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ISOMONODROMIC DEFORMATIONS OF LOGARITHMIC CONNECTIONS AND STABILITY

INDRANIL BISWAS, VIKTORIA HEU, AND JACQUES HURTUBISE

ABSTRACT. Let X_0 be a compact connected Riemann surface of genus g with $D_0 \subset X_0$ an ordered subset of cardinality n , and let E_G be a holomorphic principal G -bundle on X , where G is a reductive affine algebraic group defined over \mathbb{C} , that admits a logarithmic connection ∇_0 with polar divisor D_0 . Let (\mathcal{E}_G, ∇) be the universal isomonodromic deformation of (E_G, ∇_0) over the universal Teichmüller curve $(\mathcal{X}, \mathcal{D}) \rightarrow \text{Teich}_{g,n}$, where $\text{Teich}_{g,n}$ is the Teichmüller space for genus g Riemann surfaces with n -marked points. We prove the following (see Section 5):

- (1) Assume that $g \geq 2$ and $n = 0$. Then there is a closed complex analytic subset $\mathcal{Y} \subset \text{Teich}_{(g,n)}$, of codimension at least g , such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$, the principal G -bundle $\mathcal{E}_G|_{\mathcal{X}_t}$ is semistable, where \mathcal{X}_t is the compact Riemann surface over t .
- (2) Assume that $g \geq 1$, and if $g = 1$, then $n > 0$. Also, assume that the monodromy representation for ∇_0 does not factor through some proper parabolic subgroup of G . Then there is a closed complex analytic subset $\mathcal{Y}' \subset \text{Teich}_{(g,n)}$, of codimension at least g , such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}'$, the principal G -bundle $\mathcal{E}_G|_{\mathcal{X}_t}$ is semistable.
- (3) Assume that $g \geq 2$. Assume that the monodromy representation for ∇_0 does not factor through some proper parabolic subgroup of G . Then there is a closed complex analytic subset $\mathcal{Y}'' \subset \text{Teich}_{(g,n)}$, of codimension at least $g - 1$, such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}''$, the principal G -bundle $\mathcal{E}_G|_{\mathcal{X}_t}$ is stable.

In [He1], the second-named author proved the above results for $G = \text{GL}(2, \mathbb{C})$.

1. INTRODUCTION

Take any triple of the form $(E \rightarrow X, D, \nabla)$, where E is a holomorphic vector bundle over a smooth connected complex variety X , and ∇ is an integrable logarithmic connection on E singular over a simple normal crossing divisor $D \subset X$. The monodromy functor associates to it a representation $\rho_\nabla : \pi_1(X \setminus D, x_0) \rightarrow \text{GL}(E_{x_0})$. Altering the connection by a holomorphic automorphism of E leads to a conjugated representation. The monodromy functor produces an equivalence between the category of logarithmic connections (E, ∇) on (X, D) such that the real parts of residues lie in $[0, 1)$ and the category of equivalence classes of linear representations of $\pi_1(X \setminus D, x_0)$. Given a monodromy representation ρ , one can consider the set of all logarithmic connections $(E \rightarrow X, D, \nabla)$ (with no condition on the residues) up to holomorphic isomorphisms that produce the same monodromy representation $\rho = \rho_\nabla$ up to conjugation. All these connections are conjugated to each other by meromorphic gauge transformations with possible poles over D (see for example [Sa]).

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The classical Riemann–Hilbert problem can be formulated as follows: *Given a representation $\rho : \pi_1(\mathbb{P}_{\mathbb{C}}^1 \setminus D_0, x) \rightarrow \mathrm{GL}(r, \mathbb{C})$, is there a logarithmic connection $(E \rightarrow X, D_0, \nabla)$ such that $\rho = \rho_{\nabla}$ and E is holomorphically trivial?* The answer to this problem is

- (1) positive if rank $r = 2$ [Pl], [De],
- (2) negative in general ($r \geq 3$) [AB],
- (3) positive if ρ is irreducible [Bol1], [Ko].

On the other hand, the fundamental group $\pi_1(\mathbb{P}_{\mathbb{C}}^1 \setminus D_0, x)$ depends only on the topological and not the complex structure of $\mathbb{P}_{\mathbb{C}}^1 \setminus D_0$. So given an integrable connection, one can consider variations of the complex structure without changing the monodromy representation. More precisely, consider the Teichmüller space $\mathrm{Teich}_{0,n}$ of the n -pointed Riemann sphere together with its universal Teichmüller curve

$$\tau : (\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1 \times \mathrm{Teich}_{0,n}, \mathcal{D}) \rightarrow \mathrm{Teich}_{0,n},$$

where $n = \deg(D_0)$. Since $\mathrm{Teich}_{0,n}$ is contractible, the inclusion

$$(\mathbb{P}_{\mathbb{C}}^1, D_t) := \tau^{-1}(t) \hookrightarrow (\mathcal{X}, \mathcal{D}), \quad t \in \mathrm{Teich}_{0,n},$$

induces an isomorphism $\pi_1(\mathbb{P}_{\mathbb{C}}^1 \setminus D_t, x_t) \simeq \pi_1(\mathcal{X} \setminus \mathcal{D}, x_t)$. Hence by the Riemann–Hilbert correspondence, we can associate to any logarithmic connection (E_0, ∇_0) on $\mathbb{P}_{\mathbb{C}}^1$, with polar divisor D_0 , its *universal isomonodromic deformation*: a flat logarithmic connection (\mathcal{E}, ∇) over \mathcal{X} with polar divisor \mathcal{D} that extends the connection (E_0, ∇_0) . The monodromy representation for $\nabla|_{\tau^{-1}(t)}$ does not change as t moves over $\mathrm{Teich}_{0,n}$ (see for example [He2]).

We are led to another Riemann–Hilbert problem: *Given a logarithmic connection (E_0, ∇_0) on $\mathbb{P}_{\mathbb{C}}^1$ with polar divisor D_0 of degree n , is there a point $t \in \mathrm{Teich}_{0,n}$ such that the vector bundle $E_t = \mathcal{E}|_{\mathbb{P}_{\mathbb{C}}^1 \times \{t\}}$ underlying the universal isomonodromic deformation (\mathcal{E}, ∇) is trivial?*

A partial answer to this question is given by the following theorem of Bolibruch:

Theorem 1.1 ([Bol2]). *Let (E_0, ∇_0) be an irreducible trace-free logarithmic rank two connection with $n \geq 4$ poles on $\mathbb{P}_{\mathbb{C}}^1$ such that each singularity is resonant. There is a proper closed complex analytic subset $\mathcal{Y} \subset \mathrm{Teich}_{0,n}$ such that for all $t \in \mathrm{Teich}_{0,n} \setminus \mathcal{Y}$, the vector bundle $E_t = \mathcal{E}|_{\mathbb{P}_{\mathbb{C}}^1 \times \{t\}}$ underlying the universal isomonodromic deformation (\mathcal{E}, ∇) of (E_0, ∇_0) is trivial.*

It should be mentioned that the condition in Theorem 1.1, that each singularity is resonant, can actually be removed [He1].

From the Birkhoff–Grothendieck classification of holomorphic vector bundles on $\mathbb{P}_{\mathbb{C}}^1$ it follows immediately that the only semistable holomorphic vector bundle of degree 0 and rank r on $\mathbb{P}_{\mathbb{C}}^1$ is the trivial bundle $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}^{\oplus r}$. This leads to the following more general question: *Given a representation $\rho : \pi_1(X \setminus D, x) \rightarrow \mathrm{GL}_r(\mathbb{C})$, where X is a compact connected Riemann surface, is there a logarithmic connection $(E \rightarrow X, D, \nabla)$ such that $\rho = \rho_{\nabla}$ and E is semistable of degree zero?* The answer to this problem is

- (1) negative in general [EH],

(2) positive if ρ is irreducible [EV].

On the other hand, we can ask the following: Let $\tau : (\mathcal{X}, \mathcal{D}) \rightarrow \text{Teich}_{g,n}$ be the universal Teichmüller curve. *Given a logarithmic connection (E_0, ∇_0) , with polar divisor D_0 of degree n on a compact connected Riemann surface X_0 of genus g , is there an element $t \in \text{Teich}_{(g,n)}$ such that the vector bundle $E_t = \mathcal{E}|_{\mathcal{X}_t} \rightarrow \mathcal{X}_t = \tau^{-1}(t)$ underlying the universal isomonodromic deformation (\mathcal{E}, ∇) of (E_0, ∇_0) is semistable?*

Note that we necessarily have $\text{degree}(E_t) = \text{degree}(E_0)$. Again, Theorem 1.1 can be generalized:

Theorem 1.2 ([Hel]). *Let (E_0, ∇_0) be an irreducible logarithmic rank 2 connection with polar divisor D_0 of degree n on a compact connected Riemann surface X_0 of genus g such that $3g - 3 + n > 0$. Consider its universal isomonodromic deformation (\mathcal{E}, ∇) over $\tau : (\mathcal{X}, \mathcal{D}) \rightarrow \text{Teich}_{(g,n)}$. There is a closed complex analytic subset $\mathcal{Y} \subset \text{Teich}_{(g,n)}$ of codimension at least g (respectively, $g - 1$) such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$, the vector bundle $E_t = \mathcal{E}|_{\mathcal{X}_t}$, where $(\mathcal{X}_t, D_t) = \tau^{-1}(t)$, is semistable (respectively, stable).*

Our aim here is to prove an analog of Theorem 1.2 in the more general context of logarithmic connections on principal G -bundles over a compact connected Riemann surface (see [Boa] for logarithmic connections on principal G -bundles).

Let X_0 be a compact connected Riemann surface of genus g , and let $D_0 \subset X_0$ be an ordered subset of cardinality n . Let G be a reductive affine algebraic group over \mathbb{C} . Let E_G be a holomorphic principal G -bundle on X and ∇_0 a logarithmic connection on E_G with polar divisor D_0 . Let (\mathcal{E}_G, ∇) be the universal isomonodromic deformation of (E_G, ∇_0) over the universal Teichmüller curve $\tau : (\mathcal{X}, \mathcal{D}) \rightarrow \text{Teich}_{g,n}$. For any point $t \in \text{Teich}_{g,n}$, the restriction $\mathcal{E}_G|_{\tau^{-1}(t)} \rightarrow \mathcal{X}_t := \tau^{-1}(t)$ will be denoted by \mathcal{E}_G^t .

We prove the following (see Section 5):

Theorem 1.3.

- (1) *Assume that $g \geq 2$ and $n = 0$. There is a closed complex analytic subset $\mathcal{Y} \subset \text{Teich}_{(g,n)}$ of codimension at least g such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}$, the principal G -bundle $\mathcal{E}_G^t \rightarrow \mathcal{X}_t$ is semistable.*
- (2) *Assume that $g \geq 1$, and if $g = 1$, then $n > 0$. Also, assume that the monodromy representation for ∇_0 does not factor through some proper parabolic subgroup of G . There is a closed complex analytic subset $\mathcal{Y}' \subset \text{Teich}_{(g,n)}$ of codimension at least g such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}'$, the principal G -bundle \mathcal{E}_G^t is semistable.*
- (3) *Assume that $g \geq 2$. Assume that the monodromy representation for ∇_0 does not factor through some proper parabolic subgroup of G . There is a closed complex analytic subset $\mathcal{Y}'' \subset \text{Teich}_{(g,n)}$ of codimension at least $g - 1$ such that for any $t \in \text{Teich}_{(g,n)} \setminus \mathcal{Y}''$, the principal G -bundle \mathcal{E}_G^t is stable.*

It is known that if a holomorphic principal G -bundle E_G over an elliptic curve admits a holomorphic connection, then E_G is semistable. Therefore, a stronger version of Theorem 1.3(1) is valid when $g = 1$.

2. INFINITESIMAL DEFORMATIONS

We first recall some classical results in deformation theory, thereby setting up our notation.

2.1. Deformations of a n -pointed curve. Let X_0 be an irreducible smooth complex projective curve of genus g , with $g > 0$, and let

$$D_0 := \{x_1, \dots, x_n\} \subset X_0$$

be an ordered subset of cardinality n (it may be zero). We assume that $n > 0$ if $g = 1$. This condition implies that the pair (X_0, D_0) does not have any infinitesimal automorphism, equivalently, the automorphism group of (X_0, D_0) is finite.

Let $B := \text{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ be the spectrum of the local ring. An *infinitesimal deformation* of (X_0, D_0) is given by a quadruple

$$(\mathcal{X}, q, \mathcal{D}, f), \tag{2.1}$$

where

- $q : \mathcal{X} \rightarrow B$ is a smooth proper algebraic morphism of relative dimension one,
- $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$ is a collection of n ordered disjoint sections of q , and
- $f : X_0 \rightarrow \mathcal{X}$ is an algebraic morphism such that

$$f(X_0) \subset \mathcal{X}_0 := q^{-1}(0) \quad \text{with} \quad x_i = \mathcal{D}_i(0) \quad \forall \quad 1 \leq i \leq n,$$

and the morphism $X_0 \xrightarrow{f} \mathcal{X}_0$ is an isomorphism.

The divisor $\sum_{i=1}^n \mathcal{D}_i(B)$ on \mathcal{X} will also be denoted by \mathcal{D} . For a vector bundle \mathcal{V} on \mathcal{X} , the vector bundle $\mathcal{V} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$ will be denoted by $\mathcal{V}(-\mathcal{D})$.

The differential of f

$$df : \text{TX}_0 \rightarrow f^*\text{T}\mathcal{X}$$

produces a homomorphism $\text{TX}_0(-D_0) := \text{TX}_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0}(-D_0) \rightarrow f^*\text{T}\mathcal{X}(-\mathcal{D})$ which will also be denoted by df . Consider the following short exact sequence of sheaves on X_0 :

$$0 \rightarrow \text{TX}_0(-D_0) \xrightarrow{df} f^*\text{T}\mathcal{X}(-\mathcal{D}) \xrightarrow{h} \mathcal{O}_{X_0} \rightarrow 0; \tag{2.2}$$

note that the normal bundle to $\mathcal{X}_0 \subset \mathcal{X}$ is the pullback of T_0B by $q|_{\mathcal{X}_0}$ and hence this normal bundle is identified with \mathcal{O}_{X_0} . Consider the connecting homomorphism

$$\mathbb{C} = H^0(X_0, \mathcal{O}_{X_0}) \xrightarrow{\phi} H^1(X_0, \text{TX}_0(-D_0)) \tag{2.3}$$

in the long exact sequence of cohomologies associated to the short exact sequence in (2.2). Let 1_{X_0} denote the constant function 1 on X_0 . The cohomology element

$$\phi(1_{X_0}) \in H^1(X_0, \text{TX}_0(-D_0)), \tag{2.4}$$

where ϕ is the homomorphism in (2.3), is the *Kodaira–Spencer infinitesimal deformation class* for the family in (2.1). If this infinitesimal deformation class is zero, then the family $(\mathcal{X}, \mathcal{D}) \rightarrow B$ is isomorphic to the trivial family $(X_0 \times B, D_0 \times B) \rightarrow B$.

2.2. Deformations of a curve together with a principal bundle. Take (X_0, D_0) as before. Let G be a connected reductive affine algebraic group defined over \mathbb{C} . The Lie algebra of G will be denoted by \mathfrak{g} . Let

$$p : E_G \longrightarrow X_0 \quad (2.5)$$

be a principal G -bundle on X_0 . The infinitesimal deformations of the triple

$$(X_0, D_0, E_G) \quad (2.6)$$

are guided by the Atiyah bundle $\text{At}(E_G) \longrightarrow X_0$, the construction of which we shall briefly recall (see [At] for a more detailed exposition). Consider the direct image $p_*\text{TE}_G \longrightarrow X_0$, where TE_G is the holomorphic tangent bundle of E_G , and p is the projection in (2.5). It is a quasicohherent sheaf equipped with an action of G given by the action of G on E_G . The invariant part

$$\text{At}(E_G) := (p_*\text{TE}_G)^G \subset (p_*\text{TE}_G)$$

is a vector bundle on X_0 of rank $1 + \dim G$ which is known as the Atiyah bundle of E_G . Consequently, we have $\text{At}(E_G) = (\text{TE}_G)/G$. Let

$$\text{ad}(E_G) := E_G \times^G \mathfrak{g} \longrightarrow X_0$$

be the adjoint vector bundle associated to E_G for the adjoint action of G on its Lie algebra \mathfrak{g} . Let

$$dp : \text{TE}_G \longrightarrow p^*\text{TX}_0$$

be the differential of the map p in (2.5). Being G -equivariant it produces a homomorphism $\text{At}(E_G) \longrightarrow \text{TX}_0$ which will also be denoted by dp . Now the action of G on E_G produces an isomorphism $E_G \times \mathfrak{g} \longrightarrow \text{kernel}(dp)$. Therefore, we have $\text{kernel}(dp)/G = \text{ad}(E_G)$. In other words, the above isomorphism descends to an isomorphism

$$\text{ad}(E_G) \xrightarrow{\sim} (p_*(\text{kernel}(dp)))^G$$

that preserves the Lie algebra structure on the fibers of the two vector bundles (the Lie algebra structure on the fibers of $(p_*(\text{kernel}(dp)))^G$ is given by the Lie bracket of G -invariant vertical vector fields). Therefore, $\text{At}(E_G)$ fits in the following short exact sequence of vector bundles on X_0

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G) \xrightarrow{dp} \text{TX}_0 \longrightarrow 0, \quad (2.7)$$

which is known as the Atiyah exact sequence for E_G . The logarithmic Atiyah bundle $\text{At}(E_G, D_0)$ is defined by

$$\text{At}(E_G, D_0) := (dp)^{-1}(\text{TX}_0(-D_0)) \subset \text{At}(E_G).$$

From (2.7) we have the short exact sequence

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G, D_0) \xrightarrow{\sigma} \text{TX}_0(-D_0) \longrightarrow 0, \quad (2.8)$$

which is called the logarithmic Atiyah exact sequence. The above homomorphism σ is the restriction of dp to $\text{At}(E_G, D_0) \subset \text{At}(E_G)$.

An *infinitesimal deformation* of the triple (X_0, D_0, E_G) in (2.6) is a 6-tuple

$$(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi), \quad (2.9)$$

where

- $(\mathcal{X}, q, \mathcal{D}, f)$ in an infinitesimal deformation of the n -pointed curve (X_0, D_0) as in (2.1),
- $\mathcal{E}_G \rightarrow \mathcal{X}$ is a principal G -bundle, and
- ψ is an isomorphism

$$\psi : E_G \rightarrow f^* \mathcal{E}_G \quad (2.10)$$

of principal G -bundles.

The logarithmic Atiyah bundle

$$\text{At}(\mathcal{E}_G, \mathcal{D}) \rightarrow \mathcal{X}$$

for $(\mathcal{E}_G, \mathcal{D})$ is the inverse image, in $\text{At}(\mathcal{E}_G)$, of the subsheaf $\text{T}\mathcal{X}(-\mathcal{D}) \subset \text{T}\mathcal{X}$. We have the following short exact sequence of sheaves on X_0 :

$$0 \rightarrow \text{At}(E_G, D_0) \rightarrow f^* \text{At}(\mathcal{E}_G, \mathcal{D}) \rightarrow \mathcal{O}_{X_0} \rightarrow 0 \quad (2.11)$$

given by ψ in (2.10). Let

$$\mathbb{C} = H^0(X_0, \mathcal{O}_{X_0}) \xrightarrow{\tilde{\phi}} H^1(X_0, \text{At}(E_G, D_0))$$

be the connecting homomorphism in the long exact sequence of cohomologies associated to (2.11). The cohomology element

$$\tilde{\phi}(1_{X_0}) \in H^1(X_0, \text{At}(E_G, D_0)) \quad (2.12)$$

is the *infinitesimal deformation class* of the triple (X_0, D_0, E_G) given by (2.9). Let

$$\sigma_* : H^1(X_0, \text{At}(E_G, D_0)) \rightarrow H^1(X_0, \text{TX}_0(-D_0))$$

be the homomorphism given by the projection σ in (2.8). From the commutativity of the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{At}(E_G, D_0) & \rightarrow & f^* \text{At}(\mathcal{E}_G, \mathcal{D}) & \rightarrow & \mathcal{O}_{X_0} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \text{TX}_0(-D_0) & \rightarrow & f^* \text{T}\mathcal{X}(-\mathcal{D}) & \rightarrow & \mathcal{O}_{X_0} \rightarrow 0 \end{array}$$

where the top and bottom rows are as in (2.11) and (2.2) respectively, it follows that

$$\sigma_*(\tilde{\phi}(1_{X_0})) = \phi(1_{X_0}),$$

where $\tilde{\phi}(1_{X_0})$ and $\phi(1_{X_0})$ are constructed in (2.12) and (2.4) respectively. We note that σ_* is the forgetful map that sends an infinitesimal deformation of (X_0, D_0, E_G) to the underlying infinitesimal deformation of (X_0, D_0) .

3. OBSTRUCTION TO AN EXTENSION OF A REDUCTION OF STRUCTURE GROUP

Given a reduction of structure group of E_G to a parabolic subgroup of G , our aim in this section is to compute the obstruction for it to extend to a reduction of an infinitesimal deformation $\mathcal{E}_G \rightarrow \mathcal{X}$ as in (2.9).

Fix a parabolic subgroup $P \subset G$. The Lie algebra of P will be denoted by \mathfrak{p} . Let

$$E_P \subset E_G \quad (3.1)$$

be a holomorphic reduction of structure group of E_G to the subgroup $P \subset G$. Let

$$\mathrm{ad}(E_P) := E_P \times^P \mathfrak{p} \longrightarrow X_0$$

be the adjoint vector bundle associated to E_P for the adjoint action of P on its Lie algebra \mathfrak{p} . The vector bundle over X_0 associated to the principal P -bundle E_P for the adjoint action of P on the quotient $\mathfrak{g}/\mathfrak{p}$ will be denoted by $E_P(\mathfrak{g}/\mathfrak{p})$. The logarithmic Atiyah bundle for (E_P, D_0) will be denoted by $\mathrm{At}(E_P, D_0)$. We have

$$\mathrm{ad}(E_P) \subset \mathrm{ad}(E_G) \quad \text{and} \quad \mathrm{At}(E_P, D_0) \subset \mathrm{At}(E_G, D_0);$$

both the quotient bundles above are identified with $E_P(\mathfrak{g}/\mathfrak{p})$. In other words, we have the following commutative diagram of vector bundles on X_0

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{ad}(E_P) & \longrightarrow & \mathrm{At}(E_P, D_0) & \xrightarrow{\beta} & \mathrm{TX}_0(-D_0) \longrightarrow 0 \\ & & \downarrow & & \downarrow \xi & & \parallel \\ 0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}(E_G, D_0) & \xrightarrow{\gamma} & \mathrm{TX}_0(-D_0) \longrightarrow 0 \\ & & \downarrow \mu_1 & & \downarrow \mu & & \\ & & E_P(\mathfrak{g}/\mathfrak{p}) & = & E_P(\mathfrak{g}/\mathfrak{p}) & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (3.2)$$

Let

$$\tilde{\xi} : \mathrm{H}^1(X_0, \mathrm{At}(E_P, D_0)) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}(E_G, D_0)) \quad (3.3)$$

be the homomorphism induced by the canonical injection ξ in (3.2).

Take $(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)$ as in (2.9). Assume that the reduction $E_P \subset E_G$ in (3.1) extends to a reduction of structure group

$$\mathcal{E}_P \subset \mathcal{E}_G$$

to $P \subset G$ on \mathcal{X} . Consider the short exact sequence on X_0

$$0 \longrightarrow \mathrm{At}(E_P, D_0) \longrightarrow f^* \mathrm{At}(\mathcal{E}_P, \mathcal{D}) \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0, \quad (3.4)$$

where $\mathrm{At}(\mathcal{E}_P, \mathcal{D}) \longrightarrow \mathcal{X}$ is the logarithmic Atiyah bundle associated to the principal P -bundle \mathcal{E}_P , and f is the map in (2.1). Let

$$\theta \in \mathrm{H}^1(X_0, \mathrm{At}(E_P, D_0)) \quad (3.5)$$

be the image of the constant function 1_{X_0} by the homomorphism

$$\mathrm{H}^0(X_0, \mathcal{O}_{X_0}) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}(E_P, D_0))$$

in the long exact sequence of cohomologies associated to (3.4).

Lemma 3.1. *The cohomology class θ in (3.5) satisfies the equation*

$$\tilde{\xi}(\theta) = \tilde{\phi}(1_{X_0}),$$

where $\tilde{\xi}$ and $\tilde{\phi}(1_{X_0})$ are constructed in (3.3) and (2.12) respectively.

Proof. Consider the commutative diagram of vector bundles on X_0

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{At}(E_P, D_0) & \longrightarrow & f^*\text{At}(\mathcal{E}_P, \mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \\
& & \downarrow \xi & & \downarrow & & \parallel \\
0 & \longrightarrow & \text{At}(E_G, D_0) & \longrightarrow & f^*\text{At}(\mathcal{E}_G, \mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0
\end{array} \tag{3.6}$$

where the top and bottom rows are as in (3.4) and (2.11) respectively, and ξ is the homomorphism in (3.2); the above homomorphism $f^*\text{At}(\mathcal{E}_P, \mathcal{D}) \longrightarrow f^*\text{At}(\mathcal{E}_G, \mathcal{D})$ is the pullback of the natural homomorphism $\text{At}(\mathcal{E}_P, \mathcal{D}) \longrightarrow \text{At}(\mathcal{E}_G, \mathcal{D})$. In view of (3.6), the lemma follows by comparing the constructions of θ and $\tilde{\varphi}(1_{X_0})$. \square

4. LOGARITHMIC CONNECTIONS AND THE SECOND FUNDAMENTAL FORM

In this section we characterize those infinitesimal deformations of the principal bundle E_G on the n -pointed curve that arise from the isomonodromic deformations.

4.1. Canonical extension of a logarithmic connection. As before, let $p : E_G \longrightarrow X_0$ be a principal G -bundle. A *logarithmic connection* on E_G with polar divisor D_0 is a holomorphic splitting of the logarithmic Atiyah exact sequence in (2.8). In other words, a logarithmic connection is a homomorphism

$$\delta : \text{TX}_0(-D_0) \longrightarrow \text{At}(E_G, D_0) \tag{4.1}$$

such that $\sigma \circ \delta = \text{Id}_{\text{TX}_0(-D_0)}$, where σ is the homomorphism in (2.8). Note that given such a δ , there is a unique homomorphism

$$\delta'' : \text{At}(E_G, D_0) \longrightarrow \text{ad}(E_G) \tag{4.2}$$

such that $\delta'' \circ \delta = 0$, and the composition

$$\text{ad}(E_G) \hookrightarrow \text{At}(E_G, D_0) \xrightarrow{\delta''} \text{ad}(E_G)$$

(see (2.8)) is the identity map of $\text{ad}(E_G)$. As there are no nonzero $(2, 0)$ -forms on X_0 , all logarithmic connections on E_G are automatically integrable.

At the level of first order deformations, given a principal G -bundle $\mathcal{E}_G \longrightarrow \mathcal{X} \xrightarrow{q} B$, a logarithmic connection on \mathcal{E}_G with polar divisor \mathcal{D} is a homomorphism $\text{At}(\mathcal{E}_G, \mathcal{D}) \longrightarrow \text{ad}(\mathcal{E}_G)$ that splits the logarithmic Atiyah exact sequence for \mathcal{E}_G . We note that a connection on \mathcal{E}_G need not be integrable, as we have added an (infinitesimal) extra dimension. However, the Riemann–Hilbert correspondence for principal G -bundles yields the following:

Lemma 4.1. *Let $(\mathcal{X}, q, \mathcal{D}, f)$ be an infinitesimal deformation of (X_0, D_0) as in (2.1). Let δ be a logarithmic connection on a principal G -bundle E_G on X_0 with polar divisor D_0 . Then there exists a unique pair (\mathcal{E}_G, ∇) , where*

- \mathcal{E}_G is a principal G -bundle on \mathcal{X} , and
- ∇ is an integrable logarithmic connection on \mathcal{E}_G with polar divisor \mathcal{D} ,

such that $(f^*\mathcal{E}_G, f^*\nabla) = (E_G, \delta)$.

Let us recall a few elements of the proof of this (classical) result. Choose a covering \mathfrak{U} of $X_0 \setminus D_0$ by complex discs and a small neighborhood U_i for each $x_i \in D_0$. Since δ is integrable, we can choose local charts for E_G over \mathfrak{U} such that all transition functions are constants. Now if the curve fits into an analytic family $\mathcal{X} \rightarrow \mathcal{B}$, one can, restricting \mathcal{B} if necessary, extend all the U_i into tubular neighborhoods of \mathcal{D}_i and extend the open subsets in \mathfrak{U} such that $\mathfrak{U} \cup U_1 \cup \dots \cup U_n$ is a covering of \mathcal{X} . The isomonodromic deformation is then given by simply extending the transition maps by keeping them to be constant in deformation parameters.

The logarithmic connection δ gives a splitting of the logarithmic Atiyah bundle

$$\text{At}(E_G, D_0) = \text{ad}(E_G) \oplus \text{TX}_0(-D_0).$$

The corresponding cohomological decomposition

$$\text{H}^1(X_0, \text{At}(E_G, D_0)) = \text{H}^1(X_0, \text{ad}(E_G)) \oplus \text{H}^1(X_0, \text{TX}_0(-D_0))$$

given a splitting of the infinitesimal deformations of (X_0, D_0, E_G) into infinitesimal the deformations of (X_0, D_0) and the infinitesimal deformations E_G (keeping (X_0, D_0) fixed). In other words, let

$$\delta' : \text{H}^1(X_0, \text{TX}_0(-D_0)) \longrightarrow \text{H}^1(X_0, \text{At}(E_G, D_0))$$

be the homomorphism induced by the homomorphism $\delta : \text{TX}_0(-D_0) \rightarrow \text{At}(E_G, D_0)$ in Lemma 4.1 defining the logarithmic connection on E_G . Given an infinitesimal deformation $(\mathcal{X}, q, \mathcal{D}, f)$ of (X_0, D_0) , the above homomorphism δ' produces an infinitesimal deformation $(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)$ of (X_0, D_0, E_G) . As explained above, this principal G -bundle \mathcal{E}_G on \mathcal{X} coincides with the principal G -bundle on \mathcal{X} produced by the isomonodromic deformation in Lemma 4.1.

We will now construct the exact sequence in (2.11) corresponding to the above infinitesimal deformation $(\mathcal{X}, q, \mathcal{D}, f, \mathcal{E}_G, \psi)$. Consider the injective homomorphism

$$\text{TX}_0(-D_0) \longrightarrow \text{At}(E_G, D_0) \oplus f^*\text{T}\mathcal{X}(-\mathcal{D}), \quad v \longmapsto (\delta(v), -(df)(v)),$$

where df is the differential in (2.2). The corresponding cokernel

$$\text{At}^\delta(E_G, D_0) := (\text{At}(E_G, D_0) \oplus f^*\text{T}\mathcal{X}(-\mathcal{D})) / (\text{TX}_0(-D_0))$$

possesses a canonical projection

$$\widehat{\delta} : \text{At}^\delta(E_G, D_0) \longrightarrow \mathcal{O}_{X_0}, \quad (v, w) \longmapsto h(w), \tag{4.3}$$

where h is the homomorphism in (2.2); note that the above homomorphism $\widehat{\delta}$ is well-defined because h vanishes on the image of $\text{TX}_0(-D_0)$ in $\text{At}(E_G, D_0) \oplus f^*\text{T}\mathcal{X}(-\mathcal{D})$. The kernel of $\widehat{\delta}$ is identified with $\text{At}(E_G, D_0)$ by sending any $z \in \text{At}(E_G, D_0)$ to the image in $\text{At}^\delta(E_G, D_0)$ of $(z, 0) \in \text{At}(E_G, D_0) \oplus f^*\text{T}\mathcal{X}(-\mathcal{D})$. Therefore, we obtain the following exact sequence of vector bundles over X_0 :

$$0 \longrightarrow \text{At}(E_G, D_0) \longrightarrow \text{At}^\delta(E_G, D_0) \xrightarrow{\widehat{\delta}} \mathcal{O}_{X_0} \longrightarrow 0,$$

This exact sequence coincides with the one in (2.11).

Consider the projection

$$\mathrm{At}(E_G, D_0) \oplus f^* \mathrm{TX}(-\mathcal{D}) \longrightarrow \mathrm{ad}(E_G), \quad (z_1, z_2) \longmapsto \delta''(z_1),$$

where δ'' is constructed in (4.2) from δ . It vanishes on the image of $\mathrm{TX}_0(-D_0)$, yielding a projection

$$\lambda : \mathrm{At}^\delta(E_G, D_0) \longrightarrow \mathrm{ad}(E_G). \quad (4.4)$$

Let $\nabla'' : \mathrm{At}(\mathcal{E}_G, \mathcal{D}) \longrightarrow \mathrm{ad}(\mathcal{E}_G)$ be the homomorphism given by the logarithmic connection ∇ in Lemma 4.1. The homomorphism in (4.4) fits in the commutative diagram

$$\begin{array}{ccc} \mathrm{At}^\delta(E_G, D_0) & \xrightarrow{\lambda} & \mathrm{ad}(E_G) \\ \parallel & & \parallel \\ f^* \mathrm{At}(\mathcal{E}_G, \mathcal{D}) & \xrightarrow{f^* \nabla''} & f^* \mathrm{ad}(\mathcal{E}_G) \end{array} \quad (4.5)$$

(the vertical identifications are evident).

We summarize the above constructions in the following lemma:

Lemma 4.2. *Given $(\mathcal{X}, q, \mathcal{D}, f)$ as in (2.1), and also given a logarithmic connection δ on a principal G -bundle $E_G \longrightarrow X_0$, the exact sequence in (2.11) corresponding to the infinitesimal deformation of (X_0, D_0, E_G) in Lemma 4.1 is*

$$0 \longrightarrow \mathrm{At}(E_G, D_0) \longrightarrow \mathrm{At}^\delta(E_G, D_0) \xrightarrow{\widehat{\delta}} \mathcal{O}_{X_0} \longrightarrow 0,$$

where $\widehat{\delta}$ is constructed in (4.3).

4.2. The second fundamental form. Fix a logarithmic connection δ on (E_G, D_0) as in (4.1). Take a reduction of structure group $E_P \subset E_G$ to P as in (3.1). The composition

$$S(\delta) := \mu \circ \delta : \mathrm{TX}_0(-D_0) \longrightarrow E_P(\mathfrak{g}/\mathfrak{p}), \quad (4.6)$$

where μ is constructed in (3.2), is called the *second fundamental form* of E_P for the connection δ . We note that δ is induced by a logarithmic connection on the principal P -bundle E_P if and only if we have $S(\delta) = 0$.

Assume that E_P satisfies the condition that $S(\delta) \neq 0$. Let

$$\mathcal{L} \subset E_P(\mathfrak{g}/\mathfrak{p}) \quad (4.7)$$

be the holomorphic line subbundle generated by the image of the homomorphism $S(\delta)$ in (4.6). More precisely, \mathcal{L} is the inverse image, in $E_P(\mathfrak{g}/\mathfrak{p})$, of the torsion part of the quotient $E_P(\mathfrak{g}/\mathfrak{p})/(S(\delta)(\mathrm{TX}_0(-D_0)))$. Now consider the homomorphism

$$S(\delta)' : \mathrm{TX}_0(-D_0) \longrightarrow \mathcal{L} \quad (4.8)$$

given by the second fundamental form $S(\delta)$. Let

$$S' : H^1(X_0, \mathrm{TX}_0(-D_0)) \longrightarrow H^1(X_0, \mathcal{L}) \quad (4.9)$$

be the homomorphism induced by $S(\delta)'$ in (4.8).

Proposition 4.3. *As before, let δ be a logarithmic connection on $E_G \rightarrow X_0$ with polar divisor D_0 , and let $(\mathcal{X}, q, \mathcal{D}, f)$ be an infinitesimal deformation of (X_0, D_0) . Let $\mathcal{E}_G \rightarrow \mathcal{X}$ be the isomonodromic deformation of δ along $(\mathcal{X}, \mathcal{D})$ obtained in Lemma 4.1. Let $E_P \subset E_G$ be a reduction of structure group to P over X_0 that extends to a reduction of structure group $\mathcal{E}_P \subset \mathcal{E}_G$ to P over \mathcal{X} . Then*

$$S'(\phi(1_{X_0})) = 0,$$

where $\phi(1_{X_0})$ is the corresponding cohomology class constructed in (2.4), and S' is constructed in (4.9).

Proof. Consider the inverse images

$$\text{At}_P(E_G, D_0) := \mu^{-1}(\mathcal{L}) \subset \text{At}(E_G, D_0) \quad \text{and} \quad \text{ad}_P(E_G) := \mu_1^{-1}(\mathcal{L}) \subset \text{ad}(E_G),$$

where μ and μ_1 are the quotient maps in (3.2), and \mathcal{L} is constructed in (4.7). These two vector bundles fit in the following commutative diagram produced from (3.2):

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{ad}(E_P) & \longrightarrow & \text{At}(E_P, D_0) & \xrightarrow{\beta} & \text{TX}_0(-D_0) \longrightarrow 0 \\ & & \downarrow & & \downarrow \xi & & \parallel \\ 0 & \longrightarrow & \text{ad}_P(E_G) & \longrightarrow & \text{At}_P(E_G, D_0) & \xrightarrow{\gamma} & \text{TX}_0(-D_0) \longrightarrow 0 \\ & & \downarrow \mu_1 & & \downarrow \mu & & \\ & & \mathcal{L} & = & \mathcal{L} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (4.10)$$

By the construction of $\text{At}_P(E_G, D_0)$, the connection homomorphism δ in (4.1) factors through a homomorphism

$$\delta^1 : \text{TX}_0(-D_0) \longrightarrow \text{At}_P(E_G, D_0).$$

Consider the homomorphism

$$\delta_*^1 : \text{H}^1(X_0, \text{TX}_0(-D_0)) \longrightarrow \text{H}^1(X_0, \text{At}_P(E_G, D_0))$$

induced by the above homomorphism δ^1 , and let

$$\Phi := \delta_*^1(\phi(1_{X_0})) \in \text{H}^1(X_0, \text{At}_P(E_G, D_0)) \quad (4.11)$$

be the image of the cohomology class $\phi(1_{X_0})$ characterizing the deformation $(\mathcal{X}, \mathcal{D})$ as in (2.4).

As in the statement of the proposition, let $\mathcal{E}_P \rightarrow \mathcal{X}$ be an extension of the reduction E_P . Note that from (3.6), (3.2) and Lemma 4.2 we have

$$\begin{aligned} \text{At}(E_G, D_0)/\text{At}(E_P, D_0) &= E_P(\mathfrak{g}/\mathfrak{p}) = (f^*\text{At}(\mathcal{E}_G, \mathcal{D}))/ (f^*\text{At}(\mathcal{E}_P, \mathcal{D})) \\ &= \text{At}^\delta(E_G, D_0)/f^*\text{At}(\mathcal{E}_P, \mathcal{D}). \end{aligned}$$

Let $\mu_2 : \text{At}^\delta(E_G, D_0) \rightarrow E_P(\mathfrak{g}/\mathfrak{p})$ be the above quotient map. Define

$$\text{At}_P^\delta(E_G, D_0) := \mu_2^{-1}(\mathcal{L}) \subset \text{At}^\delta(E_G, D_0),$$

where \mathcal{L} is constructed in (4.7).

Now we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{TX}_0(-D_0) & \longrightarrow & f^*\mathrm{T}\mathcal{X}(-\mathcal{D}) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \\
& & \downarrow \delta & & \downarrow f^*\nabla & & \parallel \\
0 & \longrightarrow & \mathrm{At}_P(E_G, D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_G, \mathcal{D}) = \mathrm{At}_P^\delta(E_G, D_0) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0
\end{array} \tag{4.12}$$

where the bottom exact sequence is obtained from (3.6), and the top exact sequence is as in (2.2) (see also (4.5)). Let

$$\nu : \mathrm{H}^0(X_0, \mathcal{O}_{X_0}) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}_P(E_G, D_0))$$

be the connecting homomorphism in the long exact sequence of cohomologies associated to the bottom exact sequence in (4.12). From the commutativity of (4.12) and the construction of $\phi(1_{X_0})$ (see (2.4)) it follows that

$$\Phi = \nu(1_{X_0}) = \delta_*(\phi(1_{X_0})), \tag{4.13}$$

where $\delta_* : \mathrm{H}^1(X_0, \mathrm{TX}_0(-D_0)) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}_P(E_G, D_0))$ is the homomorphism induced by δ , and Φ is the cohomology class in (4.11).

The diagram in (3.6) produces the diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{At}(E_P, D_0) & \longrightarrow & f^*\mathrm{At}(\mathcal{E}_P, D_0) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \\
& & \downarrow \xi & & \downarrow & & \parallel \\
0 & \longrightarrow & \mathrm{At}_P(E_G, D_0) & \longrightarrow & \mathrm{At}_P^\delta(E_G, D_0) & \longrightarrow & \mathcal{O}_{X_0} \longrightarrow 0 \\
& & \downarrow \mu & & \downarrow & & \\
& & \mathcal{L} & = & \mathcal{L} & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array} \tag{4.14}$$

where ξ and μ are the homomorphisms in (4.10). Using this diagram we can check that

$$\Phi = \xi_*(\theta), \tag{4.15}$$

where θ is the cohomology classes in (3.5), and

$$\xi_* : \mathrm{H}^1(X_0, \mathrm{At}(E_P, D_0)) \longrightarrow \mathrm{H}^1(X_0, \mathrm{At}_P(E_G, D_0))$$

is the homomorphism induced by ξ in (4.14). Indeed, (4.15) follows from (4.13), the commutativity of (4.14) and the construction of θ .

Let $\mu_* : \mathrm{H}^1(X_0, \mathrm{At}_P(E_G, D_0)) \longrightarrow \mathrm{H}^1(X_0, \mathcal{L})$ be the homomorphism induced by the homomorphism μ in (4.14). It is straight-forward to check that

$$\mu_*(\Phi) = S'(\phi(1_{X_0}))$$

(see (4.9), (2.4) and (4.11) for S' , $\phi(1_{X_0})$ and Φ respectively). Indeed, this follows from (4.13) and the definition of $S(\delta)$ in (4.6). Combining this with (4.15), we have

$$S'(\phi(1_{X_0})) = \mu_*(\Phi) = \mu_*(\xi_*(\theta)).$$

Since $\mu \circ \xi = 0$ (see (4.14)), it now follows that $S'(\phi(1_{X_0})) = 0$. \square

5. LOGARITHMIC CONNECTIONS AND SEMISTABILITY OF UNDERLYING PRINCIPAL BUNDLE

Let $\mathcal{T}_{g,n}$ be the Teichmüller space of genus g compact Riemann surfaces with n ordered marked points. As before, we assume that $g > 0$, and if $g = 1$, then $n > 0$. Let

$$\tau : \mathcal{C} \longrightarrow \mathcal{T}_{g,n}$$

be the universal Teichmüller curve with n ordered sections Σ . The fiber of \mathcal{C} on any point $t \in \mathcal{T}_{g,n}$ will be denoted by \mathcal{C}_t . The ordered subset $\mathcal{C}_t \cap \Sigma \subset \mathcal{C}_t$ will be denoted by Σ_t .

Take a n -pointed Riemann surface (C_0, Σ_0) of genus g , which is represented by a point of $\mathcal{T}_{g,n}$. Let

$$\nabla_0 \tag{5.1}$$

be a logarithmic connection on a principal G -bundle $F_G \longrightarrow C_0$ with polar divisor Σ_0 . By Riemann–Hilbert correspondence, the connection ∇_0 produces a flat (isomonodromic) logarithmic connection ∇ on a holomorphic principal G -bundle $\mathcal{F}_G \longrightarrow \mathcal{C}$ with polar divisor Σ .

The following lemma is a special case of the main theorem in [GN] (see also [Sh]). Although the families of G -bundles considered in [GN] are algebraic, all arguments there go through in the analytic case of our interest with obvious modifications.

Lemma 5.1 ([GN]). *Let $\mathcal{F}_G \longrightarrow \mathcal{C} \longrightarrow \mathcal{T}_{g,n}$ be as above. For each Harder–Narasimhan type κ , the set*

$$\mathcal{Y}_\kappa := \{t \in \mathcal{T}_{g,n} \mid \mathcal{F}_G|_{\mathcal{C}_t} \text{ is of type } \kappa\}$$

is a (possibly empty) locally closed complex analytic subspace of $\mathcal{T}_{g,n}$. More precisely, for each Harder–Narasimhan type κ , the union $\mathcal{Y}_{\leq \kappa} := \bigcup_{\kappa' \leq \kappa} \mathcal{Y}_{\kappa'}$ is a closed complex analytic subset of $\mathcal{T}_{g,n}$. Moreover, the principal G -bundle

$$\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)} \longrightarrow \tau^{-1}(\mathcal{Y}_\kappa)$$

possesses a canonical reduction of structure group inducing the Harder–Narasimhan reduction of $\mathcal{F}_G|_{\mathcal{C}_t}$ for each $t \in \mathcal{Y}_\kappa$.

In the following two Sections 5.1 and 5.2, we will see that under certain assumptions, the only Harder–Narasimhan stratum \mathcal{Y}_κ of maximal dimension $\dim(\mathcal{T}_{g,n}) = 3g - 3 + n$ is the trivial one, in the sense that the Harder–Narasimhan parabolic subgroup is G itself. In other words, if the principal G -bundle F_G is not semistable, and therefore has a non-trivial Harder–Narasimhan reduction $E_P \subset E_G$ to a certain parabolic subgroup $P \subsetneq G$, then there is always an isomonodromic deformation in which direction the reduction E_P is obstructed.

5.1. **The case of $n = 0$.** In this subsection we assume that $n = 0$. So, we have $g > 1$.

Theorem 5.2. *There is a closed complex analytic subset $\mathcal{Y} \subset \mathcal{T}_{g,0}$ of codimension at least g such that for any $t \in \mathcal{T}_{g,0} \setminus \mathcal{Y}$, the holomorphic principal G -bundle $\mathcal{F}_G|_{\mathcal{C}_t} \rightarrow \mathcal{C}_t$ is semistable.*

Proof. Let $\mathcal{Y} \subset \mathcal{T}_{g,0}$ denote the (finite) union of all Harder-Narasimhan strata \mathcal{Y}_κ as in Lemma 5.1 with nontrivial Harder-Narasimhan type κ . From Lemma 5.1 we know that \mathcal{Y} is a closed complex analytic subset of $\mathcal{T}_{g,0}$.

Take any $t \in \mathcal{Y}_\kappa \subset \mathcal{Y}$. Let $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$ be the holomorphic principal G -bundle on $X_0 := \mathcal{C}_t$. The holomorphic connection on E_G obtained by restricting ∇ will be denoted by δ . Since E_G is not semistable, there is a proper parabolic subgroup $P \subsetneq G$ and a holomorphic reduction of structure group $E_P \subset E_G$ to P , such that E_P is the Harder-Narasimhan reduction [Be], [AAB]. From Lemma 5.1 we know that E_P extends to a reduction of structure group of the principal G -bundle $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$ to the subgroup P .

Let $\text{ad}(E_P)$ and $\text{ad}(E_G)$ be the adjoint vector bundles of E_P and E_G respectively. Consider the vector bundle

$$\text{ad}(E_G)/\text{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$$

(see (3.2)). We know that

$$\mu_{\max}(E_P(\mathfrak{g}/\mathfrak{p})) < 0 \tag{5.2}$$

[AAB, p. 705] (see sixth line from bottom). In particular

$$\text{degree}(E_P(\mathfrak{g}/\mathfrak{p})) < 0. \tag{5.3}$$

A holomorphic connection on E_G induces a holomorphic connection on $\text{ad}(E_G)$, hence $\text{degree}(\text{ad}(E_G)) = 0$. Combining this with (5.3) it follows that $\text{degree}(\text{ad}(E_P)) > 0$, because $\text{ad}(E_G)/\text{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$. Since $\text{degree}(\text{ad}(E_P)) \neq 0$, the principal P -bundle E_P does not admit a holomorphic connection. Consequently, the second fundamental form $S(\delta)$ of E_P for δ (see (4.6)) is nonzero.

Using the second fundamental form $S(\delta)$, construct the holomorphic line subbundle

$$\mathcal{L} \subset E_P(\mathfrak{g}/\mathfrak{p})$$

as done in (4.7). From (5.2) we have

$$\text{degree}(\mathcal{L}) < 0. \tag{5.4}$$

Consider the short exact sequence of sheaves on X_0

$$0 \rightarrow \text{TX}_0 \xrightarrow{S(\delta)'} \mathcal{L} \rightarrow T^\delta := \mathcal{L}/(S(\delta)(\text{TX}_0)) \rightarrow 0, \tag{5.5}$$

where $S(\delta)'$ is constructed in (4.8); note that T^δ is a torsion sheaf because $S(\delta)' \neq 0$ (recall that $S(\delta) \neq 0$). From (5.4) it follows that

$$\text{degree}(T^\delta) = \text{degree}(\mathcal{L}) - \text{degree}(\text{TX}_0) < -\text{degree}(\text{TX}_0) = 2g - 2.$$

So, we have

$$\dim H^0(X_0, T^\delta) = \text{degree}(T^\delta) < 2g - 2 = \dim H^1(X_0, \text{TX}_0) + 1 - g.$$

This implies that

$$\dim H^1(X_0, TX_0) - \dim H^0(X_0, T^\delta) \leq g. \quad (5.6)$$

Consider the long exact sequence of cohomologies

$$H^0(X_0, T^\delta) \longrightarrow H^1(X_0, TX_0) \xrightarrow{\zeta} H^1(X_0, \mathcal{L})$$

associated to the short exact sequence of sheaves in (5.5). From (5.6) it follows that

$$\dim \zeta(H^1(X_0, TX_0)) \geq g. \quad (5.7)$$

Since the reduction E_P extends to a reduction of structure group of the principal G -bundle $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$ to the subgroup P , combining (5.7) and Proposition 4.3 we conclude that the codimension of the complex analytic subset $\mathcal{Y}_\kappa \subset \mathcal{T}_{g,0}$ is at least g . This completes the proof of the theorem. \square

5.2. When n is arbitrary. Now we assume that $n > 0$.

A logarithmic connection η on a holomorphic principal G -bundle $F_G \rightarrow X_0$ is called *reducible* if there is pair (P, F_P) , where $P \subsetneq G$ is a parabolic subgroup and $F_P \subset F_G$ is a holomorphic reduction of structure group of F_G to P , such that η is induced by a connection on F_P . Note that η is induced by a connection on F_P if and only if the second fundamental form of F_P for η vanishes identically. A connection is called *irreducible* if it is not reducible or, equivalently, if the monodromy representation of the corresponding flat principal G -bundle does not factor through a proper parabolic subgroup of G .

Proposition 5.3. *Assume that the logarithmic connection ∇_0 in (5.1) is irreducible. Then there is a closed complex analytic subset $\mathcal{Y} \subset \mathcal{T}_{g,n}$ of codimension at least g such that for any $t \in \mathcal{T}_{g,n} \setminus \mathcal{Y}$, the holomorphic principal G -bundle $\mathcal{F}_G|_{\mathcal{C}_t}$ is semistable.*

Proof. The proof of Theorem 5.2 goes through after some obvious modifications. As before, let $\mathcal{Y} \subset \mathcal{T}_{g,n}$ be the locus of all points t such that the principal G -bundle $\mathcal{F}_G|_{\mathcal{C}_t}$ is not semistable. Take any $t \in \mathcal{Y}_\kappa \subset \mathcal{Y}$. Let $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$ be the holomorphic principal G -bundle on $X_0 := \mathcal{C}_t$. The logarithmic connection on E_G with polar divisor $D_0 := \Sigma_t$ obtained by restricting ∇ will be denoted by δ .

Let $E_P \subset E_G$ be the Harder–Narasimhan reduction. Since ∇_0 is irreducible, the second fundamental form $S(\delta)$ of E_P for δ (see (4.6)) is nonzero. Consider the short exact sequence of sheaves on X_0

$$0 \longrightarrow TX_0(-D_0) \xrightarrow{S(\delta)'} \mathcal{L} \longrightarrow T^\delta := \mathcal{L}/(S(\delta)(TX_0(-D_0))) \longrightarrow 0, \quad (5.8)$$

where $S(\delta)'$ is constructed in (4.8). As before, $\text{degree}(\mathcal{L}) < 0$, because $\mu_{\max}(E_P(\mathfrak{g}/\mathfrak{p})) < 0$. So, $\text{degree}(T^\delta) < -\text{degree}(TX_0(-D_0)) = 2g - 2 + n$. We now have

$$\dim H^0(X_0, T^\delta) = \text{degree}(T^\delta) < 2g - 2 + n = \dim H^1(X_0, TX_0(-D_0)) + 1 - g.$$

Hence the dimension of the image of the homomorphism

$$H^1(X_0, TX_0(-D_0)) \longrightarrow H^1(X_0, \mathcal{L}) \quad (5.9)$$

in the long exact sequence of cohomologies associated to (5.8) is at least g . Since the reduction E_P extends to a reduction of $\mathcal{F}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$ to P , and the dimension of the image of the homomorphism in (5.9) is at least g , from Proposition 4.3 we conclude that the codimension of $\mathcal{Y}_\kappa \subset \mathcal{T}_{g,0}$ is at least g . \square

5.3. Stability of underlying principal bundle. We now assume that $g \geq 2$.

Proposition 5.4. *Assume that the logarithmic connection ∇_0 in (5.1) is irreducible. There is a closed analytic subset $\mathcal{Y}' \subset \mathcal{T}$ of codimension at least $g-1$ such that for any $t \in \mathcal{T}_g \setminus \mathcal{Y}'$, the holomorphic principal G -bundle $\mathcal{F}_G|_{\mathcal{C}_t}$ is stable.*

Proof. The proof is identical to the proof of Proposition 5.3. If $E_G = \mathcal{F}_G|_{\mathcal{C}_t}$ is not stable, there is a maximal parabolic subgroup $P \subsetneq G$ and a holomorphic reduction of structure group $E_P \subset E_G$ to P , such that the quotient bundle

$$\mathrm{ad}(E_G)/\mathrm{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$$

is semistable of degree zero. Therefore, we have $\mathrm{degree}(\mathcal{L}) \leq 0$. This implies that

$$\mathrm{degree}(T^\delta) \leq 2g - 2 - n.$$

Hence the dimension of the image of the homomorphism $H^1(X_0, TX_0(-D_0)) \rightarrow H^1(X_0, \mathcal{L})$ in the long exact sequence of cohomologies associated to (5.8) is at least $g-1$. \square

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