On the Stability of Peakons and the Sum of Peakons for the Degasperis-Procesi Equation with Strong Dispersion

André Kabakouala

On the Stability of Peakons and the Sum of Peakons for the Degasperis-Procesi Equation with Strong Dispersion

André Kabakouala
L.M.P.T., U.F.R Sciences et Techniques, Université de Tours, Parc Grandmont, 37200 Tours, France.

Abstract

In this paper, we present a new argument (see Lemma 3.4) that allows us to simplify the proof of stability of peakons established in [9] (Theorem 1.1), [6] (Theorem 1.1) and [8] (Theorem 1.3). Also, we extend the result of stability of ordered trains of peakons obtained in [8] (Theorem 1.1), to the Degasperis-Procesi equation with strong dispersion.

1 Introduction

In this paper, we consider the Degasperis-Procesi equation with strong dispersion (DPsd)

$$u_t - u_{txx} + 4uu_x + \alpha(u_x - u_{xxx}) = 3u_xu_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \tag{1.1}$$

where α is a real constant. The case $\alpha = 0$ is called the Degasperis-Procesi equation (DP)

$$u_t - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}. \tag{1.2}$$

The DP and DPsd equations possess two identical conservation laws (note that the DP equation is completely integrable, see [2])

$$E(u) = \int_{\mathbb{R}} yv = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right), \quad F(u) = \int_{\mathbb{R}} u^3 = \int_{\mathbb{R}} \left(-v_{xx}^3 + 12vv_{xx}^2 - 48v^2v_{xx} + 64v^3\right), \tag{1.3}$$

where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$. One can remark that the conservation law $E(\cdot)$ is equivalent to $\|\cdot\|_{L^2(\mathbb{R})}^2$. Indeed, using integration by parts (we assume that $u(\pm\infty) = v(\pm\infty) = v_x(\pm\infty) = 0$), it holds

$$\|u\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u^2 = \int_{\mathbb{R}} (4v - v_{xx})^2 = \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right) \sim E(u). \tag{1.4}$$

In the sequel we will denote

$$\|u\|_{\mathcal{H}} = \sqrt{E(u)}. \tag{1.5}$$

The DPsd equation admits non-smooth solitary waves of the form (see [7], [12])

$$u(t,x) = \varphi_{c,\alpha}(x - ct) = (c - \alpha)\varphi(x - ct) = (c - \alpha)e^{-|x-ct|}, \quad \text{with} \quad c \neq \alpha, \tag{1.6}$$

and called *peakons* (see Figure 1). In the case $\alpha = 0$, we recover the peakons for the DP equation. One can notice that, thanks to the parameter α we can change the amplitude of peakons without changing

their speeds. As for the Camassa-Holm equation (CH) (see for instance [1], [4] and [3]), the peakons are not strong solutions of (1.1), but solutions in the distribution sense of the DPsd equation in the conservative form

$$u_t + \alpha\partial_x u + \frac{1}{2}\partial_x u^2 + \frac{3}{2}(1-\partial_x^2)^{-1}\partial_x u^2 = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}. \tag{1.7}$$

Our first goal is to simplify the proof given in [9] of the stability of a single peakon for the DP equation. Recall that the proof of the stability for the CH equation in [1] follows from two integral relations between two conservation laws of CH, $\max_{\mathbb{R}} u$ and fonctions related to u. In [9] the proof is more complicated, since all the local maxima and minima of $v = (4-\partial_x^2)^{-1}u$ are involved in the relations. In this paper we present a simplication of this proof, where only the maximum of v is involved in the relations. Our proof is thus closer to the proof for CH in [1]. The main idea is the following: since u is L^2-close to the peakon $\varphi_{c,\alpha}(\cdot-\xi)$, for some $\xi \in \mathbb{R}$, and $(u-u_{xx}) \in \mathcal{M}^+(\mathbb{R})$, it is easy to check that u is actually C^0-close to the peakon, and thus v is C^2-close to the *smooth-peakon* (see Figure 1)

$$\rho_{c,\alpha}(x-\xi) = (4-\partial_x^2)^{-1}\varphi_{c,\alpha}(x-\xi) = \frac{c-\alpha}{3}e^{-|x-\xi|} - \frac{c-\alpha}{6}e^{-2|x-\xi|}. \tag{1.8}$$

First, since $\rho_{c,\alpha}(\cdot-\xi)$, $\rho'_{c,\alpha}(\cdot-\xi)$ and $\rho''_{c,\alpha}(\cdot-\xi)$ are very small with respect to the amplitude $(c-\alpha)$ outside of the interval $\Theta = [\xi-6.7, \xi+6.7]$, we can restrict ourself to study v on Θ. Now we observe that $\rho''_{c,\alpha}(\cdot-\xi)$ has strictly negative values in the interval $\mathcal{V} = [\xi-\ln 2/2, \xi+\ln 2/2]$, with v_x strictly positive on $[\xi-6.7, \xi-\ln 2/2]$ and v_x strictly negative on $[\xi-\ln 2/2, \xi+6.7]$. This forces v_x to change sign only one time on Θ, and thus v has only one local extremum (which is a maximum) on Θ. This fact will considerably simplify the proof of the stability.

Our second goal is to prove that ordered trains of peakons are stable under small perturbations in the energy space $\mathcal{H}$ (equivalent to L^2). Since the proof of the stability result is principally based on energy arguments (see for instance [3], [4], [6], [8], [9], [11]), from the fact that the DP and DPsd equations have the same conservation laws, the term $\alpha(u_x - u_{xxx})$ will play a significant role only in the estimate of the speed of peakons (respectively smooth-peakons) as time is increasing (see Subsection 4.1), and in the monotonicity of the local energy (see Subsection 4.2). The rest of the arguments which lead to the stability result will be mainly deduced from the work done in [8] with the DP equation.

Let us introduce the function space where will live our class of solutions to the equation. For I a finite or infinite time interval of $\mathbb{R}_+$, we denote by $\mathcal{X}(I)$ the function space [1]

$$\mathcal{X}(I) = \left\{ u \in C\left(I; H^1(\mathbb{R})\right) \cap L^\infty\left(I; W^{1,1}(\mathbb{R})\right),\ u_x \in L^\infty\left(I; BV(\mathbb{R})\right) \right\}. \tag{1.9}$$

We have the following stability theorem.

Theorem 1.1 (Stability of the trains of peakons)**.** *Let be given $\alpha \in \mathbb{R}$ and N velocities $c_1, \ldots, c_N$ such that $\max(0,\alpha) < c_1 < \ldots < c_N$. Let $u \in \mathcal{X}([0,T[)$, with $0 < T \leq +\infty$, be a solution of the DPsd equation. There exist $C > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ only depending on the speeds $(c_i)_{i=1}^N$ and the parameter α, such that if*

$$y_0 = (1-\partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \tag{1.10}$$

and

$$\left\| u_0 - \sum_{i=1}^N \varphi_{c_i,\alpha}(\cdot - z_i^0) \right\|_{\mathcal{H}} \leq \varepsilon^2, \quad \textit{with} \quad 0 < \varepsilon < \varepsilon_0, \tag{1.11}$$

for some $z_1^0, \ldots, z_N^0$ satisfying

$$z_1^0 < \ldots < z_N^0 \quad \textit{and} \quad z_i^0 - z_{i-1}^0 \geq L, \quad \textit{with} \quad L > L_0 > 0, \quad i = 2, \ldots, N, \tag{1.12}$$

[1] $W^{1,1}(\mathbb{R})$ is the space of $L^1(\mathbb{R})$ functions with derivatives in $L^1(\mathbb{R})$ and $BV(\mathbb{R})$ is the space of function with bounded variation.

then there exist $\xi(t) = (\xi_1(t), \dots, \xi_N(t)) \in \mathbb{R}^N$ *such that*

$$\left\| u(t) - \sum_{i=1}^{N} \varphi_{c_i,\alpha}(\cdot - \xi_i(t)) \right\|_{\mathcal{H}} \leq C(\sqrt{\varepsilon} + L^{-1/8}), \quad \forall t \in [0, T[\tag{1.13}$$

and

$$\xi_i(t) - \xi_{i-1}(t) > \frac{L}{2}, \quad \forall t \in [0, T[, \quad i = 2, \dots, N, \tag{1.14}$$

see Lemmas 4.1-4.3 for the definition and the properties of $\xi(t)$.

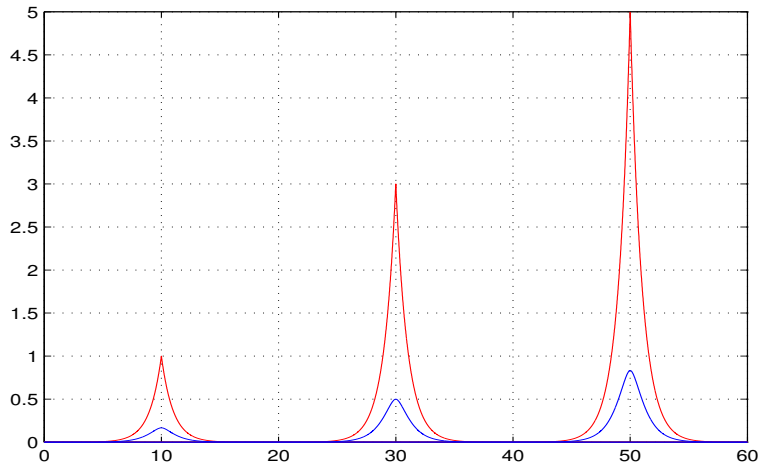


Figure 1: Sum of three red peakons $\sum_{i=1}^{3}(2i-1)\varphi[x - 10(2i-1)]$ and three blue smooth-peakons $\sum_{i=1}^{3}(2i-1)\rho[x - 10(2i-1)]$ (at time $t = 10$ with respective speeds 1, 3, 5, and $\alpha = 0$) profiles.

2 Preliminaries

In this section, we briefly recall the global well-posedness results for the Cauchy problem of the DPsd equation (see [5] and [10]), and its consequences.

Theorem 2.1 (Global weak solution; see [5] and [10])**.** *Assume that* $u_0 \in L^2(\mathbb{R})$ *with* $y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$. *Then the DPsd equation has a unique global weak solution* $u \in \mathcal{X}(\mathbb{R}_+)$ *such that*

$$y(t, \cdot) = (1 - \partial_x^2)u(t, \cdot) \in \mathcal{M}^+(\mathbb{R}), \quad \forall t \in \mathbb{R}_+ \tag{2.1}$$

and

$$|u_x(t, x)| \leq u(t, x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.2}$$

Moreover $E(\cdot)$ *and* $F(\cdot)$ *are conserved by the flow.*

Remark 2.1 (Control of L^∞ norm by L^2 norm)**.** Using the Sobolev embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and (2.2), we infer that there exists a constant $C_S > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R})} \leq C_S \|u\|_{H^1(\mathbb{R})} \leq 2C_S \|u\|_{L^2(\mathbb{R})}. \tag{2.3}$$

3 Stability of a single peakon

In this section, we present our simplification of the proof of stability of peakons.

Theorem 3.1 (Stability of peakons). *Let $u \in \mathcal{X}([0,T[)$, with $0 < T \leq +\infty$, be a solution of the DPsd equation and $\varphi_{c,\alpha}$ be the peakon defined in* (1.6)*, traveling to the right at the speed $c > \max(0,\alpha)$. There exist $C > 0$ and $\varepsilon_0 > 0$ only depending on the speed c and the parameter α, such that if*

$$y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \tag{3.1}$$

and

$$\|u_0 - \varphi_{c,\alpha}\|_{\mathcal{H}} \leq \varepsilon^2, \quad \textit{with} \quad 0 < \varepsilon < \varepsilon_0, \tag{3.2}$$

then

$$\|u(t,\cdot) - \varphi_{c,\alpha}(\cdot - \xi(t))\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \quad \forall t \in [0,T[, \tag{3.3}$$

where $\xi(t) \in \mathbb{R}$ is the only point where the function $v(t,\cdot) = (4 - \partial_x^2)^{-1}u(t,\cdot)$ attains its maximum.

We first recall the following result (see for instance [9] or [8]).

Lemma 3.1 (Control of distances between energies; see [8]). *Let $u \in L^2(\mathbb{R})$ with $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. If $\|u - \varphi_{c,\alpha}\|_{\mathcal{H}} \leq \varepsilon^2$, then*

$$|E(u) - E(\varphi_{c,\alpha})| \leq O(\varepsilon^2) \tag{3.4}$$

and

$$|F(u) - F(\varphi_{c,\alpha})| \leq O(\varepsilon^2), \tag{3.5}$$

where $O(\cdot)$ only depends on the speed c and the parameter α.

To prove Theorem 3.1, by the conservation of $E(\cdot)$, $F(\cdot)$ and the continuity of the map $t \mapsto u(t)$ from $[0,T[$ to $\mathcal{H}$ (since $\mathcal{H} \simeq L^2$), it suffices to prove that for any function $u \in L^2(\mathbb{R})$ satisfying $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, (3.4) and (3.5), if

$$\inf_{z\in\mathbb{R}} \|u - \varphi_{c,\alpha}(\cdot - z)\|_{\mathcal{H}} \leq \varepsilon^{1/4}, \tag{3.6}$$

then

$$\|u - \varphi_{c,\alpha}(\cdot - \xi)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \tag{3.7}$$

where $\xi \in \mathbb{R}$ is the only point of maximum of v.

Let us present some important properties of smooth-peakons defined in (1.8), which will play a crutial role in the proof of Theorem 3.1. The smooth-peakon $\rho_{c,\alpha}$ belong to $H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$ (by the Sobolev embedding) since $\varphi_{c,\alpha}$ belong to $H^1(\mathbb{R})$ (defined in (1.6)). It is a positive even function, which admits a single maximum $(c-\alpha)/6$ at point 0, and decays at infinity to 0 (see Figure 2a). Its derivative $\rho'_{c,\alpha}$ belong to $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ (by the Sobolev embedding), it is an odd function, which vanishes only at the origin, has positive values on $]-\infty,0]$ and negative values on $[0,+\infty[$. It admits a single maximum $(c-\alpha)/12$ at point $-\ln 2$ and a single minimum $-(c-\alpha)/12$ at point $\ln 2$, and decays at infinity to 0 (see Figure 2b). Its second derivative $\rho''_{c,\alpha}$ belong to $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ (by the Sobolev embedding), it is an even function, which vanishes at two points $\pm\ln 2$, takes positive values on $]-\infty,-\ln 2[\cup]\ln 2,+\infty[$ and negative values on $[-\ln 2,\ln 2]$. It admits a single minimum $-(c-\alpha)/3$ at point 0 and two maxima $(c-\alpha)/24$ at points $\pm\ln 4$, and decays at infinity to 0 (see Figure 2c).

Next, we will need the following estimates.

Lemma 3.2 (C^0 and C^1 approximations). *Let $u \in L^2(\mathbb{R})$ with $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. If $\|u-\varphi_{c,\alpha}\|_{\mathcal{H}} \leq \varepsilon^{1/4}$, then*

$$\|u - \varphi_{c,\alpha}\|_{C^0(\mathbb{R})} \leq O(\varepsilon^{1/8}) \tag{3.8}$$

and

$$\|v - \rho_{c,\alpha}\|_{C^1(\mathbb{R})} \leq O(\varepsilon^{1/4}). \tag{3.9}$$

Proof. Let us begin with the second estimate. From the definition of $E(\cdot)$ and $\mathcal{H}$ (see respectively (1.3) and (1.5)), one can see that $\|u\|_{\mathcal{H}}$ is equivalent to $\|v\|_{H^2(\mathbb{R})}$, since $\|v\|_{H^2(\mathbb{R})} \leq \|u\|_{\mathcal{H}} \leq 5\|v\|_{H^2(\mathbb{R})}$. Then, assumption u is $\mathcal{H}$-close to $\varphi_{c,\alpha}$ implies that v is H^2-close to $\rho_{c,\alpha}$. Now, using the Sobolev embedding of $H^2(\mathbb{R})$ into $C^1(\mathbb{R})$, we deduce (3.9).

For the first estimate, note that the assumption $y = (1-\partial_x^2)u \geq 0$ implies that $u = (1-\partial_x^2)^{-1}y \geq 0$ and satisfies $|u_x| \leq u$ on $\mathbb{R}$ (see (2.2)). Then, applying triangular inegality, and using that $|\varphi'_{c,\alpha}| = \varphi_{c,\alpha}$ on $\mathbb{R}$ and (2.3), we have

$$\begin{aligned}\|u-\varphi_{c,\alpha}\|_{H^1(\mathbb{R})} &\leq \|u\|_{H^1(\mathbb{R})} + \|\varphi_{c,\alpha}\|_{H^1(\mathbb{R})}\\ &\leq 2\|u\|_{L^2(\mathbb{R})} + 2\|\varphi_{c,\alpha}\|_{L^2(\mathbb{R})}\\ &\leq 2\|u-\varphi_{c,\alpha}\|_{L^2(\mathbb{R})} + 4\|\varphi_{c,\alpha}\|_{L^2(\mathbb{R})}\\ &\leq O(\varepsilon^{1/4}) + O(1),\end{aligned}$$

where $O(1) = 4(c-\alpha)$. Therefore, applying the Gagliardo-Nirenberg inegality and using that $\|u - \varphi_{c,\alpha}\|_{\mathcal{H}} \leq \varepsilon^{1/4}$ (with $\mathcal{H} \simeq L^2$), we obtain

$$\begin{aligned}\|u-\varphi_{c,\alpha}\|_{C^0(\mathbb{R})} &\leq C_G\|u-\varphi_{c,\alpha}\|_{L^2(\mathbb{R})}^{1/2}\|u-\varphi_{c,\alpha}\|_{H^1(\mathbb{R})}^{1/2}\\ &\leq O(\varepsilon^{1/8})\left(O(\varepsilon^{1/8}) + O(1)\right)\\ &\leq O(\varepsilon^{1/8}),\end{aligned}$$

where C_G is the constant of Gagliardo-Nirenberg. This proves the lemma.

The following lemma specifies the distance to minimize for stability.

Lemma 3.3 (Quadratic identity; see [9])**.** *For any $u \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, it holds*

$$E(u) - E(\varphi_{c,\alpha}) = \|u - \varphi_{c,\alpha}(\cdot-\xi)\|_{\mathcal{H}}^2 + 4(c-\alpha)\left(v(\xi) - \frac{c-\alpha}{6}\right), \tag{3.10}$$

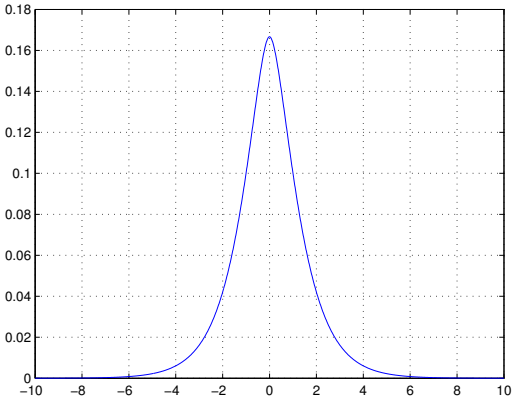
where $v = (4-\partial_x^2)^{-1}u$.

Proof. Following the idea of Constantin and Strauss with the CH equation (see [1], Lemma 1), we will present another way to establish identity (3.10). We compute

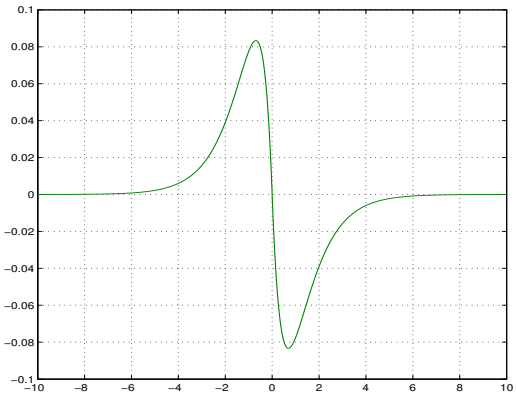
$$\begin{aligned}E(u-\varphi_{c,\alpha}) &= E(u) + E(\varphi_{c,\alpha}) - 2\left\langle(1-\partial_x^2)\varphi_{c,\alpha}(\cdot-\xi), (4-\partial_x^2)^{-1}u\right\rangle_{H^{-1},H^1}\\ &= E(u) + E(\varphi_{c,\alpha}) - 2\left\langle(1-\partial_x^2)\varphi_{c,\alpha}(\cdot-\xi), v\right\rangle_{H^{-1},H^1},\end{aligned} \tag{3.11}$$

where $\langle\cdot,\cdot\rangle_{H^{-1},H^1}$ denotes the duality $H^{-1}(\mathbb{R})$ over $H^1(\mathbb{R})$. Now, using the definition of $\varphi'_{c,\alpha}(\cdot-\xi)$ and integration by parts, we have

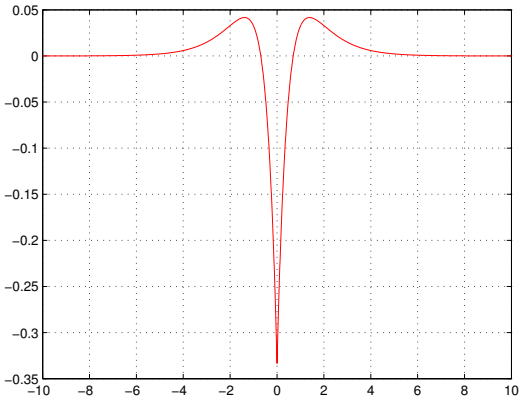
$$\begin{aligned}\left\langle(1-\partial_x^2)\varphi_{c,\alpha}(\cdot-\xi), v\right\rangle_{H^{-1},H^1} &= \int_{\mathbb{R}} v\varphi_{c,\alpha}(\cdot-\xi) + \int_{\mathbb{R}} v_x\varphi'_{c,\alpha}(\cdot-\xi)\\ &= \int_{\mathbb{R}} v\varphi_{c,\alpha}(\cdot-\xi) + \int_{-\infty}^{\xi} v_x\varphi_{c,\alpha}(\cdot-\xi) - \int_{\xi}^{+\infty} v_x\varphi_{c,\alpha}(\cdot-\xi)\\ &= 2(c-\alpha)v(\xi).\end{aligned} \tag{3.12}$$



(a) $\rho(x) = \frac{1}{3}e^{-|x|} - \frac{1}{6}e^{-2|x|}$ profile.



(b) $\rho'(x) = \left(\frac{1}{3}e^{-|x|} - \frac{1}{3}e^{-2|x|}\right)_{x<0} + \left(\frac{1}{3}e^{-2|x|} - \frac{1}{3}e^{-2|x|}\right)_{x>0}$ profile.



(c) $\rho''(x) = \frac{1}{3}e^{-|x|} - \frac{2}{3}e^{-2|x|}$ profile

Figure 2: Variation of the smooth-peakon with the amplitude 1/6 at initial time.

Recall that the energy of peakons gives us

$$\begin{aligned} E(\varphi_{c,\alpha}) &= \left\langle (1-\partial_x^2)\varphi_{c,\alpha}, (4-\partial_x^2)^{-1}\varphi_{c,\alpha} \right\rangle_{H^{-1},H^1} \\ &= \int_{\mathbb{R}} \rho_{c,\alpha}\varphi_{c,\alpha} + \int_{\mathbb{R}} \rho'_{c,\alpha}\varphi'_{c,\alpha} \\ &= \int_{\mathbb{R}} \rho_{c,\alpha}\varphi_{c,\alpha} + \int_{-\infty}^{0} \rho'_{c,\alpha}\varphi_{c,\alpha} - \int_{0}^{+\infty} \rho'_{c,\alpha}\varphi_{c,\alpha} \\ &= 2(c-\alpha)\rho_{c,\alpha}(0) \\ &= \frac{(c-\alpha)^2}{3}. \end{aligned} \tag{3.13}$$

Thus, combining (3.11)-(3.13), we obtain the lemma.

Now we will study carefully the local extrema of $v = (4-\partial_x^2)^{-1}u$. Let $u \in L^2(\mathbb{R})$ with $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, and assume that (3.6) holds for some $z \in \mathbb{R}$. We consider the interval in which the mass of smooth-peakons is concentrated, and the interval in which the mass of second derivative of smooth-

peakons is strictly negative. In the sequel of this paper, the notation $\alpha \simeq \beta$ means that $0.9 \times \beta \leq \alpha \leq 1.1 \times \beta$. We set

$$\Theta = [z - 6.7, z + 6.7], \quad \text{where} \quad 6.7 \simeq \ln\left(1 - \frac{\sqrt{399}}{20}\right)^{-1}, \tag{3.14}$$

and

$$\mathcal{V} = \left[z - \frac{\ln 2}{2}, z + \frac{\ln 2}{2}\right]. \tag{3.15}$$

One can clearly see that $\mathcal{V}$ is a subset of Θ (since $20/(20 - \sqrt{399}) > \sqrt{2}$), and we chose the values ± 6.7 such that $\rho_{c,\alpha}(\pm 6.7) \simeq (c-\alpha)/2400 \simeq 4.1 \times 10^{-4}(c-\alpha)$ as in [8]. Also, we have $\rho'_{c,\alpha}(-6.7) = -\rho'_{c,\alpha}(6.7) \simeq -4.1 \times 10^{-4}(c-\alpha)$ and $\rho''_{c,\alpha}(\pm 6.7) \simeq 4.1 \times 10^{-4}(c-\alpha)$. Then $\rho_{c,\alpha}(\cdot - z)$, $\rho'_{c,\alpha}(\cdot - z)$ and $\rho''_{c,\alpha}(\cdot - z)$ are very small with respect to the amplitude $(c - \alpha)$ on $\mathbb{R} \setminus \Theta$.

We claim the following result.

Lemma 3.4 (Uniqueness of the local maximum)**.** *Let $u \in L^2(\mathbb{R})$, with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies* (3.6) *for some $z \in \mathbb{R}$. There exist $\varepsilon_0 > 0$ only depending on the speed c and the parameter α, such that if $0 < \varepsilon < \varepsilon_0$, then the function $v = (4 - \partial_x^2)^{-1}u$ admits a unique local extremum on Θ. This extremum is a maximum, and it holds*

$$v(x) \leq \frac{c - \alpha}{300}, \quad \forall x \in \mathbb{R} \setminus \Theta, \tag{3.16}$$

$$u(x) \leq \frac{c - \alpha}{300}, \quad \forall x \in \mathbb{R} \setminus \Theta. \tag{3.17}$$

Proof. The key is to study the impact of assumption $y \in \mathcal{M}^+(\mathbb{R})$ on v. First, let us show that $|v_x| \leq 2v$ on $\mathbb{R}$. We recall that from the assumption $y \geq 0$, we have $u \geq 0$ and $v \geq 0$ on $\mathbb{R}$. According to the definition of v, we have for all $x \in \mathbb{R}$,

$$v(x) = \frac{e^{-2x}}{4} \int_{-\infty}^{x} e^{2x'} u(x')dx' + \frac{e^{2x}}{4} \int_{x}^{+\infty} e^{-2x'} u(x')dx'$$

and

$$v_x(x) = -\frac{e^{-2x}}{2} \int_{-\infty}^{x} e^{2x'} u(x')dx' + \frac{e^{2x}}{2} \int_{x}^{+\infty} e^{-2x'} u(x')dx',$$

which yields

$$|v_x(x)| \leq 2v(x), \quad \forall x \in \mathbb{R}. \tag{3.18}$$

Second, let us show that $u \leq 6v$ on $\mathbb{R}$. Using the Fourier transform, one can check that

$$\begin{aligned}(1 - \partial_x^2)^{-1}(4 - \partial_x^2)^{-1}(\cdot) &= \mathcal{F}^{-1}\left[\frac{1}{3(1 + \omega^2)} - \frac{1}{3(4 + \omega^2)}\right] \\ &= \frac{1}{3}(1 - \partial_x^2)^{-1}(\cdot) - \frac{1}{3}(4 - \partial_x^2)^{-1}(\cdot),\end{aligned} \tag{3.19}$$

and one can rewrite v as

$$v = (4 - \partial_x^2)^{-1}(1 - \partial_x^2)^{-1}y = \frac{1}{3}(1 - \partial_x^2)^{-1}y - \frac{1}{3}(4 - \partial_x^2)^{-1}y. \tag{3.20}$$

Then for all $x \in \mathbb{R}$,

$$\begin{aligned}u(x) - 6v(x) &= -(1 - \partial_x^2)^{-1}y(x) + 2(4 - \partial_x^2)^{-1}y(x) \\ &= -\frac{1}{2}\int_{\mathbb{R}} e^{-|x-x'|}y(x')dx' + \frac{1}{2}\int_{\mathbb{R}} e^{-2|x-x'|}y(x')dx' \\ &\leq 0,\end{aligned} \tag{3.21}$$

since $e^{-2|\cdot|} \leq e^{-|\cdot|}$ on $\mathbb{R}$.

Next, let us show that v_{xx} is uniformly close to $\rho''_{c,\alpha}(\cdot - z)$ on $\mathbb{R}$. From the definition of v and using (3.21), we have

$$|v_{xx}(x)| = |4v(x) - u(x)| \leq 4v(x) + u(x) \leq 10v(x), \quad \forall x \in \mathbb{R}. \tag{3.22}$$

Also, using that $|u_x| \leq u \leq 6v$ on $\mathbb{R}$ (see (2.2)), and that $|v_x| \leq 2v$ on $\mathbb{R}$ (see (3.18)), we get

$$|v_{xxx}(x)| = |4v_x(x) - u_x(x)| \leq 4|v_x(x)| + |u_x(x)| \leq 14v(x), \quad \forall x \in \mathbb{R}. \tag{3.23}$$

Then, combining (3.18), (3.22) and (3.23), we infer that

$$\|v\|_{H^3(\mathbb{R})} = \sum_{j=0}^{3} \|\partial_x^{(j)} v\|_{L^2(\mathbb{R})} \leq 27\|v\|_{L^2(\mathbb{R})}. \tag{3.24}$$

Please note that, all these properties hold with $\rho_{c,\alpha} = (4 - \partial_x^2)^{-1}\varphi_{c,\alpha}$. Now, combining (3.9) and (3.24), one can check that

$$\begin{aligned} \|v - \rho_{c,\alpha}(\cdot - z)\|_{H^3(\mathbb{R})} &\leq \|v\|_{H^3(\mathbb{R})} + \|\rho_{c,\alpha}\|_{H^3(\mathbb{R})} \\ &\leq 27\|v\|_{L^2(\mathbb{R})} + 27\|\rho_{c,\alpha}\|_{L^2(\mathbb{R})} \\ &\leq 27\|v - \rho_{c,\alpha}(\cdot - z)\|_{L^2(\mathbb{R})} + 54\|\rho_{c,\alpha}\|_{L^2(\mathbb{R})} \\ &\leq O(\varepsilon^{1/4}) + O(1), \end{aligned} \tag{3.25}$$

where $O(1) = \sqrt{594}(c - \alpha)$. Applying the Gagliardo-Nirenberg inegality, and using (3.9) and (3.25), it holds

$$\begin{aligned} \|v_{xx} - \rho''_{c,\alpha}(\cdot - z)\|_{L^\infty(\mathbb{R})} &\leq C_G \|v - \rho_{c,\alpha}(\cdot - z)\|_{H^2(\mathbb{R})}^{1/2} \|v - \rho_{c,\alpha}(\cdot - z)\|_{H^3(\mathbb{R})}^{1/2} \\ &\leq O(\varepsilon^{1/8})(O(\varepsilon^{1/8}) + O(1)) \\ &\leq O(\varepsilon^{1/8}), \end{aligned} \tag{3.26}$$

which leads to the desired result.

We are now ready to prove the uniqueness of local maxima. Since $v_x(x) = \rho'_{c,\alpha}(x - z) + O(\varepsilon^{1/4})$ for all $x \in \mathbb{R}$ (thanks to (3.9)), in particular, we have $v_x(z) = O(\varepsilon^{1/4})$, since $\rho'_{c,\alpha}(0) = 0$. Then, we will restrict our research of local extrema of v on $\mathcal{V}$. Let us study the sign of v_{xx} on $\mathcal{V}$. One can easy check that for all $x \in \mathcal{V}$,

$$\rho''_{c,\alpha}(x) \leq \frac{\sqrt{2} - 2}{6}(c - \alpha). \tag{3.27}$$

Then, combining (3.26) and (3.27), taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, we have for all $x \in \mathcal{V}$,

$$v_{xx}(x) \leq \frac{\sqrt{2} - 2}{6}(c - \alpha) + O(\varepsilon^{1/8}) \leq \frac{\sqrt{2} - 2}{600}(c - \alpha) < 0,$$

which implies that v_x is strictly decreasing on $\mathcal{V}$. Let us study the sign of v_x on $\Theta \setminus \mathcal{V}$. One can easy check that

$$\rho'_{c,\alpha}\left(-\frac{\ln 2}{2}\right) = \frac{\sqrt{2} - 1}{6}(c - \alpha) \quad \text{and} \quad \rho'_{c,\alpha}\left(\frac{\ln 2}{2}\right) = -\frac{\sqrt{2} - 1}{6}(c - \alpha), \tag{3.28}$$

and one can clearly see that $\rho'_{c,\alpha}(x) \geq 4 \times 10^{-4}(c - \alpha)$ for all $x \in [z - 6.7, z - \ln 2/2]$. Then using (3.9) and taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, we have $v_x(x) \geq 4 \times 10^{-5}(c - \alpha) > 0$ for all $x \in [z - 6.7, z - \ln 2/2]$. Proceeding in the same way, we obtain $v_x(x) \leq -4 \times 10^{-5}(c - \alpha) < 0$ for all $x \in [z + \ln 2/2, z + 6.7]$. Since v_x is strictly decreasing on $\mathcal{V}$ and changes sign, then v_x vanishes once on $\mathcal{V}$, and thus v admits a single local extemum on $\mathcal{V}$, which is a maximum since $v_{xx} < 0$ on $\mathcal{V}$.

Now, using that $\rho_{c,\alpha}(\cdot - z)$ is increasing on $]-\infty, z]$, (3.9) and taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, it holds for all $x \in]-\infty, z - 6.7[$,

$$v(x) = \rho_{c,\alpha}(x - z) + O(\varepsilon^{1/4}) \leq \frac{c-\alpha}{2400} + O(\varepsilon^{1/4}) \leq \frac{c-\alpha}{300}.$$

Proceeding in the same way for $x \in]z + 6.7, +\infty[$, we obtain (3.16).

Combining (3.8), (3.21) and proceeding as for the estimate (3.16), we get (3.17). Note that $\varphi_{c,\alpha}(\pm 6.7) \simeq 1.2 \times 10^{-3}(c - \alpha)$. This completes the proof of the lemma.

We deduce from the above lemma that v has got a unique point of global maximum on $\mathbb{R}$. In the sequel of this section, we will denote by ξ this point of global maximum and we set $M = v(\xi) = \max_{x \in \mathbb{R}} v(x)$. The next two lemmas can be directly deduced from the similar lemmas established in [9] and [8].

Lemma 3.5 (Connection between $E(\cdot)$ and M^2; see [9])**.** *Let $u \in L^2(\mathbb{R})$ and $v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R})$. Define the function g by*

$$g(x) = \begin{cases} 2v + v_{xx} - 3v_x, & x < \xi, \\ 2v + v_{xx} + 3v_x, & x > \xi. \end{cases} \tag{3.29}$$

Then it holds

$$\int_{\mathbb{R}} g^2(x)dx = E(u) - 12M^2. \tag{3.30}$$

Lemma 3.6 (Connection between $F(\cdot)$ and M^3; see [9])**.** *Let $u \in L^2(\mathbb{R})$ and $v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R})$. Define the function h by*

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi, \\ -v_{xx} + 6v_x + 16v, & x > \xi. \end{cases} \tag{3.31}$$

Then it holds

$$\int_{\mathbb{R}} h(x)g^2(x)dx = F(u) - 144M^3. \tag{3.32}$$

We can now connect the conservation laws.

Lemma 3.7 (Connection between $E(\cdot)$ and $F(\cdot)$)**.** *Let $u \in L^2(\mathbb{R})$, with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies* (3.6) *for some $z \in \mathbb{R}$. There exists $\varepsilon_0 > 0$ only depending on the speed c and the parameter α, such that if $0 < \varepsilon < \varepsilon_0$, then it holds*

$$M^3 - \frac{1}{4}E(u)M + \frac{1}{72}F(u) \leq 0. \tag{3.33}$$

Proof. The key is to show that $h \leq 18M$ on $\mathbb{R}$. Note that by (3.9) we know that $18M \geq (c - \alpha)/4$. Let us set $\lambda = z - 6.7$, $\mu = z + 6.7$, and we rewrite the function h as

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \lambda, \\ u - 6v_x + 12v, & \lambda < x < \xi, \\ u + 6v_x + 12v, & \xi < x < \mu, \\ -v_{xx} + 6v_x + 16v, & x > \mu. \end{cases}$$

If $x \in \mathbb{R} \setminus \Theta$, using that $v_{xx} = 4v - u$, (3.16) and (3.17), it holds

$$h \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c-\alpha}{9} \leq 18M.$$

If $\lambda < x < \xi$, then $v_x \geq 0$, and using that $u \leq 6v$ on $\mathbb{R}$, we have

$$h = u - 6v_x + 12v \leq 18v.$$

If $\xi < x < \mu$, then $v_x \leq 0$, and similarly using that $u \leq 6v$, we get

$$h = u + 6v_x + 12v \leq 18v.$$

Therefore, it holds

$$h(x) \leq 18 \max_{x\in\mathbb{R}} v(x) = 18M, \quad \forall x \in \mathbb{R}. \tag{3.34}$$

Now, combining (3.30), (3.32) and (3.34), we get

$$F(u) - 144M^3 = \int_{\mathbb{R}} h(x)g^2(x)dx \leq \|h\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} g^2(x)dx \leq 18M(E(u) - 12M^2),$$

and we obtain the lemma.

Proof of Theorem 3.1. As noticed after the statement of the theorem, it suffices to prove (3.7) assuming that $u \in L^2(\mathbb{R})$ satisfies (3.1), (3.2) and (3.6). We recall that $M = v(\xi) = \max_{x\in\mathbb{R}} v(x)$ and we set $\delta = (c-\alpha)/6 - M$. We first remark that if $\delta \leq 0$, combining (3.4) and (3.10), it holds

$$\|u - \varphi_{c,\alpha}(\cdot - \xi)\|_{\mathcal{H}} \leq |E(u_0) - E(\varphi_{c,\alpha})|^{1/2} \leq O(\varepsilon),$$

that yields the desired result. Now suppose that $\delta > 0$, that is the maximum of the function v is less than the maximum of $\rho_{c,\alpha}$. Combining (3.4), (3.5) and (3.33), we get

$$M^3 - \frac{1}{4}E(\varphi_{c,\alpha})M + \frac{1}{72}F(\varphi_{c,\alpha}) \leq O(\varepsilon^2).$$

Using that $E(\varphi_{c,\alpha}) = (c-\alpha)^2/3$ and $F(\varphi_{c,\alpha}) = 2(c-\alpha)^3/3$, our inequality becomes

$$\left(M - \frac{c-\alpha}{6}\right)^2 \left(M + \frac{c-\alpha}{3}\right) \leq O(\varepsilon^2).$$

Next, substituting M by $(c-\alpha)/6 - \delta$ and using that $[M + (c-\alpha)/3]^{-1} < 3/(c-\alpha)$, we obtain

$$\delta^2 \leq O(\varepsilon^2) \Rightarrow \delta \leq O(\varepsilon). \tag{3.35}$$

Finally, combining (3.4), (3.10) and (3.35), we infer that

$$\|u - \varphi_{c,\alpha}(\cdot - \xi)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon},$$

where $C > 0$ only depends on the speed c and the parameter α. This completes the proof of the stability of a single peakon.

4 Proof of Theorem 1.1

In this section, we generalize the stability result to the sum of ordered trains of peakons (respectively smooth-peakons). For $\gamma > 0$ and $L > 0$, we define the following neighborhood of all the sums of N peakons of speed $c_1, ..., c_N$ with spatial shifts z_i that satisfied $z_i - z_{i-1} \geq L$,

$$U(\gamma, L) = \left\{ u \in L^2(\mathbb{R}), \inf_{z_i - z_{i-1} > L} \left\| u - \sum_{i=1}^{N} \varphi_{c_i,\alpha}(\cdot - z_i) \right\|_{\mathcal{H}} < \gamma \right\}. \tag{4.1}$$

By the continuity of the map $t \mapsto u(t)$ from $[0, T[$ into $\mathcal{H}$ (since $\mathcal{H} \simeq L^2$), to prove Theorem 1.1 it suffices to prove that there exist $A > 0$, $\varepsilon_0 > 0$ and $L_0 > 0$ such that for all $L > L_0$ and $0 < \varepsilon < \varepsilon_0$, if u_0 satisfies (1.10)-(1.12), and if for some $0 < t_0 < T$,

$$u(t) \in U\left(A(\sqrt{\varepsilon} + L^{-1/8}), \frac{L}{2}\right), \quad \forall t \in [0, t_0], \tag{4.2}$$

then

$$u(t_0) \in U\left(\frac{A}{2}(\sqrt{\varepsilon} + L^{-1/8}), \frac{2L}{3}\right). \tag{4.3}$$

Therefore, in the sequel of this section we will assume (4.2) for some $0 < \varepsilon < \varepsilon_0$ and $L > L_0$, with A, ε_0 and L_0 to be specified later, and we will prove (4.3).

Remark 4.1 (Distance between v and the sum of N smooth-peakons)**.** Recall that $\|u\|_{\mathcal{H}}$ is equivalent to $\|v\|_{H^2(\mathbb{R})}$, where $v = (4 - \partial_x^2)^{-1}u$. Thus if $u(t) \in U(\gamma, L/2)$ on $[0, t_0]$, then $v(t)$ stays H^2-close to $\sum_{i=1}^N \rho_{c_i,\alpha}(\cdot - z_i)$ for all $t \in [0, t_0]$.

4.1 Control of the distance between the peakons (respectively the smooth-peakons)

In this subsection, we want to prove that the different bumps of u (respectively $v = (4 - \partial_x^2)^{-1}u$) that are individually close to a peakon (respectively a smooth-peakon) get away from each others as time is increasing. This is crucial in our analysis since we do not know how to manage strong interactions.

Lemma 4.1 (Modulation argument in H^2)**.** *Let u_0 satisfying* (1.10)-(1.12)*. There exist $\gamma_0 > 0$, $L_0 > 0$ and $C_0 > 0$ such that for all $0 < \gamma < \gamma_0$ and $0 < L_0 < L$, if $u(t) \in U(\gamma, L/2)$ on $[0, t_0]$ for some $0 < t_0 < T$, then there exist N C^1 functions $\tilde{x}_1, \dots, \tilde{x}_N$ defined on $[0, t_0]$ such that*

$$\left\| u(t) - \sum_{i=1}^N \varphi_{c_i,\alpha}(\cdot - \tilde{x}_i(t)) \right\|_{\mathcal{H}} < O(\gamma), \tag{4.4}$$

$$\left\| v(t) - \sum_{i=1}^N \rho_{c_i,\alpha}(\cdot - \tilde{x}_i(t)) \right\|_{C^1(\mathbb{R})} \leq O(\gamma), \tag{4.5}$$

$$\left|\dot{\tilde{x}}_i(t) - c_i\right| \leq (c_1 - \alpha)^{-2} \left(O(\gamma) + O(e^{-L/4})\right), \quad i = 1, \dots, N, \tag{4.6}$$

and

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2}, \quad i = 2, \dots, N. \tag{4.7}$$

Moreover, for $i = 1, \dots, N$, setting $J_i = [y_i(t), y_{i+1}(t)]$, with

$$\begin{cases} y_1 = -\infty, \\ y_i(t) = \dfrac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, \quad i = 2, \dots, N, \\ y_{N+1} = +\infty, \end{cases} \tag{4.8}$$

it holds

$$|\xi_i(t) - \tilde{x}_i(t)| \leq \frac{L}{12}, \quad i = 1, \dots, N, \tag{4.9}$$

where $\xi(t) = (\xi_i(t), \dots, \xi_N(t)) \in \mathbb{R}^N$ is any point such that

$$v(t, \xi_i(t)) = \max_{x \in J_i} v(t, x), \quad i = 1, \dots, N, \tag{4.10}$$

and where $v = (4 - \partial_x^2)^{-1}u$, and $O(\cdot)$ only depends on the speeds $(c_i)_{i=1}^N$ and the parameter α.

Proof. The proof is a standard application of the implicit function theorem. We will repeat the technique used in [8] taking account of the dispersion term $\alpha(u_x - u_{xxx})$, with $\alpha \neq 0$. This implies that it is necessary to show more the following *almost orthogonality* condition for all $t \in [0, t_0]$ (see (4.19)):

$$\alpha \left\langle v(t) - \sum_{j=1}^{N} \rho_{c_j,\alpha}(\cdot - \tilde{x}_j(t)), \partial_x^2 \rho_{c_j,\alpha}(\cdot - \tilde{x}_j(t)) \right\rangle_{L^2} \approx 0,$$

where $\langle \cdot, \cdot \rangle_{L^2}$ denotes the scalar product in L^2.

For $Z = (z_1, \dots, z_N) \in \mathbb{R}^N$ fixed such that $|z_i - z_{i-1}| > L/2$, we set

$$R_Z(\cdot) = \sum_{i=1}^{N} \rho_{c_i,\alpha}(\cdot - z_i) \quad \text{and} \quad S_Z(\cdot) = \sum_{i=1}^{N} \varphi_{c_i,\alpha}(\cdot - z_i). \tag{4.11}$$

For $0 < \gamma < \gamma_0$, we define the function

$$\begin{aligned} \mathcal{Y} : (-\gamma, \gamma)^N \times B_{H^2}(R_Z, \gamma) &\to \mathbb{R}^N, \\ (y_1, \dots, y_N, v) &\mapsto \big(\mathcal{Y}^1(y_1, \dots, y_N, v), \dots, \mathcal{Y}^N(y_1, \dots, y_N, v)\big) \end{aligned}$$

with

$$\mathcal{Y}^i(y_1, \dots, y_N, v) = \int_{\mathbb{R}} \left(v - \sum_{j=1}^{N} \rho_{c_j,\alpha}(\cdot - z_j - y_j) \right) \partial_x \rho_{c_i,\alpha}(\cdot - z_i - y_i).$$

Y is clearly of class C^1. For $i = 1, \dots, N$,

$$\frac{\partial \mathcal{Y}^i}{\partial y_i}(y_1, \dots, y_N, v) = -\int_{\mathbb{R}} \left(v - \sum_{\substack{1 \le j \le N \\ j \neq i}} \rho_{c_j,\alpha}(\cdot - z_j - y_j) \right) \partial_x^2 \rho_{c_i,\alpha}(\cdot - z_i - y_i)$$

and for $j \neq i$,

$$\frac{\partial \mathcal{Y}^i}{\partial y_j}(y_1, \dots, y_N, v) = \int_{\mathbb{R}} \partial_x \rho_{c_j,\alpha}(\cdot - z_j - y_j) \partial_x \rho_{c_i,\alpha}(\cdot - z_i - y_i).$$

Hence

$$\frac{\partial \mathcal{Y}^i}{\partial y_i}(0, \dots, 0, R_Z) = \|\partial_x \rho_{c_i,\alpha}\|^2_{L^2(\mathbb{R})} = \frac{(c_i - \alpha)^2}{54} \geq \frac{(c_1 - \alpha)^2}{54}$$

and for $j \neq i$, using the exponential decay of $\varphi_{c_i,\alpha}$ and that $|z_i - z_{i-1}| > L/2$, for $L > L_0 > 0$ with $L_0 \gg 1$, it holds

$$\begin{aligned} &\left| \frac{\partial \mathcal{Y}^i}{\partial y_j}(0, \dots, 0, R_Z) \right| \\ &\quad = \left| \int_{\mathbb{R}} \partial_x \rho_{c_j,\alpha}(\cdot - z_j) \partial_x \rho_{c_i,\alpha}(\cdot - z_i) \right| \\ &\quad = \left| \int_{\mathbb{R}} \rho_{c_j,\alpha}(\cdot - z_j) \partial_x^2 \rho_{c_i,\alpha}(\cdot - z_i) \right| \\ &\quad \leq \frac{1}{9}(c_i - \alpha)(c_j - \alpha) \left\{ \int_{\mathbb{R}} e^{-|x - z_j| - |x - z_i|} dx + 2 \int_{\mathbb{R}} e^{-|x - z_j| - 2|x - z_i|} dx \right. \\ &\qquad \left. + \frac{1}{2} \int_{\mathbb{R}} e^{-2|x - z_j| - |x - z_i|} dx + \int_{\mathbb{R}} e^{-2|x - z_j| - 2|x - z_i|} dx \right\} \\ &\quad \leq O(e^{-L/4}). \end{aligned}$$

We deduce that, for $L > 0$ large enough, $D_{(y_1,\dots,y_N)}\mathcal{Y}(0,\dots,0,R_Z) = D + P$ where D is an invertible diagonal matrix with $\|D^{-1}\| \leq [(c_1-\alpha)/3\sqrt{6}]^{-2}$ and $\|P\| \leq O(e^{-L/4})$. Hence there exists $L_0 > 0$ such that for $L > L_0$, $D_{(y_1,\dots,y_N)}\mathcal{Y}(0,\dots,0,R_Z)$ is invertible with an inverse matrix of norm smaller than $2[(c_1-\alpha)/3\sqrt{6}]^{-2}$. From the implicit function theorem we deduce that there exists $\beta_0 > 0$ and C^1 functions $(y_1,\dots,y_N)$ from $B_{H^2}(R_Z,\beta_0)$ to a neighborhood of $(0,\dots,0)$ which are uniquely determined such that

$$\mathcal{Y}(y_1(v),\dots,y_N(v),v) = 0, \quad \forall v \in B_{H^2}(R_Z,\beta_0).$$

In particular, there exists $C_0 > 0$ such that if $v \in B_{H^2}(R_Z,\beta)$, with $0 < \beta \leq \beta_0$, then

$$\sum_{i=1}^{N} |y_i(v)| \leq C_0\beta. \tag{4.12}$$

Note that β_0 and C_0 only depend on c_1 and L_0 and not on the point $(z_1,\dots,z_N)$. For $v \in B_{H^2}(R_Z,\beta_0)$ we set $\tilde{x}_i(v) = z_i + y_i(v)$. Assuming that $\beta_0 \leq L_0/8C_0$, $(\tilde{x}_1(v),\dots,\tilde{x}_N(v))$ are thus C^1 functions on $B_{H^2}(R_Z,\beta)$ satisfying

$$\tilde{x}_i(v) - \tilde{x}_{i-1}(v) = z_i - z_{i-1} + y_i(v) - y_{i-1}(v) > \frac{L}{2} - 2C_0\beta > \frac{L}{4}. \tag{4.13}$$

For $L > L_0$ and $0 < \gamma < \gamma_0 < \beta_0/2$ to be chosen later, we define the modulation of v in the following way: we cover the trajectory of v by a finite number of open balls in the following way:

$$\{v(t), t \in [0,t_0]\} \subset \bigcup_{k=1,\dots,M} B_{H^2}(R_{Z^k},2\gamma).$$

This is possible thanks to Remark 4.1. It is worth noticing that, since $0 < \gamma < \gamma_0 < \beta_0/2$, the functions $\tilde{x}_i(v)$ are uniquely determined for $v \in B_{H^2}(R_{Z^k},2\gamma) \cap B_{H^2}(R_{Z^{k'}},2\gamma)$. We can thus define the functions $t \mapsto \tilde{x}_i(t)$ on $[0,t_0]$ by setting $\tilde{x}_i(t) = \tilde{x}_i(v(t))$. By construction

$$\int_{\mathbb{R}} \left(v(t,\cdot) - \sum_{j=1}^{N} \rho_{c_j,\alpha}(\cdot - \tilde{x}_j(t)) \right) \partial_x \rho_{c_i,\alpha}(\cdot - \tilde{x}_i(t)) = 0. \tag{4.14}$$

For $0 < \gamma < \gamma_0$, with $\gamma_0 \ll 1$, using that $u \in U(\gamma, L/2)$ and (4.12), we have

$$\begin{aligned}
&\|u(t) - S_{\tilde{X}(t)}\|_{\mathcal{H}} \\
&\quad \leq \|u(t) - S_Z(t)\|_{\mathcal{H}} + \sum_{i=1}^{N} \|\varphi_{c_i,\alpha}(\cdot - z_i) - \varphi_{c_i,\alpha}(\cdot - z_i - y_i(v(t)))\|_{L^2(\mathbb{R})} \\
&\quad \leq \gamma + \sqrt{2}\sum_{i=1}^{N} \left(\int_{\mathbb{R}} \varphi_{c_i,\alpha}^2(x)dx - \int_{\mathbb{R}} \varphi_{c_i,\alpha}(x - z_i)\varphi_{c_i,\alpha}(x - z_i - y_i(v(t)))dx \right)^{1/2} \\
&\quad = \gamma + \sqrt{2}\sum_{i=1}^{N} (c_i - \alpha) \left(1 - e^{-|y_i(v(t))|} - |y_i(v(t))|e^{-|y_i(v(t))|} \right)^{1/2} \\
&\quad \leq \gamma + \sum_{i=1}^{N} O(|y_i(v(t))|) \\
&\quad \leq O(\gamma),
\end{aligned}$$

where we apply two time the mean value theorem with the function φ on $[0,|y_i(v(t))|]$ for substituting $(1 - e^{-|y_i(v(t))|})$ by $|y_i(v(t))|e^{-\theta(|y_i(v(t))|)}$, with $\theta(|y_i(v(t))|) \in]0,|y_i(v(t))|[$, and this proves (4.4).

To get (4.5) it suffice to use (4.4), Remark 4.1, and the Sobolev embedding of $H^2(\mathbb{R})$ into $C^1(\mathbb{R})$. Next, to prove that the speed of $\tilde{x}_i(\cdot)$ stays close to c_i, we set

$$S_j(t) = \varphi_{c_j,\alpha}(\cdot - \tilde{x}_j(t)), \quad \varepsilon_1(t) = u(t) - \sum_{j=1}^{N} S_j(t)$$

and

$$R_j(t) = \rho_{c_j,\alpha}(\cdot - \tilde{x}_j(t)), \quad \varepsilon_2(t) = v(t) - \sum_{j=1}^{N} R_j(t).$$

One can notice that

$$\partial_x^2 R_i = 4R_i - S_i, \tag{4.15}$$

and recall that from (3.19),

$$(1-\partial_x^2)^{-1}(4-\partial_x^2)^{-1}(\cdot) = \frac{1}{3}(1-\partial_x^2)^{-1}(\cdot) - \frac{1}{3}(4-\partial_x^2)^{-1}(\cdot).$$

Differentiating (4.14) with respect to time and using (4.15), we get

$$\int_{\mathbb{R}} \partial_t \varepsilon_2 \partial_x R_i = \dot{\tilde{x}}_i(t) \left(4 \int_{\mathbb{R}} \varepsilon_2 R_i - \int_{\mathbb{R}} \varepsilon_2 S_i \right)$$

and thus

$$\begin{aligned} \left| \int_{\mathbb{R}} \partial_t \varepsilon_2 \partial_x R_i \right| &\leq |\dot{\tilde{x}}_i(t)| \left(4\|\varepsilon_2\|_{L^\infty(\mathbb{R})} \|R_i\|_{L^1(\mathbb{R})} + \|\varepsilon_2\|_{L^\infty(\mathbb{R})} \|S_i\|_{L^1(\mathbb{R})} \right) \\ &\leq |\dot{\tilde{x}}_i(t) - c_i| O(\gamma) + O(\gamma). \end{aligned} \tag{4.16}$$

Substituting u by $\varepsilon_1 + \sum_{j=1}^{N} S_j$ in (1.7) and using that S_j satisfies

$$\partial_t S_j = -(\dot{\tilde{x}}_j(t) - c_j)\partial_x S_j - \alpha \partial_x S_j - \frac{1}{2}\partial_x S_j^2 - \frac{3}{2}(1-\partial_x^2)^{-1}\partial_x S_j^2,$$

we infer that ε_1 satisfies on $[0, t_0]$,

$$\begin{aligned} \partial_t \varepsilon_1 - \sum_{j=1}^{N} (\dot{\tilde{x}}_j(t) - c_j)\partial_x S_j & \\ &= -\alpha \partial_x \varepsilon_1 - \frac{1}{2}\partial_x \left[\left(\varepsilon_1 + \sum_{j=1}^{N} S_j \right)^2 - \sum_{j=1}^{N} S_j^2 \right] \\ &\quad - \frac{3}{2}\partial_x (1-\partial_x^2)^{-1} \left[\left(\varepsilon_1 + \sum_{j=1}^{N} S_j \right)^2 - \sum_{j=1}^{N} S_j^2 \right]. \end{aligned}$$

Multiplying by $(4-\partial_x^2)^{-1}$ and using (3.19), we get

$$\partial_t \varepsilon_2 - \sum_{j=1}^{N} (\dot{\tilde{x}}_j(t) - c_j)\partial_x R_j = -\alpha \partial_x \varepsilon_2 - \frac{1}{2}\partial_x (1-\partial_x^2)^{-1} \left[\left(\varepsilon_1 + \sum_{j=1}^{N} S_j \right)^2 - \sum_{j=1}^{N} S_j^2 \right].$$

Taking the L^2 scalar product with $\partial_x R_i$, and integrating by parts, we find

$$\begin{aligned}
&-(\dot{\tilde{x}}_i(t)-c_i)\int_{\mathbb{R}}(\partial_x R_i)^2 \\
&\quad=-\int_{\mathbb{R}}\partial_t\varepsilon_2\partial_x R_i+\sum_{\substack{1\le j\le N\\ j\neq i}}(\dot{\tilde{x}}_j(t)-c_j)\int_{\mathbb{R}}(\partial_x R_i)(\partial_x R_j) \\
&\qquad+\alpha\int_{\mathbb{R}}\varepsilon_2\partial_x^2 R_i+\frac{1}{2}\int_{\mathbb{R}}(1-\partial_x^2)^{-1}\left[\left(\varepsilon_1+\sum_{j=1}^N S_j\right)^2-\sum_{j=1}^N S_j^2\right]\partial_x^2 R_i. \qquad (4.17)
\end{aligned}$$

We set

$$Q=\left(\varepsilon_1+\sum_{j=1}^N S_j\right)^2-\sum_{j=1}^N S_j^2=\varepsilon_1^2+2\varepsilon_1\left(\sum_{j=1}^N S_j\right)+\sum_{\substack{1\le i,j\le N\\ j\neq i}}S_iS_j,$$

and multiplying by $(1-\partial_x^2)^{-1}$, we have

$$\begin{aligned}
(1-\partial_x^2)^{-1}Q&=(1-\partial_x^2)^{-1}\varepsilon_1^2+2\sum_{j=1}^N(1-\partial_x^2)^{-1}(\varepsilon_1 S_j)+\sum_{\substack{1\le i,j\le N\\ j\neq i}}(1-\partial_x^2)^{-1}(S_iS_j)\\
&=I+J+K.
\end{aligned}$$

We derive the following estimates

$$I=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-x'|}\varepsilon_1^2(x')\le\frac{1}{2}\|e^{-|\cdot|}\|_{L^\infty(\mathbb{R})}\|\varepsilon_1\|_{L^2(\mathbb{R})}^2\le\frac{1}{2}\|\varepsilon_1\|_{L^2(\mathbb{R})}^2,$$

$$J=\sum_{j=1}^N\int_{\mathbb{R}}e^{-|x-x'|}\varepsilon_1(x')S_j(x')\le\|e^{-|\cdot|}\|_{L^\infty(\mathbb{R})}\|\varepsilon_1\|_{L^2(\mathbb{R})}\sum_{j=1}^N\|S_j\|_{L^2(\mathbb{R})}\le\left(\sum_{j=1}^N c_j\right)\|\varepsilon_1\|_{L^2(\mathbb{R})}$$

and

$$K=\frac{1}{2}\sum_{\substack{1\le i,j\le N\\ j\neq i}}\int_{\mathbb{R}}e^{-|x-x'|}S_j(x')S_i(x')\le\frac{1}{2}\sum_{\substack{1\le i,j\le N\\ j\neq i}}\int_{\mathbb{R}}S_j(x')S_i(x').$$

Thus, using (4.4) and the exponential decay of S_j, we infer that

$$\|(1-\partial_x^2)^{-1}Q\|_{L^\infty(\mathbb{R})}\le O(\gamma)+O(e^{-L/4})$$

and then

$$\begin{aligned}
\left|\frac{1}{2}\int_{\mathbb{R}}[(1-\partial_x^2)^{-1}Q]\partial_x^2 R_i\right|&\le\frac{1}{2}\|(1-\partial_x^2)^{-1}Q\|_{L^\infty(\mathbb{R})}\|\partial_x^2 R_i\|_{L^1(\mathbb{R})}\\
&\le O(\gamma)+O(e^{-L/4}). \qquad (4.18)
\end{aligned}$$

Using (4.5), the term which depends on α give us

$$\begin{aligned}
\left|\alpha\int_{\mathbb{R}}\varepsilon_2\partial_x^2 R_i\right|&\le|\alpha|\|\varepsilon_2\|_{L^\infty(\mathbb{R})}\|\partial_x^2 R_i\|_{L^1(\mathbb{R})}\\
&\le O(\gamma). \qquad (4.19)
\end{aligned}$$

Now, combining (4.16)-(4.19) and using the exponential decay of R_i, it holds

$$\left|\dot{\tilde{x}}_i(t) - c_i\right| \|\partial_x R_i\|^2_{L^2(\mathbb{R})} \leq \left|\dot{\tilde{x}}_i(t) - c_i\right| O(\gamma) + O(\gamma) + O(e^{-L/4}),$$

then

$$\left|\dot{\tilde{x}}_i(t) - c_i\right| \left(\frac{(c_i-\alpha)^2}{54} - O(\gamma)\right) \leq O(\gamma) + O(e^{-L/4}),$$

which yields (4.6).

Taking $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, with $\gamma_0 \ll 1$ and $L_0 \gg 1$, combining (1.10)-(1.12), (4.6) and (4.13), we deduce that

$$\begin{aligned}\tilde{x}_i(t) - \tilde{x}_{i-1}(t) &= \tilde{x}_i(0) - \tilde{x}_{i-1}(0) + (c_i - c_{i-1})t \\ &\geq L - 2C_0\gamma_0 + \frac{(c_i - c_{i-1})t}{2} \\ &\geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2},\end{aligned}$$

this proves (4.7).

From (4.5), we infer that

$$v(x) = \sum_{j=1}^{N} \rho_{c_j,\alpha}(x - \tilde{x}_j) + O(\gamma), \quad \forall x \in \mathbb{R},$$

please note that, we abuse notation by writing $\varepsilon_2(x) = O(\gamma)$. Applying this formula with $x = \xi_i$ and $v(\xi_i) = \max_{x\in J_i} v(x)$, and using (4.7), it holds

$$\begin{aligned}v(\xi_i) &= \max_{x\in J_i} \left\{\sum_{j=1}^{N} \rho_{c_j,\alpha}(x - \tilde{x}_j)\right\} + O(\gamma) \\ &= \frac{c_i - \alpha}{6} + O(e^{-L/4}) + O(\gamma) \\ &\geq \frac{c_i - \alpha}{7}.\end{aligned}$$

On the other hand, for $x \in J_i \setminus [\tilde{x}_i(t) - L/12,\ \tilde{x}_i(t) + L/12]$, we get

$$v(x) \leq \frac{c_i - \alpha}{3} e^{-L/12} + O(e^{-L/4}) + O(\gamma) \leq \frac{c_i - \alpha}{8}.$$

This ensures that $\xi_i \in [\tilde{x}_i(t) - L/12,\ \tilde{x}_i(t) + L/12]$, and this concluded the proof of the lemma.

4.2 Monotonicity property

Thanks to the preceding lemma, for $\varepsilon_0 > 0$ small enough and $L_0 > 0$ large enough, one can construct N C^1 functions $\tilde{x}_1, \ldots, \tilde{x}_N$ defined on $[0, t_0]$ such that (4.4)-(4.8) are satisfied. In this subsection we state the almost monotonicity of functionals that are very close to the energy at the right of ith bump, $i = 1, \ldots, N-1$ of u (respectively $v = (4 - \partial_x^2)^{-1}u$). Let ψ be a C^∞ test-function (see Figure 3) such that

$$\begin{cases} 0 < \psi(x) < 1,\ \psi'(x) > 0, & x \in \mathbb{R}, \\ |\psi^{(q)}(x)| \leq 10\psi'(x),\ q = 2, 3, 4, 5, & x \in [-10, 10], \end{cases} \tag{4.20}$$

and

$$\psi(x) = \begin{cases} e^{-|x|}, & x < -10, \\ 1 - e^{-|x|}, & x > 10. \end{cases} \tag{4.21}$$

Setting $\psi_K = \psi(\cdot/K)$, we introduce for $i = 2, \dots, N$,

$$\mathcal{J}_{i,K}(t) = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \psi_{i,K}(t), \tag{4.22}$$

where $\psi_{i,K}(t,x) = \psi_K(x - y_i(t))$ with y_i's as in (4.8). Note that $\mathcal{J}_{i,K}(t)$ is close to $\|u(t)\|^2_{\mathcal{H}(x>y_i(t))}$ (respectively $\|v(t)\|^2_{H^2(x>y_i(t))}$) and thus measures the energy at the right of the $(i-1)$th bump of u (respectively v). Finally, we set

$$\sigma_0 = \frac{1}{4} \min\{c_1, c_2 - c_1, \dots, c_N - c_{N-1}\}. \tag{4.23}$$

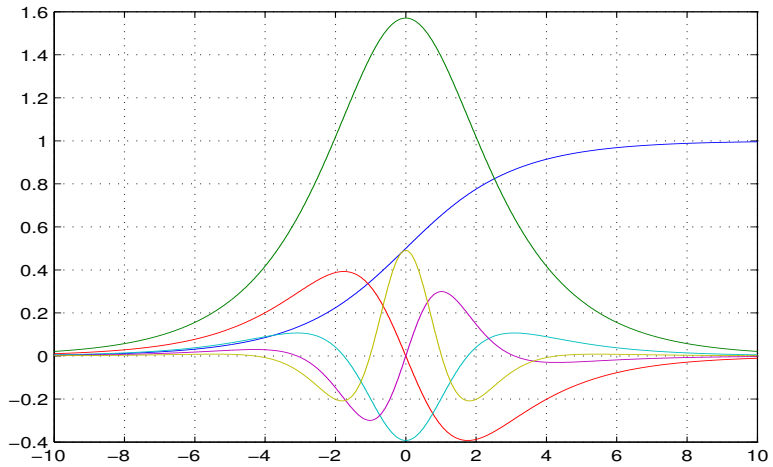


Figure 3: $\psi^{(q)}(x)$, $q = 0, 1, 2, 3, 4, 5$, profiles. Here $\psi(x) = a_0 + a_1 \tanh(a_2 x)$, with $a_0, a_1, a_2 > 0$.

We derive the following monotonicity result.

Proposition 4.1 (Exponential decay of the functional $\mathcal{J}_{i,K}(t)$)**.** *Assume that* $\max(0, \alpha) < c_1$. *Let* $u \in \mathcal{X}([0,T[)$, *with* $0 < T \le +\infty$, *be a solution of the DPsd equation that satisfies* (1.10)-(1.12) *and* (4.4)-(4.5). *There exist* $\gamma_0 > 0$ *and* $L_0 > 0$ *only depending on the speed* c_1 *and the parameter* α, *such that if* $0 < \gamma < \gamma_0$ *and* $L > L_0 > 0$, *then for any* $4 \le K \lesssim \sqrt{L}$,

$$\mathcal{J}_{i,K}(t) - \mathcal{J}_{i,K}(0) \le O(e^{-\frac{L}{8K}}), \quad \forall t \in [0, t_0], \quad i = 2, \dots, N. \tag{4.24}$$

The proof of Proposition 4.1 relies on the following Virial type identity.

Lemma 4.2 (Virial type identity)**.** *Let* $u \in \mathcal{X}([0,T[)$, *with* $0 < T \le +\infty$, *be a solution of the DPsd equation that satisfies* (1.10)-(1.12). *For any smooth space function* $g : \mathbb{R} \mapsto \mathbb{R}$, *it holds*

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) g \\
&\quad = \frac{2}{3} \int_{\mathbb{R}} u^3 g' - 4 \int_{\mathbb{R}} u^2 v g' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g''' + \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g'' + \int_{\mathbb{R}} u h g' \\
&\qquad + \frac{1}{2} \int_{\mathbb{R}} u h_x g'' - \frac{5}{2} \int_{\mathbb{R}} v h_x g'' - 2 \int_{\mathbb{R}} v_x h g'' + \frac{1}{2} \int_{\mathbb{R}} v h_x g^{(4)} \\
&\qquad\quad + \alpha \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}) g' - \frac{5\alpha}{2} \int_{\mathbb{R}} v^2 g''' - 2\alpha \int_{\mathbb{R}} v_x^2 g''' + \frac{\alpha}{2} \int_{\mathbb{R}} v^2 g^{(5)},
\end{aligned} \tag{4.25}$$

where $y = (1 - \partial_x^2) u$, $v = (4 - \partial_x^2)^{-1} u$, *and* $h = (1 - \partial_x^2)^{-1} u^2$.

Since $E(\cdot)$ is a conservation law, then by computing $\frac{d}{dt}\int \left(4v^2+5v_x^2+v_{xx}^2\right) g$ it is sure to have only integrals with the derivatives of g as in (4.25) (all terms of the form $\int(\ldots)g$ will cancel each other). Then taking $g=\psi$ (as in Figure 4), we establish the monotony of functional $\mathcal{J}_{i,K}(t)$. The full proof of Lemma 4.2 is given in the appendix.

Proof of Proposition 3.1. We first note that, combining (4.6) and (4.8), it holds for $i=2,\ldots,N$,

$$\begin{aligned}\dot{y}_i(t) &= \frac{\dot{\tilde{x}}_{i-1}(t)+\dot{\tilde{x}}_i(t)}{2}\\ &= \frac{c_{i-1}+c_i}{2}+O(\gamma^{1/2})\\ &\geq c_{i-1}+O(\gamma^{1/2})\\ &\geq c_1+O(\gamma^{1/2})\end{aligned} \tag{4.26}$$

Recall that the assumption (1.10) ensures that $u\geq 0$ and $v\geq 0$ on $\mathbb{R}$. Now, applying the Virial type identity (4.25) with $g=\psi_{i,K}$, using that the term $-4\int u^2 v\psi_{i,K}'$ is negative, and (4.26), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{J}_{i,K}(t) &= -\dot{y}_i\int_{\mathbb{R}}\left(4v^2+5v_x^2+v_{xx}^2\right)\psi_{i,K}' + \frac{2}{3}\int_{\mathbb{R}}u^3\psi_{i,K}' - 4\int_{\mathbb{R}}u^2v\psi_{i,K}' - \frac{1}{2}\int_{\mathbb{R}}u^2v\psi_{i,K}'''\\ &\quad + \frac{1}{2}\int_{\mathbb{R}}u^2v_x\psi_{i,K}'' + \int_{\mathbb{R}}uh\psi_{i,K}' + \frac{1}{2}\int_{\mathbb{R}}uh_x\psi_{i,K}'' - \frac{5}{2}\int_{\mathbb{R}}vh_x\psi_{i,K}''\\ &\quad - 2\int_{\mathbb{R}}v_xh\psi_{i,K}'' + \frac{1}{2}\int_{\mathbb{R}}vh_x\psi_{i,K}^{(4)} + \alpha\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx})\psi_{i,K}'\\ &\quad - \frac{5\alpha}{2}\int_{\mathbb{R}}v^2\psi_{i,K}''' - 2\alpha\int_{\mathbb{R}}v_x^2\psi_{i,K}''' + \frac{\alpha}{2}\int_{\mathbb{R}}v^2\psi_{i,K}^{(5)}\\ &\leq \left(\alpha-c_1+O(\gamma^{1/2})\right)\int_{\mathbb{R}}\left(4v^2+5v_x^2+v_{xx}^2\right)\psi_{i,K}' + \sum_{k=1}^{8}I_k + I_{9,\alpha}+I_{10,\alpha}+I_{11,\alpha}.\end{aligned} \tag{4.27}$$

Next, we divide $\mathbb{R}$ into two regions D_i and D_i^c with

$$D_i = \left[\tilde{x}_{i-1}(t)+\frac{L}{4}, \tilde{x}_i(t)-\frac{L}{4}\right], \quad i=2,\ldots,N.$$

Combining (4.7), (4.8) and (4.23), one can check that for $x\in D_i^c$,

$$\begin{aligned}|x-y_i(t)| &\geq \frac{\tilde{x}_i(t)-\tilde{x}_{i-1}(t)}{2}-\frac{L}{4}\\ &\geq \frac{c_i-c_{i-1}}{4}t+\frac{L}{8}\\ &\geq \sigma_0 t+\frac{L}{8}.\end{aligned} \tag{4.28}$$

Now, we claim that for $k=1,\ldots,8$, it holds

$$I_k \leq \left(O(\gamma^{1/2})+O(e^{-L/8})\right)\int_{\mathbb{R}}\left(4v^2+5v_x^2+v_{xx}^2\right)\psi_{i,K}' + \frac{C}{K}\|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{K}(\sigma_0 t+L/8)}, \tag{4.29}$$

and that for $k=9,10,11$,

$$I_{k,\alpha}\leq O(L^{-1})\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\psi_{i,K}'. \tag{4.30}$$

The estimates (4.29) was completed in detail in [8], so we will present a brief proof of some of them. Let

us begin by the estimate of I_1. Using (4.28) and the exponential decay of $\psi'_{i,K}$ on D_i^c, we have

$$\begin{aligned}\frac{2}{3}\int_{\mathbb{R}} u^3\psi'_{i,K} &= \frac{2}{3}\int_{D_i} u^3\psi'_{i,K} + \frac{2}{3}\int_{D_i^c} u^3\psi'_{i,K}\\ &\le \frac{2}{3}\|u\|_{L^\infty(D_i)}\int_{\mathbb{R}} u^2\psi'_{i,K} + \frac{2}{3}\|\psi'_{i,K}\|_{L^\infty(D_i^c)}\|u\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R})}\\ &\le \frac{2}{3}\|u\|_{L^\infty(D_i)}\int_{\mathbb{R}} u^2\psi'_{i,K} + \frac{C}{K}\|u\|^3_{L^2(\mathbb{R})}e^{-\frac{1}{K}(\sigma_0 t+L/8)}. \end{aligned} \tag{4.31}$$

Now, using that $u = 4v - v_{xx}$, $|\psi'''_{i,K}| \le (10/K^2)\psi'_{i,K}$ on D_i and $|\psi'''_{i,K}|$ decays exponentially on D_i^c, and that $\|v\|_{L^2(\mathbb{R})} \le \|u\|_{\mathcal{H}} = \|u_0\|_{\mathcal{H}}$, and $K \ge 4$, we have

$$\begin{aligned}\int_{\mathbb{R}} u^2\psi'_{i,K} &= \int_{\mathbb{R}} (4v - v_{xx})^2\psi'_{i,K}\\ &= 16\int_{\mathbb{R}} v^2\psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2\psi'_{i,K} - 8\int_{\mathbb{R}} vv_{xx}\psi'_{i,K}\\ &= 16\int_{\mathbb{R}} v^2\psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2\psi'_{i,K} + 8\int_{\mathbb{R}} v_x^2\psi'_{i,K} + 4\int_{\mathbb{R}} \partial_x(v^2)\psi''_{i,K}\\ &= 16\int_{\mathbb{R}} v^2\psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2\psi'_{i,K} + 8\int_{\mathbb{R}} v_x^2\psi'_{i,K} - 4\int_{\mathbb{R}} v^2\psi'''_{i,K}\\ &\le \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right)\psi'_{i,K} + 4\int_{D_i} v^2|\psi'''_{j,K}| + 4\int_{D_i^c} v^2|\psi'''_{j,K}|\\ &\le \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right)\psi'_{i,K} + \frac{40}{K^2}\int_{\mathbb{R}} v^2\psi'_{i,K} + \frac{C}{K^3}\|u_0\|^2_{\mathcal{H}}e^{-\frac{1}{K}(\sigma_0 t+L/8)}\\ &\le 5\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right)\psi'_{i,K} + \frac{C}{K^3}\|u_0\|^2_{\mathcal{H}}e^{-\frac{1}{K}(\sigma_0 t+L/8)}. \end{aligned} \tag{4.32}$$

Next, applying the triangular inequality, and using (4.4) and that $|\varphi'_{c_i,\alpha}| = \varphi_{c_i,\alpha}$ on $\mathbb{R}$, we have

$$\begin{aligned}\|\varepsilon_1\|_{H^1(\mathbb{R})} &\le \|u\|_{H^1(\mathbb{R})} + \sum_{j=1}^N \|\varphi_{c_j,\alpha}\|_{H^1(\mathbb{R})}\\ &\le 2\|u\|_{L^2(\mathbb{R})} + 2\sum_{j=1}^N \|\varphi_{c_j,\alpha}\|_{L^2(\mathbb{R})}\\ &\le 2\|\varepsilon_1\|_{L^2(\mathbb{R})} + 4\sum_{j=1}^N \|\varphi_{c_j,\alpha}\|_{L^2(\mathbb{R})}\\ &\le O(\gamma) + O(1), \end{aligned} \tag{4.33}$$

where $\varepsilon_1 = u - \sum_{j=1}^N \varphi_{c_j,\alpha}(\cdot - \tilde{x}_j)$ on $\mathbb{R}$ as in Lemma 4.1, and $O(1) = 4\sum_{j=1}^N (c_j - \alpha)$. Then, applying the Gagliardo-Nirenberg inequality, and using (4.33), we obtain

$$\begin{aligned}\|\varepsilon_1\|_{L^\infty(\mathbb{R})} &\le C_G\|\varepsilon_1\|^{1/2}_{L^2(\mathbb{R})}\|\varepsilon_1\|^{1/2}_{H^1(\mathbb{R})}\\ &\le O(\gamma^{1/2})(O(\gamma^{1/2}) + O(1))\\ &\le O(\gamma^{1/2}). \end{aligned} \tag{4.34}$$

Thus, using (4.34) and the exponential decay of $\varphi_{c_i,\alpha}$ on D_i, it holds

$$\begin{aligned}\|u\|_{L^\infty(D_i)} &\leq \|\varepsilon_1\|_{L^\infty(D_i)} + \sum_{j=1}^{N} \left\|\varphi_{c_j,\alpha}(\cdot - \tilde{x}_j(t))\right\|_{L^\infty(D_i)} \\ &\leq O(\gamma^{1/2}) + O(e^{-L/8}). \end{aligned} \tag{4.35}$$

Therefore, combining (4.31), (4.32) and (4.35), we deduce that I_1 satisfies (4.29).

As the second example, let us do the estimate of I_4. On D_i^c we obtain

$$\begin{aligned}\int_{D_i^c} uh\psi'_{i,K} &\leq \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \int_{\mathbb{R}} uh \\ &= \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \int_{\mathbb{R}} u[(1-\partial_x^2)^{-1}u^2] \\ &= \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \int_{\mathbb{R}} u^2[(1-\partial_x^2)^{-1}u] \\ &\leq \|\psi'_{i,K}\|_{L^\infty(D_j^c)} \|(1-\partial_x^2)^{-1}u\|_{L^\infty(\mathbb{R})} \|u\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Applying the Hölder inegality, we have for all $x \in \mathbb{R}$,

$$(1-\partial_x^2)^{-1}u(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-x'|}u(x')dx' \leq \frac{1}{2}\|e^{-|\cdot|}\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})} = \frac{1}{2}\|u\|_{L^2(\mathbb{R})},$$

and thus, using the exponential decay of $\psi'_{i,K}$ on D_i^c, we infer that

$$\int_{D_i^c} uh\psi'_{i,K} \leq \frac{1}{2K}\|u\|^3_{L^2(\mathbb{R})} e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \tag{4.36}$$

On D_i the estimate of I_4 give us

$$\begin{aligned}\int_{D_i} uh\psi'_{i,K} &\leq \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} [(1-\partial_x^2)^{-1}u^2]\psi'_{i,K} \\ &= \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2[(1-\partial_x^2)^{-1}\psi'_{i,K}]. \end{aligned} \tag{4.37}$$

On the other hand, using that $|\psi'''_{i,K}| \leq (10/K^2)\psi'_{i,K}$ on $\mathbb{R}$, we have

$$(1-\partial_x^2)\psi'_{i,K}(x) = \psi'_{i,K}(x) - \frac{1}{K^2}\psi'''_{i,K}(x) \geq \left(1 - \frac{10}{K^2}\right)\psi'_{i,K}(x), \quad \forall x \in \mathbb{R},$$

and since $K \geq 4$, it holds

$$(1-\partial_x^2)^{-1}\psi'_{i,K}(x) \leq \left(1 - \frac{10}{K^2}\right)^{-1}\psi'_{i,K}(x), \quad \forall x \in \mathbb{R}. \tag{4.38}$$

Therefore, combining (4.35)-(4.38), we infer that I_4 also satisfies (4.29).

Let us tackle now the estimate of I_6. First, noticing that for all $x \in \mathbb{R}$,

$$h(x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{x'}u^2(x')dx' + \frac{e^x}{2}\int_x^{+\infty} e^{-x'}u^2(x')dx'$$

and

$$h_x(x) = -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{x'}u^2(x')dx' + \frac{e^x}{2}\int_x^{+\infty} e^{-x'}u^2(x')dx',$$

and we infer that

$$|h_x(x)| \leq h(x), \quad \forall x \in \mathbb{R}. \tag{4.39}$$

Using (4.39), the estimate of I_6 on D_i^c leads to

$$\begin{aligned}
\left|\frac{5}{2}\int_{D_i^c} v h_x \psi''_{i,K}\right| &\leq \frac{5}{2}\int_{D_i^c} v|h_x||\psi''_{i,K}| \\
&\leq \frac{5}{2}\|\psi''_{i,K}\|_{L^\infty(D_i^c)}\int_{\mathbb{R}} v|h_x| \\
&\leq \frac{5}{2}\|\psi''_{i,K}\|_{L^\infty(D_i^c)}\int_{\mathbb{R}} vh \\
&= \frac{5}{2}\|\psi''_{i,K}\|_{L^\infty(D_i^c)}\int_{\mathbb{R}} [(1-\partial_x^2)^{-1}v]u^2 \\
&\leq \frac{5}{2}\|\psi''_{i,K}\|_{L^\infty(D_i^c)}\|(1-\partial_x^2)^{-1}v\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

With the aid of identity (3.19), we have for all $x \in \mathbb{R}$,

$$\begin{aligned}
(1-\partial_x^2)^{-1}v(x) &\leq \frac{1}{3}|(1-\partial_x^2)^{-1}u(x)| + \frac{1}{3}|(4-\partial_x^2)^{-1}u(x)| \\
&\leq \frac{1}{6}\|u\|_{L^2(\mathbb{R})} + \frac{1}{12\sqrt{2}}\|u\|_{L^2(\mathbb{R})} \\
&= \frac{4+\sqrt{2}}{24}\|u\|_{L^2(\mathbb{R})},
\end{aligned} \tag{4.40}$$

Then, using (4.40) we deduce that

$$\left|\frac{5}{2}\int_{D_i^c} v h_x \psi''_{i,K}\right| \leq \frac{20-5\sqrt{2}}{48K^2}\|u\|^3_{L^2(\mathbb{R})} e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \tag{4.41}$$

Next, using that $|\psi''_{i,K}| \leq (10/K)\psi'_{i,K}$ on $\mathbb{R}$ and (4.39), the estimate of I_6 on D_i yields

$$\begin{aligned}
\left|\frac{5}{2}\int_{D_i} v h_x \psi''_{i,K}\right| &\leq \frac{5}{2}\|v\|_{L^\infty(D_i)}\int_{\mathbb{R}} |h_x||\psi''_{i,K}| \\
&\leq \frac{25}{K}\|v\|_{L^\infty(D_i)}\int_{\mathbb{R}} h\psi'_{i,K} \\
&= \frac{25}{K}\|v\|_{L^\infty(D_i)}\int_{\mathbb{R}} u^2[(1-\partial_x^2)^{-1}\psi'_{i,K}].
\end{aligned} \tag{4.42}$$

Now, using the exponential decay of $\rho_{c_i,\alpha}$ on D_i and (4.4), it holds

$$\begin{aligned}
\|v\|_{L^\infty(D_i)} &\leq \|\varepsilon_2\|_{L^\infty(D_i)} + \sum_{j=1}^N \left\|\rho_{c_j,\alpha}(\cdot - \tilde{x}_j(t))\right\|_{L^\infty(D_i)} \\
&\leq O(\gamma) + O(e^{-L/8}),
\end{aligned} \tag{4.43}$$

where $\varepsilon_2 = v - \sum_{j=1}^N \rho_{c_i,\alpha}(\cdot - \tilde{x}_j)$ on $\mathbb{R}$ as in Lemma 4.1. Therefore, combining (4.38), (4.41)-(4.43), we deduce that I_6 also satisfies (4.29).

For the estimate of I_7, we have two possibility. First, one can use that $|v_x| \leq 2v$ on $\mathbb{R}$, and proceed as for the estimate of I_6. Second, using the exponential decay of $\rho_{c_i,\alpha}$ and $|\rho'_{c_i,\alpha}|$ on D_i, and (4.5), it holds

$$\begin{aligned}
\|v\|_{C^1(D_i)} &\leq \|\varepsilon_2\|_{C^1(D_i)} + \sum_{j=1}^N \left\|\rho_{c_j,\alpha}(\cdot - \tilde{x}_j(t))\right\|_{C^1(D_i)} \\
&\leq O(\gamma) + O(e^{-L/8}).
\end{aligned} \tag{4.44}$$

Then, using (4.44) and proceeding as for the estimate of I_6, one can obtain the estimate of I_7. In the same way, one can establish the estimates of the remaining terms. Please note that, since $u \leq 6v$ on $\mathbb{R}$ (see (3.21)) and using (4.44), one can obtain the estimates of I_1, I_2, I_3, I_4 and I_5 more easily. Also, in the estimates (4.29) the error term becomes $O(\gamma)$ instead of $O(\gamma^{1/2})$.

Let us now estimate the terms which depend on the parameter α. By the definition of ψ, we have $|\psi_{i,K}'''| \leq (10/K^2)\psi_{i,K}'$ and $|\psi_{i,K}^{(5)}| \leq (10/K^4)\psi_{i,K}'$ on $\mathbb{R}$. Thus, using that $K = O(L^{1/2})$ and $\|v\|_{H^1(\mathbb{R})} \leq \|u\|_{\mathcal{H}}$, we easy deduce that (4.30) holds for $k = 9, 10, 11$.

Finally, combining (4.27), (4.29) and (4.30), it holds actually

$$\frac{d}{dt}\mathcal{J}_{i,K}(t) \leq \Big(\alpha - c_1 + O(\gamma^{1/2}) + O(L^{-1})\Big)\int_{\mathbb{R}}(4v^2 + 5v_x^2 + v_{xx}^2)\psi_{i,K}' + \frac{C}{K}\|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

Therefore, taking $\max(0,\alpha) < c_1$, $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, with $\gamma_0 \ll 1$ and $L_0 \gg 1$, and integrating between 0 and t, we obtain

$$\begin{aligned}\mathcal{J}_{i,K}(t) - \mathcal{J}_{i,K}(0) &\leq \frac{C}{K}\|u_0\|_{\mathcal{H}}^3\left(-\frac{K}{\sigma_0}e^{-\frac{1}{K}(\sigma_0 t + L/8)} + \frac{K}{\sigma_0}e^{-\frac{L}{8K}}\right)\\ &\leq \frac{C}{\sigma_0}\|u_0\|_{\mathcal{H}}^3 e^{-\frac{L}{8K}},\end{aligned}$$

and this proves the proposition for smooth initial solutions.

For $u \in \mathcal{X}([0,T[)$, we will use that for any $T_0 > 0$ and any sequence $(u_{0,n})_{n\geq 1} \subset L^2(\mathbb{R})$ such that $(u_{0,n} - \partial_x^2 u_{0,n})_{n\geq 1} \subset \mathcal{M}^+(\mathbb{R})$ and $u_{0,n} \to u_0$ in $L^2(\mathbb{R})$, the sequence of emanating global weak solutions $(u_n)_{n\geq 1}$ to the DPsd equation satisfies

$$u_n \underset{n\to+\infty}{\longrightarrow} u \ \text{ in } \ C\left([0,T_0];L^2(\mathbb{R})\right), \tag{4.45}$$

where u is the global weak solution emanating from u_0. This fact can be easily deduced from the proof of the existence of the global weak solutions in [5]. Indeed, by the same arguments developed in this proof, we obtain that, up to a subsequence, $(u_n)_{n\geq 1}$ converges in $C\left([0,T_0];L^2(\mathbb{R})\right)$ towards a solution of the DPsd equation emanating from u_0. (4.45) then follows by the uniqueness result. Finally, combinig (4.45) and Remark 4.1, it holds

$$v_n \underset{n\to+\infty}{\longrightarrow} v \ \text{ in } \ C\left([0,T_0];H^2(\mathbb{R})\right), \tag{4.46}$$

where $v = (4 - \partial_x^2)^{-1}u$. Therefore, combining (4.45) and (4.46), it is not too hard to check that (see [8], Proposition 4.1, for details)

$$\lim_{n\to+\infty}\sup_{0\leq t<T}|\mathcal{J}_{i,K}(u_n(t)) - \mathcal{J}_{i,K}(u(t))| = 0. \tag{4.47}$$

Thanks to this continuity property, the monotonicity formula (4.24) holds for any $u \in \mathcal{X}([0,T[)$, with $0 < T \leq +\infty$.

4.3 A localized and a global estimate

Let $K = \sqrt{L}/8$ and define the function $\phi_i = \phi_i(t,x)$ (see Figure 4) by

$$\begin{cases}\phi_1 = 1 - \psi_{2,K} = 1 - \psi_K(\cdot - y_2(t)),\\ \phi_i = \psi_{i,K} - \psi_{i+1,K} = \psi_K(\cdot - y_i(t)) - \psi_K(\cdot - y_{i+1}(t)), \quad i = 2,\dots,N-1,\\ \phi_N = \psi_{N,K} = \psi_K(\cdot - y_N(t)),\end{cases} \tag{4.48}$$

where $\psi_{i,K}$'s and y_i's are defined in Subsection 4.2. One can see that the ϕ_i's are positive functions and that $\sum_{i=1}^N \phi_i = 1$. We take $L/K > 4$ so that ϕ_i satisfies for $i = 1, \dots, N$,

$$|1 - \phi_i| \leq 2e^{-\frac{L}{8K}} \quad \text{on} \quad \left]y_i + \frac{L}{8}, y_{i+1} - \frac{L}{8}\right[\tag{4.49}$$

and

$$|\phi_i| \leq 2e^{-\frac{L}{8K}} \quad \text{on} \quad \mathbb{R} \setminus \left]y_i - \frac{L}{8}, y_{i+1} + \frac{L}{8}\right[. \tag{4.50}$$

We will use the following localized version of the conservation laws defined for $i = 1, \dots, N$ by

$$E_i(u(t)) = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \phi_i(t), \quad F_i(u(t)) = \int_{\mathbb{R}} \left(-v_{xx}^3 + 12vv_{xx}^2 - 48v^2v_{xx} + 64v^3\right) \phi_i(t). \tag{4.51}$$

One can remark that the functionals $E_i(\cdot)$ and $F_i(\cdot)$ do not depend on time in the statement below since we fix $-\infty = y_1 < y_2 < \ldots < y_N < y_{N+1} = +\infty$.

For $i = 1, \dots, N$, we set $\Omega_i =]y_i - L/8, y_{i+1} + L/8[$, the interval in which the mass of each peakon $\varphi_{c_i,\alpha}$ (and smooth-peakon $\rho_{c_i,\alpha}$) is concentrated. One can see that

$$\sum_{j=1}^N \rho_{c_j}(x - \tilde{x}_j) = \rho_{c_i}(x - \tilde{x}_i) + O(e^{-L/4}), \quad \forall x \in \Omega_i, \tag{4.52}$$

we abuse notation by writing $\rho_{c_i,\alpha}(x - \tilde{x}_i) = O(e^{-L/4})$ for all $x \in \mathbb{R} \setminus \Omega_i$. We will decompose Ω_i as in Section 3 by setting

$$\Theta_i = [\tilde{x}_i - 6.7, \tilde{x}_i + 6.7], \quad \text{where} \quad 6.7 \simeq \ln\left(1 - \frac{\sqrt{399}}{20}\right)^{-1}, \tag{4.53}$$

and note that

$$\rho_{c_i,\alpha}(\pm 6.7) \simeq \frac{c_i - \alpha}{2400}. \tag{4.54}$$

Then, repeating the proof of Lemma 3.4 on each Ω_i, we deduce the following result.

Lemma 4.3 (Uniqueness of local maxima)**.** *Let $u \in L^2(\mathbb{R})$, with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies (4.4)-(4.5). There exist $\gamma_0 > 0$ and $L_0 > 0$ only depending on the speeds $(c_i)_{i=1}^N$ and the parameter α, such that if $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, then for $i = 1, \dots, N$, the function $v = (4 - \partial_x^2)^{-1}u$ admits a unique local extremum on Θ_i. This extremum is a maximum, and it holds*

$$v(x) \leq \frac{c_i - \alpha}{300}, \quad \forall x \in \Omega_i \setminus \Theta_i, \tag{4.55}$$

$$u(x) \leq \frac{c_i - \alpha}{300}, \quad \forall x \in \Omega_i \setminus \Theta_i, \tag{4.56}$$

and

$$\Theta_i \subset \left]y_i + \frac{L}{8}, y_{i+1} - \frac{L}{8}\right[. \tag{4.57}$$

In the sequel of Subsection 4.3, we will denote by $(\xi_i)_{i=1}^N$ the point with local maximum values of v on $\bigcup_{i=1}^N \Theta_i$, and the corresponding local maximum by $M_i = \max_{x \in \Theta_i} v(x) = \max_{x \in J_i \supset \Omega_i} v(x) = v(\xi_i)$ as in Lemma 4.1.

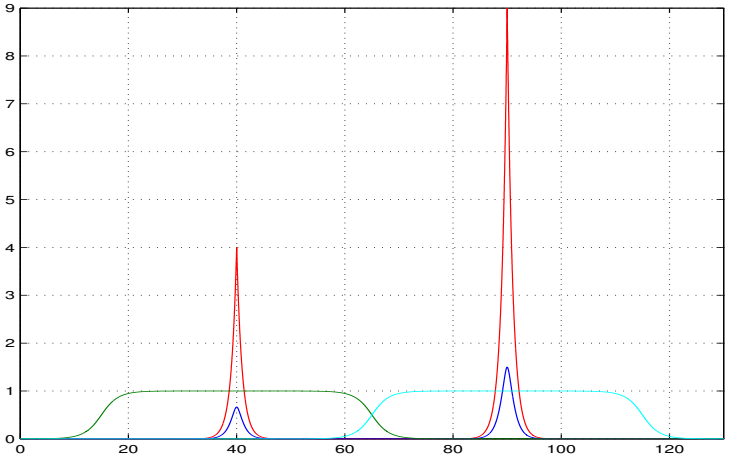


Figure 4: Locatization-functions $\phi_{green}(x) = \psi(x-15) - \psi(x-65)$, $\phi_{cyan}(x) = \psi(x-65) - \psi(x-115)$ (at time $t = 10$) profiles. Also the sum of two peakons $4\varphi(x-40) + 9\varphi(x-90)$ and two smooth-peakons $4\rho(x-40) + 9\rho(x-90)$ (at time $t = 10$, with respective speeds 4, 9, and $\alpha = 0$) profiles. In this example, one can see that ϕ_{green} (respectively ϕ_{cyan}) is close to 1 in $]25, 55[$ (respectively in $]75, 105[$) and decays exponentially in $\mathbb{R}\backslash]10, 70[$ (respectively in $\mathbb{R}\backslash]60, 120[$) .

Lemma 4.4 (Connection between $E_i(\cdot)$ and M_i^2; see [8]). *Let $u \in L^2(\mathbb{R})$ and $v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R})$. For $i = 1, \dots, N$, define the function g_i by*

$$g_i(x) = \begin{cases} 2v + v_{xx} - 3v_x, & x < \xi_i, \\ 2v + v_{xx} + 3v_x, & x > \xi_i. \end{cases} \tag{4.58}$$

Then it holds

$$\int_{\mathbb{R}} g_i^2(x)\phi_i(x) = E_i(u) - 12M_i^2\phi_i(\xi_i) + \|u\|_{\mathcal{H}}^2 O(L^{-1/2}). \tag{4.59}$$

Lemma 4.5 (Connection between $F_i(\cdot)$ and M_i^3; see [8]). *Let $u \in L^2(\mathbb{R})$ and $v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R})$. For $i = 1, \dots, N$, define the function h_i by*

$$h_i(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi_i, \\ -v_{xx} + 6v_x + 16v, & x > \xi_i. \end{cases} \tag{4.60}$$

Then it holds

$$\int_{\mathbb{R}} h_i(x)g_i^2(x)\phi_i(x) = F_i(u) - 144M_i^3\phi_i(\xi_i) + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}). \tag{4.61}$$

These results follows directly by repeating the proof of similar lemmas in [8] with a single point ξ_i with local maximum value of v on each Θ_i, for $i = 1, \dots, N$.

Lemma 4.6 (Connection between $E_i(\cdot)$ and $F_i(\cdot)$; see [8]). *Let $u \in L^2(\mathbb{R})$, with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies (4.4)-(4.5). Let be given $N - 1$ real numbers $-\infty = y_1 < y_2 < \ldots < y_N < y_{N+1} = +\infty$ with $y_i - y_{i-1} \geq 2L/3$. There exist $\gamma_0 > 0$ and $L_0 > 0$ only depending on the speeds $(c_i)_{i=1}^N$ and the parameter α, such that if $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, then defining the functional $E_i(\cdot)$'s and $F_i(\cdot)$'s as in (4.48)-(4.51), it holds*

$$F_i(u) \leq 18M_iE_i(u) - 72M_i^3 + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}), \quad i = 1, \dots, N. \tag{4.62}$$

Proof. Combining (4.49), (4.57) and (4.59) with $K = \sqrt{L}/8$, we get

$$\int_{\mathbb{R}} g_i^2(x)\phi_i(x) = E_i(u) - 12M_i^2 + \|u\|_{\mathcal{H}}^2 O(L^{-1/2}). \tag{4.63}$$

Similarly, combining (4.49), (4.57) and (4.61), we get

$$\int_{\mathbb{R}} h_i(x)g_i^2(x)\phi_i(x)dx = F_i(u) - 144M_i^3 + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}). \tag{4.64}$$

Now, let us show that $h_i \leq 18M_i$ on Ω_i. Note that, using (4.4) and (4.52), one can check that $18M_i \geq (c_i - \alpha)/4$. We set $\lambda_i = \tilde{x}_i - 6.7$, $\mu_i = \tilde{x}_i + 6.7$, and we rewrite the function h_i as

$$h_i(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \lambda_i, \\ u - 6v_x + 12v, & \lambda_i < x < \xi_i, \\ u + 6v_x + 12v, & \xi_i < x < \mu_i, \\ -v_{xx} + 6v_x + 16v, & x > \mu_i. \end{cases}$$

Then, if $x \in \Omega_i \setminus \Theta_i$, using that $v_{xx} = 4v - u$, $|v_x| \leq 2v$ on $\mathbb{R}$, (4.55) and (4.56), it holds

$$h_i \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c_i - \alpha}{9} \leq 18M_i.$$

If $\lambda_i < x < \xi_i$, then $v_x \geq 0$, and using that $u \leq 6v$ on $\mathbb{R}$, we have

$$h_i = u - 6v_x + 12v \leq 18v.$$

If $\xi_i < x < \mu_i$, then $v_x \leq 0$, and similarly using that $u \leq 6v$ on $\mathbb{R}$, we obtain

$$h_i = u + 6v_x + 12v \leq 18v.$$

Therefore, it holds

$$h_i(x) \leq 18 \max_{x \in \Omega_i} v(x) = 18M_i, \quad \forall x \in \Omega_i. \tag{4.65}$$

Next, taking $\phi_i \equiv 1$ on $\mathbb{R}$ in (4.59), we have $\|g_i\|_{L^2(\mathbb{R})} \leq \|u\|_{\mathcal{H}}$. Also, one can see that $\|h_i\|_{L^\infty(\mathbb{R})} \leq 38\|v\|_{L^\infty(\mathbb{R})} \leq O(\|u\|_{\mathcal{H}})$. Thus, combining (4.50) and (4.63)-(4.65), we obtain

$$\begin{aligned} F_i(u) - 144M_i^3 &= \int_{\mathbb{R}} h_i(x)g_i^2(x)\phi_i(x)dx + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\ &= \int_{\Omega_i} h_i(x)g_i^2(x)\phi_i(x)dx + \int_{\Omega_i^c} h_i(x)g_i^2(x)\phi_i(x)dx + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\ &\leq 18M_i \int_{\Omega_i} g_i^2(x)\phi_i(x)dx + \|h_i\|_{L^\infty(\mathbb{R})}\|g_i\|_{L^2(\mathbb{R})}^2\|\phi_i\|_{L^\infty(\Omega_i^c)} + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\ &\leq 18M_i E_i(u) - 216M_i^3 + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}), \end{aligned}$$

and we deduce the lemma.

Lemma 4.7 (Quadratic identity; see [8])**.** *Let $Z = (z_i)_{i=1}^N \in \mathbb{R}^N$ with $|z_i - z_{i-1}| \geq L/2$, and $u \in L^2(\mathbb{R})$. It holds*

$$E(u) - \sum_{i=1}^N E(\varphi_{c_i,\alpha}) = \|u - S_Z\|_{\mathcal{H}}^2 + 4\sum_{i=1}^N (c_i - \alpha)\left[v(z_i) - \frac{c_i - \alpha}{6}\right] + O(e^{-L/4}), \tag{4.66}$$

where S_Z is defined in (4.11)*, and $O(\cdot)$ only depends on the speeds $(c_i)_{i=1}^N$ and the parameter α.*

Proof. We compute

$$E(u-S_Z) = E(u) + E(S_Z) - 2\left\langle (1-\partial_x^2)S_Z, (4-\partial_x^2)^{-1}u\right\rangle_{H^{-1},H^1}$$
$$= E(u) + E(S_Z) - 2\sum_{i=1}^{N}\left\langle (1-\partial_x^2)\varphi_{c_i,\alpha}(\cdot - z_i), v\right\rangle_{H^{-1},H^1}. \tag{4.67}$$

Now, using that $|\varphi'_{c_i,\alpha}(\cdot - z_i)| = \varphi_{c_i,\alpha}(\cdot - z_i)$ and integration by parts, we have

$$\begin{aligned}\left\langle (1-\partial_x^2)\varphi_{c_i,\alpha}(\cdot - z_i), v\right\rangle_{H^{-1},H^1} &= \int_{\mathbb{R}} v\varphi_{c_i,\alpha}(\cdot - z_i) + \int_{\mathbb{R}} v_x\varphi'_{c_i,\alpha}(\cdot - z_i)\\ &= \int_{\mathbb{R}} v\varphi_{c_i,\alpha}(\cdot - z_i) + \int_{-\infty}^{z_i} v_x\varphi_{c_i,\alpha}(\cdot - z_i) - \int_{z_i}^{+\infty} v_x\varphi_{c_i,\alpha}(\cdot - z_i)\\ &= 2(c_i-\alpha)v(z_i). \end{aligned}\tag{4.68}$$

A similar calculation leads to

$$\begin{aligned}E(S_Z) &= \left\langle (1-\partial_x^2)S_Z, (4-\partial_x^2)^{-1}S_Z\right\rangle_{H^{-1},H^1}\\ &= \int_{\mathbb{R}} R_Z S_Z + \int_{\mathbb{R}} R'_Z S'_Z\\ &= \int_{\mathbb{R}} R_Z S_Z + \sum_{i=1}^{N}\left(\int_{-\infty}^{z_i} R'_Z\varphi_{c_i,\alpha}(\cdot - z_i) - \int_{z_i}^{+\infty} R'_Z\varphi_{c_i,\alpha}(\cdot - z_i)\right)\\ &= 2\sum_{i=1}^{N}(c_i-\alpha)R_Z(z_i)\\ &= 2\sum_{i=1}^{N}(c_i-\alpha)\rho_{c_i,\alpha}(0) + 2\sum_{\substack{1\le i,j\le N\\ i\neq j}}(c_i-\alpha)\rho_{c_j,\alpha}(z_i-z_j)\\ &= \frac{1}{3}\sum_{i=1}^{N}(c_i-\alpha)^2 + O(e^{-L/4}),\end{aligned}\tag{4.69}$$

where we also use that $|z_i - z_j| > L/4$, the exponential decay of $\rho_{c_j,\alpha}(\cdot - z_j)$, and that $R_Z = (4-\partial_x^2)^{-1}S_Z$ as in Lemma 4.1 (see (4.11)). Thus, combining (4.67)-(4.69), we obtain the lemma.

4.4 End of the proof of Theorem 1.1

Let $u \in \mathcal{X}([0,T[)$, with $0 < T \le +\infty$, be a solution of the DPsd equation satisfying (1.10)-(1.12) and (4.2) for some $t_0 \in]0,T[$. Let us recall that $M_i = v(t_0,\xi_i(t_0)) = \max_{x\in J_i} v(t_0,x)$, with J_i's as in (4.9), and set $\delta_i = (c_i-\alpha)/6 - M_i$. First, from (4.7) and (4.9), we know that for $i = 2,\dots,N$,

$$\xi_i(t_0) - \xi_{i-1}(t_0) \ge 2L/3 > L/2, \tag{4.70}$$

and from (1.11) it is easy to check that (see [8], Lemma 4.7)

$$\left| E(u_0) - \sum_{i=1}^{N} E(\varphi_{c_i,\alpha})\right| \le O(\varepsilon^2) + O(e^{-L/4}). \tag{4.71}$$

Applying (4.66) with $u(t_0)$ and using (4.71), we get

$$\left\| u(t_0) - \sum_{i=1}^{N}\varphi_{c_i,\alpha}(\cdot - \xi_i(t_0))\right\|_{\mathcal{H}}^2 \le 4\sum_{i=1}^{N}(c_i-\alpha)\delta_i + O(\varepsilon^2) + O(e^{-L/4}). \tag{4.72}$$

In the same way, from (4.62) we get

$$F_i(u(t_0)) \leq 18 M_i E_i(u(t_0)) - 72 M_i^3 + O(L^{-1/2}),$$

which leads to

$$F(u(t_0)) = \sum_{i=1}^{N} F_i(u(t_0)) \leq 18 \sum_{i=1}^{N} M_i E_i(u(t_0)) - 72 \sum_{i=1}^{N} M_i^3 + O(L^{-1/2}), \tag{4.73}$$

by summing over $i \in \{1, \dots, N\}$.

Now, we will use the following notation: for a function $f : \mathbb{R}_+ \mapsto \mathbb{R}$, we set

$$\Delta_0^{t_0} f = f(t_0) - f(0). \tag{4.74}$$

From (4.73) and the fact that $E(\cdot)$ and $F(\cdot)$ are conservation laws for u, we obtain

$$\begin{aligned} 0 = \Delta_0^{t_0} F(u) = \sum_{i=1}^{N} \Delta_0^{t_0} F_i(u) &\leq 18 \sum_{i=1}^{N} M_i \Delta_0^{t_0} E_i(u) \\ &+ \sum_{i=1}^{N} \left[-72 M_i^3 + 18 M_i E_i(u_0) - F_i(u_0) \right] + O(L^{-1/2}). \end{aligned} \tag{4.75}$$

By (1.11), the exponential decay of $\varphi_{c_i,\alpha}$'s and the ϕ_i's, and the definition of $E_i(\cdot)$ and $F_i(\cdot)$, it is easy to check that (see [8], Lemma 4.7)

$$|E_i(u_0) - E(\varphi_{c_i,\alpha})| + |F_i(u_0) - F(\varphi_{c_i,\alpha})| \leq O(\varepsilon^2) + O(e^{-\sqrt{L}}), \quad i = 1, \dots, N. \tag{4.76}$$

Then it holds

$$\sum_{i=1}^{N} \left[-72 M_i^3 + 18 M_i E_i(u_0) - F_i(u_0) \right] = -72 \sum_{i=1}^{N} \delta_i^2 \left[M_i + \frac{c_i - \alpha}{3} \right] + O(\varepsilon^2) + O(e^{-\sqrt{L}}). \tag{4.77}$$

Combining (4.75) and (4.77), we get

$$\sum_{i=1}^{N} \delta_i^2 \left[M_i + \frac{c_i - \alpha}{3} \right] \leq \frac{1}{4} \sum_{i=1}^{N} M_i \Delta_0^{t_0} E_i(u) + O(\varepsilon^2) + O(L^{-1/2}),$$

and using the Abel transformation with $M_0 = 0$, we obtain

$$\sum_{i=1}^{N} \delta_i^2 \left[M_i + \frac{c_i - \alpha}{3} \right] \leq \frac{1}{4} \sum_{i=1}^{N} (M_i - M_{i-1}) \Delta_0^{t_0} \mathcal{J}_{i,K}(u) + O(\varepsilon^2) + O(L^{-1/2}), \tag{4.78}$$

where $\mathcal{J}_{i,K}(t)$ is defined in (4.22).

From (4.2) we know that $u(t_0) \in U(\gamma, L/2)$, on account of Lemma 4.1 there exists $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_N)$ with $\tilde{x}_i \in J_i$ such that $E\left(u(t_0) - S_{\tilde{X}}\right) \leq O(\gamma^2)$, where $S_{\tilde{X}}$ is defined in (4.11). Since $v(t_0, \xi_i(t_0)) = \max_{x \in J_i} v(t_0, x)$ and using (4.66), we obtain $E\left(u(t_0) - S_{\xi}\right) \leq O(\gamma^2) + O(e^{-L/4})$, with $\xi = (\xi_1, \dots, \xi_N)$. From (4.5) , we deduce that

$$\left\| v(t_0) - \sum_{j=1}^{N} \rho_{c_j,\alpha}(\cdot - \xi_j(t_0)) \right\|_{C^0(\mathbb{R})} \leq O(\gamma) + O(e^{-L/8}).$$

Thus

$$v(x) = \sum_{j=1}^{N} \rho_{c_j,\alpha}(\cdot - \xi_j(t_0)) + O(\gamma) + O(e^{-L/8}), \quad \forall x \in \mathbb{R},$$

and applying this formula with $x = \xi_i(t_0)$ and using (4.70), we get

$$\begin{aligned} v(\xi_i(t_0)) &= \sum_{j=1}^{N} \rho_{c_j,\alpha}(\xi_i(t_0) - \xi_j(t_0)) + O(\gamma) + O(e^{-L/8}) \\ &= \frac{c_i - \alpha}{6} + \sum_{\substack{1 \le j \le N \\ j \neq i}} \rho_{c_j,\alpha}(\xi_i(t_0) - \xi_j(t_0)) + O(\gamma) + O(e^{-L/8}) \\ &= \frac{c_i - \alpha}{6} + O(\gamma) + O(e^{-L/8}). \end{aligned}$$

We take $\gamma = A\left(\sqrt{\varepsilon} + L^{-1/8}\right)$, then $M_i = (c_i - \alpha)/6 + O(\varepsilon^{1/2}) + O(L^{-1/8})$. Therefore, for $0 < \varepsilon < \varepsilon_0$ and $L > L_0 > 0$, with $\varepsilon_0 \ll 1$ and $L_0 \gg 1$, it holds

$$0 < M_1 < \ldots < M_N. \tag{4.79}$$

Combining (4.78), (4.79) and using the monotonicity estimate (4.24), it holds

$$\sum_{i=1}^{N} \delta_i^2 \left[M_i + \frac{c_i - \alpha}{3} \right] \le O(\varepsilon^2) + O(L^{-1/2}).$$

Therefore, using that $[M_i + (c_i - \alpha)/3]^{-1} < 3/(c_i - \alpha)$, there exists $C > 0$ only depending on α and $(c_i)_{i=1}^N$ such that

$$\delta_i \le C(\varepsilon + L^{-1/4}), \quad i = 1, \ldots, N. \tag{4.80}$$

Now, combining (4.72) and (4.80), we obtain

$$\left\| u(t_0) - \sum_{i=1}^{N} \varphi_{c_i,\alpha}(\cdot - \xi_i(t_0)) \right\|_{\mathcal{H}} \le C(\sqrt{\varepsilon} + L^{-1/8}),$$

and the theorem follows by choosing $A = 2C$.

Appendix. Proof of Lemma 4.2

The aim of this subsection is to establish the Virial type identity (4.25). Let us first assume that u is a smooth solution. The case $u \in \mathcal{X}([0,T[)$ will follow by a density argument. A part of this calculation was performed in [8] (Lemma 4.2), we will use these results and focus on terms that are dependent of the parameter α.

We recall that in [8] (Lemma 4.2) we establish that

$$\int_{\mathbb{R}} yvg = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) g - \frac{5}{2} \int_{\mathbb{R}} v^2 g'' - 2 \int_{\mathbb{R}} v_x^2 g'' + \frac{1}{2} \int_{\mathbb{R}} v^2 g^{(4)} \tag{4.81}$$

and

$$\frac{d}{dt} \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) g = \frac{d}{dt} \int_{\mathbb{R}} yvg + 2 \int_{\mathbb{R}} u^2 v_x g'' - \frac{5}{2} \int_{\mathbb{R}} v h_x g'' - 2 \int_{\mathbb{R}} v_x h g'' + \frac{1}{2} \int_{\mathbb{R}} v h_x g^{(4)}. \tag{4.82}$$

Let us compute the variation of the energy

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} yvg &= \int_{\mathbb{R}} y_t vg + \int_{\mathbb{R}} yv_t g \\ &= I + J.\end{aligned}$$

Applying the operator $(1-\partial_x^2)$ on both sides of equation (1.7), we get

$$y_t = -\frac{1}{2}(1-\partial_x^2)\partial_x u^2 - \frac{3}{2}\partial_x u^2 - \alpha(1-\partial_x^2)\partial_x u.$$

Substituting y_t by this value, I becomes

$$\begin{aligned}I &= -\frac{1}{2}\int_{\mathbb{R}} \left[(1-\partial_x^2)\partial_x u^2\right] vg - \frac{3}{2}\int_{\mathbb{R}} \left[\partial_x(u^2)\right] vg - \alpha\int_{\mathbb{R}} \left[(1-\partial_x^2)\partial_x u\right] vg \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Thanks to the calculations done in [8] (Lemma 4.2), the terms I_1 and I_2 (independent of α) are known, and give us

$$I_1 + I_2 = \frac{4}{3}\int_{\mathbb{R}} u^3 g' - 4\int_{\mathbb{R}} u^2 vg' - \frac{3}{2}\int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2}\int_{\mathbb{R}} u^2 vg'''. \tag{4.83}$$

Let us now compute the term which depends on α:

$$\begin{aligned}I_3 &= \alpha\int_{\mathbb{R}} \left[(1-\partial_x^2)u\right]\partial_x(vg) \\ &= \alpha\int_{\mathbb{R}} \left[(1-\partial_x^2)u\right] v_x g + \alpha\int_{\mathbb{R}} \left[(1-\partial_x^2)u\right] vg' \\ &= I_4 + I_5\end{aligned}$$

with

$$\begin{aligned}I_4 &= \alpha\int_{\mathbb{R}} uv_x g - \alpha\int_{\mathbb{R}} u_{xx}v_x g \\ &= \alpha\int_{\mathbb{R}} uv_x g + \alpha\int_{\mathbb{R}} u_x\partial_x(v_x g) \\ &= \alpha\int_{\mathbb{R}} uv_x g + \alpha\int_{\mathbb{R}} u_x v_{xx} g + \alpha\int_{\mathbb{R}} u_x v_x g' \\ &= \alpha\int_{\mathbb{R}} uv_x g - \alpha\int_{\mathbb{R}} u\partial_x(v_{xx} g) - \alpha\int_{\mathbb{R}} u\partial_x(v_x g') \\ &= \alpha\int_{\mathbb{R}} uv_x g - \alpha\int_{\mathbb{R}} uv_{xxx} g - 2\alpha\int_{\mathbb{R}} uv_{xx} g' - \alpha\int_{\mathbb{R}} uv_x g''\end{aligned} \tag{4.84}$$

and

$$\begin{aligned}I_5 &= \alpha\int_{\mathbb{R}} uvg' - \alpha\int_{\mathbb{R}} u_{xx}vg' \\ &= \alpha\int_{\mathbb{R}} uvg' + \alpha\int_{\mathbb{R}} u_x\partial_x(vg') \\ &= \alpha\int_{\mathbb{R}} uvg' + \alpha\int_{\mathbb{R}} u_x v_x g' + \alpha\int_{\mathbb{R}} u_x vg'' \\ &= \alpha\int_{\mathbb{R}} uvg' - \alpha\int_{\mathbb{R}} u\partial_x(v_x g') - \alpha\int_{\mathbb{R}} u\partial_x(vg'') \\ &= \alpha\int_{\mathbb{R}} uvg' - \alpha\int_{\mathbb{R}} uv_{xx} g' - 2\alpha\int_{\mathbb{R}} uv_x g'' - \alpha\int_{\mathbb{R}} uvg'''.\end{aligned} \tag{4.85}$$

Adding (4.84) and (4.85), we get

$$\begin{aligned} I_3 &= \alpha \int_{\mathbb{R}} uv_x g - \alpha \int_{\mathbb{R}} uv_{xxx} g + \alpha \int_{\mathbb{R}} uvg' - 3\alpha \int_{\mathbb{R}} uv_{xx} g' - 3\alpha \int_{\mathbb{R}} uv_x g'' - \alpha \int_{\mathbb{R}} uvg''' \\ &= \alpha \int_{\mathbb{R}} u \left[(1-\partial_x^2)v_x\right] g + \alpha \int_{\mathbb{R}} u \left[(1-3\partial_x^2)v\right] g' - 3\alpha \int_{\mathbb{R}} uv_x g'' - \alpha \int_{\mathbb{R}} uvg'''. \end{aligned}$$

The first two integrals lead to

$$\alpha \int_{\mathbb{R}} u \left[(1-\partial_x^2)v_x\right] g = \alpha \int_{\mathbb{R}} u \left[(4-\partial_x^2)v_x - 3v_x\right] g = \alpha \int_{\mathbb{R}} uu_x g - 3\alpha \int_{\mathbb{R}} uv_x g = -\frac{\alpha}{2} \int_{\mathbb{R}} u^2 g' - 3\alpha \int_{\mathbb{R}} uv_x g$$

and

$$\alpha \int_{\mathbb{R}} u \left[(1-3\partial_x^2)v\right] g' = 3\alpha \int_{\mathbb{R}} u \left[(4-\partial_x^2)v - \frac{11}{3}v\right] g' = 3\alpha \int_{\mathbb{R}} u^2 g' - 11\alpha \int_{\mathbb{R}} uvg'.$$

Therefore

$$I_3 = -3\alpha \int_{\mathbb{R}} uv_x g + \frac{5\alpha}{2} \int_{\mathbb{R}} u^2 g' - 11\alpha \int_{\mathbb{R}} uvg' - 3\alpha \int_{\mathbb{R}} uv_x g'' - \alpha \int_{\mathbb{R}} uvg'''. \tag{4.86}$$

Finally, combining (4.83) and (4.86), we get

$$\begin{aligned} I = \frac{4}{3} \int_{\mathbb{R}} u^3 g' - 4 \int_{\mathbb{R}} u^2 vg' - \frac{3}{2} \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2} \int_{\mathbb{R}} u^2 vg''' \\ - 3\alpha \int_{\mathbb{R}} uv_x g + \frac{5\alpha}{2} \int_{\mathbb{R}} u^2 g' - 11\alpha \int_{\mathbb{R}} uvg' - 3\alpha \int_{\mathbb{R}} uv_x g'' - \alpha \int_{\mathbb{R}} uvg'''. \end{aligned} \tag{4.87}$$

Now, applying the operator $(4-\partial_x^2)^{-1}$ on both sides of equation (1.7) and using (3.19), we get

$$\begin{aligned} v_t &= -\frac{1}{2}(4-\partial_x^2)^{-1}\partial_x u^2 - \frac{3}{2}(4-\partial_x^2)^{-1}(1-\partial_x^2)^{-1}\partial_x u^2 - \alpha(4-\partial_x^2)^{-1}\partial_x u \\ &= -\frac{1}{2}(1-\partial_x^2)^{-1}\partial_x u^2 - \alpha(4-\partial_x^2)^{-1}\partial_x u. \end{aligned}$$

Substitute v_t by this value, J becomes

$$\begin{aligned} J &= -\frac{1}{2} \int_{\mathbb{R}} \left[(1-\partial_x^2)^{-1}\partial_x u^2\right] yg - \alpha \int_{\mathbb{R}} \left[(4-\partial_x^2)^{-1}\partial_x u\right] yg \\ &= J_1 + J_2. \end{aligned}$$

Setting $h = (1-\partial_x^2)^{-1}u^2$ and using the calculations done in [8] (Lemma 4.2), we know that

$$J_1 = -\frac{2}{3} \int_{\mathbb{R}} u^3 g' + \int_{\mathbb{R}} uhg' + \frac{1}{2} \int_{\mathbb{R}} uh_x g''. \tag{4.88}$$

Let us compute the term which depends on α:

$$\begin{aligned} J_2 &= \alpha \int_{\mathbb{R}} \partial_x(yg)v \\ &= \alpha \int_{\mathbb{R}} y_x vg + \alpha \int_{\mathbb{R}} yvg' \\ &= J_3 + J_4 \end{aligned}$$

with

$$\begin{aligned}
J_3 &= \alpha \int_{\mathbb{R}} (u_x - u_{xxx}) vg \\
&= \alpha \int_{\mathbb{R}} u_x vg - \alpha \int_{\mathbb{R}} u_{xxx} vg \\
&= -\alpha \int_{\mathbb{R}} u \partial_x (vg) + \alpha \int_{\mathbb{R}} u_{xx} \partial_x (vg) \\
&= -\alpha \int_{\mathbb{R}} uvg' - \alpha \int_{\mathbb{R}} uv_x g + \alpha \int_{\mathbb{R}} u_{xx} vg' + \alpha \int_{\mathbb{R}} u_{xx} v_x g \\
&= -\alpha \int_{\mathbb{R}} uvg' - \alpha \int_{\mathbb{R}} uv_x g + \alpha \int_{\mathbb{R}} u_{xx} vg' - \alpha \int_{\mathbb{R}} u_x \partial_x (v_x g) \\
&= -\alpha \int_{\mathbb{R}} uvg' - \alpha \int_{\mathbb{R}} uv_x g + \alpha \int_{\mathbb{R}} u_{xx} vg' - \alpha \int_{\mathbb{R}} u_x v_{xx} g - \alpha \int_{\mathbb{R}} u_x v_x g' \\
&= -\alpha \int_{\mathbb{R}} uvg' - \alpha \int_{\mathbb{R}} uv_x g + \alpha \int_{\mathbb{R}} u_{xx} vg' + \alpha \int_{\mathbb{R}} u \partial_x (v_{xx} g) + \alpha \int_{\mathbb{R}} u \partial_x (v_x g') \\
&= -\alpha \int_{\mathbb{R}} uvg' - \alpha \int_{\mathbb{R}} uv_x g + \alpha \int_{\mathbb{R}} u_{xx} vg' + \alpha \int_{\mathbb{R}} uv_x g'' + 2\alpha \int_{\mathbb{R}} uv_{xx} g' + \alpha \int_{\mathbb{R}} uv_{xxx} g
\end{aligned} \tag{4.89}$$

and

$$J_4 = \alpha \int_{\mathbb{R}} (u - u_{xx}) vg' = \alpha \int_{\mathbb{R}} uvg' - \alpha \int_{\mathbb{R}} u_{xx} vg'. \tag{4.90}$$

Adding (4.89) and (4.90), we get

$$J_2 = -\alpha \int_{\mathbb{R}} uv_x g + \alpha \int_{\mathbb{R}} uv_x g'' + 2\alpha \int_{\mathbb{R}} uv_{xx} g' + \alpha \int_{\mathbb{R}} uv_{xxx} g.$$

Next, using that $v_{xx} = 4v - u$ and $v_{xxx} = 4v_x - u_x$, we have

$$2\alpha \int_{\mathbb{R}} uv_{xx} g' = 2\alpha \int_{\mathbb{R}} u(4v - u) g' = 8\alpha \int_{\mathbb{R}} uvg' - 2\alpha \int_{\mathbb{R}} u^2 g'$$

and

$$\alpha \int_{\mathbb{R}} uv_{xxx} g = \alpha \int_{\mathbb{R}} ug(4v_x - u_x) = 4\alpha \int_{\mathbb{R}} uv_x g - \alpha \int_{\mathbb{R}} uu_x g = 4\alpha \int_{\mathbb{R}} uv_x g + \frac{\alpha}{2} \int_{\mathbb{R}} u^2 g'.$$

Thus

$$J_2 = 3\alpha \int_{\mathbb{R}} uv_x g + 8\alpha \int_{\mathbb{R}} uvg' - \frac{3\alpha}{2} \int_{\mathbb{R}} u^2 g' + \alpha \int_{\mathbb{R}} uv_x g''. \tag{4.91}$$

Finally, combining (4.88) and (4.91), we get

$$J = -\frac{2}{3} \int_{\mathbb{R}} u^3 g' + \int_{\mathbb{R}} uhg' + \frac{1}{2} \int_{\mathbb{R}} uh_x g'' + 3\alpha \int_{\mathbb{R}} uv_x g + 8\alpha \int_{\mathbb{R}} uvg' - \frac{3\alpha}{2} \int_{\mathbb{R}} u^2 g' + \alpha \int_{\mathbb{R}} uv_x g''. \tag{4.92}$$

Therefore, combining (4.82), (4.87) and (4.92), it holds

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) g \\
&\quad = \frac{2}{3} \int_{\mathbb{R}} u^3 g' - 4 \int_{\mathbb{R}} u^2 vg' - \frac{1}{2} \int_{\mathbb{R}} u^2 vg''' + \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g'' + \int_{\mathbb{R}} uhg' \\
&\qquad + \frac{1}{2} \int_{\mathbb{R}} uh_x g'' - \frac{5}{2} \int_{\mathbb{R}} vh_x g'' - 2 \int_{\mathbb{R}} v_x hg'' + \frac{1}{2} \int_{\mathbb{R}} vh_x g^{(4)} \\
&\qquad\quad + \alpha \int_{\mathbb{R}} u^2 g' - 3\alpha \int_{\mathbb{R}} uvg' - 2\alpha \int_{\mathbb{R}} uv_x g'' - \alpha \int_{\mathbb{R}} uvg'''.
\end{aligned} \tag{4.93}$$

The four last integrals which depend on α give us

$$\begin{aligned}
K_1 &= \alpha \int_{\mathbb{R}} u^2 g' \\
&= \alpha \int_{\mathbb{R}} (4v - v_{xx})^2 g' \\
&= 16\alpha \int_{\mathbb{R}} v^2 g' + \alpha \int_{\mathbb{R}} v_{xx}^2 g' - 8\alpha \int_{\mathbb{R}} v v_{xx} g' \\
&= 16\alpha \int_{\mathbb{R}} v^2 g' + \alpha \int_{\mathbb{R}} v_{xx}^2 g' + 8\alpha \int_{\mathbb{R}} v_x^2 g' + 4\alpha \int_{\mathbb{R}} \partial_x(v^2) g'' \\
&= \alpha \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right) g' - 4\alpha \int_{\mathbb{R}} v^2 g''',
\end{aligned}$$

$$\begin{aligned}
K_2 &= -3\alpha \int_{\mathbb{R}} uvg' \\
&= -3\alpha \int_{\mathbb{R}} (4v - v_{xx}) v g' \\
&= -12\alpha \int_{\mathbb{R}} v^2 g' - 3\alpha \int_{\mathbb{R}} \partial_x(vg') v_x \\
&= -12\alpha \int_{\mathbb{R}} v^2 g' - 3\alpha \int_{\mathbb{R}} v_x^2 g' - 3\alpha \int_{\mathbb{R}} v v_x g'' \\
&= -\alpha \int_{\mathbb{R}} \left(12v^2 + 3v_x^2\right) g' + \frac{3\alpha}{2} \int_{\mathbb{R}} v^2 g''',
\end{aligned}$$

$$\begin{aligned}
K_3 &= -2\alpha \int_{\mathbb{R}} u v_x g'' \\
&= -2\alpha \int_{\mathbb{R}} (4v - v_{xx}) v_x g'' \\
&= -8\alpha \int_{\mathbb{R}} v v_x g'' + 2\alpha \int_{\mathbb{R}} v_x v_{xx} g'' \\
&= -4\alpha \int_{\mathbb{R}} \partial_x(v^2) g'' + \alpha \int_{\mathbb{R}} \partial_x(v_x^2) g'' \\
&= 4\alpha \int_{\mathbb{R}} v^2 g''' - \alpha \int_{\mathbb{R}} v_x^2 g''',
\end{aligned}$$

and proceeding as for K_2, we get

$$K_4 = -\alpha \int_{\mathbb{R}} uvg''' = -4\alpha \int_{\mathbb{R}} v^2 g''' - \alpha \int_{\mathbb{R}} v_x^2 g''' + \frac{\alpha}{2} \int_{\mathbb{R}} v^2 g^{(5)}.$$

Summing over $j \in \{1, 2, 3, 4\}$ and using (4.81), it holds

$$\sum_{j=1}^{4} K_j = \alpha \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}) g' - \frac{5\alpha}{2} \int_{\mathbb{R}} v^2 g''' - 2\alpha \int_{\mathbb{R}} v_x^2 g''' + \frac{\alpha}{2} \int_{\mathbb{R}} v^2 g^{(5)} = \alpha \int_{\mathbb{R}} yvg'. \tag{4.94}$$

The lemma follows by combining (4.93) and (4.94).

Acknowledgements. The author would like to thank his PhD advisor Luc Molinet for his help and his careful reading of this manuscript.

References

[1] Adrian Constantin and Walter A. Strauss. Stability of peakons. *Comm. Pure Appl. Math.*, 53(5):603–610, 2000.

[2] A. Degasperis, D. D. Kholm, and A. N. I. Khon. A new integrable equation with peakon solutions. *Teoret. Mat. Fiz.*, 133(2):170–183, 2002.

[3] Khaled El Dika and Luc Molinet. Stability of multi antipeakon-peakons profile. *Discrete Contin. Dyn. Syst. Ser. B*, 12(3):561–577, 2009.

[4] Khaled El Dika and Luc Molinet. Stability of multipeakons. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(4):1517–1532, 2009.

[5] Joachim Escher, Yue Liu, and Zhaoyang Yin. Global weak solutions and blow-up structure for the Degasperis-Procesi equation. *J. Funct. Anal.*, 241(2):457–485, 2006.

[6] Yanggeng Fu, Zhengrong Liu, and Xingyu Yang. Orbital stability of peakons for the Degasperis-Procesi equation with strong dispersion. *Nonlinear Anal.*, 73(2):538–546, 2010.

[7] Fei Guo. Global weak solutions and breaking waves to the Degasperis-Procesi equation with linear dispersion. *J. Math. Anal. Appl.*, 360(2):345–362, 2009.

[8] André Kabakouala. Stability in the energy space of the sum of N peakons for the Degasperis–Procesi equation. *J. Differential Equations*, 259(5):1841–1897, 2015.

[9] Zhiwu Lin and Yue Liu. Stability of peakons for the Degasperis-Procesi equation. *Comm. Pure Appl. Math.*, 62(1):125–146, 2009.

[10] Yue Liu and Zhaoyang Yin. Global existence and blow-up phenomena for the Degasperis-Procesi equation. *Comm. Math. Phys.*, 267(3):801–820, 2006.

[11] Yvan Martel, Frank Merle, and Tai-Peng Tsai. Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. *Comm. Math. Phys.*, 231(2):347–373, 2002.

[12] Jiangbo Zhou and Lixin Tian. Solitons, peakons, and periodic cuspons of a generalized Degasperis-Procesi equation. *Math. Probl. Eng.*, pages Art. ID 249361, 13, 2009.