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To cite this version:

Laetitia Giraldi, Jean-Baptiste Pomet. Local Controllability of the Two-Link Magneto-Elastic Micro-Swimmer. IEEE Transactions on Automatic Control, 2017, 62, pp.2512-2518. 10.1109/TAC.2016.2600158 . hal-01145537v2

HAL Id: hal-01145537

https://hal.science/hal-01145537v2

Submitted on 3 Dec 2016

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LOCAL CONTROLLABILITY OF THE TWO-LINK MAGNETO-ELASTIC MICRO-SWIMMER

LAETITIA GIRALDI JEAN-BAPTISTE POMET

Abstract. A recent promising technique for robotic micro-swimmers is to endow them with a magnetization and apply an external magnetic field to provoke their deformation. In this note we consider a simple planar micro-swimmer model made of two magnetized segments connected by an elastic joint, controlled via a magnetic field. After recalling the analytical model, we establish a local controllability result around the straight position of the swimmer.
Keywords: micro-swimmer, controllability, return-method

I. Introduction

Micro-scale robotic swimmers have potential high impact applications. For instance, they could be used in new therapeutic and diagnostic procedures such as targeted drug delivery or minimized invasive microsurgical operations [16]. One of the main challenges is to design a controlled micro-robot able to swim through a narrow channel. In this context, This note states a controllability result for a (simplified model of) micro-robot which is controlled by an external magnetic field that provokes deformations on the magnetic body.

Swimming is the ability for a body to move through a fluid by performing self-deformations. In general, the fluid is modeled by Navier-Stokes equations and the coupling with the swimming body gives rise to a very complex model. It is well known [17] that the locomotion of microscopic bodies in fluids like water (or of macroscopic bodies in a very viscous fluid) is characterized by a low Reynolds number (a dimensionless ratio between inertia effects and viscous effects, of the order of 10^{-6} for common micro-organisms), and that it is then legitimate to consider that the fluid is governed by the Stokes equations, so that hydrodynamic forces applied to the swimmer can be derived linearly with respect to its speed, i.e., the associated Dirichlet-to-Neumann mapping is linear, see for more details [3, 5, 13].

The absence of inertia in the model, that we assume from now on, makes mobility more difficult: a typical obstruction, known as the scallop theorem [17], imposes to use non reciprocal swimming strategies for achieving their self-propulsion. It means that the swimmer is not able to move by using a periodic change of shape with only one degree of freedom.

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Date: to appear in *IEEE Trans. Autom. Control*, 2017 (electronically: 2016).
DOI: 10.1109/TAC.2016.2600158.
L. Giraldi was partially funded by the labex LMH through the grant ANR-11-LABX-0056-LMH.

The Resistive Force Theory [11] consists in approximating the Dirichlet-to-Neumann mapping by an explicit dependence of the hydrodynamic force on the relative speed at each point of the boundary. Following [2, 12], the presnt note uses the simplified model resulting from this approximation.

As far as controllability of these devices is concerned, most known results, starting with the pioneering work of A. Shapere and F. Wilczek [18], focus on the case where the control is the rate of deformation of the swimmer's shape. One then deals with a driftless control-affine system and controllability derives, at least when the shape has a finite number of degrees of freedom, from Lie algebraic methods, see for instance in [8]; more generaly these control problems are related to non-holonomy of distributions and sub-Riemannian geometry [15], where not only controllability but also optimality may be studied, as in [4, 10] where the shape has two degrees of freedom and equations derive from the Resistive Force Theory (Purcell's swimmer). A model with more degrees of freedom is considered in [1]. In [3, 9, 14, 6], the model is more complex, based on explicit solutions of Stokes equations and not on the Resistive Force Theory.

In [2], for the first time, the case where the filament is not fully controlled was considered: the filament is discretized into line segment with a certain magnetization; the elasticity is represented by a torsional spring at each joint and the control is, instead of the rate of deformation, an external magnetic field that provokes deformation (and also some movement) of the magnetized shape. In that paper, by exploiting sinusoidal external magnetic fields, the authors show numerically that the swimmer can be steered along one direction by using prescribed sinusoidal magnetic field. Here we simplify the latter model by considering a swimmer made of only two magnetized segments connected by an elastic joint. This reduced model has already been addressed in [12], in which the authors studied the effect of a prescribed sinusoidal magnetic field by expressing the leader term of the displacement with respect to small external magnetic fields.

We go further by stating a local controllability result including reorientation and not only movement along the longitudinal direction. This is, to our knowledge, the first true local controllability result for these partially actuated systems. We point certain degenerate values of the model's parameters (lengths, magnetizations, spring constant) for which controllability does not hold and prove local controllability for other values.

The model is derived in Section II; this part only uses ideas taken from [2], but is needed to obtain more precise expressions of the equations of motion. The controllability results are stated, commented and proved in Section III. Finally, Section IV briefly states perspectives of this result.

II. Modeling

The present note considers the same swimmer model as [12]; it consists of 2 magnetized segments of length ℓ_1 and ℓ_2, with a magnetic moment M_1 and M_2 respectively, connected by a joint equipped with a torsional spring of stiffness κ that tends to align the segments with one another. It is constrained to move in a plane and is subject to an external magnetic

field $\mathbf{H}$ as well as hydrodynamic forces due to the ambient fluid (these are characterized later).

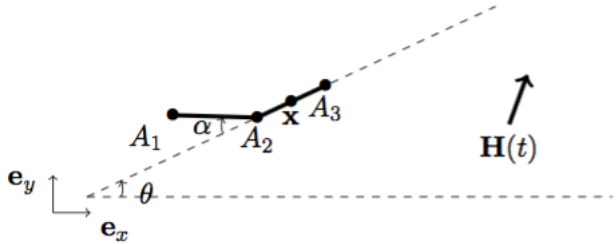

FIGURE 1. Magneto-elastic 2-link swimmer in plane subject to an external magnetic field $\mathbf{H}(t)$.

Under the low Reynolds number assumption introduced and justified in the introduction, inertia is negligible and the equations are obtained, as in [5, 13], by a simple balance of forces and torques (see [2]):

$$\left\{\begin{array}{lcl} \mathbf{F}_1^h + \mathbf{F}_2^h & = & 0\,, \\ \mathbf{T}_1^h + \mathbf{T}_2^h + \mathbf{T}_1^m + \mathbf{T}_2^m & = & 0\,, \\ \mathbf{T}_2^h + \mathbf{T}_2^m + \mathbf{T}_2^{el} & = & 0\,, \end{array}\right. \tag{1}$$

where

- the two first equations state that the balance of exterior forces and exterior moments for the whole system is zero, denoting by $\mathbf{F}_i^h$ the hydrodynamic force applied by the fluid on the ith link, and by $\mathbf{T}_i^h$ and $\mathbf{T}_i^m$ the moment (with respect to any point, so we chose it to be A_2) applied on the ith link by the fluid and the magnetic field respectively,
- the last equation states that the moment with respect to A_2 of the forces and torques applied the subsystem consisting of the second link $[A_2, A_3]$ is zero, $\mathbf{T}_2^{el}$ being the elastic torque applied by the first link on the second one (the moment of the contact force is zero because it is applied at A_2).

This only contains four non-trivial relations because the first equation takes place in the horizontal plane and the other two on the vertical axis.

Let $(\mathbf{e}_x, \mathbf{e}_y)$ be a fixed frame spanning the 2d-plane in which the robot evolves and set $\mathbf{e}_z := \mathbf{e}_x \times \mathbf{e}_y$. We call $\mathbf{x} = (x, y)$ the coordinates in the frame $(\mathbf{e}_x, \mathbf{e}_y)$ of the central point of the second segment, θ the angle that it forms with the x-axis, α the relative angle between the first and second segments (see Figure 1). The position and orientation of the swimmer are characterized by the triplet (x, y, θ), and its shape by α. We denote by

$$\mathbf{e}_{1,\|} = \begin{pmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{pmatrix}, \quad \mathbf{e}_{2,\|} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

the unit vectors aligned with segments $[A_1, A_2]$ and $[A_2, A_3]$, their orthogonal vectors by

$$\mathbf{e}_{1,\perp} = \begin{pmatrix} -\sin(\theta+\alpha) \\ \cos(\theta+\alpha) \end{pmatrix}, \quad \mathbf{e}_{2,\perp} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

We assume that the external magnetic field $\mathbf{H}$ is horizontal in such a way that the motion holds in the plane generate by $\mathbf{e}_x$ and $\mathbf{e}_y$ and we call $H_\parallel$ and $H_\perp$ its coordinates into the moving frame

$$\mathbf{H} = H_\parallel\, \mathbf{e}_{2,\parallel} + H_\perp\, \mathbf{e}_{2,\perp}.$$

Let us now compute the different contributions in (1).

Elastic effects. The torsional spring delivers the following torque to segment $[A_2, A_3]$:

$$\mathbf{T}_2^{el} = \kappa\, \alpha\, \mathbf{e}_z\,.$$

Magnetic effects. The magnetic torque applied to the ith segment is

$$\mathbf{T}_i^m = M_i \left(\mathbf{e}_{i,\parallel} \times \mathbf{H}\right), \quad i = 1, 2\,,$$

with the notations defined above.

Hydrodynamic effects. The force applied to the swimmer by the fluid depends on their relative speed. As announced in the introduction, we use the Resistive Force Theory [11], that assumes that the hydrodynamic drag force of each segment is linear with respect to its velocity. More precisely, if the point of abscissa s on segment i ($i = 1, 2$) has velocity $\mathbf{u}_i(s) = u_{i,\parallel}(s)\, \mathbf{e}_{i,\parallel} + u_{i,\perp}(s)\, \mathbf{e}_{i,\perp}$, then the drag force applied to that point is given by

$$\mathbf{f}_i(s) = -\,\xi_i\, u_{i,\parallel}(s)\, \mathbf{e}_{i,\parallel} \;-\; \eta_i\, u_{i,\perp}(s)\, \mathbf{e}_{i,\perp}\,, \quad i = 1, 2\,, \tag{2}$$

with ξ_i and η_i the constant positive drag coefficients.

Denote by $\mathbf{R}_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ the matrix of the rotation of angle φ, for any φ. The matrix $\mathbf{R}_{\theta+\alpha}$ sends the basis $(\mathbf{e}_x, \mathbf{e}_y)$ onto the basis $(\mathbf{e}_{1,\parallel}, \mathbf{e}_{1,\perp})$ and $\mathbf{R}_\theta$ sends the basis $(\mathbf{e}_x, \mathbf{e}_y)$ onto the basis $(\mathbf{e}_{2,\parallel}, \mathbf{e}_{2,\perp})$, hence relation (2) translates into

$$\begin{aligned} \mathbf{f}_1(s) &= -\mathbf{R}_{(\theta+\alpha)}\, \mathbf{D}_1\, \mathbf{R}_{-(\theta+\alpha)}\, \mathbf{u}_1(s)\,, \\ \mathbf{f}_2(s) &= -\mathbf{R}_\theta\, \mathbf{D}_2\, \mathbf{R}_{-\theta}\, \mathbf{u}_2(s)\,, \end{aligned} \tag{3}$$

with $\mathbf{D}_i$ the matrix $\begin{pmatrix} \xi_i & 0 \\ 0 & \eta_i \end{pmatrix}$ for $i = 1, 2$ and where the vectors are coordinates in the bases $(\mathbf{e}_x, \mathbf{e}_y)$.

If the origin of the abscissa s is the point A_2 on segment 1 and the point $\mathbf{x}$ on segment 2, one has

$$\mathbf{x}_1(s) = \mathbf{x} - \frac{\ell_2}{2}\, \mathbf{e}_{2,\parallel} \;-\, s\, \mathbf{e}_{1,\parallel}\,, \qquad 0 \le s \le \ell_1\,, \tag{4}$$

$$\mathbf{x}_2(s) = \mathbf{x} + s\, \mathbf{e}_{2,\parallel}\,, \qquad -\frac{\ell_2}{2} \le s \le \frac{\ell_2}{2}\,, \tag{5}$$

hence

$$\mathbf{u}_1(s) = \dot{\mathbf{x}} - \frac{\ell_2}{2}\dot{\theta}\mathbf{e}_{2,\perp} - s\left(\dot{\alpha} + \dot{\theta}\right)\mathbf{e}_{1,\perp}\,, \qquad 0 \le s \le \ell_1\,, \tag{6}$$

$$\mathbf{u}_2(s) = \dot{\mathbf{x}} + s\dot{\theta}\mathbf{e}_{2,\perp}\,, \qquad -\frac{\ell_2}{2} \le s \le \frac{\ell_2}{2}\,. \tag{7}$$

The total hydrodynamic force acting on the first and second segments are given by $\mathbf{F}_1^h = \int_0^{\ell_1} \mathbf{f}_1(s)\mathrm{d}s$, $\mathbf{F}_2^h = \int_{-\frac{\ell_2}{2}}^{\frac{\ell_2}{2}} \mathbf{f}_2(s)\mathrm{d}s$, and a straightforward integration yields

$$\mathbf{F}_1^h = \mathbf{R}_{\theta+\alpha}\mathbf{D}_1 \left(-\ell_1 \mathbf{R}_{-(\theta+\alpha)}\, \dot{\mathbf{x}} + \frac{\ell_1\ell_2}{2}\dot{\theta}\, \mathbf{R}_{-\alpha}\mathbf{e}_y + \frac{{\ell_1}^2}{2}\left(\dot{\theta}+\dot{\alpha}\right)\mathbf{e}_y \right), \tag{8}$$

$$\mathbf{F}_2^h = -\ell_2 \mathbf{R}_\theta \mathbf{D}_2 \mathbf{R}_{-\theta}\dot{\mathbf{x}}\,. \tag{9}$$

The moment with respect to point A_2 of the hydrodynamic forces generated by the ith segment has the expression

$$\mathbf{T}_i^h := \int_{\text{i-th segment}} (\mathbf{x}_i(s) - \mathbf{x} + \frac{\ell_2}{2}\,\mathbf{e}_{2,\|}) \times \mathbf{f}_i(\mathbf{s})\,\mathrm{d}\mathbf{s}\,, \qquad i = 1, 2\,,$$

where $\times$ stands for the cross product, the expressions of $\mathbf{x}_i(s)$ and $\mathbf{f}_i(\mathbf{s})$ are given by (3) through (7) and integration takes part for $s \in [0, \ell_1]$ on the first link and $s \in [-\ell_2/2, \ell_2/2]$ on the second link. This yields

$$\begin{aligned}\mathbf{T}_1^h = \eta_1 {\ell_1}^2 \Big(&\tfrac{1}{2}\left(-\sin\alpha\, \dot{x}_\theta + \cos\alpha\, \dot{y}_\theta\right) \\ &-\tfrac{1}{4}\ell_2 \cos\alpha\, \dot{\theta} - \tfrac{1}{3}\eta_1\left(\dot{\theta}+\dot{\alpha}\right)\Big)\,\mathbf{e}_z\,,\end{aligned} \tag{10}$$

$$\mathbf{T}_2^h = -\tfrac{1}{2}\eta_2 {\ell_2}^2 \left(\dot{y}_\theta + \tfrac{1}{6}\ell_2\,\dot{\theta}\right)\mathbf{e}_z\,, \tag{11}$$

where $\dot{x}_\theta$ and $\dot{y}_\theta$ are defined as: $\begin{pmatrix}\dot{x}_\theta \\ \dot{y}_\theta\end{pmatrix} = \mathbf{R}_{-\theta}\dot{\mathbf{x}}$.

Substituting the elements in equations (1) with their just computed expressions yields

$$\mathbf{M}^h(\theta,\alpha)\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\alpha}\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -M_1\left(\cos\alpha H_\perp - \sin\alpha H_\|\right) - M_2 H_\perp \\ -\kappa\alpha - M_2 H_\perp \end{pmatrix} \tag{12}$$

where

$$\mathbf{M}^h(\theta,\alpha) = \left(\begin{array}{c|c}\mathbf{R}_{\theta+\alpha} & 0 \\ \hline 0 & \mathbf{I}_2\end{array}\right)\mathbf{E}(\alpha)\left(\begin{array}{c|c}\mathbf{R}_{-\theta} & 0 \\ \hline 0 & \mathbf{I}_2\end{array}\right), \tag{13}$$

with

$$\mathbf{E}(\alpha) = \left(\begin{array}{c|c}E_{11}(\alpha) & E_{12}(\alpha) \\ \hline E_{21}(\alpha) & E_{22}(\alpha)\end{array}\right), \tag{14}$$

and

$$E_{11} = \begin{pmatrix} -\left(\xi_1\ell_1 + \xi_2\ell_2\right)\cos\alpha & -\left(\xi_1\ell_1 + \eta_2\ell_2\right)\sin\alpha \\ \left(\eta_1\ell_1 + \xi_2\ell_2\right)\sin\alpha & -\left(\eta_1\ell_1 + \eta_2\ell_2\right)\cos\alpha \end{pmatrix},$$

$$E_{12} = \begin{pmatrix} \frac{1}{2}\xi_1\ell_1\ell_2\sin\alpha & 0 \\ \frac{1}{2}\eta_1\ell_1\left(\ell_1 + \ell_2\cos\alpha\right) & \frac{1}{2}\eta_1{\ell_1}^2 \end{pmatrix},$$

$$E_{21} = \tfrac{1}{2}\begin{pmatrix}\eta_1\ell_1^2 & \eta_2\ell_2^2 \\ 0 & \eta_2\ell_2^2\end{pmatrix}\begin{pmatrix}-\sin\alpha & \cos\alpha \\ 0 & -1\end{pmatrix},$$

$$E_{22} = -\begin{pmatrix}\eta_1\ell_1^2 & \eta_2\ell_2^2 \\ 0 & \eta_2\ell_2^2\end{pmatrix}\begin{pmatrix}\frac{1}{4}\,\ell_2\cos\alpha + \frac{1}{3}\,\ell_1 & \frac{1}{3}\,\ell_1 \\ \frac{1}{12}\,\ell_2 & 0\end{pmatrix}.$$

The determinant of $E(\alpha)$ is given by

$$-\tfrac{1}{9}\eta_1\eta_2\,\ell_1^3\ell_2^3\left(\tfrac{1}{4}\left(\xi_1\ell_1+\xi_2\ell_2\right)\left(\eta_1\ell_1+\eta_2\ell_2\right)\cos^2\alpha \right.$$
$$\left. +\left(\xi_1\ell_1+\tfrac{1}{4}\,\eta_2\ell_2\right)\left(\tfrac{1}{4}\,\eta_1\ell_1+\xi_2\ell_2\right)\sin^2\alpha\right),$$

hence it remains negative, $E(\alpha)$ is invertible for all α, so is M_h, and the dynamics (12) of the swimmer can be written as a control system

$$\dot{\mathbf{z}} = \mathbf{F}_0(\mathbf{z}) + H_\parallel\,\mathbf{F}_1(\mathbf{z}) + H_\perp\,\mathbf{F}_2(\mathbf{z}) \quad \text{with } \mathbf{z} = \begin{pmatrix} x \\ y \\ \theta \\ \alpha \end{pmatrix}, \tag{15}$$

affine with respect to the controls $H_\parallel$ and $H_\perp$ where $\mathbf{F}_0$, $\mathbf{F}_1$, $\mathbf{F}_2$ are vector fields on $\mathbb{R}^2 \times S^1 \times S^1$ expressed as follows.

Proposition II.1. *The vector fields* $\mathbf{F}_0$, $\mathbf{F}_1$, $\mathbf{F}_2$ *of system* (15) *are given by*

$$\begin{aligned} \mathbf{F}_0(\mathbf{z}) &= \kappa\alpha\mathbf{X}_4\,, \quad \mathbf{F}_1(\mathbf{z}) = M_1\sin\alpha\mathbf{X}_3\,, \\ \mathbf{F}_2(\mathbf{z}) &= -\left(M_1\cos\alpha + M_2\right)\mathbf{X}_3 - M_2\mathbf{X}_4\,, \end{aligned} \tag{16}$$

where $\mathbf{X}_3$ *and* $\mathbf{X}_4$ *are the vector fields whose vector of coordinates in the "rotating" basis*

$$(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\alpha})$$

are respectively the third and fourth columns of $E(\alpha)^{-1}$.

Proof. This is easily derived from (12) and (13). Note that the matrix depending on $\theta+\alpha$ in (13) plays no role in (12) because it leaves the right-hand side invariant. □

III. Controllability result

According to (16) and the fact that the vector field $\mathbf{X}_4$ does not vanish (its components are one column of an invertible matrix), the zeroes of $\mathbf{F}_0$ are exactly described by $(x, y, \theta) \in \mathbb{R}^2 \times S^1$ arbitrary and $\alpha = 0$. Hence these are the equilibrium positions of system (15) when the control, namely the magnetic field, is zero: $H_\parallel = H_\perp = 0$. Local controllability describes how all points (x, y, θ, α) sufficiently close to a fixed equilibrium $(x^e, y^e, \theta^e, 0)$ can be reached by applying *small* magnetic fields for a *small duration* using a trajectory that *remains close to* $(x^e, y^e, \theta^e, 0)$. The local controllability result stated below does not ensure that the control can be chosen arbitrarily small, i.e. this result is not local with respect to the control; it however gives an explicit bound on the needed control.

A. Main result. Let us first point two cases where controllability cannot hold. In the first case, the variables α and θ may be controlled but $x(t)$ and $y(t)$ are related to $\alpha(t)$ and $\theta(t)$ by a formula, valid everywhere, that does not depend on the control: the system is nowhere controllable. In the second case, the variables x, y, θ may be controlled but $\alpha(0) = 0$ implies $\alpha(t) = 0$ for all t, irrespective of the control, thus forbidding local controllability around any $(x^e, y^e, \theta^e, 0)$ but possibly not away from $\{\alpha = 0\}$.

Proposition III.1.

- *If $\eta_1 - \xi_1 = \eta_2 - \xi_2 = 0$, then there exists two constants x^0, y^0, that depend on the initial conditions of the state but not on the control (the magnetic field H), such that solutions of* (15) *satisfy, for all t,*

$$x(t) = x^0 + \frac{\eta_1\ell_1\left(\ell_1\cos(\alpha(t)+\theta(t)) + \ell_2\cos\alpha(t)\right)}{2(\eta_1\ell_1+\eta_2\ell_2)}\,,$$
$$y(t) = y^0 + \frac{\eta_1\ell_1\left(\ell_1\sin(\alpha(t)+\theta(t)) + \ell_2\sin\alpha(t)\right)}{2(\eta_1\ell_1+\eta_2\ell_2)}\,.$$

- *If*

$$\left(3+4\frac{\ell_2}{\ell_1}+\frac{\eta_2{\ell_2}^2}{\eta_1{\ell_1}^2}\right)M_1 - \left(3+4\frac{\ell_1}{\ell_2}+\frac{\eta_1{\ell_1}^2}{\eta_2{\ell_2}^2}\right)M_2 = 0\,, \tag{17}$$

then the set $\{\alpha = 0\}$ is invariant for equations (15).

Proof. A simple computation shows that if $\eta_1 - \xi_1 = \eta_2 - \xi_2 = 0$, then the time-derivatives of

$$x - \frac{\eta_1\ell_1\left(\ell_1\cos(\alpha+\theta) + \ell_2\cos\alpha\right)}{2(\eta_1\ell_1+\eta_2\ell_2)}$$

and

$$y - \frac{\eta_1\ell_1\left(\ell_1\sin(\alpha+\theta) + \ell_2\sin\alpha\right)}{2(\eta_1\ell_1+\eta_2\ell_2)}$$

are zero (i.e. these are first intergrals of the system), hence the first point. Another computation (see details further, a few lines after (42)) shows that $\dot\alpha$ is zero when (17) is satisfied and $\alpha = 0$, hence proving the second point. □

From now on, we make the following assumption:

Assumption III.2. *The constants ℓ_1, ℓ_2, ξ_1, ξ_2, η_1, η_2, M_1, M_2, κ characterizing the system are such that ℓ_1, ℓ_2, ξ_1, ξ_2, η_1, η_2 and κ are positive, M_1 and M_2 are nonzero, and*

$$(\,\eta_1 - \xi_1\,,\,\eta_2 - \xi_2\,) \neq (0,0)\,, \tag{18}$$
$$\eta_1 \geq \xi_1\,,\quad \eta_2 \geq \xi_2\,, \tag{19}$$
$$\left(3+4\frac{\ell_2}{\ell_1}+\frac{\eta_2{\ell_2}^2}{\eta_1{\ell_1}^2}\right)M_1 - \left(3+4\frac{\ell_1}{\ell_2}+\frac{\eta_1{\ell_1}^2}{\eta_2{\ell_2}^2}\right)M_2 \neq 0\,. \tag{20}$$

Remark III.3. *Conditions* (18) *and* (20) *exactly exclude the cases in which Proposition III.1 applies. We have added conditions* (19). *Physically, these inequalities are usually satisfied: they state that the normal drag force is more important than the tangential one. Technically, they avoid numerous sub-cases. For instance,* (24) *is not satisfied if $\eta_1 l_1(\eta_2-\xi_2)+\eta_2 l_2(\eta_1-\xi_1) = 0$, that does not contradict* (18) *but contradicts* (18) *and* (20)*; the same happens in the proof of Lemma III.14. Theorem III.4 however still holds without* (19), *that we added to make the proofs lighter.*

Let us now state our main result.

Theorem III.4 (Local controllability)**.** *Let Assumption III.2 hold. Fix an equilibrium $\mathbf{z}^e = (x^e, y^e, \theta^e, 0)$. Let $\mathcal{W}$ be a neighborhood of $\mathbf{z}^e$ in $\mathbb{R}^2\times S^1\times S^1$ and T, ε positive numbers, then there exists another neighborhood $\mathcal{V}\subset\mathcal{W}$ of*

$\mathbf{z}^e$ such that, for any $\mathbf{z}^i = (\mathbf{x}^i, \theta^i, \alpha^i)$ and $\mathbf{z}^f = (\mathbf{x}^f, \theta^f, \alpha^f)$ in $\mathcal{V}$, there exist bounded measurable functions $H_\parallel$ and $H_\perp$ in $L^\infty([0,T]), I\!R)$ such that

$$\|H_\perp\|_\infty < \varepsilon\,, \quad \|H_\parallel\|_\infty < 2\,\kappa \left|\frac{M_2 + M_1}{M_2\,M_1}\right| + \varepsilon \tag{21}$$

and, if $t \mapsto \mathbf{z}(t) = (\mathbf{x}(t), \theta(t), \alpha(t))$ is the solution of (15) starting at $\mathbf{z}^i$, then $\mathbf{z}(T) = \mathbf{z}^f$ and $\mathbf{z}(t) \in \mathcal{W}$ for all time $t \in [0,T]$.

We postpone the proof of this result to make further comments. They are made simpler by restricting to one special equilibrium:

Proposition III.5 (Invariance)**.** *If Theorem III.4 (resp. any kind of local controllability like STLC, see Definition III.6 below) holds in the special case where $\mathbf{z}^e$ is "the origin"*

$$\mathbf{O} = (0,0,0,0) \in \mathbb{R}^2 \times S^1 \times S^1\,, \tag{22}$$

i.e. when $x^e = y^e = \theta^e = 0$, then the same holds for arbitrary $\mathbf{z}^e = (x^e, y^e, \theta^e, 0)$.

Proof. Solutions of (15) are invariant under the transformations

$$\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \theta, \alpha, H_\parallel, H_\perp\right) \mapsto \left(\mathbf{R}_{\bar\theta}\left(\begin{smallmatrix} x+\bar x \\ y+\bar y \end{smallmatrix}\right), \theta + \bar\theta, \alpha, H_\parallel, H_\perp\right)\,,$$

hence everything may be carried from a neighborhood of $\mathbf{O}$ to a neighborhood of an arbitrary $(x^e, y^e, \theta^e, 0)$. □

B. Discussion. We assume $(x^e, y^e, \theta^e, 0) = (0,0,0,0) = \mathbf{O}$ without loss of generality. The strongest notion of local controllability is the following (definition taken from [8, Definition 3.2]). It is interesting and natural in that that it is local both in control and in space.

Definition III.6 (STLC)**.** *The system (15) is said to be* small time locally controllable *(STLC) at equilibrium $\mathbf{z}^e = (x^e, y^e, \theta^e, 0)$ and control $(0,0)$ if and only if, for any neighborhood $\mathcal{W}$ of $\mathbf{z}^e$, any $T > 0$, and any $\varepsilon > 0$, there exists another neighborhood $\mathcal{V} \subset \mathcal{W}$ of $\mathbf{z}^e$ such that, for any $\mathbf{z}^i = (\mathbf{x}^i, \theta^i, \alpha^i)$ and $\mathbf{z}^f = (\mathbf{x}^f, \theta^f, \alpha^f)$ in $\mathcal{V}$, there exist bounded measurable functions $H_\parallel$ and $H_\perp$ in $L^\infty([0,T]), I\!R)$ such that $\|H_\perp\|_\infty < \varepsilon$, $\|H_\parallel\|_\infty < \varepsilon$, and, if $t \mapsto \mathbf{z}(t) = (\mathbf{x}(t), \theta(t), \alpha(t))$ is the solution of (15) starting at $\mathbf{z}^i$, then $\mathbf{z}(T) = \mathbf{z}^f$ and $\mathbf{z}(t) \in \mathcal{W}$ for all time $t \in [0,T]$.*

Theorem III.4 establishes a rather strong form of local controllability with an *explicit* bound on the controls. It is however weaker than STLC because the bound on the controls remains larger than $2\,\kappa\,\frac{|M_1+M_2|}{|M_2\,M_1|}$, hence it does not go to zero when $\mathcal{V}$ becomes smaller and smaller. Theorem III.4 establishes STLC only if $M_1 + M_2 = 0$, and we do not know whether STLC holds or not when $M_1 + M_2 \neq 0$ (physically, we expect to have some sort of lower-bound on the magnetic field to deform the swimmer, but this is not formalized).

Let us review the classical ways to establish STLC, namely the "linear test" and the theorem on "bad and good brackets" due to H. Sussmann [19], and explain why they fail. The notion of linearized system along a trajectory is instrumental.

Definition III.7. *The linearized control system of* (15) *around a trajectory* $t \mapsto (\mathbf{z}^*(t),\ H_\perp^*(t), H_\parallel^*(t))$ *defined on* $[0,T]$ *for some* $T > 0$ *is the time-varying control system*

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y} + \mathbf{B}(t)\mathbf{v}\,, \tag{23}$$

where $\mathbf{A}(t)$ *is the Jacobian of* $\mathbf{z} \mapsto \mathbf{F}_0(\mathbf{z}) + H_\perp^*(t)\mathbf{F}_2(\mathbf{z}) + H_\parallel^*(t)\mathbf{F}_1(\mathbf{z})$ *with respect to* $\mathbf{z}$ *at* $\mathbf{z} = \mathbf{z}^*(t)$ *and the two columns of* $\mathbf{B}(t)$ *are* $\mathbf{F}_1(\mathbf{z}^*(t))$ *and* $\mathbf{F}_2(\mathbf{z}^*(t))$.

If the trajectory is simply the equilibrium point $\mathbf{O}$ and the reference controls are zero, $\mathbf{A}$ and $\mathbf{B}$ do not depend on time and $\mathbf{A}$ is simply the Jacobian of $\mathbf{z} \mapsto \mathbf{F}_0(\mathbf{z})$ at $\mathbf{z} = \mathbf{O}$.

Proposition III.8. *The controllability matrix* $\mathcal{C} = \big[\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \mathbf{A}^3\mathbf{B}\big]$ *(8 columns, 4 lines) for the linearized control system of* (15) *at the equilibrium* $\mathbf{O}$ *has rank at most 2.*

Proof. One deduces from (16) that on the one hand the Jacobian of $\mathbf{z} \mapsto \mathbf{F}_0(\mathbf{z})$ at $\mathbf{O}$ has only one nonzero column, proportional to $\mathbf{X}_4(\mathbf{O})$, and on the other hand the vector field $\mathbf{F}_1$ is zero at $\mathbf{O}$. Hence the rank of $\mathcal{C}$ is at most the rank of $\{\mathbf{X}_4(\mathbf{O}), \mathbf{F}_2(\mathbf{O})\}$. □

The linear test for local controllability [8, Theorem 3.8] states that a nonlinear control system is STLC at an equilibrium if its linearized control system at this point is controllable. According to the Kalman rank condition [8, Theorem 1.16], the latter linearized system is not controllable because the rank of $\mathcal{C}$ is stricly less than 4. Hence, the linear test cannot be applied.

A more general sufficient condition was introduced in [19, section 7] and recalled in [8, section 3.4]. It requires the following notions.

Definition III.9 (LARC)**.** *System* (15) *satisfies the LARC (Lie Algebra Rank Condition) at* $\mathbf{O}$ *if and only if the values at* $\mathbf{O}$ *of all iterated Lie brackets of the vector fields* $\mathbf{F}_0$, $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *span a vector space of dimension 4.*

Now, for θ, η positive numbers and h an iterated Lie bracket of the vector fields $\mathbf{F}_0$, $\mathbf{F}_1$, $\mathbf{F}_2$, let

- $\sigma(h)$ be the sum of h and the iterated Lie bracket obtained by exchanging $\mathbf{F}_1$ and $\mathbf{F}_2$ in h,
- $\delta_i(h) \in \mathbb{N}$ $(i = 0, 1, 2)$ be the number of times the vector field $\mathbf{F}_i$ appears in h,
- $\rho_\theta(h)$ be given by $\rho_\theta(h) = \theta\delta_0(h) + \delta_1(h) + \delta_2(h)$,
- G_η be the vector subspace spanned by all vectors $g(\mathbf{O})$ where g is an iterated bracket of $\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2$ such that $\rho_\theta(g) < \eta$.

Note: the LARC is equivalent to G_η having dimension 4 for large enough η.

Definition III.10 (Sussman's condition $S(\theta)$ [19])**.** *Let* θ *be a number,* $0 \le \theta \le 1$. *System* (15) *satisfies the condition* $S(\theta)$ *at* $\mathbf{O}$ *if and only if it satisfies the LARC and any iterated Lie bracket* h *of the vector fields* $\mathbf{F}_0$, $\mathbf{F}_1$, $\mathbf{F}_2$ *such that* $\delta_0(h)$ *is odd and both* $\delta_1(h)$ *and* $\delta_2(h)$ *are even ("bad" brackets) satisfies* $\sigma(h)(\mathbf{O}) \in G_{\rho_\theta(h)}$.

The main theorem in [19] states that system (15) is STLC if the condition $S(\theta)$ holds for at least one θ in $[0,1]$. Proposition III.11 below shows that this sufficient condition cannot be applied, except if $M_1 + M_2 = 0$. This is consistant with our Theorem III.4, that establishes STLC only in this case.

Proposition III.11. *Assume that the parameters of the system* (15) *satisfy Assumption III.2. Then*

(1) *the LARC is satisfied at* $\mathbf{O}$,
(2) *if* $M_1 + M_2 \neq 0$, *then* $S(\theta)$ *is not satisfied at* $\mathbf{O}$ *for any* θ *in* $[0,1]$,
(3) *if* $M_1 + M_2 = 0$, *then* $S(1)$ *is satisfied at* $\mathbf{O}$.

Proof. In order to save space, we denote by $(\cdots)$ any coefficient whose value does not matter and by $f_{i_1 i_2 .. i_m}$ or $X_{i_1 i_2 .. i_m}$ the value at $\mathbf{O}$ of the iterated Lie bracket $[\mathbf{F}_{i_1}, [\mathbf{F}_{i_2}, \cdots \mathbf{F}_{i_m}] \cdots]]$ or $[\mathbf{X}_{i_1}, [\mathbf{X}_{i_2}, \cdots \mathbf{X}_{i_m}] \cdots]]$: for example, $f_0 = \mathbf{F}_0(\mathbf{O})$, $X_{34} = [\mathbf{X}_3, \mathbf{X}_4](\mathbf{O})$, $f_{1021} = [\mathbf{F}_1, [\mathbf{F}_0, [\mathbf{F}_2, \mathbf{F}_1]]](\mathbf{O})$.

Computing Lie brackets with a computer algebra software (Maple), taking their value at $\mathbf{O}$ and forming determinants, we show that

$$\det(X_3, X_4, X_{34}, X_{334}) \quad \text{and} \quad \det(X_3, X_4, X_{34}, X_{434})$$

cannot be both zero if (18) and (19) hold. This proves:

$$\mathrm{Rank}\{X_3, X_4, X_{34}, X_{334}, X_{434}\} = 4\,. \tag{24}$$

Point (1) follows because, with L a function of the constants that is nonzero if and only if (20) is met, one has

$$f_{02} = \kappa L\, X_4\,,\ f_{12} = M_1 L\, X_3\,,\ f_{212} = 2L\, X_{34} + (\cdots) X_3\,, \tag{25}$$

and, modulo a linear combination of X_3, X_4 and X_{34},

$$f_{12212} - f_{21212} = 2M_1 L^2\, X_{334}\,,\ f_{02212} - f_{20212} = 2\kappa L^2\, X_{434}\,. \tag{26}$$

These are obtained from (16), the expressions of $\mathbf{X}_3$ and $\mathbf{X}_4$ are needed to compute the number L.

To prove point (2), we use the "bad" bracket $h = [\mathbf{F}_2, [\mathbf{F}_0, \mathbf{F}_2]]$. Since $f_{101} = 0$, one has $\sigma(h)(\mathbf{O}) = f_{202}$. Computing f_{202} yields:

$$\sigma(h)(\mathbf{O}) = -2\kappa(M_1 + M_2) L\, X_{34} + (\cdots) X_4\,. \tag{27}$$

One has $\rho_\theta(h) = 2 + \theta$. $G_{2+\theta}$ is, by definition, the vector space spanned by f_0, f_1, f_2, f_{01}, f_{02}, f_{12}, that are linear combinations of X_3, X_4 (see (16) and (25)), and, depending on the value of θ, by some $f_{0\cdots 01}$, that are all zero, and by some $f_{0\cdots 02}$, that are all colinear to X_4. Hence $G_{2+\theta}$ is spanned by X_3 and X_4. Considering equation (27) where $M_1 + M_2 \neq 0$ is assumed, one then has $\sigma(h)(\mathbf{O}) \notin G_{2+\theta}$, proving that the condition $S(\theta)$ does not hold.

For point (3), we assume $M_1 + M_2 = 0$ and take $\theta = 1$ so that $\rho_\theta(h)$ is the order of the iterated Lie bracket h. According to (25) and (26), Lie brackets of order 5 generate the whole space, i.e. G_η is the whole tangent space $\mathbb{R}^4$ if $\eta > 5$. Besides $[\mathbf{F}_1, [\mathbf{F}_0, \mathbf{F}_1]]$ and $[\mathbf{F}_2, [\mathbf{F}_0, \mathbf{F}_2]]$, the bad Lie brackets of order at most 5 are these that contain three times $\mathbf{F}_0$ and two times either $\mathbf{F}_1$ or $\mathbf{F}_2$, these that contain one time $\mathbf{F}_0$, two times $\mathbf{F}_1$ and two times $\mathbf{F}_2$, and these that contain one time $\mathbf{F}_0$ and four times either $\mathbf{F}_1$ or $\mathbf{F}_2$; it can be checked that they all belong to G_5, spanned by X_3, X_4 X_{34} and X_{434}. $\square$

C. Proof of Theorem III.4. This proof relies on the *return method*, introduced by J.-M. Coron in [7] for stabilization purposes, and exposed in [8, chapter 6]. It has mostly been used to establish controllability results for infinite dimensional control systems (PDEs). The idea of the method is to find a trajectory ("loop") of system (15) such that it starts and ends at the equilibrium $\mathbf{O}$ and the linearized control system around this trajectory is controllable, and then conclude by using the implicit function theorem that one can go from any state close to the equilibrium to any other final state close to the equilibrium. The proof relies on Lemma III.12, that identifies a family of bounded controls producing "loops" from $\mathbf{O}$ to $\mathbf{O}$ and on Lemma III.13 that shows controllability of the linearized system (15) around some of these loop trajectories.

Lemma III.12 (return trajectory)**.** *Let Assumption III.2 hold. There exist positive numbers k, $\overline{T}$ and $\overline{H}$ with the following property: for any T, $0 < T \leq \overline{T}$, and any measurable control $t \mapsto H(t) = (H_\perp(t), H_\parallel(t))$ defined on $[0, T/2]$ and bounded by $\overline{H}$, there is a bounded measurable control $t \mapsto H^*(t) = (H^*_\perp(t), H^*_\parallel(t))$ defined on $[0, T]$ such that*

$$H^*_\perp(t) = H_\perp(t) \text{ and } H^*_\parallel(t) = H_\parallel(t)\,, \qquad 0 \leq t \leq \frac{T}{2}\,, \tag{28}$$

$$\|H^*_\perp(.)\|_\infty \leq k\,\|H(.)\|_\infty\,, \tag{29}$$

$$\|H^*_\parallel(.)\|_\infty \leq 2\kappa \left|\frac{M_1 + M_2}{M_1\,M_2}\right| + k\,\|H(.)\|_\infty\,, \tag{30}$$

and, if $t \mapsto \mathbf{z}^*(t) = (\mathbf{x}^*(t), \theta^*(t), \alpha^*(t))$ is the solution of system (15) with control H^* and initial condition $\mathbf{z}^*(0) = \mathbf{O}$, then

$$\mathbf{z}^*(t) = \mathbf{z}^*(T - t)\,, \quad 0 \leq t \leq T\,, \tag{31}$$

and $\alpha^*(t)$ remains in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, for all t.

Lemma III.13 (linear controllability along trajectories)**.** *For any number β, denote by $t \mapsto \mathbf{z}^\beta(t)$ the solution of (15) with initial condition $\mathbf{z}^\beta(0) = \mathbf{O}$ and (constant) controls*

$$H^\beta_\perp(t) = \beta\,, \quad H^\beta_\parallel(t) = 0\,. \tag{32}$$

It is defined on $[0, +\infty)$. Under Assumption III.2, there exist arbitrarily small positive values of β such that the linearized system (23) of (15) around $(\mathbf{z}^\beta(.),\, H^\beta_\perp(.), H^\beta_\parallel(.))$ is controllable on $[0, \tau]$ for any positive τ.

These two lemmas are proved later. Let us first use them.

Proof of Theorem III.4. Let us first prove the theorem assuming $\mathbf{z}^i = \mathbf{O}$. Let $\varepsilon > 0$, $T > 0$, the equilibrium $\mathbf{O}$ and its neighborhood $\mathcal{W}$ be given. Lemma III.12 provides some constants k, $\overline{H}$, $\overline{T}$ that depend only on $\mathbf{O}$ and the constants of the problem.

In the sequel we assume $T \leq \overline{T}$ without loss of generality; indeed, if $T > \overline{T}$, first solve the problem for $T = \overline{T}$ (and the same ε, $\mathbf{O}$) and denote by $t \mapsto H^{\overline{T}}(t) = (H^{\overline{T}}_\parallel(t), H^{\overline{T}}_\perp(t))$ the control solving that problem, then the

solution for the actual $T > \overline{T}$ is given by

$$H(t) = \begin{cases} 0 & \text{if } 0 \le t < T - \overline{T}\,, \\ H^{\overline{T}}(t - T + \overline{T}) & \text{if } T - \overline{T} \le t \le T\,. \end{cases}$$

From Lemma III.13, there exists some $\beta > 0$ such that $\beta < \varepsilon/2k$, $\beta < \overline{H}$, and the linearized system (23) of (15) around $\mathbf{z}^\beta(.) = (\mathbf{x}^\beta(.),\, H_\perp^\beta(.), H_\parallel^\beta(.))$ (defined in the lemma) is controllable on $[0, \tau]$ for any positive τ. Since $\mathbf{z}^\beta(0) = \mathbf{O}$, there is some $\overline{\overline{T}} > 0$ such that $\mathbf{z}^*([0, \overline{\overline{T}}]) \subset \mathcal{W}$. We assume that $T \le \overline{\overline{T}}$ without loss of generality for the same reason that allowed us to assume $T \le \overline{T}$ above.

Now apply Lemma III.12 with

$$\begin{aligned} H_\perp(t) &= H_\perp^\beta(t) = \beta\,, \\ H_\parallel(t) &= H^\beta(t) = 0\,, \quad 0 \le t \le \frac{T}{2}\,. \end{aligned} \tag{33}$$

This yields a control $t \mapsto (H_\perp^*(t), H_\parallel^*(t))$ associated with a solution $t \mapsto \mathbf{z}^*(t)$ of (15), both defined on $[0, T]$, such that

$$\begin{aligned} &\|H_\perp^*\|_\infty < \frac{\varepsilon}{2}\,,\ \|H_\parallel^*\|_\infty < \frac{2\kappa\,|M_1 + M_2|}{M_1\,M_2} + \frac{\varepsilon}{2}\,, \\ &\mathbf{z}^*(0) = \mathbf{z}^*(T) = \mathbf{O}\,, \quad \mathbf{z}^*(t) \in \mathcal{W}\,,\ t \in [0, T]\,, \end{aligned} \tag{34}$$

and the linear approximation of (15) along this solution is controllable. Note that $\mathbf{z}^*(t)$ remains in $\mathcal{W}$ because $T \le \overline{\overline{T}}$.

Let the end-point mapping $\mathcal{E} : L^\infty([0, T], \mathbb{R}^2) \to \mathbb{R}^4$ be the one that maps a control on $[0, T]$ to the point $\mathbf{z}(T)$ with $\mathbf{z}(.)$ the solution of the system (15) with this control and initial value $\mathbf{z}(0) = \mathbf{O}$. We have $\mathcal{E}(H^*(.)) = \mathbf{O}$ and linear controllability amounts to $\mathcal{E}$ being a submersion at this point; hence $\mathcal{E}$ sends any neighborhood of H^* in $L^\infty([0, T], \mathbb{R}^2)$ to a neighborhood of $\mathbf{O}$; this yields all the properties of the theorem restricted to $\mathbf{z}^i = \mathbf{O}$.

To obtain the theorem for arbitrary $\mathbf{z}^i, \mathbf{z}^f$, apply twice the restricted theorem we just proved: once with $\mathbf{z}^i, \mathbf{z}^f$ replaced by $\mathbf{O}, \mathbf{z}^f$, and once with $\mathbf{z}^i, \mathbf{z}^f$ replaced by $\mathbf{O}, \mathbf{z}^i$ and $T < 0$ (the proof can be adapted *mutatis mutandis* to $T < 0$); then concatenate the two controls in order to go from $\mathbf{z}^i$ to $\mathbf{O}$ first and then from $\mathbf{O}$ to $\mathbf{x}^f$. □

Proof of Lemma III.12. Taking $\overline{T}$ and $\overline{H}$ small enough, any solution $\mathbf{z}$ of (15) with $\mathbf{z}(0) = \mathbf{O}$ with a control H bounded by $\overline{H}$ satisfies $\alpha(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for all $t \in [0, \overline{T}]$. Consider T positive smaller than $\overline{T}$ and an arbitrary control $H : t \mapsto (H_\perp(t), H_\parallel(t))$ defined on $[0, T/2]$ bounded by $\overline{H}$. Let $t \mapsto \mathbf{z}^*(t) = (\mathbf{x}^*(t), \theta^*(t), \alpha^*(t))$ be the solution of system (15) on $[0, \frac{T}{2}]$ associated with the control H^* and starting at $\mathbf{O}$. We now consider a piecewise function $H_\perp^*(.)$ and $H_\parallel^*(.)$ defined as

$$\begin{aligned} &H_\perp^*(t) = H_\perp(t)\,, \quad H_\parallel^*(t) = H_\parallel(t)\,, \quad t \in [0, \tfrac{T}{2}]\,, \\ &H_\perp^*(t) = \tfrac{2\,\kappa}{M_2}\alpha^*(t) - H_\perp(T - t)\,, \quad t \in \left[\tfrac{T}{2}, T\right]\,, \\ &H_\parallel^*(t) = \tfrac{2\,\kappa\,\alpha^*(t)}{M_2\,M_1\,\sin\alpha^*(t)}\,(M_2 + M_1\cos\alpha^*(t)) - H_\parallel(T - t)\,,\ t \in [\tfrac{T}{2}, T]. \end{aligned} \tag{35}$$

This particular control leads the swimmer to come back to the starting point $\mathbf{O}$ at time T by using the same path. Indeed, the latter condition reads

$$\mathbf{z}^*(t) = \mathbf{z}^*(T-t)\,, \quad t \in \left[\frac{T}{2}, T\right]\,. \tag{36}$$

Differentiating the latter with respect to time, we get:

$$2\,\mathbf{F}_0(\mathbf{z}^*) + \mathbf{F}_1(\mathbf{z}^*)\left(H_\parallel^* + H_\parallel\right) + \mathbf{F}_2(\mathbf{z}^*)\,(H_\perp^* + H_\perp) = 0\,, \quad t \in \left[\frac{T}{2}, T\right]. \tag{37}$$

By using the expression (16) for $\mathbf{F}_0$, $\mathbf{F}_1$ and $\mathbf{F}_2$ and because $\mathbf{X}_3$ and $\mathbf{X}_4$ are linearly independent, the projection onto the vector space generated by $\mathbf{X}_i$, $i = 3, 4$ of the previous equality has to vanish. The two controls $H_\parallel^*$ and $H_\perp^*$ defined in (35) solve the latter system of linear equation (37). Note that the latter control functions is well-defined since $\alpha^*(t)$ remains in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Moreover, by Gronwall Lemma, there exists a constant k' (which depends on $\overline{T}$) such that

$$|\alpha^*(t)| \leq k' \|H\|_\infty\,, \quad t \in [0, T]\,.$$

Using the fact that $\frac{\alpha^*}{\sin\alpha^*} \leq 1 + \frac{(\alpha^*)^2}{2}$ (since $\alpha^* < \frac{\pi}{2}$), we get equations (29)-(30) from equations (35) with k larger than $1 + \frac{2\kappa}{M_2}k'$ and $1 + \frac{M_1+M_2}{M_1 M_2}k'^2\overline{H}$. □

The proof of Lemma III.13 requires a more technical lemma:

Lemma III.14. *Let Assumption III.2 hold. There exist arbitrarily small values of β such that for a certain $\overline{\alpha} > 0$, the distribution spanned by the vector fields $\mathbf{X}_3$, $\mathbf{X}_4$, $[\mathbf{X}_3, \mathbf{X}_4]$ and $\mathbf{X}_5^\beta$, with*

$$\mathbf{X}_5^\beta = -\beta\,(M_2 + M_1\cos\alpha)\,[\mathbf{X}_3, [\mathbf{X}_3, \mathbf{X}_4]] + (\kappa\alpha - \beta M_2)\,[\mathbf{X}_4, [\mathbf{X}_3, \mathbf{X}_4]]\,, \tag{38}$$

has rank 4 at all points in

$$\{\mathbf{z} = (x, y, \theta, \alpha) \in \mathbb{R}^4\,,\ |\alpha| < \overline{\alpha},\ \alpha \neq 0\}\,. \tag{39}$$

Proof. The determinant of $\mathbf{X}_3$, $\mathbf{X}_4$, $[\mathbf{X}_3, \mathbf{X}_4]$, $\mathbf{X}_5^\beta$ depends only on α and β (and the fixed parameters of the system), We computed it using a symbolic computation software (Maple). It is a polynomial with respect to $\cos\alpha$ of degree at most 12, whose coefficients are affine with respect to α and β. Its leading coefficient does not depend on α or β and is zero only if $\eta_i = \xi_i$ or $\eta_i = 4\,\xi_i$, for $i = 1$ or $i = 2$. In all these particular cases, and using (18)-(20), this determinant is a polynomial in $\cos\alpha$ with degree less than 12 but at most 1. Hence, for arbitrarily small values of β, the analytic function of one variable $\alpha \mapsto \det(\mathbf{X}_3, \mathbf{X}_4, [\mathbf{X}_3, \mathbf{X}_4], \mathbf{X}_5^\beta)$ is non zero which ensures the existence of $\overline{\alpha}$. □

Proof of Lemma III.13. Define the matrices $\mathbf{B}_j(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{A}(t)\right)^j \mathbf{B}(t)$, with $\mathbf{A}(t)$ and $\mathbf{B}(t)$ given by (23). According to [8, Theorem 1.18], the linear system (23) is controllable on $[0, \tau]$ if there is at least one t, $0 < t < \tau$, such that

$$\mathcal{S}p(t) := \mathrm{Span}\left\{\mathbf{B}_j(t)\mathbf{v};\quad \mathbf{v} \in \mathbb{R}^2;\quad j \geq 0\right\} = \mathbb{R}^4\,. \tag{40}$$

By a simple computation, it turns out that, with the constant controls $H_{\|}^*(t) = 0$ and $H_{\perp}^*(t) = \beta$, the ith column of $\mathbf{B}_j(t)$ is the column of coordinates of the vector field $\mathbf{C}_{i,j}^{\beta}$ at point $\mathbf{z}^*(t)$, with

$$\mathbf{C}_{i,0}^{\beta} = \mathbf{F}_i\,,\ \mathbf{C}_{i,1}^{\beta} = [\mathbf{F}_0 + \beta\,\mathbf{F}_1, \mathbf{C}_{i,0}^{\beta}]\,, \mathbf{C}_{i,2}^{\beta} = [\mathbf{F}_0 + \beta\,\mathbf{F}_1, \mathbf{C}_{i,1}^{\beta}] \tag{41}$$

and so on (we do not need $j > 2$). We claim that

$$\left.\begin{array}{l}\text{for any } \beta > 0, \text{ there exists } \beta^*,\ 0 < \beta^* \le \beta, \text{ and } \overline{\alpha} > 0, \text{ such that}\\ \text{the distribution spanned by } \mathbf{C}_{1,0}^{\beta}, \mathbf{C}_{2,0}^{\beta}, \mathbf{C}_{1,1}^{\beta}, \mathbf{C}_{2,1}^{\beta}, \mathbf{C}_{1,2}^{\beta}, \mathbf{C}_{2,2}^{\beta}\\ \text{has rank 4 at all points of (39).}\end{array}\right\} \tag{42}$$

This claim implies Lemma III.13. Indeed, take β^* in the lemma to be the one given by (42). Along the trajectory $t \mapsto \mathbf{z}^{\beta^*}(t) = (x^*(t), y^*(t), \theta^*(t), \alpha^*(t))$, one has $\alpha^*(0) = 0$, and this implies $\dot{\alpha}^*(0) \neq 0$ because (20) is satisfied, $H_{\perp}$ is nonzero, and a straightforward computation from (15) shows that

$$\dot{\alpha} = 3\frac{\left(3 + 4\frac{\ell_2}{\ell_1} + \frac{\eta_2 {\ell_2}^2}{\eta_1 {\ell_1}^2}\right) M_1 - \left(3 + 4\frac{\ell_1}{\ell_2} + \frac{\eta_1 {\ell_1}^2}{\eta_2 {\ell_2}^2}\right) M_2}{\ell_1\,\ell_2\,(\eta_1\ell_1 + \eta_2\ell_2)}\,H_{\perp}\,.$$

when $\alpha = 0$. Hence there exists $\bar{t} > 0$ (that depends on β^*) such that

$$0 < t < \bar{t} \quad \Rightarrow \quad \alpha(t) \neq 0 \text{ and } |\alpha(t)| < \overline{\alpha}\,. \tag{43}$$

According to the above remark and if the claim holds, (40) holds for all t, $0 < t < \bar{t}$; linear controllability on $[0, \tau]$ in the lemma follows from (40) at some positive t smaller than $\min\{\bar{t}, \tau\}$.

Let us now prove (42). Recall that

$$\mathbf{F}_0 + \beta\,\mathbf{F}_1 = -\beta\,(M_2 + M_1\cos\alpha)\,\mathbf{X}_3 + (\kappa\alpha - \beta M_2)\,\mathbf{X}_4\,. \tag{44}$$

According to (16) and (41), and since $M_1 \neq 0$ and $M_2 \neq 0$, $\mathbf{C}_{1,0}^{\beta}$ and $\mathbf{C}_{2,0}^{\beta}$ span the same distribution as $\mathbf{X}_3$ and $\mathbf{X}_4$ at points where $\alpha \neq 0$. Hence $\mathbf{C}_{1,0}^{\beta}, \mathbf{C}_{2,0}^{\beta}, \mathbf{C}_{1,1}^{\beta}, \mathbf{C}_{2,1}^{\beta}$ span the same distribution as $\mathbf{X}_3$, $\mathbf{X}_4$, $[\mathbf{F}_0 + \beta\,\mathbf{F}_1, \mathbf{X}_3]$ and $[\mathbf{F}_0 + \beta\,\mathbf{F}_1, \mathbf{X}_4]$, that is, according to (44), $\mathbf{X}_3$, $\mathbf{X}_4$, $(\beta M_2 - \kappa\alpha)\,[\mathbf{X}_3, \mathbf{X}_4]$, and $-\beta\,(M_2 + M_1\cos\alpha)\,[\mathbf{X}_3, \mathbf{X}_4]$ i.e., if, in addition, $(M_2 + M_1\cos\alpha,\ \kappa\alpha - \beta M_2) \neq (0, 0)$, the same distribution as $\{\mathbf{X}_3,\ \mathbf{X}_4,\ [\mathbf{X}_3, \mathbf{X}_4]\}$, and finally $\mathbf{C}_{1,0}^{\beta}$, $\mathbf{C}_{2,0}^{\beta}$, $\mathbf{C}_{1,1}^{\beta}$, $\mathbf{C}_{2,1}^{\beta}$, $\mathbf{C}_{1,2}^{\beta}$, $\mathbf{C}_{2,2}^{\beta}$ span the same distribution as $\mathbf{X}_3$, $\mathbf{X}_4$, $[\mathbf{X}_3, \mathbf{X}_4]$ and $\mathbf{X}_5^{\beta}$ given by (38) at points where $\alpha \neq 0$ and $(M_2 + M_1\cos\alpha, \kappa\alpha - \beta M_2) \neq (0, 0)$. This property, together with Lemma III.14, proves the claim (42), hence Lemma III.13. □

IV. Perspectives

For a micro-swimmer model made by two magnetized segments connected by an elastic joint, controlled by an external magnetic field, this note establishes local controllability *around the straight position* by controls that cannot be made arbitrarily small, but an explicit bound on the controls is given. This raises two natural questions.

On the one hand one would like to decide whether the controls can be taken arbitrarily small, thus proving STLC in the sense of [8], or the (nonzero) bound is sharp, and hence the system is *not* STLC. On the other

hand, since controllability around non-straight positions is easier, it is natural to address the question of global controllability as an extension of this note.

Further perspectives, that are currently under our investigation, are to extend our result to more realistic swimmers, for instance by considering additional segments. An other important issue is to study the optimal control problem i.e., finding the magnetic fields in such a way that the swimmer reaches a desired configuration as quick as possible; this has already been done for swimmers that are controlled by the velocity at each of their joints [10] but the study for the present system where theses velocities are indirectly controlled by a magnetic field is quite different.

Aknowlegement

The authors thank Clément Moreau, student at ENS Cachan, France, for carefully reading the manuscript.

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