(\varepsilon),$$

that yields the desired result. Now suppose that $\delta > 0$, that is the maximum of the function v is less than the maximum of ρ_c. Combining (3.4), (3.5) and (3.28), we get

$$M^3 - \frac{1}{4} E(\varphi_c) M + \frac{1}{72} F(\varphi_c) \leq O(\varepsilon^2).$$

Using that $E(\varphi_c) = c^2/3$ and $F(\varphi_c) = 2c^3/3$, our inequality becomes

$$\left(M - \frac{c}{6}\right)^2 \left(M + \frac{c}{3}\right) \leq O(\varepsilon^2).$$

Next, substituting M by $c/6 - \delta$ and using that $[M + c/3]^{-1} < 3/c$, we obtain

$$\delta^2 \leq O(\varepsilon^2) \Rightarrow \delta \leq O(\varepsilon). \tag{3.30}$$

Finally, combining (3.4), (3.10) and (3.30), we infer that

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon},$$

where $C > 0$ only depends on the speed c. This completes the proof of the stability of peakons.

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