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Aymen Braghtha

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ON THE NUMBER OF ISOLATED ZEROS OF PSEUDO-ABELIAN INTEGRALS: DEGENERACIES OF THE CUSPIDAL TYPE

Aymen Braghtha

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Abstract

We consider a multivalued function of the form $H_\varepsilon = P_\varepsilon^{\alpha_0} \prod_{i=1}^k P_i^{\alpha_i}$, $P_i \in \mathbb{R}[x, y]$, $\alpha_i \in \mathbb{R}_+^*$, which is a Darboux first integral of polynomial one-form $\omega = M_\varepsilon \frac{dH_\varepsilon}{H_\varepsilon} = 0$, $M_\varepsilon = P_\varepsilon \prod_{i=1}^k P_i$. We assume, for $\varepsilon = 0$, that the polycycle $\{H_0 = H = 0\}$ has only cuspidal singularity which we assume at the origin and other singularities are saddles.

We consider families of Darboux first integrals unfolding H_ε (and its cuspidal point) and pseudo-Abelian integrals associated to these unfolding. Under some conditions we show the existence of uniform local bound for the number of zeros of these pseudo-Abelian integrals.

Keywords. integrable systems, blowing-up, singular foliations, singularities, abelian functions

1 Formulation of main results

In this paper, we study a non generic case. Other non generic cases have been studied in [1,3,4,5]. Pseudo-Abelian integrals appear as the linear principal part of the displacement function in polynomial perturbation of Darboux integrable case.

More precisely consider Darboux integrable system ω given by

$$\omega = M d \log H, \tag{1}$$

where

$$M = \prod_{i=0}^k P_i, \quad H = \prod_{i=0}^k P_i^{\alpha_i}, \quad \alpha_i > 0, \quad P_i \in \mathbb{R}[x, y]. \tag{2}$$

Now we consider an unfolding ω_ε of Darboux integrable system ω , where ω_ε are one-forms with first integral

$$H_\varepsilon = P_\varepsilon^\alpha \prod_{i=1}^k P_i^{\alpha_i}, \quad \omega_\varepsilon = M_\varepsilon d \log H_\varepsilon, \quad M_\varepsilon = P_\varepsilon \prod_{i=1}^k P_i. \tag{3}$$

where the polynomial P_0 has a cuspidal singularity at $p_0 = (0, 0)$, i.e. $P_0(x, y) = y^2 - x^3 + \mathcal{O}((x, y)^4)$. For non zero ε , the polynomial $P_\varepsilon = y^2 - x^3 - \varepsilon x^2 + \mathcal{O}((x, y, \varepsilon)^4)$.

Choose a limit periodic set i.e. bounded component of $\mathbb{R}^2 \setminus \{\prod_{i=0}^k P_i = 0\}$ filled cycles $\gamma(h) \subset \{H = h\}$, $h \in (0, a)$. Denote by $D \subset H^{-1}(0)$ the polycycle which is in the boundary of this limit periodic set.

Consider the unfolding $\omega_\varepsilon = M_\varepsilon d \log H_\varepsilon$ of the form ω . The foliation ω_ε has a maximal nest of cycles $\gamma(\varepsilon, h) \subset \{H_\varepsilon = h\}$, $h \in (0, a(\varepsilon))$, filling a connected component of $\mathbb{R}^2 \setminus \{H_\varepsilon = 0\}$ whose boundary is a polycycle D_ε close to D . Assume moreover that the foliation $\omega_\varepsilon = 0$ has no singularities on $\text{Int}D_\varepsilon$.

Consider pseudo-Abelian integrals of the form

$$I(\varepsilon, h) := \int_{\gamma(\varepsilon, h)} \eta_2, \quad \eta_2 = \frac{\eta_1}{M_\varepsilon} \tag{4}$$

where η_1 is a polynomial one-form of degree at most n .

This integral appears as the linear term with respect to β of the displacement function of a polynomial perturbation

$$\omega_{\varepsilon,\beta} = \omega_\varepsilon + \beta\eta_1 = 0. \quad (5)$$

We assume the following genericity assumptions

1. The level curves $P_i = 0, i = 1, \dots, k$ are smooth and $P_i(0, 0) \neq 0$.
2. The level curves $P_\varepsilon = 0, P_i = 0, i = 1, \dots, k$, intersect transversally two by two.

Theorem 1. *Under the genericity assumptions there exists a bound for the number of isolated zeros of the pseudo-Abelian integrals $I(\varepsilon, h) = \int_{\gamma(\varepsilon, h)} \eta_2$ in $(0, a(\varepsilon))$. The bound is locally uniform with respect to all parameters in particular in ε .*

Let $\mathcal{F}_1 : \{\omega_\varepsilon = 0\}, \mathcal{F}_2 : \{d\varepsilon = 0\}$ are the foliations of dimension two in complex space of dimension three with coordinates (x, y, ε) .

Let \mathcal{F} be the foliation of dimension one on the complex space of dimension three with coordinates (x, y, ε) which is given by the intersection of leaves of \mathcal{F}_1 and \mathcal{F}_2 (i.e. given by the 2-form $\Omega = \omega_\varepsilon \wedge d\varepsilon$). This foliation has a cuspidal singularity at the origin (a cusp).

We want to study the analytical properties of the foliation \mathcal{F} in a neighborhood of the cusp. For this reason we make a global blowing-up of the cusp of the product space (x, y, ε) of phase and parameter spaces. We want our blow-up to separate the two branches of the cusp. This requirement leads to the quasi-homogeneous blowing-up of weight $(2, 3, 2)$.

Remark 1. *In term of first integrals, the foliation \mathcal{F} is given by two first integrals*

$$H(x, y, \varepsilon) = h, \quad \varepsilon = s.$$

2 Quasi-homogeneous blowing-up of \mathcal{F}

Recall the construction of the quasi-homogeneous blowing-up. We define the weighted projective space $\mathbb{C}\mathbb{P}_{2:3:2}^2$ as factor space of \mathbb{C}^3 by the \mathbb{C}^* action $(x, y, \varepsilon) \mapsto (t^2x, t^3y, t^2\varepsilon)$. The quasi-homogeneous blowing-up of \mathbb{C}^3 at the origin is defined as the incidence three dimensional manifold $W = \{(p, q) \in \mathbb{C}\mathbb{P}_{2:3:2}^2 \times \mathbb{C}^3 : \exists t \in \mathbb{C} : (q_1, q_2, q_3) = (t^2p_1, t^3p_2, t^2p_3)\}$, where $(q_1, q_2, q_3) \in \mathbb{C}$ and $[(p_1, p_2, p_3)] \in \mathbb{C}\mathbb{P}_{2:3:2}^2$.

The quasi-homogeneous blowing-up $\sigma : W \rightarrow \mathbb{C}^3$ is just the restriction to W of the projection $\mathbb{C}\mathbb{P}_{2:3:2}^2 \times \mathbb{C}^3$.

We will need explicit formula for the blow-up in the standard affine charts of W . The projective space $\mathbb{C}\mathbb{P}_{2:3:2}^2$ is covered by three affine charts: $U_1 = \{x \neq 0\}$ with coordinates (y_1, z_1) , $U_2 = \{y \neq 0\}$ with coordinates (x_2, z_2) and $U_3 = \{\varepsilon \neq 0\}$ with coordinates (x_3, y_3) .

The transition formula follow from the requirement that the points $(1, y_1, z_1), (x_2, 1, z_2)$ and $(x_3, y_3, 1)$ lie on the same orbit of the action:

$$F_2 : (y_1, z_1) \mapsto \left(x_2 = 1/y_1^{2/3}, z_2 = z_1/y_1\sqrt{y_1} \right)$$

$$F_3 : (y_1, z_1) \mapsto \left(x_3 = 1/z_1, y_3 = y_1/z_1\sqrt{z_1} \right).$$

These affine charts define affine charts on W , with coordinates $(y_1, z_1, t_1), (x_2, z_2, t_2)$ and (x_3, y_3, t_3) . The blow-up σ is written as

$$\sigma_1 : \quad x = t_1^2, \quad y = t_1^3 y_1, \quad \varepsilon = t_1^2 z_1 \quad (6)$$

$$\sigma_2 : \quad x = t_2^2 x_2, \quad y = t_2^3, \quad \varepsilon = t_2^2 z_2 \quad (7)$$

$$\sigma_3 : \quad x = t_3^2 x_3, \quad y = t_3^3 y_3, \quad \varepsilon = t_3^2. \quad (8)$$

We apply this blow-up σ to the one-dimensional foliation \mathcal{F} . Let $\sigma^{-1}\mathcal{F}$ the lifting of the foliation \mathcal{F} to the complement. This foliation has a cuspidal singularity at the origin. The pull-back foliation $\sigma^*\mathcal{F}$ will be called the strict transform of the foliation \mathcal{F} is defined by the pull-back $\sigma^*\Omega = \sigma^*(\omega_\varepsilon \wedge d\varepsilon)$ divided by a suitable power of the function defining the exceptional divisor. In this charts $U_j, j = 1, 2, 3$ we have

$$\sigma_1^*\Omega = x^2\Omega_1, \quad \sigma_2^*\Omega = y^3\Omega_2, \quad \sigma_3^*\Omega = \varepsilon^2\Omega_3,$$

where

$$\Omega_1 = (6y_1^2 - 6 - 4z_1)dx \wedge dz_1 + 4y_1z_1dy_1 \wedge dx + 2xy_1dy_1 \wedge dz_1, \quad (9)$$

$$\Omega_2 = (6 - 6x_2^3 - 4x_2^2z_2)dy \wedge dz_2 + (-6z_2x_2^2 - 4x_2z_2^2)dx_2 \wedge dy \quad (10)$$

$$+ (-3yx_2^2 - 2yx_2z_2)dx_2 \wedge dz_2, \quad (11)$$

$$\Omega_3 = (-6x_3^2 - 4x_3)dx_3 \wedge d\varepsilon + 4y_3dy_3 \wedge d\varepsilon. \quad (12)$$

Remark 2. In term of first integrals, the foliation $\sigma^*\mathcal{F}$ is given by two first integrals

$$\sigma^*H(x, y, \varepsilon) = h, \quad \sigma^*\varepsilon = s,$$

In particular in a neighborhood of the exceptional divisor the restrictions of the foliation $\sigma^*\mathcal{F}$ to the charts U_1 and U_3 are given respectively, by

$$\psi_1 = H(t_1^2, t_1^3y_1, t_1^2z_1) = x^3(y_1^2 - 1) = h, \quad \varphi_1 = xz_1 = s, \quad (13)$$

$$\psi_3 = H(t_3^2x_3, t_3^3y_3, t_3^2) = \varepsilon^3(y_3^2 - x_3^2 - x_3^3) = h, \quad \varphi_3 = \varepsilon = s, \quad (14)$$

where $\{x = 0\}$ and $\{\varepsilon = 0\}$ are local equations of the exceptional divisor respectively.

3 Singular locus of the foliation $\sigma^*\mathcal{F}$

In this section, we compute the singular locus of the pull-back $\sigma^*\Omega$ in a neighborhood of the exceptional divisor $\mathbb{CP}_{2:3:2}^2$. We check it in each chart separately.

In the chart U_1 , the zeros locus of the form Ω_1 in a neighborhood of the exceptional divisor $\{x = 0\}$ consists of germs of two curves $\{y_1 = \pm 1, z_1 = 0\}$ and a two singular points $p_1 = (0, 1, 0), p_2 = (0, -1, 0)$ generated by the quasi-homogeneous blowing-up.

In the chart U_3 , the zeros locus of the form Ω_3 in a neighborhood of the exceptional divisor $\{\varepsilon = 0\}$ consists of $p_3 = (0, 0, 0)$ (Morse point) and $p_4 = (-\frac{2}{3}, 0, 0)$ (center). The singularities of this foliation are the line of Morse points $x_3 = 0, y_3 = 0$, the lines of centers $x_3 = -\frac{2}{3}, y_3 = 0$ and the transform strict of $\{y^2 - x^3 - x^2\varepsilon = 0\}$.

Proposition 1. *The singularities of $\sigma^*\mathcal{F}$ are located at the points p_1, p_2, p_3 and p_4 . The points p_1, p_2 and p_3 are linearisable saddles and the point p_4 is a center.*

Proof. Since $\sigma : W \rightarrow \mathbb{C}^3$ is a biholomorphism outside the exceptional divisor $\mathbb{CP}_{2:3:2}^2$, all singularities of $\sigma^*\mathcal{F}$ on $\mathbb{C}^3 \setminus \{x = 0\}$ correspond to singularities of \mathcal{F} . Thus, it suffices to compute the singularities of $\sigma^*\mathcal{F}$ on the exceptional divisor $\{x = 0\}$. More precisely, we consider the foliation on neighborhood of $\mathbb{CP}_{(2:3:2)}^2$ (the exceptional divisor) generated by the blown-up one-form $\sigma^*\Omega$. Let ψ_1, ψ_3 are the functions given in (13) and (14).

(1) In the chart U_1 , near the divisor exceptional and for $|z_1| \leq \epsilon$ for ϵ sufficiently small, the foliation $\sigma^*\mathcal{F}$ is given by two first integrals

$$G_1 = \varphi_1^3\psi_1^{-1} = z_1^3(y_1^2 - (1 + z_1))^{-1}V^{-1} = s^3h^{-1}, \quad \varphi_1 = xz_1 = s.$$

where V is analytic function such that $V(0,0,0) \neq 0$. In particular on the exceptional divisor $\{x=0\}$ the foliation $\sigma^*\mathcal{F}$ is given by the levels $G_1 = s^3h^{-1} = t$.

Now we calculate the eigenvalues at p_1 and p_2 . The vector field V_1 generating the foliation $\sigma^*\mathcal{F}$ is given by

$$V_1(x, y_1, z_1) = \beta_1 x \frac{\partial}{\partial x} + \beta_2 y_1 \frac{\partial}{\partial y_1} + \beta_3 z_1 \frac{\partial}{\partial z_1},$$

where the vector $(\beta_1, \beta_2, \beta_3)$ satisfies the following equations

$$\langle (\beta_1, \beta_2, \beta_3), (3, 1, 0) \rangle = 0, \langle (\beta_1, \beta_2, \beta_3), (1, 0, 1) \rangle = 0$$

here \langle, \rangle be the usual scalar product on \mathbb{C}^3 . By simple computation, we obtain $\beta_1 = 1, \beta_2 = -3$ and $\beta_3 = -1$.

(2) In the chart U_3 , near the exceptional divisor $\{\varepsilon = 0\}$, the foliation $\sigma^*\mathcal{F}$ is given by

$$G_3 = \varphi_3^3 \psi_3^{-1} = (y_3^2 - x_3^2(1 + x_3))^{-1} = s^3 h^{-1}, \quad \varphi_3 = \varepsilon = s.$$

In particular the restriction of this foliation to the exceptional divisor $\{\varepsilon = 0\}$, by Morse lemma we can put the function $1/G_3$ to the normal form $y_3^2 - z_3^2$ in a neighborhood of p_3 (we put the variable change $z_3 = \pm x_3(1 + x_3)^{1/2}$). On other hand the Hessian matrix of $1/G_3$ at the point p_4 has two positive eigenvalues. \square

4 The different scaled variations of $\delta(s, t)$

In this section, we compute the scaled variations with respect to different variables s and t of the integrals of the blown-up one form $\sigma_1^*\eta_2$ along the different relative cycles using the same technics of [5].

Proposition 2. *The computation of the different scaled variations of the cycle $\delta(s, t)$ us gives*

1. For $t \in [0, 2N]$, the cycle $\delta(s, t)$ satisfies a iterated scaled variations with respect to t of the form

$$\mathcal{V}ar_{(t,3)} \circ \mathcal{V}ar_{(t,-1)} \circ \mathcal{V}ar_{(t,-\alpha_1)} \circ \dots \circ \mathcal{V}ar_{(t,-\alpha_k)} \delta(s, t) = 0. \quad (15)$$

2. For $t \in [N, +\infty]$, the cycle $\delta(s, t)$ satisfies a iterated scaled variations with respect to $1/t$ of the form

$$\mathcal{V}ar_{(1/t,-3)} \circ \mathcal{V}ar_{(1/t,1)} \circ \mathcal{V}ar_{(1/t,1)} \circ \mathcal{V}ar_{(1/t,\alpha_1)} \circ \dots \circ \mathcal{V}ar_{(1/t,\alpha_k)} \delta(s, 1/t) = 0. \quad (16)$$

3. Near $s = 0$, we have

$$\mathcal{V}ar_{(s,1)} \circ \mathcal{V}ar_{(s,1)} \delta(s, t) = \mathcal{V}ar_{(s,1)}(\tilde{\delta}(s, t)) = 0, \quad (17)$$

where $\mathcal{V}ar_{(s,1)} \delta(s, t) = \tilde{\delta}(s, t)$ is a figure eight cycle.

Proof. As in [5], there exist a some local chart with coordinates (u, v, w) defined in a some neighborhood of each separatrix of polycycle such that the foliation $\sigma^*\mathcal{F}$ is defined by two first integrals. Precisely:

1. for $t \in [0, 2N]$, there exist a local chart $(V_{div}, (u, v, w))$ defined in neighborhood of the separatrix δ_{div} such the foliation $\sigma^*\mathcal{F}$ by two first integrals

$$F_1 = w^3(v-1)^{-1}(v+1)^{-1} = t, \quad F_2 = uw = s,$$

2. for $t \in [N, +\infty]$, there exists a local chart $(V_{div}^+, (u, v, w))$ defined in neighborhood of the separatrix δ_{div}^+ such that the foliation $\sigma_1^* \mathcal{F}$ is defined by two first integrals

$$F_1 = w^3(v+2)^{-1}v^{-1} = t, \quad F_2 = uw = s,$$

3. for $t \in [N, +\infty]$, there exists a local chart $(V_{div}^-, (u, v, w))$ defined in a neighborhood of the separatrix δ_{div}^- such that the foliation $\sigma_1^* \mathcal{F}$ is defined by two first integrals

$$F_1 = w^3(v-2)^{-1}v^{-1} = t, \quad F_2 = uw = s.$$

In second step we prove that each relative cycle can be chosen as a lift of a path contained in the separatrix associated to this relative cycle. Precisely:

1. on the chart $(V_{div}, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is every where transverse to the levels of the foliation $\sigma^* \mathcal{F}$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left(st^{-\frac{1}{3}}(v-1)^{-\frac{1}{3}}(v+1)^{-\frac{1}{3}}, t^{\frac{1}{3}}(v-1)^{\frac{1}{3}}(v+1)^{\frac{1}{3}} \right),$$

2. on the chart $(V_{div}^+, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is every where transverse to the levels of the foliation $\sigma^* \mathcal{F}$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left(st^{-\frac{1}{3}}v^{-\frac{1}{3}}(v+2)^{-\frac{1}{3}}, t^{\frac{1}{3}}v^{\frac{1}{3}}(v+2)^{\frac{1}{3}} \right),$$

3. on the chart $(V_{div}^-, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is every where transverse to the levels of the foliation $\sigma^* \mathcal{F}$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left(st^{-\frac{1}{3}}v^{-\frac{1}{3}}(v-2)^{-\frac{1}{3}}, t^{\frac{1}{3}}v^{\frac{1}{3}}(v-2)^{\frac{1}{3}} \right).$$

In third step, we compute the different scaled variations of relative cycles using the local expression of two first integrals F_1 and F_2 above near the singular points p_1, p_2 and p_3 . Recall that the scaled variation of a relative cycle $\delta(s)$ is given by

$$\mathcal{V}ar_{(s,\beta)} \delta(s) = \delta(se^{i\pi\beta}) - \delta(se^{-i\pi\beta}).$$

In the local chart $(V_{div}^+, (u, v, w))$, the restriction of the blown-up foliation $\sigma_1^* \mathcal{F}$ to the transversals sections $\Sigma_{div}^- = \{w = 1\}$ (near the point p_3) and $\Omega_+ = \{u = 1\}$ (near the point p_1) is given respectively by

$$\begin{aligned} F_1|_{\Sigma_{div}^-} &= \frac{1}{v} = t, & F_2|_{\Sigma_{div}^-} &= u = s, \\ F_1|_{\Omega_+} &= \frac{w^3}{v} = t, & F_2|_{\Omega_+} &= w = s. \end{aligned}$$

Let us fix $t \in [N, +\infty]$. We observe that the restriction of the foliation $\sigma_1^* \mathcal{F}$ to the transversal section $\Sigma_{div}^+ = \{w = 1\}$ is analytic with respect to s . Then, after taking an scaled variation with respect to s , the relative cycle $\delta_{div}^+(s, t)$ is replaced by a loop θ_1 , modulo homotopy, which consists of line segment $\ell_{31} = [p_3, p_1]$ connecting the Morse point p_3 with the point p_1 encircling the latter along a small counterclockwise circular arc α_1 and then returning along the segment $\ell_{13} = [p_1, p_3]$. The loop θ_1 can be moved along the complex curve $\{u = w = 0\}$. Then, we have

$$\mathcal{V}ar_{(s,1)} \delta_{div}^+(s, t) = \theta_1 = \ell_{31} \alpha_1 \ell_{13}.$$

The same computation of the scaled variation with respect to s for the relative cycle $\delta_{div}^-(s, t)$ gives us a loop θ_3 , modulo homotopy, which can be moved along the complex plane $\{u = w = 0\}$. The loop θ_3

consists of line segment $\ell_{32} = [p_3, p_2]$ connecting the point p_3 with the point p_2 encircling the latter along a small counterclockwise circular arc α_3 and then returning along the segment $\ell_{23} = [p_2, p_3]$. Then, we have

$$\mathcal{V}ar_{(s,1)}\delta_{div}^-(s,t) = \theta_3 = \ell_{32}\alpha_3\ell_{23}.$$

In the local chart $(V_{div}, (u, v, w))$, we define the transversal section $\Omega_+ = \{u = 1\}$ (resp $\Omega_+ = \{u = 1\}$) near p_1 (resp near p_2). The restriction of the foliation $\sigma_1^*\mathcal{F}$ to the transversal section Ω_+ is given by

$$F_1|_{\Omega_+} = \frac{w^3}{v} = t, \quad F_2|_{\Omega_+} = w = s.$$

On the second step let us fix $t \in [0, 2N]$. After taking an scaled variation with respect to s , the relative cycle $\delta_{div}(s,t)$ is replaced by a figure eight cycle which can be moved along the complex line $C_{div}^t = \{x = 0, G_1 = t\}$ of the foliation $\sigma_1^*\mathcal{F}$. This case is similar to the classical situation which is studied by Bobieński and Mardešić in [2].

Now using the analyticity of the lifting $\sigma^{-1}\mathcal{F}$ with respect to s , the scaled variation of the cycle of integration $\delta(s,t)$ with respect to s is equal to the scaled variation with respect to s of the following difference $\delta_{div}^+(s,t) - \delta_{div}^-(s,t)$ which is equal, modulo homotopy, to the cycle $\theta_1\theta_3^{-1}$, where θ_3^{-1} is the inverse of the loop θ_3 . Schematically, the loop $\theta_1\theta_3^{-1}$ is a figure eight cycle. \square

Remark 3.

- In the local chart $(V_{div}^+, (u, v, w))$ (resp $V_{div}^-, (u, v, w)$), the loop θ_1 (resp θ_3) generating the fundamental group of the complex plane $\{u = w = 0\} \setminus \{p_1\}$ (resp $\{u = w = 0\} \setminus \{p_2\}$) with base point p_3 .
- By the univalence of the blown-up one form $\sigma_1^*\eta_2$, we have

$$\mathcal{V}ar_{(t,\alpha)} \int_{\delta(s,t)} \sigma_1^*\eta_2 = \int_{\mathcal{V}ar_{(t,\alpha)}\delta(s,t)} \sigma_1^*\eta_2.$$

5 Proof of the Theorem

In this section we first take benefit from the blowing-up in the family to prove our principal theorem. the proof is analogous of the following :

Theorem 2. *There exists a bound of the number of zeros of the function $t \mapsto J(s,t)$, for $t \in [0, +\infty]$ and $s > 0$ sufficiently small. This bound is locally with respect to all parameters uniform, in particular with respect to s .*

Let $\beta = (\beta_1, \dots, \beta_{k+2})$ where $\beta_1 = 3, \beta_2 = -1, \beta_3 = -\alpha_1, \dots, \beta_{k+2} = -\alpha_k$. Let D_1 is slit annulus in the complex plane \mathbb{C}_t^* with boundary ∂D_1 . This boundary is decomposed as follows $\partial D_1 = C_{R_1} \cup C_{r_1} \cup C^\pm$, where $C_{R_1} = \{|t| = R_1, |\arg t| \leq \alpha\pi\}$, $C^\pm = \{r_1 < |t| < R_1, |\arg t| = \pm\alpha\}$ and $C_{r_1} = \{|t| = r_1, |\arg t| \leq \alpha\pi\}$.

Petrov's method gives us that the number of zeros $\#Z(J(s,t))$ of the function $J(s,t)$ in slit annulus D_1 is bounded by the increment of the argument of $J(s,t)$ along ∂D_1 divided by 2π i.e.

$$\begin{aligned} \#Z(J(s,t)|_{D_1}) &\leq \frac{1}{2\pi} \Delta \arg(J(s,t)|_{\partial D_1}) = \frac{1}{2\pi} \Delta \arg(J(s,t)|_{C_{R_1}}) \\ &\quad + \frac{1}{2\pi} \Delta \arg(J(s,t)|_{C^\pm}) + \frac{1}{2\pi} \Delta \arg(J(s,t)|_{C_{r_1}}) \end{aligned}$$

(A) The increment of argument $\Delta \arg(J(s,t)|_{C_{R_1}})$ is uniformly bounded by Gabrielov's theorem [6].

(B) We use the Schwartz's principle

$$\operatorname{Im}(J(s, t))|_{C^\pm} = \mp 2i \operatorname{Var}_{(t, \alpha)} J(s, t).$$

Thus, the increments of argument along segments C^\pm are bounded by zeros of the variation $\operatorname{Var}_{(t, \alpha)} J(s, t)$ on segment (r, R) . By identity (18), the function $\operatorname{Var}_{(t, \beta_i)} J(s, t)$ can be written as follows

$$\begin{aligned} \operatorname{Var}_{(t, \beta_i)} J(s, t) &= K(t^{\frac{\beta_1}{\beta_i}}, \dots, t^{\frac{\beta_{k+\mu}}{\beta_i}}, s; \log s) \\ &= K(e^{\frac{\beta_1}{\beta_i} \log t}, \dots, e^{\frac{\beta_{k+\mu}}{\beta_i} \log t}, e^{\log s}; \log s) \end{aligned}$$

where K is a meromorphic function. The function $\operatorname{Var}_{(t, \beta_i)} J(s, t)$ is logarithmico-analytic function of type 1 in the variable s (see [9]). Then, there exist a finite cover of $\mathbb{R}^{k+\mu+1} \times \mathbb{R}$ by a logarithmico-exponential cylinders, using Rolin-Lion's theorem [9], such that on each cylinder of this family we have

$$\operatorname{Var}_{(t, \beta_i)} J(s, t) = y_0^{r_0} y_1^{r_1} A(t) U(t, y_0, y_1),$$

with $y_0 = s - \theta_0(t)$, $y_1 = \log y_0 - \theta_1(t)$, where θ_0, θ_1, A are logarithmico-exponential functions and U is a logarithmico-exponential unity function. As the number of zeros of a logarithmico-exponential function is bounded, the number of zeros of $\operatorname{Var}_{(t, \beta_i)} J(s, t)$ is bounded.

(C) Finally, we estimate the increment of argument of J along the small arc C_{r_1} . Then, it is necessarily to study the increment of argument of the leading term of the function J at $t = 0$.

Lemma 1. *The increment of the argument of $J(s, t)$ along the small circular arc C_{r_1} can be estimated by the increment of the argument of a some meromorphic function $F(s, t)$.*

Proof. The problem of the estimation of the increment of the argument of $J(s, t)$ along the circular arc C_{r_1} consist that the principal part of the function J contains the term $\log s \rightarrow -\infty$ as $s \rightarrow 0$. To resolve this problem we make a blowing-up at the origin in the total space with coordinates (x, y, z) where

$$x = J_1(s, t), \quad y = J_2(s, t), \quad z = (\log s)^{-1}.$$

The function $J(s, t)$ can be rewritten as follows

$$J(s, t) = J_1(s, t) + J_2(s, t) \log s = ((\log s)^{-1} J_1(s, t) + J_2(s, t)) \log s = (zx + y)z^{-1}.$$

Thus, for $z^{-1} \in \mathbb{R}$ be sufficiently small, we have

$$\arg(J(s, t)) = \arg((zx + y)z^{-1}) = \arg(zx + y).$$

To estimate the increment of argument of $zx + y$ uniformly with respect to $s > 0$ we make a quasi-homogeneous blowing-up π_1 with weight $(\frac{1}{2}, 1, \frac{1}{2})$ of the polynomial $zx + y$ at $C_1 = \{x = y = z = 0\}$ (the centre of blowing-up). The explicit formula of the quasi-homogeneous blowing-up π_1 in the affine charts $T_1 = \{x \neq 0\}$, $T_2 = \{y \neq 0\}$ and $T_3 = \{z \neq 0\}$ is written respectively as

$$\begin{aligned} \pi_{11} : x &= \sqrt{x_1}, \quad y = y_1 x_1, \quad z = z_1 \sqrt{x_1}, \\ \pi_{12} : x &= x_2 \sqrt{y_2}, \quad y = y_2, \quad z = z_2 \sqrt{y_2}, \\ \pi_{13} : x &= x_3 \sqrt{z_3}, \quad y = y_3 z_3, \quad z = \sqrt{z_3}. \end{aligned}$$

The pull-back $\pi_1^*(zx + y)$ is given, in different charts, by

$$\begin{aligned} \pi_{11}^*(zx + y) &= x_1(z_1 + y_1) = d_1 P_1(x_1, y_1, z_1), \\ \pi_{12}^*(zx + y) &= y_2(x_2 z_2 + 1) = d_2 P_2(x_2, y_2, z_2), \\ \pi_{13}^*(zx + y) &= z_3(x_3 + y_3) = d_3 P_3(x_3, y_3, z_3). \end{aligned}$$

where $d_i = 0$ and $P_i = 0$ are equations of exceptional divisor and the strict transform of $zx + y = 0$ respectively.

Observe that $P_i = 0, i = 1, 3$ has not a normal crossing with the exceptional divisor $d_i = 0, i = 1, 3$. To resolve this problem we make a second blowing-up π_2 with centre a subvariety C_2 which is given, in different charts, as following:

1. In the chart T_1 , choose a local coordinate chart with coordinates (x_1, y_1, z_1) in which $C_2 = \{y_1 = z_1 = 0\}$. Then $\pi_2^{-1}(C_2)$ is covered by two coordinate charts U_{y_1} and U_{z_1} with coordinate $(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1)$ where in y_1 -chart U_{y_1} the blowing-up π_2 is given by $x_1 = \tilde{x}_1, y_1 = \tilde{y}_1, z_1 = \tilde{z}_1\tilde{y}_1$ and in z_1 -chart U_{z_1} the blowing-up π_2 is given by $x_1 = \tilde{x}_1, y_1 = \tilde{y}_1\tilde{z}_1, z_1 = \tilde{z}_1$.
2. In the chart T_2 , the blowing-up π_2 is a biholomorphism (π_2 is a proper map).
3. In this chart T_3 , choose a local coordinate chart with coordinates (x_3, y_3, z_3) in which $C_2 = \{x_3 = y_3 = 0\}$. Then $\pi_2^{-1}(C_2)$ is covered by two coordinate charts U_{x_3} and U_{y_3} with coordinate $(\tilde{x}_3, \tilde{y}_3, \tilde{z}_3)$ where in x_3 -chart U_{x_3} the blowing-up π_2 is given by $x_3 = \tilde{x}_3, y_3 = \tilde{y}_3\tilde{x}_3, z_3 = \tilde{z}_3$ and in y_3 -chart U_{y_3} the blowing-up π_2 is given by $x_3 = \tilde{x}_3\tilde{y}_3, y_3 = \tilde{y}_3, z_3 = \tilde{z}_3$.

The pull-back $\pi_1^*(zx + y)$ is given, in different charts, by

- In the y_1 -chart U_{y_1} , the transformation of the pull-back $\pi_1^*(zx + y)$ by the blowing-up π_2 is given by

$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_1P_1(x_1, y_1, z_1)) = \tilde{x}_1\tilde{y}_1(\tilde{z}_1 + 1) \stackrel{0}{\approx} \tilde{x}_1\tilde{y}_1 = J_2(s, t) = F(s, t).$$

- In the z_1 -chart U_{z_1} , the transformation of the pull-back $\pi_1^*(zx + y)$ by the blowing-up π_2 is given by

$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_1P_1(x_1, y_1, z_1)) = \tilde{z}_1\tilde{x}_1(\tilde{y}_1 + 1) \stackrel{0}{\approx} \tilde{x}_1\tilde{z}_1 = (\log s)^{-1}J_1(s, t) = F(s, t).$$

- In the chart T_2 , we have

$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_2P_2(x_2, y_2, z_2)) = d_2P_2(x_2, y_2, z_2) = (\log s)^{-1}J_1(s, t) + J_2(s, t) = F(s, t).$$

- In the x_3 -chart U_{x_3} , the transformation of the pull-back $\pi_1^*(zx + y)$ by the blowing-up π_2 is given by

$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_3P_3(x_3, y_3, z_3)) = \tilde{z}_3\tilde{x}_3(\tilde{y}_3 + 1) \stackrel{0}{\approx} \tilde{x}_3\tilde{z}_3 = (\log s)^{-1}J_1(s, t) = F(s, t).$$

- In the y_3 -chart U_{y_3} , the transformation of the pull-back $\pi_1^*(zx + y)$ by the blowing-up π_2 is given by

$$\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_3P_3(x_3, y_3, z_3)) = \tilde{z}_3\tilde{y}_3(\tilde{x}_3 + 1) \stackrel{0}{\approx} \tilde{y}_3\tilde{z}_3 = J_2(s, t) = F(s, t).$$

Finally, we distinguish three cases:

1. $\arg_{C_{r_1}} J(s, t) = \arg_{C_{r_1}} ((\log s)^{-1}J_1(s, t)) = \arg_{C_{r_1}} J_1(s, t), ((\log s)^{-1} \in \mathbb{R})$
2. $\arg_{C_{r_1}} J(s, t) = \arg_{C_{r_1}} J_2(s, t),$
3. In the chart T_2 , the function $F(s, t) = ((\log s)^{-1}J_1(s, t) + J_2(s, t))$ is meromorphic.

□

Now we define the functional space \mathcal{P}_β which are formed of coefficients of the polynomials P_i of the Darboux first integral H , the coefficients of the polynomials R, S of the perturbative one forme η , exponents α_i and degrees $n_i = \deg P_i, n = \max(\deg R, \deg S)$. Consider the following finite dimensional functional space \mathcal{P}_β

$$\mathcal{P}_\beta(m_\beta, M_\beta; \beta_1, \dots, \beta_{k+2}) = \left\{ \sum_{j=1}^{k+2} \sum_{n, \ell} A_{j\ell n}(s) t^{\beta_j n} s^m \log^\ell(t) : \right. \\ \left. A_{j\ell n}(s) \in \mathbb{C}, m_\beta < A_{j\ell n} < M_\beta, 0 \leq \ell \leq k+1 \right\}.$$

For the first two cases, the function $J_i(s, t), i = 1, 2$ satisfies the following iterated variations equation with respect to t

$$\mathcal{V}ar_{(t, \beta_1)} \circ \dots \circ \mathcal{V}ar_{(t, \beta_{k+2})} J_i(s, t) = 0.$$

Thus, by Lemma 4.8 from [2], there exists a non zero leading term $P_{i\beta} \in \mathcal{P}_\beta$ of $J_i(s, t), i=1,2$ at $t = 0$ such that $|J_i(s, t) - P_{i\beta}(s, t)| = O(t^{\mu_1}), \mu_1 > 0$, uniformly in s . Moreover, the function $J_i(s, t), i = 1, 2$ satisfies the iterated variation equation

$$\mathcal{V}ar_{(s, 1)} J_i(s, t) = 0.$$

Thus, we have $J_i(s, t) = O(s^{\mu_2}), \mu_2 > 0$, uniformly in t .

For each element in the parameter space, we can choose the leading term of $P_{i\beta}$. The increment of argument of this leading term is bounded by a constant $C(M_\beta, k+2, \beta_{k+2})$. Since the leading term of $P_{i\beta}$ is also the leading term of $J_i(s, t)$, the limit $\lim_{r_1 \rightarrow 0} \Delta \arg(J_i(s, t)|_{C_{r_1}}) \leq C(M_\beta, k+2, \beta_{k+2})$.

In the chart T_2 , the function F is meromorphic. Thus, this function can be rewritten as following

$$F(s, t) = (\log s)^{-1} J_1(s, t) + J_2(s, t) = G(t^{\beta_1}, \dots, t^{\beta_k}, s, (\log s)^{-1})$$

where G is meromorphic function. The number $\#Z(G)$ of zeros of the function G is uniformly bounded. The latter claim is a direct application of fewnomials theory of Khovanskii [8]: since the functions $\epsilon_i(t) = t^{\beta_i}, \epsilon(s) = (\log s)^{-1}$ are Pfaffian functions (solutions of Pfaffian equations $t d\epsilon_i - \beta_i \epsilon_i dt = 0$ and $s d\epsilon + \epsilon^2 ds$, respectively), the upper bound for this number of zeros can be given, using Rolle-Khovanskii arguments of [7], in terms of the number of zeros of some polynomial and its derivatives. The latter are uniformly bounded by Gabrielov's theorem [6].

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**Burgundy University, Burgundy Institute of Mathematics,
U.M.R. 5584 du C.N.R.S., B.P. 47870, 21078 Dijon
Cedex - France.
E-mail adress: aymenbraghtha@yahoo.fr**