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# ISOMONODROMIC DEFORMATIONS OF LOGARITHMIC CONNECTIONS AND STABILITY

INDRANIL BISWAS, VIKTORIA HEU, AND JACQUES HURTUBISE

ABSTRACT. Let  $X_0$  be a compact connected Riemann surface of genus  $g$  with  $D_0 \subset X_0$  an ordered subset of cardinality  $n$ , and let  $E_G$  be a holomorphic principal  $G$ -bundle on  $X$ , where  $G$  is a reductive affine algebraic group defined over  $\mathbb{C}$ , that admits a logarithmic connection  $\nabla_0$  with polar divisor  $D_0$ . Let  $(\mathcal{E}_G, \nabla)$  be the universal isomonodromic deformation of  $(E_G, \nabla_0)$  over the universal Teichmüller curve  $(\mathcal{X}, \mathcal{D}) \rightarrow \text{Teich}_{g,n}$ , where  $\text{Teich}_{g,n}$  is the Teichmüller space for genus  $g$  Riemann surfaces with  $n$ -marked points. We prove the following (see Section 5):

- (1) Assume that  $g \geq 2$  and  $n = 0$ . Then there is a closed complex analytic subset  $\mathcal{Y} \subset \text{Teich}_{(g,n)}$ , of codimension at least  $g$ , such that for any  $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$ , the principal  $G$ -bundle  $\mathcal{E}_G|_{\mathcal{X}_t}$  is semistable, where  $\mathcal{X}_t$  is the compact Riemann surface over  $t$ .
- (2) Assume that  $g \geq 1$ , and if  $g = 1$ , then  $n > 0$ . Also, assume that the monodromy representation for  $\nabla_0$  does not factor through some proper parabolic subgroup of  $G$ . Then there is a closed complex analytic subset  $\mathcal{Y}' \subset \text{Teich}_{(g,n)}$ , of codimension at least  $g$ , such that for any  $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}'$ , the principal  $G$ -bundle  $\mathcal{E}_G|_{\mathcal{X}_t}$  is semistable.
- (3) Assume that  $g \geq 2$ . Assume that the monodromy representation for  $\nabla_0$  does not factor through some proper parabolic subgroup of  $G$ . Then there is a closed complex analytic subset  $\mathcal{Y}'' \subset \text{Teich}_{(g,n)}$ , of codimension at least  $g - 1$ , such that for any  $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}''$ , the principal  $G$ -bundle  $\mathcal{E}_G|_{\mathcal{X}_t}$  is stable.

In [He1], the second-named author proved the above results for  $G = \text{GL}(2, \mathbb{C})$ .

## 1. INTRODUCTION

Take any triple of the form  $(E \rightarrow X, D, \nabla)$ , where  $E$  is a holomorphic vector bundle over a smooth connected complex variety  $X$ , and  $\nabla$  is an integrable logarithmic connection on  $E$  singular over a simple normal crossing divisor  $D \subset X$ . The monodromy functor associates to it a representation  $\rho_\nabla : \pi_1(X \setminus D, x_0) \rightarrow \text{GL}(E_{x_0})$ . Altering the connection by a holomorphic automorphism of  $E$  leads to a conjugated representation. The monodromy functor produces an equivalence between the category of logarithmic connections  $(E, \nabla)$  on  $(X, D)$  such that the real parts of residues lie in  $[0, 1)$  and the category of equivalence classes of linear representations of  $\pi_1(X \setminus D, x_0)$ . Given a monodromy representation  $\rho$ , one can consider the set of all logarithmic connections  $(E \rightarrow X, D, \nabla)$  (with no condition on the residues) up to holomorphic isomorphisms that produce the same monodromy representation  $\rho = \rho_\nabla$  up to conjugation. All these connections are conjugated to each other by meromorphic gauge transformations with possible poles over  $D$  (see for example [Sa]).

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The classical Riemann–Hilbert problem can be formulated as follows: *Given a representation  $\rho : \pi_1(\mathbb{P}_{\mathbb{C}}^1 \setminus D_0, x) \rightarrow \mathrm{GL}(r, \mathbb{C})$ , is there a logarithmic connection  $(E \rightarrow X, D_0, \nabla)$  such that  $\rho = \rho_{\nabla}$  and  $E$  is holomorphically trivial?* The answer to this problem is

- (1) positive if rank  $r = 2$  [Pl], [De],
- (2) negative in general ( $r \geq 3$ ) [AB],
- (3) positive if  $\rho$  is irreducible [Bol1], [Ko].

On the other hand, the fundamental group  $\pi_1(\mathbb{P}_{\mathbb{C}}^1 \setminus D_0, x)$  depends only on the topological and not the complex structure of  $\mathbb{P}_{\mathbb{C}}^1 \setminus D_0$ . So given an integrable connection, one can consider variations of the complex structure without changing the monodromy representation. More precisely, consider the Teichmüller space  $\mathrm{Teich}_{0,n}$  of the  $n$ -pointed Riemann sphere together with its universal Teichmüller curve

$$\tau : (\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1 \times \mathrm{Teich}_{0,n}, \mathcal{D}) \rightarrow \mathrm{Teich}_{0,n},$$

where  $n = \deg(D_0)$ . Since  $\mathrm{Teich}_{0,n}$  is contractible, the inclusion

$$(\mathbb{P}_{\mathbb{C}}^1, D_t) := \tau^{-1}(t) \hookrightarrow (\mathcal{X}, \mathcal{D}), \quad t \in \mathrm{Teich}_{0,n},$$

induces an isomorphism  $\pi_1(\mathbb{P}_{\mathbb{C}}^1 \setminus D_t, x_t) \simeq \pi_1(\mathcal{X} \setminus \mathcal{D}, x_t)$ . Hence by the Riemann–Hilbert correspondence, we can associate to any logarithmic connection  $(E_0, \nabla_0)$  on  $\mathbb{P}_{\mathbb{C}}^1$ , with polar divisor  $D_0$ , its *universal isomonodromic deformation*: a flat logarithmic connection  $(\mathcal{E}, \nabla)$  over  $\mathcal{X}$  with polar divisor  $\mathcal{D}$  that extends the connection  $(E_0, \nabla_0)$ . The monodromy representation for  $\nabla|_{\tau^{-1}(t)}$  does not change as  $t$  moves over  $\mathrm{Teich}_{0,n}$  (see for example [He2]).

We are led to another Riemann–Hilbert problem: *Given a logarithmic connection  $(E_0, \nabla_0)$  on  $\mathbb{P}_{\mathbb{C}}^1$  with polar divisor  $D_0$  of degree  $n$ , is there a point  $t \in \mathrm{Teich}_{0,n}$  such that the vector bundle  $E_t = \mathcal{E}|_{\mathbb{P}_{\mathbb{C}}^1 \times \{t\}}$  underlying the universal isomonodromic deformation  $(\mathcal{E}, \nabla)$  is trivial?*

A partial answer to this question is given by the following theorem of Bolibruch:

**Theorem 1.1** ([Bol2]). *Let  $(E_0, \nabla_0)$  be an irreducible trace-free logarithmic rank two connection with  $n \geq 4$  poles on  $\mathbb{P}_{\mathbb{C}}^1$  such that each singularity is resonant. There is a proper closed complex analytic subset  $\mathcal{Y} \subset \mathrm{Teich}_{0,n}$  such that for all  $t \in \mathrm{Teich}_{0,n} \setminus \mathcal{Y}$ , the vector bundle  $E_t = \mathcal{E}|_{\mathbb{P}_{\mathbb{C}}^1 \times \{t\}}$  underlying the universal isomonodromic deformation  $(\mathcal{E}, \nabla)$  of  $(E_0, \nabla_0)$  is trivial.*

It should be mentioned that the condition in Theorem 1.1, that each singularity is resonant, can actually be removed [He1].

From the Birkhoff–Grothendieck classification of holomorphic vector bundles on  $\mathbb{P}_{\mathbb{C}}^1$  it follows immediately that the only semistable holomorphic vector bundle of degree 0 and rank  $r$  on  $\mathbb{P}_{\mathbb{C}}^1$  is the trivial bundle  $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}^{\oplus r}$ . This leads to the following more general question: *Given a representation  $\rho : \pi_1(X \setminus D, x) \rightarrow \mathrm{GL}_r(\mathbb{C})$ , where  $X$  is a compact connected Riemann surface, is there a logarithmic connection  $(E \rightarrow X, D, \nabla)$  such that  $\rho = \rho_{\nabla}$  and  $E$  is semistable of degree zero?* The answer to this problem is

- (1) negative in general [EH],

(2) positive if  $\rho$  is irreducible [EV].

On the other hand, we can ask the following: Let  $\tau : (\mathcal{X}, \mathcal{D}) \rightarrow \text{Teich}_{g,n}$  be the universal Teichmüller curve. *Given a logarithmic connection  $(E_0, \nabla_0)$ , with polar divisor  $D_0$  of degree  $n$  on a compact connected Riemann surface  $X_0$  of genus  $g$ , is there an element  $t \in \text{Teich}_{(g,n)}$  such that the vector bundle  $E_t = \mathcal{E}|_{\mathcal{X}_t} \rightarrow \mathcal{X}_t = \tau^{-1}(t)$  underlying the universal isomonodromic deformation  $(\mathcal{E}, \nabla)$  of  $(E_0, \nabla_0)$  is semistable?*

Note that we necessarily have  $\text{degree}(E_t) = \text{degree}(E_0)$ . Again, Theorem 1.1 can be generalized:

**Theorem 1.2** ([Hel]). *Let  $(E_0, \nabla_0)$  be an irreducible logarithmic rank 2 connection with polar divisor  $D_0$  of degree  $n$  on a compact connected Riemann surface  $X_0$  of genus  $g$  such that  $3g - 3 + n > 0$ . Consider its universal isomonodromic deformation  $(\mathcal{E}, \nabla)$  over  $\tau : (\mathcal{X}, \mathcal{D}) \rightarrow \text{Teich}_{(g,n)}$ . There is a closed complex analytic subset  $\mathcal{Y} \subset \text{Teich}_{(g,n)}$  of codimension at least  $g$  (respectively,  $g - 1$ ) such that for any  $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$ , the vector bundle  $E_t = \mathcal{E}|_{\mathcal{X}_t}$ , where  $(\mathcal{X}_t, D_t) = \tau^{-1}(t)$ , is semistable (respectively, stable).*

Our aim here is to prove an analog of Theorem 1.2 in the more general context of logarithmic connections on principal  $G$ -bundles over a compact connected Riemann surface (see [Boa] for logarithmic connections on principal  $G$ -bundles).

Let  $X_0$  be a compact connected Riemann surface of genus  $g$ , and let  $D_0 \subset X_0$  be an ordered subset of cardinality  $n$ . Let  $G$  be a reductive affine algebraic group over  $\mathbb{C}$ . Let  $E_G$  be a holomorphic principal  $G$ -bundle on  $X$  and  $\nabla_0$  a logarithmic connection on  $E_G$  with polar divisor  $D_0$ . Let  $(\mathcal{E}_G, \nabla)$  be the universal isomonodromic deformation of  $(E_G, \nabla_0)$  over the universal Teichmüller curve  $\tau : (\mathcal{X}, \mathcal{D}) \rightarrow \text{Teich}_{g,n}$ . For any point  $t \in \text{Teich}_{g,n}$ , the restriction  $\mathcal{E}_G|_{\tau^{-1}(t)} \rightarrow \mathcal{X}_t := \tau^{-1}(t)$  will be denoted by  $\mathcal{E}_G^t$ .

We prove the following (see Section 5):

**Theorem 1.3.**

- (1) *Assume that  $g \geq 2$  and  $n = 0$ . There is a closed complex analytic subset  $\mathcal{Y} \subset \text{Teich}_{(g,n)}$  of codimension at least  $g$  such that for any  $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$ , the principal  $G$ -bundle  $\mathcal{E}_G^t \rightarrow \mathcal{X}_t$  is semistable.*
- (2) *Assume that  $g \geq 1$ , and if  $g = 1$ , then  $n > 0$ . Also, assume that the monodromy representation for  $\nabla_0$  does not factor through some proper parabolic subgroup of  $G$ . There is a closed complex analytic subset  $\mathcal{Y}' \subset \text{Teich}_{(g,n)}$  of codimension at least  $g$  such that for any  $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}'$ , the principal  $G$ -bundle  $\mathcal{E}_G^t$  is semistable.*
- (3) *Assume that  $g \geq 2$ . Assume that the monodromy representation for  $\nabla_0$  does not factor through some proper parabolic subgroup of  $G$ . There is a closed complex analytic subset  $\mathcal{Y}'' \subset \text{Teich}_{(g,n)}$  of codimension at least  $g - 1$  such that for any  $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}''$ , the principal  $G$ -bundle  $\mathcal{E}_G^t$  is stable.*

It is known that if a holomorphic principal  $G$ -bundle  $E_G$  over an elliptic curve admits a holomorphic connection, then  $E_G$  is semistable. Therefore, a stronger version of Theorem 1.3(1) is valid when  $g = 1$ .

## 2. INFINITESIMAL DEFORMATIONS

We first recall some classical results in deformation theory, thereby setting up our notation.

**2.1. Deformations of a  $n$ -pointed curve.** Let  $X_0$  be an irreducible smooth complex projective curve of genus  $g$ , with  $g > 0$ , and let

$$D_0 := \{x_1, \dots, x_n\} \subset X_0$$

be an ordered subset of cardinality  $n$  (it may be zero). We assume that  $n > 0$  if  $g = 1$ . This condition implies that the pair  $(X_0, D_0)$  does not have any infinitesimal automorphism, equivalently, the automorphism group of  $(X_0, D_0)$  is finite.

Let  $B := \text{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$  be the spectrum of the local ring. An *infinitesimal deformation* of  $(X_0, D_0)$  is given by a quadruple

$$(\mathcal{X}, q, \mathcal{D}, f), \tag{2.1}$$

where

- $q : \mathcal{X} \rightarrow B$  is a smooth proper algebraic morphism of relative dimension one,
- $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$  is a collection of  $n$  ordered disjoint sections of  $q$ , and
- $f : X_0 \rightarrow \mathcal{X}$  is an algebraic morphism such that

$$f(X_0) \subset \mathcal{X}_0 := q^{-1}(0) \quad \text{with} \quad x_i = \mathcal{D}_i(0) \quad \forall \quad 1 \leq i \leq n,$$

and the morphism  $X_0 \xrightarrow{f} \mathcal{X}_0$  is an isomorphism.

The divisor  $\sum_{i=1}^n \mathcal{D}_i(B)$  on  $\mathcal{X}$  will also be denoted by  $\mathcal{D}$ . For a vector bundle  $\mathcal{V}$  on  $\mathcal{X}$ , the vector bundle  $\mathcal{V} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$  will be denoted by  $\mathcal{V}(-\mathcal{D})$ .

The differential of  $f$

$$df : \text{TX}_0 \rightarrow f^*\text{T}\mathcal{X}$$

produces a homomorphism  $\text{TX}_0(-D_0) := \text{TX}_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0}(-D_0) \rightarrow f^*\text{T}\mathcal{X}(-\mathcal{D})$  which will also be denoted by  $df$ . Consider the following short exact sequence of sheaves on  $X_0$ :

$$0 \rightarrow \text{TX}_0(-D_0) \xrightarrow{df} f^*\text{T}\mathcal{X}(-\mathcal{D}) \xrightarrow{h} \mathcal{O}_{X_0} \rightarrow 0; \tag{2.2}$$

note that the normal bundle to  $\mathcal{X}_0 \subset \mathcal{X}$  is the pullback of  $T_0B$  by  $q|_{\mathcal{X}_0}$  and hence this normal bundle is identified with  $\mathcal{O}_{X_0}$ . Consider the connecting homomorphism

$$\mathbb{C} = H^0(X_0, \mathcal{O}_{X_0}) \xrightarrow{\phi} H^1(X_0, \text{TX}_0(-D_0)) \tag{2.3}$$

in the long exact sequence of cohomologies associated to the short exact sequence in (2.2). Let  $1_{X_0}$  denote the constant function 1 on  $X_0$ . The cohomology element

$$\phi(1_{X_0}) \in H^1(X_0, \text{TX}_0(-D_0)), \tag{2.4}$$

where  $\phi$  is the homomorphism in (2.3), is the *Kodaira–Spencer infinitesimal deformation class* for the family in (2.1). If this infinitesimal deformation class is zero, then the family  $(\mathcal{X}, \mathcal{D}) \rightarrow B$  is isomorphic to the trivial family  $(X_0 \times B, D_0 \times B) \rightarrow B$ .

**2.2. Deformations of a curve together with a principal bundle.** Take  $(X_0, D_0)$  as before. Let  $G$  be a connected reductive affine algebraic group defined over  $\mathbb{C}$ . The Lie algebra of  $G$  will be denoted by  $\mathfrak{g}$ . Let

$$p : E_G \longrightarrow X_0 \quad (2.5)$$

be a principal  $G$ -bundle on  $X_0$ . The infinitesimal deformations of the triple

$$(X_0, D_0, E_G) \quad (2.6)$$

are guided by the Atiyah bundle  $\text{At}(E_G) \longrightarrow X_0$ , the construction of which we shall briefly recall (see [At] for a more detailed exposition). Consider the direct image  $p_*\text{TE}_G \longrightarrow X_0$ , where  $\text{TE}_G$  is the holomorphic tangent bundle of  $E_G$ , and  $p$  is the projection in (2.5). It is a quasicoherent sheaf equipped with an action of  $G$  given by the action of  $G$  on  $E_G$ . The invariant part

$$\text{At}(E_G) := (p_*\text{TE}_G)^G \subset (p_*\text{TE}_G)$$

is a vector bundle on  $X_0$  of rank  $1 + \dim G$  which is known as the Atiyah bundle of  $E_G$ . Consequently, we have  $\text{At}(E_G) = (\text{TE}_G)/G$ . Let

$$\text{ad}(E_G) := E_G \times^G \mathfrak{g} \longrightarrow X_0$$

be the adjoint vector bundle associated to  $E_G$  for the adjoint action of  $G$  on its Lie algebra  $\mathfrak{g}$ . Let

$$dp : \text{TE}_G \longrightarrow p^*\text{TX}_0$$

be the differential of the map  $p$  in (2.5). Being  $G$ -equivariant it produces a homomorphism  $\text{At}(E_G) \longrightarrow \text{TX}_0$  which will also be denoted by  $dp$ . Now the action of  $G$  on  $E_G$  produces an isomorphism  $E_G \times \mathfrak{g} \longrightarrow \text{kernel}(dp)$ . Therefore, we have  $\text{kernel}(dp)/G = \text{ad}(E_G)$ . In other words, the above isomorphism descends to an isomorphism

$$\text{ad}(E_G) \xrightarrow{\sim} (p_*(\text{kernel}(dp)))^G$$

that preserves the Lie algebra structure on the fibers of the two vector bundles (the Lie algebra structure on the fibers of  $(p_*(\text{kernel}(dp)))^G$  is given by the Lie bracket of  $G$ -invariant vertical vector fields). Therefore,  $\text{At}(E_G)$  fits in the following short exact sequence of vector bundles on  $X_0$

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G) \xrightarrow{dp} \text{TX}_0 \longrightarrow 0, \quad (2.7)$$

which is known as the Atiyah exact sequence for  $E_G$ . The logarithmic Atiyah bundle  $\text{At}(E_G, D_0)$  is defined by

$$\text{At}(E_G, D_0) := (dp)^{-1}(\text{TX}_0(-D_0)) \subset \text{At}(E_G).$$

From (2.7) we have the short exact sequence

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G, D_0) \xrightarrow{\sigma} \text{TX}_0(-D_0) \longrightarrow 0, \quad (2.8)$$

which is called the logarithmic Atiyah exact sequence. The above homomorphism  $\sigma$  is the restriction of  $dp$  to  $\text{At}(E_G, D_0) \subset \text{At}(E_G)$ .

An *infinitesimal deformation* of the triple  $(X_0, D_0, E_G)$  in (2.6) is a 6-tuple

$$(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi), \quad (2.9)$$

where

- $(\mathcal{X}, q, \mathcal{D}, f)$  in an infinitesimal deformation of the  $n$ -pointed curve  $(X_0, D_0)$  as in (2.1),
- $\mathcal{E}_G \rightarrow \mathcal{X}$  is a principal  $G$ -bundle, and
- $\psi$  is an isomorphism

$$\psi : E_G \rightarrow f^* \mathcal{E}_G \quad (2.10)$$

of principal  $G$ -bundles.

The logarithmic Atiyah bundle

$$\text{At}(\mathcal{E}_G, \mathcal{D}) \rightarrow \mathcal{X}$$

for  $(\mathcal{E}_G, \mathcal{D})$  is the inverse image, in  $\text{At}(\mathcal{E}_G)$ , of the subsheaf  $\text{T}\mathcal{X}(-\mathcal{D}) \subset \text{T}\mathcal{X}$ . We have the following short exact sequence of sheaves on  $X_0$ :

$$0 \rightarrow \text{At}(E_G, D_0) \rightarrow f^* \text{At}(\mathcal{E}_G, \mathcal{D}) \rightarrow \mathcal{O}_{X_0} \rightarrow 0 \quad (2.11)$$

given by  $\psi$  in (2.10). Let

$$\mathbb{C} = H^0(X_0, \mathcal{O}_{X_0}) \xrightarrow{\tilde{\phi}} H^1(X_0, \text{At}(E_G, D_0))$$

be the connecting homomorphism in the long exact sequence of cohomologies associated to (2.11). The cohomology element

$$\tilde{\phi}(1_{X_0}) \in H^1(X_0, \text{At}(E_G, D_0)) \quad (2.12)$$

is the *infinitesimal deformation class* of the triple  $(X_0, D_0, E_G)$  given by (2.9). Let

$$\sigma_* : H^1(X_0, \text{At}(E_G, D_0)) \rightarrow H^1(X_0, \text{TX}_0(-D_0))$$

be the homomorphism given by the projection  $\sigma$  in (2.8). From the commutativity of the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{At}(E_G, D_0) & \rightarrow & f^* \text{At}(\mathcal{E}_G, \mathcal{D}) & \rightarrow & \mathcal{O}_{X_0} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \text{TX}_0(-D_0) & \rightarrow & f^* \text{T}\mathcal{X}(-\mathcal{D}) & \rightarrow & \mathcal{O}_{X_0} \rightarrow 0 \end{array}$$

where the top and bottom rows are as in (2.11) and (2.2) respectively, it follows that

$$\sigma_*(\tilde{\phi}(1_{X_0})) = \phi(1_{X_0}),$$

where  $\tilde{\phi}(1_{X_0})$  and  $\phi(1_{X_0})$  are constructed in (2.12) and (2.4) respectively. We note that  $\sigma_*$  is the forgetful map that sends an infinitesimal deformation of  $(X_0, D_0, E_G)$  to the underlying infinitesimal deformation of  $(X_0, D_0)$ .

### 3. OBSTRUCTION TO AN EXTENSION OF A REDUCTION OF STRUCTURE GROUP

Given a reduction of structure group of  $E_G$  to a parabolic subgroup of  $G$ , our aim in this section is to compute the obstruction for it to extend to a reduction of an infinitesimal deformation  $\mathcal{E}_G \rightarrow \mathcal{X}$  as in (2.9).

Fix a parabolic subgroup  $P \subset G$ . The Lie algebra of  $P$  will be denoted by  $\mathfrak{p}$ . Let

$$E_P \subset E_G \quad (3.1)$$

be a holomorphic reduction of structure group of  $E_G$  to the subgroup  $P \subset G$ . Let

$$\mathrm{ad}(E_P) := E_P \times^P \mathfrak{p} \longrightarrow X_0$$

be the adjoint vector bundle associated to  $E_P$  for the adjoint action of  $P$  on its Lie algebra  $\mathfrak{p}$ . The vector bundle over  $X_0$  associated to the principal  $P$ -bundle  $E_P$  for the adjoint action of  $P$  on the quotient  $\mathfrak{g}/\mathfrak{p}$  will be denoted by  $E_P(\mathfrak{g}/\mathfrak{p})$ . The logarithmic Atiyah bundle for  $(E_P, D_0)$  will be denoted by  $\mathrm{At}(E_P, D_0)$ . We have

$$\mathrm{ad}(E_P) \subset \mathrm{ad}(E_G) \quad \text{and} \quad \mathrm{At}(E_P, D_0) \subset \mathrm{At}(E_G, D_0);$$

both the quotient bundles above are identified with  $E_P(\mathfrak{g}/\mathfrak{p})$ . In other words, we have the following commutative diagram of vector bundles on  $X_0$

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{ad}(E_P) & \longrightarrow & \mathrm{At}(E_P, D_0) & \xrightarrow{\beta} & \mathrm{TX}_0(-D_0) \longrightarrow 0 \\ & & \downarrow & & \downarrow \xi & & \parallel \\ 0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}(E_G, D_0) & \xrightarrow{\gamma} & \mathrm{TX}_0(-D_0) \longrightarrow 0 \\ & & \downarrow \mu_1 & & \downarrow \mu & & \\ & & E_P(\mathfrak{g}/\mathfrak{p}) & = & E_P(\mathfrak{g}/\mathfrak{p}) & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (3.2)$$

Let

$$\tilde{\xi} : \mathrm{H}^1(X_0, \mathrm{At}(E_P, D_0)) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}(E_G, D_0)) \quad (3.3)$$

be the homomorphism induced by the canonical injection  $\xi$  in (3.2).

Take  $(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)$  as in (2.9). Assume that the reduction  $E_P \subset E_G$  in (3.1) extends to a reduction of structure group

$$\mathcal{E}_P \subset \mathcal{E}_G$$

to  $P \subset G$  on  $\mathcal{X}$ . Consider the short exact sequence on  $X_0$

$$0 \longrightarrow \mathrm{At}(E_P, D_0) \longrightarrow f^* \mathrm{At}(\mathcal{E}_P, \mathcal{D}) \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0, \quad (3.4)$$

where  $\mathrm{At}(\mathcal{E}_P, \mathcal{D}) \longrightarrow \mathcal{X}$  is the logarithmic Atiyah bundle associated to the principal  $P$ -bundle  $\mathcal{E}_P$ , and  $f$  is the map in (2.1). Let

$$\theta \in \mathrm{H}^1(X_0, \mathrm{At}(E_P, D_0)) \quad (3.5)$$

be the image of the constant function  $1_{X_0}$  by the homomorphism

$$\mathrm{H}^0(X_0, \mathcal{O}_{X_0}) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}(E_P, D_0))$$

in the long exact sequence of cohomologies associated to (3.4).

**Lemma 3.1.** *The cohomology class  $\theta$  in (3.5) satisfies the equation*

$$\tilde{\xi}(\theta) = \tilde{\phi}(1_{X_0}),$$

where  $\tilde{\xi}$  and  $\tilde{\phi}(1_{X_0})$  are constructed in (3.3) and (2.12) respectively.



*Proof.* Consider the commutative diagram of vector bundles on  $X_0$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{At}(E_P, D_0) & \longrightarrow & f^*\text{At}(\mathcal{E}_P, \mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \\ & & \downarrow \xi & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{At}(E_G, D_0) & \longrightarrow & f^*\text{At}(\mathcal{E}_G, \mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \end{array} \quad (3.6)$$

where the top and bottom rows are as in (3.4) and (2.11) respectively, and  $\xi$  is the homomorphism in (3.2); the above homomorphism  $f^*\text{At}(\mathcal{E}_P, \mathcal{D}) \longrightarrow f^*\text{At}(\mathcal{E}_G, \mathcal{D})$  is the pullback of the natural homomorphism  $\text{At}(\mathcal{E}_P, \mathcal{D}) \longrightarrow \text{At}(\mathcal{E}_G, \mathcal{D})$ . In view of (3.6), the lemma follows by comparing the constructions of  $\theta$  and  $\tilde{\varphi}(1_{X_0})$ .  $\square$

#### 4. LOGARITHMIC CONNECTIONS AND THE SECOND FUNDAMENTAL FORM

In this section we characterize those infinitesimal deformations of the principal bundle  $E_G$  on the  $n$ -pointed curve that arise from the isomonodromic deformations.

**4.1. Canonical extension of a logarithmic connection.** As before, let  $p : E_G \longrightarrow X_0$  be a principal  $G$ -bundle. A *logarithmic connection* on  $E_G$  with polar divisor  $D_0$  is a holomorphic splitting of the logarithmic Atiyah exact sequence in (2.8). In other words, a logarithmic connection is a homomorphism

$$\delta : \text{TX}_0(-D_0) \longrightarrow \text{At}(E_G, D_0) \quad (4.1)$$

such that  $\sigma \circ \delta = \text{Id}_{\text{TX}_0(-D_0)}$ , where  $\sigma$  is the homomorphism in (2.8). Note that given such a  $\delta$ , there is a unique homomorphism

$$\delta'' : \text{At}(E_G, D_0) \longrightarrow \text{ad}(E_G) \quad (4.2)$$

such that  $\delta'' \circ \delta = 0$ , and the composition

$$\text{ad}(E_G) \hookrightarrow \text{At}(E_G, D_0) \xrightarrow{\delta''} \text{ad}(E_G)$$

(see (2.8)) is the identity map of  $\text{ad}(E_G)$ . As there are no nonzero  $(2, 0)$ -forms on  $X_0$ , all logarithmic connections on  $E_G$  are automatically integrable.

At the level of first order deformations, given a principal  $G$ -bundle  $\mathcal{E}_G \longrightarrow \mathcal{X} \xrightarrow{q} B$ , a logarithmic connection on  $\mathcal{E}_G$  with polar divisor  $\mathcal{D}$  is a homomorphism  $\text{At}(\mathcal{E}_G, \mathcal{D}) \longrightarrow \text{ad}(\mathcal{E}_G)$  that splits the logarithmic Atiyah exact sequence for  $\mathcal{E}_G$ . We note that a connection on  $\mathcal{E}_G$  need not be integrable, as we have added an (infinitesimal) extra dimension. However, the Riemann–Hilbert correspondence for principal  $G$ -bundles yields the following:

**Lemma 4.1.** *Let  $(\mathcal{X}, q, \mathcal{D}, f)$  be an infinitesimal deformation of  $(X_0, D_0)$  as in (2.1). Let  $\delta$  be a logarithmic connection on a principal  $G$ -bundle  $E_G$  on  $X_0$  with polar divisor  $D_0$ . Then there exists a unique pair  $(\mathcal{E}_G, \nabla)$ , where*

- $\mathcal{E}_G$  is a principal  $G$ -bundle on  $\mathcal{X}$ , and
- $\nabla$  is an integrable logarithmic connection on  $\mathcal{E}_G$  with polar divisor  $\mathcal{D}$ ,

such that  $(f^*\mathcal{E}_G, f^*\nabla) = (E_G, \delta)$ .

Let us recall a few elements of the proof of this (classical) result. Choose a covering  $\mathfrak{U}$  of  $X_0 \setminus D_0$  by complex discs and a small neighborhood  $U_i$  for each  $x_i \in D_0$ . Since  $\delta$  is integrable, we can choose local charts for  $E_G$  over  $\mathfrak{U}$  such that all transition functions are constants. Now if the curve fits into an analytic family  $\mathcal{X} \rightarrow \mathcal{B}$ , one can, restricting  $\mathcal{B}$  if necessary, extend all the  $U_i$  into tubular neighborhoods of  $\mathcal{D}_i$  and extend the open subsets in  $\mathfrak{U}$  such that  $\mathfrak{U} \cup U_1 \cup \dots \cup U_n$  is a covering of  $\mathcal{X}$ . The isomonodromic deformation is then given by simply extending the transition maps by keeping them to be constant in deformation parameters.

The logarithmic connection  $\delta$  gives a splitting of the logarithmic Atiyah bundle

$$\text{At}(E_G, D_0) = \text{ad}(E_G) \oplus \text{TX}_0(-D_0).$$

The corresponding cohomological decomposition

$$\text{H}^1(X_0, \text{At}(E_G, D_0)) = \text{H}^1(X_0, \text{ad}(E_G)) \oplus \text{H}^1(X_0, \text{TX}_0(-D_0))$$

given a splitting of the infinitesimal deformations of  $(X_0, D_0, E_G)$  into infinitesimal the deformations of  $(X_0, D_0)$  and the infinitesimal deformations  $E_G$  (keeping  $(X_0, D_0)$  fixed). In other words, let

$$\delta' : \text{H}^1(X_0, \text{TX}_0(-D_0)) \longrightarrow \text{H}^1(X_0, \text{At}(E_G, D_0))$$

be the homomorphism induced by the homomorphism  $\delta : \text{TX}_0(-D_0) \rightarrow \text{At}(E_G, D_0)$  in Lemma 4.1 defining the logarithmic connection on  $E_G$ . Given an infinitesimal deformation  $(\mathcal{X}, q, \mathcal{D}, f)$  of  $(X_0, D_0)$ , the above homomorphism  $\delta'$  produces an infinitesimal deformation  $(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)$  of  $(X_0, D_0, E_G)$ . As explained above, this principal  $G$ -bundle  $\mathcal{E}_G$  on  $\mathcal{X}$  coincides with the principal  $G$ -bundle on  $\mathcal{X}$  produced by the isomonodromic deformation in Lemma 4.1.

We will now construct the exact sequence in (2.11) corresponding to the above infinitesimal deformation  $(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)$ . Consider the injective homomorphism

$$\text{TX}_0(-D_0) \longrightarrow \text{At}(E_G, D_0) \oplus f^*\text{T}\mathcal{X}(-\mathcal{D}), \quad v \longmapsto (\delta(v), -(df)(v)),$$

where  $df$  is the differential in (2.2). The corresponding cokernel

$$\text{At}^\delta(E_G, D_0) := (\text{At}(E_G, D_0) \oplus f^*\text{T}\mathcal{X}(-\mathcal{D})) / (\text{TX}_0(-D_0))$$

possesses a canonical projection

$$\widehat{\delta} : \text{At}^\delta(E_G, D_0) \longrightarrow \mathcal{O}_{X_0}, \quad (v, w) \longmapsto h(w), \tag{4.3}$$

where  $h$  is the homomorphism in (2.2); note that the above homomorphism  $\widehat{\delta}$  is well-defined because  $h$  vanishes on the image of  $\text{TX}_0(-D_0)$  in  $\text{At}(E_G, D_0) \oplus f^*\text{T}\mathcal{X}(-\mathcal{D})$ . The kernel of  $\widehat{\delta}$  is identified with  $\text{At}(E_G, D_0)$  by sending any  $z \in \text{At}(E_G, D_0)$  to the image in  $\text{At}^\delta(E_G, D_0)$  of  $(z, 0) \in \text{At}(E_G, D_0) \oplus f^*\text{T}\mathcal{X}(-\mathcal{D})$ . Therefore, we obtain the following exact sequence of vector bundles over  $X_0$ :

$$0 \longrightarrow \text{At}(E_G, D_0) \longrightarrow \text{At}^\delta(E_G, D_0) \xrightarrow{\widehat{\delta}} \mathcal{O}_{X_0} \longrightarrow 0,$$

This exact sequence coincides with the one in (2.11).

Consider the projection

$$\mathrm{At}(E_G, D_0) \oplus f^* \mathrm{TX}(-\mathcal{D}) \longrightarrow \mathrm{ad}(E_G), \quad (z_1, z_2) \longmapsto \delta''(z_1),$$

where  $\delta''$  is constructed in (4.2) from  $\delta$ . It vanishes on the image of  $\mathrm{TX}_0(-D_0)$ , yielding a projection

$$\lambda : \mathrm{At}^\delta(E_G, D_0) \longrightarrow \mathrm{ad}(E_G). \quad (4.4)$$

Let  $\nabla'' : \mathrm{At}(\mathcal{E}_G, \mathcal{D}) \longrightarrow \mathrm{ad}(\mathcal{E}_G)$  be the homomorphism given by the logarithmic connection  $\nabla$  in Lemma 4.1. The homomorphism in (4.4) fits in the commutative diagram

$$\begin{array}{ccc} \mathrm{At}^\delta(E_G, D_0) & \xrightarrow{\lambda} & \mathrm{ad}(E_G) \\ \parallel & & \parallel \\ f^* \mathrm{At}(\mathcal{E}_G, \mathcal{D}) & \xrightarrow{f^* \nabla''} & f^* \mathrm{ad}(\mathcal{E}_G) \end{array} \quad (4.5)$$

(the vertical identifications are evident).

We summarize the above constructions in the following lemma:

**Lemma 4.2.** *Given  $(\mathcal{X}, q, \mathcal{D}, f)$  as in (2.1), and also given a logarithmic connection  $\delta$  on a principal  $G$ -bundle  $E_G \longrightarrow X_0$ , the exact sequence in (2.11) corresponding to the infinitesimal deformation of  $(X_0, D_0, E_G)$  in Lemma 4.1 is*

$$0 \longrightarrow \mathrm{At}(E_G, D_0) \longrightarrow \mathrm{At}^\delta(E_G, D_0) \xrightarrow{\widehat{\delta}} \mathcal{O}_{X_0} \longrightarrow 0,$$

where  $\widehat{\delta}$  is constructed in (4.3).

**4.2. The second fundamental form.** Fix a logarithmic connection  $\delta$  on  $(E_G, D_0)$  as in (4.1). Take a reduction of structure group  $E_P \subset E_G$  to  $P$  as in (3.1). The composition

$$S(\delta) := \mu \circ \delta : \mathrm{TX}_0(-D_0) \longrightarrow E_P(\mathfrak{g}/\mathfrak{p}), \quad (4.6)$$

where  $\mu$  is constructed in (3.2), is called the *second fundamental form* of  $E_P$  for the connection  $\delta$ . We note that  $\delta$  is induced by a logarithmic connection on the principal  $P$ -bundle  $E_P$  if and only if we have  $S(\delta) = 0$ .

Assume that  $E_P$  satisfies the condition that  $S(\delta) \neq 0$ . Let

$$\mathcal{L} \subset E_P(\mathfrak{g}/\mathfrak{p}) \quad (4.7)$$

be the holomorphic line subbundle generated by the image of the homomorphism  $S(\delta)$  in (4.6). More precisely,  $\mathcal{L}$  is the inverse image, in  $E_P(\mathfrak{g}/\mathfrak{p})$ , of the torsion part of the quotient  $E_P(\mathfrak{g}/\mathfrak{p})/(S(\delta)(\mathrm{TX}_0(-D_0)))$ . Now consider the homomorphism

$$S(\delta)' : \mathrm{TX}_0(-D_0) \longrightarrow \mathcal{L} \quad (4.8)$$

given by the second fundamental form  $S(\delta)$ . Let

$$S' : H^1(X_0, \mathrm{TX}_0(-D_0)) \longrightarrow H^1(X_0, \mathcal{L}) \quad (4.9)$$

be the homomorphism induced by  $S(\delta)'$  in (4.8).

**Proposition 4.3.** *As before, let  $\delta$  be a logarithmic connection on  $E_G \rightarrow X_0$  with polar divisor  $D_0$ , and let  $(\mathcal{X}, q, \mathcal{D}, f)$  be an infinitesimal deformation of  $(X_0, D_0)$ . Let  $\mathcal{E}_G \rightarrow \mathcal{X}$  be the isomonodromic deformation of  $\delta$  along  $(\mathcal{X}, \mathcal{D})$  obtained in Lemma 4.1. Let  $E_P \subset E_G$  be a reduction of structure group to  $P$  over  $X_0$  that extends to a reduction of structure group  $\mathcal{E}_P \subset \mathcal{E}_G$  to  $P$  over  $\mathcal{X}$ . Then*

$$S'(\phi(1_{X_0})) = 0,$$

where  $\phi(1_{X_0})$  is the corresponding cohomology class constructed in (2.4), and  $S'$  is constructed in (4.9).

*Proof.* Consider the inverse images

$$\text{At}_P(E_G, D_0) := \mu^{-1}(\mathcal{L}) \subset \text{At}(E_G, D_0) \quad \text{and} \quad \text{ad}_P(E_G) := \mu_1^{-1}(\mathcal{L}) \subset \text{ad}(E_G),$$

where  $\mu$  and  $\mu_1$  are the quotient maps in (3.2), and  $\mathcal{L}$  is constructed in (4.7). These two vector bundles fit in the following commutative diagram produced from (3.2):

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{ad}(E_P) & \longrightarrow & \text{At}(E_P, D_0) & \xrightarrow{\beta} & \text{TX}_0(-D_0) \longrightarrow 0 \\ & & \downarrow & & \downarrow \xi & & \parallel \\ 0 & \longrightarrow & \text{ad}_P(E_G) & \longrightarrow & \text{At}_P(E_G, D_0) & \xrightarrow{\gamma} & \text{TX}_0(-D_0) \longrightarrow 0 \\ & & \downarrow \mu_1 & & \downarrow \mu & & \\ & & \mathcal{L} & = & \mathcal{L} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (4.10)$$

By the construction of  $\text{At}_P(E_G, D_0)$ , the connection homomorphism  $\delta$  in (4.1) factors through a homomorphism

$$\delta^1 : \text{TX}_0(-D_0) \longrightarrow \text{At}_P(E_G, D_0).$$

Consider the homomorphism

$$\delta_*^1 : \text{H}^1(X_0, \text{TX}_0(-D_0)) \longrightarrow \text{H}^1(X_0, \text{At}_P(E_G, D_0))$$

induced by the above homomorphism  $\delta^1$ , and let

$$\Phi := \delta_*^1(\phi(1_{X_0})) \in \text{H}^1(X_0, \text{At}_P(E_G, D_0)) \quad (4.11)$$

be the image of the cohomology class  $\phi(1_{X_0})$  characterizing the deformation  $(\mathcal{X}, \mathcal{D})$  as in (2.4).

As in the statement of the proposition, let  $\mathcal{E}_P \rightarrow \mathcal{X}$  be an extension of the reduction  $E_P$ . Note that from (3.6), (3.2) and Lemma 4.2 we have

$$\begin{aligned} \text{At}(E_G, D_0)/\text{At}(E_P, D_0) &= E_P(\mathfrak{g}/\mathfrak{p}) = (f^*\text{At}(\mathcal{E}_G, \mathcal{D}))/ (f^*\text{At}(\mathcal{E}_P, \mathcal{D})) \\ &= \text{At}^\delta(E_G, D_0)/f^*\text{At}(\mathcal{E}_P, \mathcal{D}). \end{aligned}$$

Let  $\mu_2 : \text{At}^\delta(E_G, D_0) \rightarrow E_P(\mathfrak{g}/\mathfrak{p})$  be the above quotient map. Define

$$\text{At}_P^\delta(E_G, D_0) := \mu_2^{-1}(\mathcal{L}) \subset \text{At}^\delta(E_G, D_0),$$

where  $\mathcal{L}$  is constructed in (4.7).

Now we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{TX}_0(-D_0) & \longrightarrow & f^*\mathrm{T}\mathcal{X}(-\mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \\
& & \downarrow \delta & & \downarrow f^*\nabla & & \parallel \\
0 & \longrightarrow & \mathrm{At}_P(E_G, D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_G, \mathcal{D}) = \mathrm{At}_P^\delta(E_G, D_0) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0
\end{array} \tag{4.12}$$

where the bottom exact sequence is obtained from (3.6), and the top exact sequence is as in (2.2) (see also (4.5)). Let

$$\nu : \mathrm{H}^0(X_0, \mathcal{O}_{X_0}) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}_P(E_G, D_0))$$

be the connecting homomorphism in the long exact sequence of cohomologies associated to the bottom exact sequence in (4.12). From the commutativity of (4.12) and the construction of  $\phi(1_{X_0})$  (see (2.4)) it follows that

$$\Phi = \nu(1_{X_0}) = \delta_*(\phi(1_{X_0})), \tag{4.13}$$

where  $\delta_* : \mathrm{H}^1(X_0, \mathrm{TX}_0(-D_0)) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}_P(E_G, D_0))$  is the homomorphism induced by  $\delta$ , and  $\Phi$  is the cohomology class in (4.11).

The diagram in (3.6) produces the diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{At}(E_P, D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_P, D_0) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \\
& & \downarrow \xi & & \downarrow & & \parallel \\
0 & \longrightarrow & \mathrm{At}_P(E_G, D_0) & \longrightarrow & \mathrm{At}_P^\delta(E_G, D_0) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \\
& & \downarrow \mu & & \downarrow & & \\
& & \mathcal{L} & = & \mathcal{L} & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array} \tag{4.14}$$

where  $\xi$  and  $\mu$  are the homomorphisms in (4.10). Using this diagram we can check that

$$\Phi = \xi_*(\theta), \tag{4.15}$$

where  $\theta$  is the cohomology classes in (3.5), and

$$\xi_* : \mathrm{H}^1(X_0, \mathrm{At}(E_P, D_0)) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}_P(E_G, D_0))$$

is the homomorphism induced by  $\xi$  in (4.14). Indeed, (4.15) follows from (4.13), the commutativity of (4.14) and the construction of  $\theta$ .

Let  $\mu_* : \mathrm{H}^1(X_0, \mathrm{At}_P(E_G, D_0)) \longrightarrow \mathrm{H}^1(X_0, \mathcal{L})$  be the homomorphism induced by the homomorphism  $\mu$  in (4.14). It is straight-forward to check that

$$\mu_*(\Phi) = S'(\phi(1_{X_0}))$$

(see (4.9), (2.4) and (4.11) for  $S'$ ,  $\phi(1_{X_0})$  and  $\Phi$  respectively). Indeed, this follows from (4.13) and the definition of  $S(\delta)$  in (4.6). Combining this with (4.15), we have

$$S'(\phi(1_{X_0})) = \mu_*(\Phi) = \mu_*(\xi_*(\theta)).$$

Since  $\mu \circ \xi = 0$  (see (4.14)), it now follows that  $S'(\phi(1_{X_0})) = 0$ .  $\square$

## 5. LOGARITHMIC CONNECTIONS AND SEMISTABILITY OF UNDERLYING PRINCIPAL BUNDLE

Let  $\mathcal{T}_{g,n}$  be the Teichmüller space of genus  $g$  compact Riemann surfaces with  $n$  ordered marked points. As before, we assume that  $g > 0$ , and if  $g = 1$ , then  $n > 0$ . Let

$$\tau : \mathcal{C} \longrightarrow \mathcal{T}_{g,n}$$

be the universal Teichmüller curve with  $n$  ordered sections  $\Sigma$ . The fiber of  $\mathcal{C}$  on any point  $t \in \mathcal{T}_{g,n}$  will be denoted by  $\mathcal{C}_t$ . The ordered subset  $\mathcal{C}_t \cap \Sigma \subset \mathcal{C}_t$  will be denoted by  $\Sigma_t$ .

Take a  $n$ -pointed Riemann surface  $(C_0, \Sigma_0)$  of genus  $g$ , which is represented by a point of  $\mathcal{T}_{g,n}$ . Let

$$\nabla_0 \tag{5.1}$$

be a logarithmic connection on a principal  $G$ -bundle  $F_G \longrightarrow C_0$  with polar divisor  $\Sigma_0$ . By Riemann–Hilbert correspondence, the connection  $\nabla_0$  produces a flat (isomonodromic) logarithmic connection  $\nabla$  on a holomorphic principal  $G$ -bundle  $\mathcal{F}_G \longrightarrow \mathcal{C}$  with polar divisor  $\Sigma$ .

The following lemma is a special case of the main theorem in [GN] (see also [Sh]). Although the families of  $G$ -bundles considered in [GN] are algebraic, all arguments there go through in the analytic case of our interest with obvious modifications.

**Lemma 5.1** ([GN]). *Let  $\mathcal{F}_G \longrightarrow \mathcal{C} \longrightarrow \mathcal{T}_{g,n}$  be as above. For each Harder–Narasimhan type  $\kappa$ , the set*

$$\mathcal{Y}_\kappa := \{t \in \mathcal{T}_{g,n} \mid \mathcal{F}_G|_{\mathcal{C}_t} \text{ is of type } \kappa\}$$

*is a (possibly empty) locally closed complex analytic subspace of  $\mathcal{T}_{g,n}$ . More precisely, for each Harder–Narasimhan type  $\kappa$ , the union  $\mathcal{Y}_{\leq \kappa} := \bigcup_{\kappa' \leq \kappa} \mathcal{Y}_{\kappa'}$  is a closed complex analytic subset of  $\mathcal{T}_{g,n}$ . Moreover, the principal  $G$ -bundle*

$$\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)} \longrightarrow \tau^{-1}(\mathcal{Y}_\kappa)$$

*possesses a canonical reduction of structure group inducing the Harder–Narasimhan reduction of  $\mathcal{F}_G|_{\mathcal{C}_t}$  for each  $t \in \mathcal{Y}_\kappa$ .*

In the following two Sections 5.1 and 5.2, we will see that under certain assumptions, the only Harder–Narasimhan stratum  $\mathcal{Y}_\kappa$  of maximal dimension  $\dim(\mathcal{T}_{g,n}) = 3g - 3 + n$  is the trivial one, in the sense that the Harder–Narasimhan parabolic subgroup is  $G$  itself. In other words, if the principal  $G$ -bundle  $F_G$  is not semistable, and therefore has a non-trivial Harder–Narasimhan reduction  $E_P \subset E_G$  to a certain parabolic subgroup  $P \subsetneq G$ , then there is always an isomonodromic deformation in which direction the reduction  $E_P$  is obstructed.

5.1. **The case of  $n = 0$ .** In this subsection we assume that  $n = 0$ . So, we have  $g > 1$ .

**Theorem 5.2.** *There is a closed complex analytic subset  $\mathcal{Y} \subset \mathcal{T}_{g,0}$  of codimension at least  $g$  such that for any  $t \in \mathcal{T}_{g,0} \setminus \mathcal{Y}$ , the holomorphic principal  $G$ -bundle  $\mathcal{F}_G|_{\mathcal{C}_t} \rightarrow \mathcal{C}_t$  is semistable.*

*Proof.* Let  $\mathcal{Y} \subset \mathcal{T}_{g,0}$  denote the (finite) union of all Harder-Narasimhan strata  $\mathcal{Y}_\kappa$  as in Lemma 5.1 with nontrivial Harder-Narasimhan type  $\kappa$ . From Lemma 5.1 we know that  $\mathcal{Y}$  is a closed complex analytic subset of  $\mathcal{T}_{g,0}$ .

Take any  $t \in \mathcal{Y}_\kappa \subset \mathcal{Y}$ . Let  $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$  be the holomorphic principal  $G$ -bundle on  $X_0 := \mathcal{C}_t$ . The holomorphic connection on  $E_G$  obtained by restricting  $\nabla$  will be denoted by  $\delta$ . Since  $E_G$  is not semistable, there is a proper parabolic subgroup  $P \subsetneq G$  and a holomorphic reduction of structure group  $E_P \subset E_G$  to  $P$ , such that  $E_P$  is the Harder-Narasimhan reduction [Be], [AAB]. From Lemma 5.1 we know that  $E_P$  extends to a reduction of structure group of the principal  $G$ -bundle  $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$  to the subgroup  $P$ .

Let  $\text{ad}(E_P)$  and  $\text{ad}(E_G)$  be the adjoint vector bundles of  $E_P$  and  $E_G$  respectively. Consider the vector bundle

$$\text{ad}(E_G)/\text{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$$

(see (3.2)). We know that

$$\mu_{\max}(E_P(\mathfrak{g}/\mathfrak{p})) < 0 \tag{5.2}$$

[AAB, p. 705] (see sixth line from bottom). In particular

$$\text{degree}(E_P(\mathfrak{g}/\mathfrak{p})) < 0. \tag{5.3}$$

A holomorphic connection on  $E_G$  induces a holomorphic connection on  $\text{ad}(E_G)$ , hence  $\text{degree}(\text{ad}(E_G)) = 0$ . Combining this with (5.3) it follows that  $\text{degree}(\text{ad}(E_P)) > 0$ , because  $\text{ad}(E_G)/\text{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$ . Since  $\text{degree}(\text{ad}(E_P)) \neq 0$ , the principal  $P$ -bundle  $E_P$  does not admit a holomorphic connection. Consequently, the second fundamental form  $S(\delta)$  of  $E_P$  for  $\delta$  (see (4.6)) is nonzero.

Using the second fundamental form  $S(\delta)$ , construct the holomorphic line subbundle

$$\mathcal{L} \subset E_P(\mathfrak{g}/\mathfrak{p})$$

as done in (4.7). From (5.2) we have

$$\text{degree}(\mathcal{L}) < 0. \tag{5.4}$$

Consider the short exact sequence of sheaves on  $X_0$

$$0 \rightarrow \text{TX}_0 \xrightarrow{S(\delta)'} \mathcal{L} \rightarrow T^\delta := \mathcal{L}/(S(\delta)(\text{TX}_0)) \rightarrow 0, \tag{5.5}$$

where  $S(\delta)'$  is constructed in (4.8); note that  $T^\delta$  is a torsion sheaf because  $S(\delta)' \neq 0$  (recall that  $S(\delta) \neq 0$ ). From (5.4) it follows that

$$\text{degree}(T^\delta) = \text{degree}(\mathcal{L}) - \text{degree}(\text{TX}_0) < -\text{degree}(\text{TX}_0) = 2g - 2.$$

So, we have

$$\dim H^0(X_0, T^\delta) = \text{degree}(T^\delta) < 2g - 2 = \dim H^1(X_0, \text{TX}_0) + 1 - g.$$

This implies that

$$\dim H^1(X_0, TX_0) - \dim H^0(X_0, T^\delta) \leq g. \quad (5.6)$$

Consider the long exact sequence of cohomologies

$$H^0(X_0, T^\delta) \longrightarrow H^1(X_0, TX_0) \xrightarrow{\zeta} H^1(X_0, \mathcal{L})$$

associated to the short exact sequence of sheaves in (5.5). From (5.6) it follows that

$$\dim \zeta(H^1(X_0, TX_0)) \geq g. \quad (5.7)$$

Since the reduction  $E_P$  extends to a reduction of structure group of the principal  $G$ -bundle  $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$  to the subgroup  $P$ , combining (5.7) and Proposition 4.3 we conclude that the codimension of the complex analytic subset  $\mathcal{Y}_\kappa \subset \mathcal{T}_{g,0}$  is at least  $g$ . This completes the proof of the theorem.  $\square$

**5.2. When  $n$  is arbitrary.** Now we assume that  $n > 0$ .

A logarithmic connection  $\eta$  on a holomorphic principal  $G$ -bundle  $F_G \rightarrow X_0$  is called *reducible* if there is pair  $(P, F_P)$ , where  $P \subsetneq G$  is a parabolic subgroup and  $F_P \subset F_G$  is a holomorphic reduction of structure group of  $F_G$  to  $P$ , such that  $\eta$  is induced by a connection on  $F_P$ . Note that  $\eta$  is induced by a connection on  $F_P$  if and only if the second fundamental form of  $F_P$  for  $\eta$  vanishes identically. A connection is called *irreducible* if it is not reducible or, equivalently, if the monodromy representation of the corresponding flat principal  $G$ -bundle does not factor through a proper parabolic subgroup of  $G$ .

**Proposition 5.3.** *Assume that the logarithmic connection  $\nabla_0$  in (5.1) is irreducible. Then there is a closed complex analytic subset  $\mathcal{Y} \subset \mathcal{T}_{g,n}$  of codimension at least  $g$  such that for any  $t \in \mathcal{T}_{g,n} \setminus \mathcal{Y}$ , the holomorphic principal  $G$ -bundle  $\mathcal{F}_G|_{\mathcal{C}_t}$  is semistable.*

*Proof.* The proof of Theorem 5.2 goes through after some obvious modifications. As before, let  $\mathcal{Y} \subset \mathcal{T}_{g,n}$  be the locus of all points  $t$  such that the principal  $G$ -bundle  $\mathcal{F}_G|_{\mathcal{C}_t}$  is not semistable. Take any  $t \in \mathcal{Y}_\kappa \subset \mathcal{Y}$ . Let  $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$  be the holomorphic principal  $G$ -bundle on  $X_0 := \mathcal{C}_t$ . The logarithmic connection on  $E_G$  with polar divisor  $D_0 := \Sigma_t$  obtained by restricting  $\nabla$  will be denoted by  $\delta$ .

Let  $E_P \subset E_G$  be the Harder–Narasimhan reduction. Since  $\nabla_0$  is irreducible, the second fundamental form  $S(\delta)$  of  $E_P$  for  $\delta$  (see (4.6)) is nonzero. Consider the short exact sequence of sheaves on  $X_0$

$$0 \longrightarrow TX_0(-D_0) \xrightarrow{S(\delta)'} \mathcal{L} \longrightarrow T^\delta := \mathcal{L}/(S(\delta)(TX_0(-D_0))) \longrightarrow 0, \quad (5.8)$$

where  $S(\delta)'$  is constructed in (4.8). As before,  $\text{degree}(\mathcal{L}) < 0$ , because  $\mu_{\max}(E_P(\mathfrak{g}/\mathfrak{p})) < 0$ . So,  $\text{degree}(T^\delta) < -\text{degree}(TX_0(-D_0)) = 2g - 2 + n$ . We now have

$$\dim H^0(X_0, T^\delta) = \text{degree}(T^\delta) < 2g - 2 + n = \dim H^1(X_0, TX_0(-D_0)) + 1 - g.$$

Hence the dimension of the image of the homomorphism

$$H^1(X_0, TX_0(-D_0)) \longrightarrow H^1(X_0, \mathcal{L}) \quad (5.9)$$



in the long exact sequence of cohomologies associated to (5.8) is at least  $g$ . Since the reduction  $E_P$  extends to a reduction of  $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$  to  $P$ , and the dimension of the image of the homomorphism in (5.9) is at least  $g$ , from Proposition 4.3 we conclude that the codimension of  $\mathcal{Y}_\kappa \subset \mathcal{T}_{g,0}$  is at least  $g$ .  $\square$

**5.3. Stability of underlying principal bundle.** We now assume that  $g \geq 2$ .

**Proposition 5.4.** *Assume that the logarithmic connection  $\nabla_0$  in (5.1) is irreducible. There is a closed analytic subset  $\mathcal{Y}' \subset \mathcal{T}$  of codimension at least  $g-1$  such that for any  $t \in \mathcal{T}_g \setminus \mathcal{Y}'$ , the holomorphic principal  $G$ -bundle  $\mathcal{F}_G|_{\mathcal{C}_t}$  is stable.*

*Proof.* The proof is identical to the proof of Proposition 5.3. If  $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$  is not stable, there is a maximal parabolic subgroup  $P \subsetneq G$  and a holomorphic reduction of structure group  $E_P \subset E_G$  to  $P$ , such that the quotient bundle

$$\mathrm{ad}(E_G)/\mathrm{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$$

is semistable of degree zero. Therefore, we have  $\mathrm{degree}(\mathcal{L}) \leq 0$ . This implies that

$$\mathrm{degree}(T^\delta) \leq 2g - 2 - n.$$

Hence the dimension of the image of the homomorphism  $H^1(X_0, TX_0(-D_0)) \rightarrow H^1(X_0, \mathcal{L})$  in the long exact sequence of cohomologies associated to (5.8) is at least  $g-1$ .  $\square$

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