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Witten non abelian localization for equivariant K -theory, and the $[Q, R] = 0$ theorem

Paul-Émile PARADAN, Michèle VERGNE

April 23, 2015

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1 Introduction

Let M be a manifold provided with an action of a compact Lie group K with Lie algebra \mathfrak{k} . Our main additional data is a K -invariant map $\Phi : M \rightarrow \mathfrak{k}^*$. This gives rise to a K -invariant vector field κ_Φ , the Kirwan vector field, on M . Following an idea of E. Witten, we will use the vector field κ_Φ to deform an element σ of the equivariant \mathbf{K} -theory of T^*M . It localizes the study of the equivariant index of $\sigma \in \mathbf{K}_K^0(T^*M)$ to the study of the equivariant index of the deformed element on a neighborhood of the zeroes of the map Φ , together with the study of the same scenario on lower dimensional manifolds. In this deformation, an elliptic symbol is deformed in a sum of transversally elliptic symbols.

In the Hamiltonian framework, this non-abelian localization procedure has been studied by the first author in [29] after the first input of the second author in [39]. This localization has been transposed recently in the setting of equivariant K -homology by Y. Song [36]. The analytic approach of the Witten deformation has been studied by several people, see [38, 11, 23, 24].

The purpose of the present paper is two-fold. First, we obtain a non-abelian localization theorem, when M is any even dimensional oriented compact manifold, Φ a moment map in a weak sense, and σ a symbol of any twisted Dirac operator. Second, we use this general approach to reprove the $[Q, R] = 0$ theorem of Meinrenken-Sjamaar [27] in the Hamiltonian case, and we obtain mild generalizations to almost complex manifolds. In other words, we clarify the strategy used by the first author in [29]. Furthermore, this non-abelian localization theorem can be used to obtain a geometric description of the multiplicities of the index of general twisted Dirac operators in [32].

Before describing the localization result, and its consequences, in more details, let us present some motivation and previous results. Let A be a K -invariant elliptic operator on M with principal symbol σ and let $\text{Index}_K^M(\sigma)(k)$ its index, a central function of K . Write $\text{Index}_K^M(\sigma)(k) = \sum_{\mu \in \hat{K}} m(\mu, \sigma) \mathbf{Tr}_\mu(k)$ its Fourier expansion in terms of irreducible characters of K . The numbers $m(\mu, \sigma) \in \mathbb{Z}$ are the multiplicities of the irreducible representation V_μ in the virtual space of solutions of A . In [15], Guillemin-Sternberg formulated the $[Q, R] = 0$ principle (quantization commutes with reduction), which gave a conjectural geometric recipe to compute the multiplicities $m(\mu, \sigma_L)$, in the case where M is a Hamiltonian manifold equipped with a Kostant-Souriau line bundle L , in terms of the fibers of the moment map $\Phi : M \rightarrow \mathfrak{k}^*$ given by the Hamiltonian structure. Here σ_L is the symbol of the Dolbeaut-Dirac operator with coefficients in L . Introduce the space $M_{red} = \Phi^{-1}(0)/K$ (and

assume to simplify that M_{red} is smooth). The space M_{red} is the reduced Marsden-Weinstein symplectic space. It plays an important role, as for example many moduli spaces appear naturally as Marsden-Weinstein quotients. Also when M is a projective manifold, the reduced manifold M_{red} is the GIT quotient $M//K_{\mathbb{C}}$ of M , where $K_{\mathbb{C}}$ is the complexification of K . Thus the multiplicity of the trivial representation of K , according to Guillemin-Sternberg conjecture, is obtained as the Riemann-Roch number of M_{red} . Using symplectic cuts, Meinrenken-Sjamaar [27] gave the first proof of this fundamental theorem.

In view of describing cohomology rings of moduli spaces, Witten showed that integrals of cohomology classes on M_{red} can be computed as the value at 0 of the Fourier transform of a function on \mathfrak{k} , obtained by integrating an equivariant cohomology class on the manifold M , using a deformation argument. This result suggested that a similar deformation argument could be used to study the Fourier expansion of equivariant indices. Indeed, in [29], the first author gave another proof of the $[Q, R] = 0$ theorem, using deformation of the symbol σ_L . This proof is straightforward, once some functorial properties of the deformation are established. In this article, we describe functorial properties of this deformation procedure in a general framework and we obtain a localization formula associated to a map $\Phi : M \rightarrow \mathfrak{k}^*$ for the equivariant index of a general invariant elliptic operator on any K -manifold, not necessarily Hamiltonian. We indeed believe that this technique can be useful to study equivariant indices in many different contexts. We see, for example, that a version of $[Q, R] = 0$ theorem is true in this general context, in the asymptotic sense. Furthermore, we use the results of this article to prove a $[Q, R] = 0$ exact principle in the context of Spin^c -manifolds [32].

Let us now describe more precisely our context. Assume M is even dimensional and consider a graded Clifford bundle $\mathcal{E} \rightarrow M$ for the tangent bundle TM . We denote by $\mathbf{c}_{\mathcal{E}_m}$ the Clifford action of $T_m M$ on the fiber \mathcal{E}_m . The symbol $\sigma(\mathcal{E})$ attached to the graded Clifford bundle is

$$\sigma(\mathcal{E})(m, \nu) = \mathbf{c}_{\mathcal{E}_m}(\tilde{\nu}) : \mathcal{E}_m^+ \longrightarrow \mathcal{E}_m^-, \quad (m, \nu) \in T^*M,$$

where $\nu \in T^*M \mapsto \tilde{\nu} \in TM$ is an identification given by a K -invariant Riemannian structure on M . Note that such a symbol is invertible outside the zero section of T^*M . Hence, when M is compact, the symbol $\sigma(\mathcal{E})$ determines a class in the \mathbf{K} -group $\mathbf{K}_K^0(T^*M)$, and we denote $\mathcal{Q}_K(M, \mathcal{E})$ its equivariant index. If $K = \{1\}$ is the trivial group, $\mathcal{Q}_K(M, \mathcal{E}) \in \mathbb{Z}$ is an integer, denoted simply by $\mathcal{Q}(M, \mathcal{E})$.

We attach the Kirwan vector field κ_Φ to any equivariant map $\Phi : M \rightarrow \mathfrak{k}^*$: for $m \in M$, the vector $-\kappa_\Phi(m)$ is equal to the tangent vector $\Phi(m) \cdot m$ associated to the infinitesimal action of $\Phi(m)$ on m . Here we have identified \mathfrak{k} and \mathfrak{k}^* by choosing a K -invariant inner product. We denote by Z_Φ the set of zeroes of κ_Φ . Clearly Z_Φ contains $\Phi^{-1}(0)$. The deformed symbol $\sigma(\mathcal{E}, \Phi)$ is defined by

$$\sigma(\mathcal{E}, \Phi)(m, \nu) = \mathbf{c}_{\mathcal{E}_m}(\tilde{\nu} + \Phi(m) \cdot m) : \mathcal{E}_m^+ \longrightarrow \mathcal{E}_m^-.$$

The symbol $\sigma(\mathcal{E}, \Phi)$ can be restricted to a neighborhood of the set Z_Φ of zeroes of κ_Φ as a transversally elliptic operator. If Z is a union of connected components of Z_Φ , we denote by $\mathcal{Q}_K(M, \mathcal{E}, Z, \Phi)$ the equivariant index of the symbol $\sigma(\mathcal{E}, \Phi)$ restricted to a neighborhood of Z .

In this paper, an equivariant map $\Phi : M \rightarrow \mathfrak{k}^*$ is called¹ a “moment map” if there is a closed two form Ω (possibly degenerate) such that the data (Ω, Φ) satisfies the Kostant relations (6.14).

In Section 8, we prove the following non-abelian localization theorem (Theorem 8.6). When Φ is a moment map, the set $C_\Phi = \Phi(Z_\Phi)$ is a finite union of coadjoint orbits, and we define $Z_\mathcal{O} = Z_\Phi \cap \Phi^{-1}(\mathcal{O})$ for any $\mathcal{O} \subset C_\Phi$. We then have, for any K -equivariant graded Clifford bundle \mathcal{E} ,

$$(1.1) \quad \mathcal{Q}_K(M, \mathcal{E}) = \sum_{\mathcal{O} \subset C_\Phi} \mathcal{Q}_K(M, \mathcal{E}, Z_\mathcal{O}, \Phi).$$

When 0 is a regular value of Φ , the character $\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)$ can be described explicitly in terms of indices of elliptic symbols on the orbifold $M_{red} = \Phi^{-1}(0)/K$ (see Subsection 8.1). In particular we have that $[\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)]^K = \mathcal{Q}(M_{red}, \mathcal{E}_{red})$, where \mathcal{E}_{red} is a graded Clifford bundle on M_{red} induced by \mathcal{E} .

Let us describe the other components associated to $\mathcal{O} = K\beta$, with $\beta \neq 0$. Let M^β be the set of zeroes of the vector field β , and K_β the centralizer of β in K with Lie algebra \mathfrak{k}_β . The graded Clifford bundle \mathcal{E} on M induces a graded Clifford bundle $\mathfrak{d}_\beta(\mathcal{E})$ on M^β . The action of β provides a complex structure on the normal bundle to M^β , and we denote by \mathcal{N}_{J_β} this complex vector bundle. Let $\text{Sym}(\mathcal{N}_{J_\beta}) = \bigoplus_{k=0}^{\infty} \text{Sym}^k(\mathcal{N}_{J_\beta})$. Then $\mathcal{Q}_K(M, \mathcal{E}, Z_\mathcal{O}, \Phi)$ is the induced representation from K_β to K of the representation

$$\mathcal{Q}_{K_\beta}(M^\beta, \mathfrak{d}_\beta(\mathcal{E}) \otimes \text{Sym}(\mathcal{N}_{J_\beta}), \Phi^{-1}(\beta), \Phi|_{M^\beta}) \otimes \bigwedge (\mathfrak{k}/\mathfrak{k}_\beta)_\mathbb{C}.$$

¹It corresponds to the notion of “abstract moment map” of Karshon.

Here again, if β is a regular value of the map $\Phi : M^\beta \rightarrow \mathfrak{k}_\beta^*$, this index can be explicitly described in terms of $(\Phi^{-1}(\beta) \cap M^\beta)/K_\beta$.

Starting in Section 9, we analyze the implications of the non-abelian localization theorem to the geometric study of the multiplicities. Let us remark that we do not assume before Section 12 that nor our group K neither our manifold M are connected. Indeed the study of the ring of invariants for classical groups such as $O(n)$ is an instance of the $[Q, R] = 0$ theorem, and we want to include these cases.

We say that the $[Q, R] = 0$ principle of Guillemin-Sternberg holds for the data (\mathcal{E}, Φ) if the multiplicity of the trivial representation in the equivariant index $\mathcal{Q}_K(M, \mathcal{E}, Z_\mathcal{O}, \Phi)$ vanishes if $\mathcal{O} \neq \{0\}$. Thanks to (1.1), this implies the identity

$$[\mathcal{Q}_K(M, \mathcal{E})]^K = [\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)]^K,$$

and this last space is described explicitly as $\mathcal{Q}(M_{red}, \mathcal{E}_{red})$ when 0 is a regular value.

We introduce the notion of Φ -positivity for equivariant vector bundles and Clifford bundles. When the Clifford bundle \mathcal{E} is strictly Φ -positive, we show that the $[Q, R] = 0$ principle holds for the data (\mathcal{E}, Φ) : the multiplicity of trivial representation in the characters $\mathcal{Q}_K(M, \mathcal{E})$ and $\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)$ are equal. If furthermore 0 is a regular value of Φ , we get finally that

$$[\mathcal{Q}_K(M, \mathcal{E})]^K = \mathcal{Q}(M_{red}, \mathcal{E}_{red}).$$

In Section 10, we specialize the study to the case of a K -manifold M equipped with an invariant almost complex structure J . Then a natural Clifford bundle on M is $\bigwedge_J TM$, and we consider the Riemann-Roch character

$$RR_K^J(M, \mathcal{F}) := \mathcal{Q}_K(M, \bigwedge_J TM \otimes \mathcal{F})$$

that is attached to any equivariant vector bundle \mathcal{F} .

When Φ is a moment map associated to a closed 2-form Ω , we show that the Clifford bundle $\bigwedge_J TM$ is weakly Φ -positive when the quadratic form $v \mapsto \Omega(v, Jv)$ is non negative (we say that J is adapted to Ω). Hence the Clifford bundle $\mathcal{E} := \bigwedge_J TM \otimes \mathcal{F}$ is strictly Φ -positive when J is adapted to Ω , and the vector bundle \mathcal{F} is strictly Φ -positive.

An interesting example are the *semi-ample* line bundles L on (M, J) : we define those as line bundles L for which there exists a K -invariant Hermitian connection ∇^L with curvature $(\nabla^L)^2 = i\Omega_L$ such that $\Omega_L(v, Jv)$ is semi-positive definite. If Φ_L is the moment map attached to ∇^L , we see then that

the Clifford bundle $\bigwedge_J TM \otimes L$ is strictly Φ_L -positive. In particular we get that $[RR_K^J(M, L^{\otimes k})]^K = 0$ for any $k \geq 1$ if $\Phi_L^{-1}(0) = \emptyset$.

In Section 12, we return to the case of a K -Hamiltonian manifold (M, Ω) with a Kostant-Souriau line bundle L and moment map Φ . Here we choose an almost complex structure J that is compatible with the non degenerate 2-form Ω : here the quadratic form $v \mapsto \Omega(v, Jv)$ is positive definite. The corresponding quantized space $RR_K(M, L) = \sum_{\mu \in \hat{K}} m_\mu(L) V_\mu$ is a representation of K . As we have already seen in the case of semi-ample line bundles, the number $m_0(L) = [RR_K(M, L)]^K$ is equal to

$$(1.2) \quad [\mathcal{Q}_K(M, \bigwedge_J TM \otimes L, \Phi^{-1}(0), \Phi)]^K.$$

If 0 is regular, we thus obtain that $[RR_K(M, L)]^K$ is the Riemann-Roch number $RR(M_0, L_0)$ of the orbifold bundle $L_0 = L|_{\Phi^{-1}(0)}/K$ on the reduced symplectic orbifold $M_0 = \Phi^{-1}(0)/K$. Recall that this striking result was obtained by Meinrenken [26] using symplectic cutting. The core of Meinrenken-Sjamaar fundamental theorem is that (1.2) can be computed geometrically as the Riemann-Roch number of the (possibly singular) reduced space $M_0 = \Phi^{-1}(0)/K$:

- Either directly when 0 is a quasi-regular value of Φ . The computation², which is quite involved when 0 is quasi-regular, but not regular, is done in Section 12.2.3. In this case, the reduced space M_0 is again an orbifold.
- Or by moving the localization from $\Phi^{-1}(0)$ to the fiber of a quasi-regular point close enough to 0. This deformation argument is explained in Section 12.4.

Up to this point, our compact group K could be not connected. As usual, for K connected, one deals with the other multiplicities $m_\mu(L)$ by using the shifting trick.

At the end of the paper we state results on the locally quasi-polynomial behavior of the coefficients $m_\mu(L)$, as well as their “reduction behavior” on faces. Indeed, it seems that these consequences of the $[Q, R] = 0$ theorem were unnoticed, and other interesting proofs were given for interesting cases by several authors [21, 35, 13]. For example, the quasi polynomial behavior in k of stretched Littlewood-Richardson coefficients [13], or of Kronecker coefficients [28], the reduction properties of these coefficients on faces [35], the factorization theorem of LR coefficients [19], the locally quasi-polynomial behavior of plethysm coefficients [17], are consequences of the geometric description of multiplicities obtained by Meinrenken-Sjamaar.

²The quasi-regular case was not considered in [29].

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2 Notations

- Let K be a compact (non necessarily connected) Lie group with Lie algebra \mathfrak{k} . We denote by \hat{K} the set of isomorphism classes of irreducible complex finite dimensional representations of K . If $\mu \in \hat{K}$, we may also denote μ by V_μ (a vector space with an action of K). If $\mu \in \hat{K}$, we denote by $\mu^* \in \hat{K}$ the dual representation of K in $V_{\mu^*} = V_\mu^*$.

Any finite dimensional representation E of K is isomorphic to a direct sum $E = \bigoplus_{\mu \in \hat{K}} m(\mu) V_\mu$, and we say that $m(\mu)$ is the multiplicity of μ in E . We denote by $[E]^K$ the subspace of E formed by the K -fixed vectors. Thus $m(\mu) = \dim[E \otimes V_\mu^*]^K$.

If H is a subgroup of K , we denote by $E|_H$ the space E with the action of H .

- We denote by $R(K)$ the representation ring of K : an element $E \in R(K)$ can be represented as a finite sum $E = \sum_{\mu \in \hat{K}} m(\mu) V_\mu$, with $m(\mu) \in \mathbb{Z}$. We also say that E is a virtual representation of K . The ring structure on $R(K)$ is induced by the tensor product and we denote by $\chi \otimes \mu$ the product. If H is a subgroup of K , we denote by $\chi \rightarrow \chi|_H$ the restriction $R(K) \rightarrow R(H)$.

We denote by $\hat{R}(K)$ the space of \mathbb{Z} -valued functions on \hat{K} . An element $E \in \hat{R}(K)$ can be represented as an infinite sum $E = \sum_{\mu \in \hat{K}} m(\mu) V_\mu$, with $m(\mu) \in \mathbb{Z}$. We say that $m(\mu)$ is the multiplicity of μ in E . Then $[E]^K \in \mathbb{Z}$ and is the multiplicity of the trivial representation in E .

The space $\hat{R}(K)$ is a $R(K)$ -module. If H is a closed subgroup of K , the induction map $\text{Ind}_H^K : \hat{R}(H) \rightarrow \hat{R}(K)$ is the dual of the restriction morphism $R(K) \rightarrow R(H)$. If $\chi \in \hat{R}(H)$, then $\text{Ind}_H^K(\chi) = \sum_{\lambda \in \hat{K}} [\chi \otimes V_\lambda^*|_H]^H V_\lambda$. In particular $[\text{Ind}_H^K(E)]^K = [E]^H$.

- We consider a positive definite K invariant scalar product on \mathfrak{k} . This defines an isomorphism between \mathfrak{k} and its dual vector space \mathfrak{k}^* .

- If \mathfrak{s} is a subalgebra of \mathfrak{k} , we denote by (\mathfrak{s}) the conjugacy class of \mathfrak{s} in the set of subalgebras of \mathfrak{k} . When K acts on a set \mathcal{X} , the stabilizer subgroup of $x \in \mathcal{X}$ is denoted $K_x := \{k \in K \mid k \cdot x = x\}$. The Lie algebra of K_x is denoted by \mathfrak{k}_x . We denote by $\mathcal{X}_\mathfrak{s}$ the set of elements x of \mathcal{X} such $\mathfrak{k}_x = \mathfrak{s}$, and by $\mathcal{X}_{(\mathfrak{s})} = K\mathcal{X}_\mathfrak{s}$ the set of elements x of \mathcal{X} such $\mathfrak{k}_x \in (\mathfrak{s})$.

When K acts on a manifold M , we denote by $X_M(m) := \frac{d}{dt}|_{t=0} e^{-tX} \cdot m$ the vector field generated by $-X \in \mathfrak{k}$. We may denote also $X_M(m)$ by $-X \cdot m$. We denote by M^X the set of zeroes of the vector field X_M . This is a smooth manifold.

In particular, if $b \in \mathfrak{k}$ and E is a representation space of K , then E^b is the subspace of vectors in E invariant by the one parameter subgroup $\exp tb$ of K .

- When V is a real vector space, we equip the direct sum $V \oplus V^*$ with the symplectic structure Ω defined by : $\Omega(v, v') = \Omega(f, f') = 0$ and $\Omega(v, f) = \langle f, v \rangle$, for $v, v' \in V$ and $f, f' \in V^*$. An Euclidean structure on V induces a isomorphism $v \in V \mapsto \tilde{v} \in V^*$, and a complex structure J_V on $V \oplus V^*$ defined by $J_V v = \tilde{v}$. Note that $\Omega(-, J_V -)$ is the scalar product on $V \oplus V^*$.

- Let $\mathcal{V} \rightarrow M$ be a vector bundle. An element n of \mathcal{V} may be written as (m, v) with $m \in M, v \in \mathcal{V}|_m$.

- The letter E will denote a complex finite dimensional (eventually Hermitian) vector space, and $\mathcal{E} \rightarrow M$ a complex vector bundle. The space of smooth sections of \mathcal{E} is denoted by $\Gamma(M, \mathcal{E})$.

Most of the times, the space E will be $\mathbb{Z}/2\mathbb{Z}$ graded, $E = E^+ \oplus E^-$. We may leave the $\mathbb{Z}/2\mathbb{Z}$ grading implicit. We denote by E^{op} the shift of parity on E : we interchange E^+ and E^- . Ungraded spaces N will be considered as graded, with $N^+ = N$ and $N^- = \{0\}$. Thus $(-1)^k E$ denotes the same graded vector space E , but where the change of parity has been performed k times.

If N is a complex (ungraded) vector space, then the exterior space $\bigwedge N$ will be $\mathbb{Z}/2\mathbb{Z}$ graded in even and odd elements. If $E = E^+ \oplus E^-$ and $F = F^+ \oplus F^-$ are two $\mathbb{Z}/2\mathbb{Z}$ graded Hermitian vector spaces, the tensor product $E \otimes F$ is $\mathbb{Z}/2\mathbb{Z}$ -graded with $(E \otimes F)^+ = E^+ \otimes F^+ \oplus E^- \otimes F^-$ and $(E \otimes F)^- = E^- \otimes F^+ \oplus E^+ \otimes F^-$. If $f \in \text{hom}(E^+, E^-)$, $g \in \text{hom}(F^+, F^-)$, we define $f \boxtimes g \in \text{hom}((E \otimes F)^+, (E \otimes F)^-)$ by

$$(2.3) \quad f \boxtimes g = \begin{pmatrix} f \otimes 1_{F^+} & -1_{E^-} \otimes g^* \\ 1_{E^+} \otimes g & f^* \otimes 1_{F^-} \end{pmatrix}.$$

- If M is a K -manifold, and E a representation space for K , we denote by $[E]$ the trivial vector bundle $M \times E$, with diagonal action of K .

3 Elliptic and transversally elliptic symbols

In this section, we recall the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1].

Let M be a compact K -manifold with cotangent bundle T^*M . Let $p : T^*M \rightarrow M$ be the projection. If \mathcal{E} is a vector bundle on M , we may denote still by \mathcal{E} the vector bundle $p^*\mathcal{E}$ on the cotangent bundle T^*M . If $\mathcal{E}^+, \mathcal{E}^-$ are K -equivariant complex vector bundles over M , a K -equivariant morphism $\sigma \in \Gamma(T^*M, \text{hom}(\mathcal{E}^+, \mathcal{E}^-))$ is called a *symbol* on M . For $x \in M$, and $\nu \in T_x^*M$, thus $\sigma(x, \nu) : \mathcal{E}|_x^+ \rightarrow \mathcal{E}|_x^-$ is a linear map. The subset of all $(x, \nu) \in T^*M$ where the map $\sigma(x, \nu)$ is not invertible is called the *characteristic set* of σ , and is denoted by $\text{Char}(\sigma)$. A symbol is elliptic if its characteristic set is compact.

Define the exterior product of symbols. Let M_1, M_2 be two manifolds with action of the compact groups K_1, K_2 and let $M = M_1 \times M_2$. We write an element of $T^*M = T^*M_1 \times T^*M_2$ as $((x_1, \nu_1), (x_2, \nu_2))$. Let $f \in \Gamma(T^*M_1, \text{hom}(\mathcal{E}^+, \mathcal{E}^-))$ and $g \in \Gamma(T^*M_2, \text{hom}(\mathcal{F}^+, \mathcal{F}^-))$ be two symbols. Using K -invariant Hermitian metrics on the bundles $\mathcal{E}^\pm, \mathcal{F}^\pm$ we define the symbol $f \boxtimes g \in \Gamma(T^*M, (\mathcal{E} \otimes \mathcal{F})^+, (\mathcal{E} \otimes \mathcal{F})^-)$ by

$$(f \boxtimes g)((x_1, \nu_1), (x_2, \nu_2)) = f(x_1, \nu_1) \boxtimes g(x_2, \nu_2).$$

The symbol $f \boxtimes g$ is called the exterior product of the symbols f, g . Note that $\text{Char}(f \boxtimes g) = \text{Char}(f) \times \text{Char}(g)$.

Consider now $M_1 = M_2 = M$ with an action of K . Let f, g be two symbols on M . We still denote by $f \boxtimes g$ the restriction of the exterior product (a symbol on $M \times M$) to M embedded as the diagonal in $M \times M$.

The product of a symbol σ by a K -equivariant complex vector bundle $\mathcal{F} \rightarrow M$ is the symbol $\sigma \otimes \mathcal{F}$ defined by

$$(\sigma \otimes \mathcal{F})(x, \nu) = \sigma(x, \nu) \otimes 1_{\mathcal{F}_x}.$$

If $[E]$ is the vector bundle $M \times E$ associated to a representation space E of K , we denote $\sigma \otimes [E]$ simply by $\sigma \otimes E$.

If $E = E^+ \oplus E^-$ is $\mathbb{Z}/2\mathbb{Z}$ graded, then

$$\sigma \otimes E = \begin{pmatrix} \sigma \otimes 1_{E^+} & 0 \\ 0 & \sigma^* \otimes 1_{E^-} \end{pmatrix}.$$

An elliptic symbol σ on M defines an element $[\sigma]$ in the equivariant \mathbf{K} -theory of T^*M with compact support, which is denoted by $\mathbf{K}_K^0(T^*M)$. The class $[\sigma^*]$ is the inverse of $[\sigma]$ in \mathbf{K} -theory: $[\sigma^*] = -[\sigma]$. The index of σ is a virtual finite dimensional representation of K , that we denote by $\text{Index}_K^M(\sigma) \in R(K)$ [3, 4, 5, 6]. Recall its definition. Let A be a pseudo-differential operator $A : \Gamma(M, \mathcal{E}^+) \rightarrow \Gamma(M, \mathcal{E}^-)$ from the space $\Gamma(M, \mathcal{E}^+)$ of

smooth sections of \mathcal{E}^+ to the space $\Gamma(M, \mathcal{E}^-)$ of smooth sections of \mathcal{E}^- , which commutes with the action of K . To the pseudo-differential operator A , one associates its principal *symbol* σ_p which is defined outside the zero section of T^*M . The operator A is said to be elliptic if its principal symbol $\sigma_p(x, \nu)$ is invertible for all $(x, \nu) \in T^*M$ such that $\nu \neq 0$. Then the space $\ker(A)$ of smooth solutions of A is a finite dimensional representation space for K . Using a K -invariant function χ on T^*M identically equal to 1 in a neighborhood of M and compactly supported, then $\sigma(x, \nu) := (1 - \chi(x, \nu))\sigma_p(x, \nu)$ is defined on the whole space T^*M and is an elliptic symbol. The class of σ in $\mathbf{K}_K^0(T^*M)$ does not depend of the choice of χ . We still say that this class σ is the symbol of A . We choose K -invariant Hermitian structures on $\mathcal{E}^+, \mathcal{E}^-$. Then the formal adjoint $A^* : \Gamma(M, \mathcal{E}^-) \rightarrow \Gamma(M, \mathcal{E}^+)$ is also an elliptic pseudodifferential operator. The index of A is the virtual representation space $\ker(A) - \ker(A^*) \in R(K)$. This virtual representation *depends only of the class of the symbol* σ of A in $\mathbf{K}_K^0(T^*M)$, and by definition,

$$\text{Index}_K^M(\sigma) = \ker(A) - \ker(A^*).$$

Recall the notion of *transversally elliptic symbol*. Let T_K^*M be the following K -invariant closed subset of T^*M

$$T_K^*M = \{(x, \nu) \in T^*M, \langle \nu, X \cdot x \rangle = 0 \text{ for all } X \in \mathfrak{k}\}.$$

Its fiber over a point $x \in M$ is formed by all the cotangent vectors $\nu \in T_x^*M$ which vanish on the tangent space to the orbit of x under K , in the point x . Thus each fiber $(T_K^*M)_x$ is a linear subspace of T_x^*M . In general the dimension of $(T_K^*M)_x$ is not constant and this space is not a vector bundle. A symbol σ is *K -transversally elliptic* if the restriction of σ to T_K^*M is invertible outside a compact subset of T_K^*M (i.e. $\text{Char}(\sigma) \cap T_K^*M$ is compact).

A *K -transversally elliptic* symbol σ defines an element of $\mathbf{K}_K^0(T_K^*M)$, and the index of σ defines an element $\text{Index}_K^M(\sigma)$ of $\hat{R}(K)$. Recall its definition. A transversally elliptic operator $A : \Gamma(M, \mathcal{E}^+) \rightarrow \Gamma(M, \mathcal{E}^-)$ is a pseudo-differential operator which is *elliptic* in the directions transversal to the orbits of K : its principal symbol $\sigma_p(x, \nu)$ is invertible for all $(x, \nu) \in T_K^*M$ such that $\nu \neq 0$. Thus, by restriction to T_K^*M , the symbol $\sigma(x, \nu) = (1 - \chi(x, \nu))\sigma_p(x, \nu)$ defines a K -theory class³ $[\sigma]$ in the topological equivariant K -theory group $\mathbf{K}_K^0(T_K^*M)$ which does not depend of

³To simplify the notations, we will also denote $\sigma \in \mathbf{K}_K^0(T_K^*M)$ the class defined by a transversally elliptic symbol σ .

the choice of χ . The *index map* associates to a transversally elliptic operator A an element of $\hat{R}(K)$ constructed as follows. For every $\mu \in \hat{K}$, the space $\text{hom}_K(V_\mu, \ker(A))$ is a finite dimensional vector space of dimension $m(V_\mu, A)$. Thus $m(V_\mu, A)$ is the multiplicity of V_μ in the space $\ker(A)$ of smooth solutions of A . The number $m(V_\mu, A) - m(V_\mu, A^*)$ depends only of the class of the symbol σ in $\mathbf{K}_K^0(\mathbb{T}_K^*M)$ and by definition

$$\text{Index}_K^M(\sigma) = \sum_{\mu \in \hat{K}} (m(V_\mu, A) - m(V_\mu, A^*))V_\mu.$$

In particular two symbols σ_1 and σ_2 having the same restriction to \mathbb{T}_K^*M have the same index.

Remark 3.1 *By definition, $\text{Index}_K^M(\sigma)$ is isomorphic to a graded subspace of smooth (and K -finite) sections of \mathcal{E} , and we will not consider how to complete $\text{Index}_K^M(\sigma)$ as an Hilbert space.*

Example 3.2 *A trivial, but important, example is the following. Consider $M = K$, with action of $K \times K$ by left and right translations, $\mathcal{E}^+ = M \times \mathbb{C}$ the trivial vector bundle, $\mathcal{E}^- = M \times \{0\}$. Then $\Gamma(M, \mathcal{E}^+) = C^\infty(K)$, $\Gamma(M, \mathcal{E}^-) = 0$, and the operator $A = 0$ on $C^\infty(K)$ is transversally elliptic. Its symbol is the symbol $\sigma_0(k, \nu) = 0$, and Peter-Weyl theorem shows that*

$$\text{Index}_{K \times K}^M(\sigma_0) = \sum_{\mu \in \hat{K}} V_\mu^* \otimes V_\mu.$$

The index map $\text{Index}_K^M : \mathbf{K}_K^0(\mathbb{T}_K^*M) \rightarrow \hat{R}(K)$ is a morphism of $R(K)$ module : for E a representation space of K ,

$$(3.4) \quad \text{Index}_K^M(\sigma \otimes E) = \text{Index}_K^M(\sigma) \otimes E.$$

Any elliptic symbol is K -transversally elliptic, hence we have a restriction map $\mathbf{K}_K^0(\mathbb{T}^*M) \rightarrow \mathbf{K}_K^0(\mathbb{T}_K^*M)$, and a commutative diagram

$$(3.5) \quad \begin{array}{ccc} \mathbf{K}_K^0(\mathbb{T}^*M) & \longrightarrow & \mathbf{K}_K^0(\mathbb{T}_K^*M) \\ \text{Index}_K^M \downarrow & & \downarrow \text{Index}_K^M \\ R(K) & \longrightarrow & \hat{R}(K) . \end{array}$$

Using the *excision property*, one can easily show that the index map $\text{Index}_K^M : \mathbf{K}_K^0(\mathbb{T}_K^*\mathcal{U}) \rightarrow \hat{R}(K)$ is still defined when \mathcal{U} is a K -invariant relatively compact open subset of a K -manifold (see [29][section 3.1]).

Finally the index map $\text{Index}_K^M : \mathbf{K}_K^0(\mathbb{T}_K^*M) \rightarrow \hat{R}(K)$ can be still defined when M is a *non-compact* manifold. Any class $\sigma \in \mathbf{K}_K^0(\mathbb{T}_K^*M)$ is represented by a symbol on M with a characteristic set $\text{Char}(\sigma) \subset \mathbb{T}^*M$ intersecting \mathbb{T}_K^*M in a compact set. Let \mathcal{U} be a K -invariant relatively compact open subset of M such that $\text{Char}(\sigma) \cap \mathbb{T}_K^*M \subset \mathbb{T}^*\mathcal{U}$. The restriction $\sigma|_{\mathcal{U}}$ defines a K -transversally elliptic symbol on \mathcal{U} , and we take

$$\text{Index}_K^M(\sigma) := \text{Index}_K^{\mathcal{U}}(\sigma|_{\mathcal{U}}).$$

Remark : In the following, the manifold M will carry a K -invariant Riemannian metric and we will denote by $\nu \in \mathbb{T}^*M \mapsto \tilde{\nu} \in \mathbb{T}M$ the corresponding identification.

4 Functoriality

4.1 Invariant part

We will use the following obvious remark. Let $\sigma \in \Gamma(\mathbb{T}^*M, \text{hom}(\mathcal{E}^+, \mathcal{E}^-))$ be a K -transversally elliptic symbol on M . We assume that an invariant element $b \in \mathfrak{k}$ acts trivially on M . We can then define the subbundles $[\mathcal{E}^\pm]^b \subset \mathcal{E}^\pm$ and the K -transversally elliptic symbol $\sigma^b \in \Gamma(\mathbb{T}^*M, \text{hom}([\mathcal{E}^+]^b, [\mathcal{E}^-]^b))$ which is induced by σ .

Lemma 4.1 *We have $[\text{Index}_K^M(\sigma)]^K = [\text{Index}_K^M(\sigma^b)]^K$. In particular $[\text{Index}_K^M(\sigma)]^K = 0$ if $[\mathcal{E}^\pm]^b = 0$.*

Proof. We use a K -equivariant decomposition of vector bundles $\mathcal{E}^\pm = [\mathcal{E}^\pm]^b \oplus \mathcal{F}^\pm$. The symbol σ is thus equal to $\sigma^b \oplus \sigma'$, where the symbol $\sigma' \in \Gamma(\mathbb{T}^*M, \text{hom}(\mathcal{F}^+, \mathcal{F}^-))$ is also K -transversally elliptic. The index map is additive so $[\text{Index}_K^M(\sigma)]^K = [\text{Index}_K^M(\sigma^b)]^K + [\text{Index}_K^M(\sigma')]^K$. The space $[\text{Index}_K^M(\sigma')]^K$ is constructed as the (virtual) subspace of invariant C^∞ -sections of the bundle \mathcal{F}^\pm which are solutions of a K -invariant pseudo-differential operator on M with symbol σ' . But, as the action of b is trivial on the basis, and $[\mathcal{F}^\pm]^b = \{0\}$, the space of b -invariant C^∞ -sections of the bundle \mathcal{F}^\pm is reduced to 0. Hence $[\text{Index}_K^M(\sigma')]^K = 0$.

4.2 Free action

Let G and K be two compact Lie groups. Let P be a manifold provided with an action of $G \times K$. We assume that the action of K is free. Then the manifold $M := P/K$ is provided with an action of G and the quotient map

$\pi : P \rightarrow M$ is a G -equivariant fibration. The action of K on the bundle \mathbb{T}_K^*P is free and the quotient $(\mathbb{T}_K^*P)/K$ admits a canonical identification with \mathbb{T}^*M . Then we still denote by

$$q : \mathbb{T}_K^*P \rightarrow \mathbb{T}^*M$$

the quotient map by K . The quotient map q induces an isomorphism

$$q^* : \mathbf{K}_G^0(\mathbb{T}_G^*M) \rightarrow \mathbf{K}_{G \times K}^0(\mathbb{T}_{G \times K}^*P)$$

that makes $\mathbf{K}_G^0(\mathbb{T}_G^*M)$ into a $R(K)$ -module. For V_μ an irreducible representation of K , we denote by \mathcal{V}_μ the complex vector bundle $P \times_K V_\mu$ over M .

Theorem 4.2 (Free action property) *We have the following equality in $\hat{R}(G \times K)$*

$$\text{Index}_{G \times K}^P(q^*\sigma) = \sum_{\mu \in \hat{K}} \text{Index}_G^M(\sigma \otimes \mathcal{V}_\mu^*) \otimes V_\mu.$$

4.3 Multiplicative property

Let us recall the multiplicative property of the index for the product of manifolds. Consider a compact Lie group G_2 acting on two manifolds M_1 and M_2 , and assume that another compact Lie group G_1 acts on M_1 commuting with the action of G_2 . The external product of symbols induces a multiplication (see (2.3)):

$$\boxtimes : \mathbf{K}_{G_1 \times G_2}^0(\mathbb{T}_{G_1}^*M_1) \times \mathbf{K}_{G_2}^0(\mathbb{T}_{G_2}^*M_2) \longrightarrow \mathbf{K}_{G_1 \times G_2}^0(\mathbb{T}_{G_1 \times G_2}^*(M_1 \times M_2)).$$

Since $\mathbb{T}_{G_1 \times G_2}^*(M_1 \times M_2) \neq \mathbb{T}_{G_1}^*M_1 \times \mathbb{T}_{G_2}^*M_2$ in general, the definition of the product $[\sigma_1] \boxtimes [\sigma_2]$ of transversally elliptic symbols needs some care: we have to take a representative σ_2 of $[\sigma_2]$ which is almost homogeneous.

Theorem 4.3 (Multiplicative property) *For any $\sigma_1 \in \mathbf{K}_{G_1 \times G_2}^0(\mathbb{T}_{G_1}^*M_1)$ and any $\sigma_2 \in \mathbf{K}_{G_2}^0(\mathbb{T}_{G_2}^*M_2)$, we have*

$$\text{Index}_{G_1 \times G_2}^{M_1 \times M_2}(\sigma_1 \boxtimes \sigma_2) = \text{Index}_{G_1 \times G_2}^{M_1}(\sigma_1) \text{Index}_{G_2}^{M_2}(\sigma_2).$$

In the last theorem, the product of $\text{Index}_{G_1 \times G_2}^{M_1}(\sigma_1) \in \hat{R}(G_1 \times G_2)$ with $\text{Index}_{G_2}^{M_2}(\sigma_2) \in \hat{R}(G_2)$ is well defined since $\text{Index}_{G_1 \times G_2}^{M_1}(\sigma_1)$ is *smooth* relatively to G_2 .

4.4 Fiber product

Let $\mathcal{V} \rightarrow M$ be a K -equivariant Euclidean vector bundle.

Suppose that a central connected subgroup $\mathbb{T} \subset K$ acts trivially on M , so the torus \mathbb{T} acts linearly on the fibers of \mathcal{V} . Let $V := \mathcal{V}|_{m_o}$ be a fiber which is equipped with a linear action of \mathbb{T} , and an \mathbb{T} -invariant scalar product. We denote by $O \subset O(V)$ the group formed by orthogonal endomorphisms of V commuting with the \mathbb{T} -action. Let $P_O \rightarrow M$ be the O -principal bundle formed by the orthogonal linear maps $f : V \rightarrow \mathcal{V}|_m$ commuting with the \mathbb{T} -action. Then P_O is a $K \times O$ manifold with free action of O .

Remark 4.4 *If $\mathcal{V} \rightarrow M$ is a K -equivariant Hermitian vector bundle, we consider the group $U \subset U(V)$ formed by the unitary endomorphisms of V commuting with the \mathbb{T} -action, and the U -principal bundle $P_U \rightarrow M$ formed by the unitary linear maps $f : V \rightarrow \mathcal{V}|_m$ commuting with the \mathbb{T} -action.*

We have two principal fibrations : $q_M : P_O \rightarrow M := P_O/O$ and $q_V : P_O \times V \rightarrow \mathcal{V} := (P_O \times V)/O$. These quotient maps induce the following isomorphisms : $q_M^* : \mathbf{K}_K^0(\mathbb{T}_K^* M) \rightarrow \mathbf{K}_{K \times O}^0(\mathbb{T}_{K \times O}^* P_O)$ and $q_V^* : \mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{V}) \rightarrow \mathbf{K}_{K \times O}^0(\mathbb{T}_{K \times O}^*(P_O \times V))$.

Let us use the multiplicative property (see Section 4.3) with the groups $G_2 = K \times O, G_1 = \mathbb{T}$ acting on the manifolds $M_1 = V, M_2 = P_O$ (the subgroup K of G_2 acting trivially on V).

We have a product

$$\boxtimes : \mathbf{K}_{K \times O}^0(\mathbb{T}_{K \times O}^* P_O) \times \mathbf{K}_{\mathbb{T} \times O}^0(\mathbb{T}_{\mathbb{T}}^* V) \longrightarrow \mathbf{K}_{\mathbb{T} \times K \times O}^0(\mathbb{T}_{\mathbb{T} \times K \times O}^*(P_O \times V)).$$

Let us consider the isomorphism

$$(4.6) \quad q_V^* : \mathbf{K}_{\mathbb{T} \times K}^*(\mathbb{T}_{\mathbb{T} \times K}^* \mathcal{V}) \rightarrow \mathbf{K}_{\mathbb{T} \times K \times O}^*(\mathbb{T}_{\mathbb{T} \times K \times O}^*(P_O \times V)).$$

Since the action of \mathbb{T} on \mathcal{V} comes from the K -action, we have $\mathbb{T}_{\mathbb{T} \times K}^* \mathcal{V} = \mathbb{T}_K^* \mathcal{V}$, and then we can define a forgetful map $F : \mathbf{K}_{\mathbb{T} \times K}^0(\mathbb{T}_K^* \mathcal{V}) \rightarrow \mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{V})$.

Definition 4.5 *The fiber product*

$$(4.7) \quad \diamond : \mathbf{K}_K^0(\mathbb{T}_K^* M) \times \mathbf{K}_{\mathbb{T} \times O}^0(\mathbb{T}_{\mathbb{T}}^* V) \longrightarrow \mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{V})$$

is defined by the relation

$$\sigma \diamond \tau := F \circ (q_V^*)^{-1} (q_M^*(\sigma) \boxtimes \tau).$$

A symbol representing $\sigma \diamond \tau$ can be constructed as follows. We write an element $n \in \mathcal{V}$ as (x, v) with $x \in M$ and $v \in \mathcal{N}|_x$. We choose a K -invariant connection on the bundle $\mathcal{N} \rightarrow M$, which allows us to parameterize $\mathbb{T}^*\mathcal{N} = \{(n, (\xi, \nu))\}$, with $\xi \in \mathbb{T}_x^*M$, $\nu \in \mathcal{N}^*|_x$ (here ξ is lifted to an element of $\mathbb{T}_n^*\mathcal{N}$ using the connection). Let $\sigma(x, \xi)$ be a symbol on M representing $[\sigma]$ and $\tau(v, \eta)$ an almost homogeneous symbol on V representing τ . Then

$$(\sigma \diamond \tau)((x, v), (\xi, \nu)) = \sigma(x, \xi) \boxtimes \tau(v, \eta).$$

If E is a representation space of the group O , we denote by $[[E]]$ the vector bundle $P_O \times_O E$ over M , and by $\sigma \otimes [[E]]$ the symbol σ twisted by the vector bundle $[[E]]$.

Consider $\text{Index}_O^V(\tau)$ as a representation space for O . As τ is \mathbb{T} -transversally elliptic, we have $\text{Index}_O^V(\tau) = \sum_{\alpha \in \hat{\mathbb{T}}} E_\alpha$ where each E_α is a finite dimensional representation of O where the action of \mathbb{T} on E_α is scalar and acts via the character α . We still denote by $[[\text{Index}_O^V(\tau)]]$ the sum $\sum_{\alpha \in \hat{\mathbb{T}}} [[E_\alpha]]$ of finite dimensional K -vector bundles on M .

If we combine Theorems 4.2 and 4.3, we get

Lemma 4.6 $\text{Index}_K^V(\sigma \diamond \tau)$ is equal to $\text{Index}_K^M(\sigma \otimes [[\text{Index}_O^V(\tau)]])$ in $\hat{R}(K)$.

Let us explain briefly why $\text{Index}_K^M(\sigma \otimes [[\text{Index}_O^V(\tau)]])$ is well defined in $\hat{R}(K)$. By definition $\text{Index}_K^M(\sigma \otimes [[\text{Index}_O^V(\tau)]])$ is equal to the infinite sum

$$(4.8) \quad \sum_{\alpha \in \hat{\mathbb{T}}} \text{Index}_K^M(\sigma \otimes [[E_\alpha]]).$$

Since \mathbb{T} acts trivially on M , and commutes with K , it is not hard to see that for each $\mu \in \hat{K}$ the multiplicity of V_μ in $\text{Index}_K^M(\sigma \otimes [[E_\alpha]])$ is zero when $\|\alpha\|$ is large enough. This shows that the sum (4.8) converges in $\hat{R}(K)$.

An interesting case is already when the group \mathbb{T} is trivial (see Section 5.4). Any elliptic element $\tau \in \mathbf{K}_{O(V)}^0(\mathbb{T}^*V)$ defines a pushforward map $\mathbf{K}_K^0(\mathbb{T}_K^*M) \rightarrow \mathbf{K}_K^0(\mathbb{T}_K^*\mathcal{V})$, $\sigma \mapsto \sigma \diamond \tau$ such that $\text{Index}_K^V(\sigma \diamond \tau) = \text{Index}_K^M(\sigma \otimes [[\text{Index}_{O(V)}^V(\tau)]])$.

4.5 Induction property of the index

Let Y be a H -manifold and let $M := K \times_H Y$. Since, at the level of the tangent spaces, we have the decomposition $\text{TM}|_Y \simeq \text{TY} \oplus Y \times \mathfrak{k}/\mathfrak{h}$, one sees that \mathbb{T}_K^*M is diffeomorphic to $K \times_H \mathbb{T}_H^*Y$. Hence we have an isomorphism $\mathbf{K}_K^0(\mathbb{T}_K^*M) \simeq \mathbf{K}_H^0(\mathbb{T}_H^*Y)$: if σ is a K -transversally elliptic symbol on M , we denoted $\sigma|_Y$ the corresponding H -transversally elliptic symbol on Y .

Proposition 4.7 *For any K -transversally elliptic symbol σ on M*

$$\text{Index}_K^M(\sigma) = \text{Ind}_H^K(\text{Index}_H^Y(\sigma|_Y)).$$

Proof. Let us consider $N = K \times Y$ with action of H on $K \times Y$ by $(k, y) \cdot h = (kh^{-1}, hy)$. Then $T^*N = T^*K \times T^*Y$, and σ defines a $K \times H$ transversally elliptic symbol $\tilde{\sigma}$ on N . The restriction of the symbol $\tilde{\sigma}$ to $T_{K \times H}^*N = (K \times \{0\}) \times T_H^*Y$ is equal to $\sigma_0 \boxtimes \sigma_Y$, where σ_0 is the symbol of the zero operator on $C^\infty(K) \simeq \sum_{\lambda \in \hat{K}} V_\lambda \otimes V_\lambda^*$. Here we consider $C^\infty(K)$ with left action of K and right action of H . Thus in $\hat{R}(K) \otimes \hat{R}(H)$, we have

$$\text{Index}_{K \times H}^N(\tilde{\sigma}) = \text{Index}_{K \times H}^N(\sigma_0 \boxtimes \sigma_Y) = \sum_{\lambda \in \hat{K}} V_\lambda \otimes (V_\lambda^*|_H \otimes \text{Index}_H^Y(\sigma|_Y)).$$

The index of σ is obtained by taking the H -invariant part of $\text{Index}_{K \times H}^N(\tilde{\sigma})$. We thus obtain the proposition.

Conversely, a H -equivariant symbol σ_Y on Y allows to define an induced K -equivariant symbol σ on $K \times_H Y$ with restriction σ_Y , via $\sigma(y, q, \eta) = \sigma(y, \eta)$ on $T^*M|_Y \simeq \mathfrak{q}^* \oplus T_y^*Y$. Then we have

$$\text{Index}_K^M(\sigma) = \text{Ind}_H^K(\text{Index}_H^Y(\sigma_Y)).$$

5 Clifford bundles and Dirac operators

5.1 Clifford algebra

Let V be an Euclidean space. We denote by $\text{Cl}(V)$ its Clifford algebra. We follow the conventions of [8]. Denote by (v, w) the scalar product of two vectors v, w of V . Then, in $\text{Cl}(V)$, $vw + wv = -2(v, w)$. In particular $v^2 = -\|v\|^2$ for any $v \in V \subset \text{Cl}(V)$.

When V is even dimensional and oriented, we define $\Gamma = i^{n/2}e_1e_2 \cdots e_n$ in $\text{Cl}(V) \otimes \mathbb{C}$ modulo the choice of an oriented orthonormal basis e_1, \dots, e_n . We remark that Γ depends only of the orientation. Moreover we have $\Gamma^2 = 1$ and $\Gamma v = -v\Gamma$ for any $v \in V$.

If E is a $\text{Cl}(V)$ -module, the Clifford map is denoted $\mathbf{c}_E : \text{Cl}(V) \rightarrow \text{End}(E)$: we have $\mathbf{c}_E(v)^2 = -\|v\|^2 \text{Id}_E$ for any $v \in V$. A $\mathbb{Z}/2\mathbb{Z}$ -graduation on E is a vector space decomposition $E = E^+ \oplus E^-$ such that the map $\mathbf{c}_E(v) : E \rightarrow E$ are odd maps for any $v \in V$. Equivalently, a grading on E is defined by an involution $\theta : E \rightarrow E$ such that $\mathbf{c}_E(v) \circ \theta = -\mathbf{c}_E(v)$ for any $v \in V$.

For example, when V is even dimensional and oriented, we may consider the $\mathbb{Z}/2\mathbb{Z}$ -graduation on E defined by the involution $\Gamma_E := \mathbf{c}_E(\Gamma)$. This graduation will be called the canonical graduation of the $\text{Cl}(V)$ -module E .

Let us give the first example of Clifford module. Consider the real vector space $\bigwedge_{\mathbb{R}} V$. We denote by $m(v) : \bigwedge_{\mathbb{R}} V \rightarrow \bigwedge_{\mathbb{R}} V$ the exterior multiplication by an element of V , and by $\iota(v) : \bigwedge_{\mathbb{R}} V \rightarrow \bigwedge_{\mathbb{R}} V$ the contraction: $\iota(v)$ is the odd derivation of $\bigwedge_{\mathbb{R}} V$ such that $\iota(v_2)v_1 = (v_2, v_1)$ for $v_1 \in V \subset \bigwedge_{\mathbb{R}} V$. Then $\mathbf{c}(v) = m(v) - \iota(v)$ satisfies $\mathbf{c}(v)^2 = -\|v\|^2$. Thus the complexified space $(\bigwedge_{\mathbb{R}} V) \otimes \mathbb{C} \simeq \bigwedge_{\mathbb{C}}(V \otimes \mathbb{C})$ is a Clifford module.

If E is a graded Clifford module, and W a graded complex vector space, the space $E \otimes W$ with Clifford action $\mathbf{c}_E(v) \otimes \text{Id}_W$ is again a graded Clifford module where $(E \otimes W)^+ = E^+ \otimes W^+ \oplus E^- \otimes W^-$ and $(E \otimes W)^- = E^- \otimes W^+ \oplus E^+ \otimes W^-$. It will be called the twisted Clifford module of E with twisting space W .

If E is a graded Clifford module, it is convenient to denote $-E$ the Clifford module with opposite grading. We can see $-E$ as the tensor of E with the graded vector space W where $W^+ = \{0\}$ and $W^- = \mathbb{C}$.

Proposition 5.1 *Let V be an Euclidean space of even dimension $n = 2\ell$. There exists a $\text{Cl}(V)$ -module S of such that the map $\mathbf{c}_S : \text{Cl}(V) \rightarrow \text{End}(S)$ extends to an isomorphism of complex algebras :*

$$\text{Cl}(V) \otimes \mathbb{C} \simeq \text{End}(S).$$

If an orientation of V is given, we define $S^+ = \{s \in S \mid \Gamma s = s\}$ and $S^- = \{s \in S \mid \Gamma s = -s\}$.

Then S is unique, up to isomorphism. We call S the spinor module. There is, up to a multiplicative constant, a unique Hermitian form on S such that the operators $\mathbf{c}_S(v)$ are skew-adjoint.

For example if $V = \mathbb{R}^2$, with standard inner product, then S is $\mathbb{C} \oplus \mathbb{C}$, with Clifford action

$$\mathbf{c}(xe_1 + ye_2) = \begin{pmatrix} 0 & -(x - iy) \\ x + iy & 0 \end{pmatrix}.$$

Let $\text{Spin}(V)$ be the spinor group. We denote by $\eta : \text{Spin}(V) \rightarrow \text{SO}(V)$ the covering map. The kernel of τ has two elements. The spinor group $\text{Spin}(V)$ acts in S , preserving S^+, S^- . We denote by ρ the spin representation of $\text{Spin}(V)$. It satisfies $\rho(g)\mathbf{c}(v)\rho(g^{-1}) = \mathbf{c}(\eta(g)v)$.

Here is a construction of the spinor module. Consider $V \otimes \mathbb{C}$ and extend the form $(,)$ by complex bilinearity to $V \otimes \mathbb{C}$. Choose a decomposition

$V \otimes \mathbb{C} = L_1 \oplus L_2$ in two totally isotropic subspaces for the form $(,)$. Thus L_1, L_2 are of dimension $n/2$. We consider on the \mathbb{Z}_2 -graded complex vector space $\bigwedge L_1$ the following odd operators: the exterior multiplication $m(v_1)$ by an element of L_1 , and the contraction $\iota(v_2)$ by an element of L_2 : $\iota(v_2)$ is the odd derivation of $\bigwedge L_1$ such that $\iota(v_2)v_1 = (v_2, v_1)$ for $v_1 \in L_1$. The Clifford action of $V_{\mathbb{C}}$ on $\bigwedge L_1$ is defined by the formula

$$(5.9) \quad \mathbf{c}(v_1 + v_2) = \sqrt{2}(m(v_1) - \iota(v_2)).$$

Then $\mathbf{c}(v)$ is an odd operator on $\bigwedge L_1$ such that $\mathbf{c}(v)^2 = -(v, v)\text{Id}$. It is easy to see that $\bigwedge L_1$ is an irreducible Clifford module. Thus $\bigwedge L_1$ with action $\mathbf{c}(v)$ as above is “the” spinor module.

Let J be an endomorphism of V such that $J^2 = -1$ (J defines a complex structure on V) and such that $(Jv, Jv) = (v, v)$. Equivalently $h(v, w) = g(v, w) - ig(Jv, w)$ is a Hermitian structure on the complex vector space (V, J) . Then J defines also an orientation $o(J)$ of V . If $e_1, e_2, \dots, e_{n/2}$ is a basis of V as a complex space, then we take as oriented basis of V the basis $e_1, Je_1, e_2, Je_2, \dots, e_{n/2}, Je_{n/2}$. We denote by V_J the space V considered as a complex vector space and we may also denote by $\bigwedge_J V = \bigwedge V_J$ the exterior space of the complex space V_J .

We denote by

$$V^+ = \{v \in V \otimes \mathbb{C} \mid Jv = iv\}, \quad V^- = \{v \in V \otimes \mathbb{C} \mid Jv = -iv\}.$$

These are two totally isotropic spaces for $(,)$ and we can take $S = \bigwedge V^+$ as the irreducible Clifford module. In complex geometry, it is more customary to take $S = \bigwedge (V^-)^*$, but the space $(V^-)^*$ is isomorphic to V^+ by the duality $g(,)$.

The map $v \rightarrow v - iJv$ from V to V^+ gives an isomorphism of the complex vector spaces V_J with V^+ . Then $\bigwedge V_J$ is isomorphic to $\bigwedge V^+$, and we can realize S as $\bigwedge V_J$. The grading of S induced by the orientation $o(J)$ of V_J is then $S^+ = \bigwedge^{\text{even}} V_J$ and $S^- = \bigwedge^{\text{odd}} V_J$.

If H is a Hermitian vector space, in some situation (see Section 6.3), it is convenient to choose as spinor space $\bigwedge \overline{H}$, where \overline{H} denote the space H , with complex structure $-J$. Then the grading $\bigwedge^{\text{even}} \overline{H} \oplus \bigwedge^{\text{odd}} \overline{H}$ corresponds to the orientation $o(-J) = (-1)^{\dim H} o(J)$.

Consider the subgroup $U(V)$ of $SO(V)$ consisting of complex transformations of (V, J) . An element of $U(V)$ leaves stable the space V^+ , and we obtain a natural action λ of $U(V)$ in $\bigwedge V^+$. Let us compare the natural action λ and the restriction ρ of the spin representation of $\text{Spin}(V)$ to $\tau^{-1}(U(V))$.

Lemma 5.2 For $g \in \tau^{-1}(U(V)) \subset \text{Spin}(V)$, $\lambda(\tau(g)) = \delta(g)\rho(g)$, where $\delta(g)$ is the character of $\tau^{-1}(U(V))$ such that $\delta(g)^2 = \det_{V^+}(g)$.

We often will use complex structures on real vector spaces defined by the following procedure.

Definition 5.3 Let N be a real vector space and β a transformation of N , such that $-\beta^2$ is diagonalizable with non negative eigenvalues. Define

- the transformation $|\beta|$ of N by $|\beta| = \sqrt{-\beta^2}$,
- and the complex structure J_β on $N/\ker(\beta)$ by $J_\beta = \beta|\beta|^{-1}$.

For example if V is an Euclidean space and β a skew-symmetric transformation of V , then $-\beta^2$ is diagonalizable with non negative eigenvalues. By definition of J_β , the transformation β induces on $N/\ker(\beta) \simeq \text{Image}(\beta)$ a complex diagonalizable transformation, and the list of its complex eigenvalues is $[ia_1, \dots, ia_\ell]$ where the a_k are strictly positive real numbers.

Similarly if $\mathcal{N} \rightarrow B$ is a Euclidean vector bundle with a fibrewise bijective linear endomorphism $\mathcal{L}(\beta)$ which is anti-symmetric relatively to the metric, we denote by \mathcal{N}_{J_β} the vector bundle \mathcal{N} over B equipped with the complex structure J_β . Then \mathcal{N}_{J_β} is a Hermitian vector bundle over B , and we say that \mathcal{N}_{J_β} is the complex vector bundle \mathcal{N} polarized by β .

The following proposition follows from the Schur lemma. It is used to “divide” a Clifford module by the irreducible Clifford module S .

Proposition 5.4 When $\dim V$ is even, any $\text{Cl}(V)$ -module E has the following decomposition

$$(5.10) \quad E \simeq S \otimes \text{hom}_{\text{Cl}(V)}(S, E)$$

where $\text{hom}_{\text{Cl}(V)}(S, E)$ is the vector space of $\text{Cl}(V)$ -complex linear maps from S to E . The grading of E , and a grading on S coming from an orientation on V induces a grading on $\text{hom}_{\text{Cl}(V)}(S, E)$ so that the isomorphism (5.10) is an isomorphism of graded Clifford module.

The space $W = \text{hom}_{\text{Cl}(V)}(S, E)$ is called the twisting space. We write $E = S \otimes W$ and the Clifford action is just on the spinor module S :

$$\mathbf{c}_E(v) = \mathbf{c}_S(v) \otimes \text{Id}_W.$$

Example 5.5 • If J_1, J_2 are two complex structures on V , we may compare the spinor bundle $S_k = \bigwedge V_{J_k}$. We have $S_1 \simeq \epsilon S_2$ where $\epsilon \in \{\pm 1\}$ is the ratio between the orientations $o(J_1)$ and $o(J_2)$.

• If V is even dimensional, the twisting space W for the Clifford module $\bigwedge_{\mathbb{R}} V \otimes \mathbb{C}$ is $-S$.

We implicitly always choose Hermitian products on E obtained by tensor product of the Hermitian structure on S and a Hermitian structure on W . Then the operators $\mathbf{c}(v)$ are skew-adjoint.

If $V = V_1 \oplus V_2$ is an orthogonal decomposition, we have $\text{Cl}(V) \simeq \text{Cl}(V_1) \otimes \text{Cl}(V_2)$. So, if E_1, E_2 are graded Clifford modules for V_1, V_2 , then $E = E_1 \otimes E_2$ is a graded Clifford module for V , with Clifford action $\mathbf{c}(v_1 \oplus v_2) = \mathbf{c}(v_1) \boxtimes \mathbf{c}(v_2)$.

Reciprocally, if V_1 is even dimensional and oriented, any graded Clifford module E on $V = V_1 \oplus V_2$ has the decomposition $E \simeq S_1 \otimes E_2$, where S_1 is the spinor module for $\text{Cl}(V_1)$ with its canonical grading, and $E_2 = \text{hom}_{\text{Cl}(V_1)}(S_1, E)$ is a graded $\text{Cl}(V_2)$ -module.

5.2 Euclidean vector bundles and Clifford bundles

Consider now the case of an Euclidean vector bundle $\mathcal{V} \rightarrow M$ over a manifold M . Let $\text{Cl}(\mathcal{V}) \rightarrow M$ be the associated Clifford algebra bundle. A complex $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle $\mathcal{E} \rightarrow M$ is a $\text{Cl}(\mathcal{V})$ -bundle if there is a bundle algebra odd morphism $\mathbf{c}_{\mathcal{E}} : \text{Cl}(\mathcal{V}) \rightarrow \text{End}(\mathcal{E})$. Thus, for any $x \in M, v \in \mathcal{V}|_x$, $\mathbf{c}_{\mathcal{E}_x}(v) : \mathcal{E}|_x^{\pm} \rightarrow \mathcal{E}|_x^{\mp}$ is such that $\mathbf{c}_{\mathcal{E}_x}(v)^2 = -\|v\|^2 \text{Id}_{\mathcal{E}|_x}$.

Assume that M is a K -manifold. Let \mathcal{V} be a K -equivariant Euclidean vector bundle on M , and \mathcal{E} a $\text{Cl}(\mathcal{V})$ -bundle. Then we will always assume that $\mathcal{E}^{\pm} \rightarrow M$ are K -equivariant vector bundles and that $\mathbf{c}_{\mathcal{E}}$ is a morphism of K -equivariant vector bundles. We will say that \mathcal{E} is a K -equivariant Clifford bundle.

If \mathcal{W} is any K -equivariant graded complex vector bundle on M , and \mathcal{E} a graded Clifford bundle, then $\mathcal{E} \otimes \mathcal{W}$ with Clifford action $\mathbf{c}_{\mathcal{E}} \otimes \text{Id}_{\mathcal{W}}$ will be called the twisted Clifford bundle of \mathcal{E} by \mathcal{W} .

Definition 5.6 A Spin^c bundle on \mathcal{V} is a $\text{Cl}(\mathcal{V})$ -bundle \mathcal{S} such that, for any $x \in M$, $\mathcal{S}|_x$ is an irreducible Clifford module for $\mathcal{V}|_x$. An orientation on the vector bundle \mathcal{V} induces a (canonical) grading on \mathcal{S} .

If \mathcal{S} is a Spin^c -bundle on \mathcal{V} equipped with its canonical grading, we can, as in the linear case, divide any $\text{Cl}(\mathcal{V})$ -bundle \mathcal{F} by \mathcal{S} : we write $\mathcal{F} = \mathcal{S} \otimes \mathcal{W}$, and the twisting bundle $\mathcal{W} = \text{hom}_{\text{Cl}(\mathcal{V})}(\mathcal{S}, \mathcal{F})$ inherits a graduation.

Let K be a compact Lie group acting on a manifold M . Consider a K -equivariant Riemannian metric on M and let $\nu \in T^*M \mapsto \tilde{\nu} \in TM$ be the corresponding bundle isomorphism.

Definition 5.7 *A Clifford bundle on M is a graded Clifford bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ for the Clifford algebra $\text{Cl}(TM)$.*

*Let \mathcal{E} be a Clifford bundle on M . We define the symbol $\sigma(M, \mathcal{E}) \in \Gamma(T^*M, \text{hom}(\mathcal{E}^+, \mathcal{E}^-))$ by*

$$\sigma(M, \mathcal{E})(x, \nu) = \mathbf{c}_{\mathcal{E}_x}(\tilde{\nu}) : \mathcal{E}_x^+ \rightarrow \mathcal{E}_x^-.$$

We always choose K -invariant Hermitian structures on \mathcal{E}^\pm so that $c_x(\tilde{\nu})$ is skew-adjoint. Thus the adjoint of the symbol $\sigma(M, \mathcal{E})$ is the symbol $-\sigma(M, \mathcal{E}^{op})$ (in \mathcal{E}^{op} , we have interchanged the parity on \mathcal{E}^+ and \mathcal{E}^-). We have

$$\sigma(M, \mathcal{E})(x, \tilde{\nu})\sigma(M, \mathcal{E}^{op})(x, \tilde{\nu}) = -\|\tilde{\nu}\|_x^2$$

so that, if M is compact, $\sigma(M, \mathcal{E})$ is an elliptic symbol.

Remark 5.8 *By Bott periodicity theorem, any K -equivariant elliptic symbol on a compact even dimensional oriented manifold M is equal to a symbol $\sigma(M, \mathcal{E})$, for some equivariant Clifford bundle \mathcal{E} , in the equivariant K -theory of TM . We will prove a localization formula for the symbols $\sigma(M, \mathcal{E})$. They could be rephrased for any elliptic symbol, but this would require more notational framework.*

5.3 Choice of metrics

Let us consider the role of the metric in our setting. Let $(M, \mathcal{V}, \mathcal{E})$ be a K -equivariant Clifford data : $\mathcal{V} \rightarrow M$ is an Euclidean vector bundle and $\mathcal{E} \rightarrow M$ is a graded $\text{Cl}(\mathcal{V})$ -bundle. Let us change the metric on the real vector bundle \mathcal{V} and denote by \mathcal{V}' the new Euclidean bundle. Then we can equip \mathcal{E} with a structure of $\text{Cl}(\mathcal{V}')$ -bundle, denoted \mathcal{E}' , such that $(M, \mathcal{V}, \mathcal{E})$ and $(M, \mathcal{V}', \mathcal{E}')$ are isomorphic. Indeed, if g_0, g_1 be two K -invariant metrics on \mathcal{V} , $g_t = tg_1 + (1-t)g_0$ is a K -invariant metric on \mathcal{V} for any $t \in [0, 1]$. If $\mathcal{E} \rightarrow M$ is a Clifford bundle for \mathcal{V} with the Euclidean structure given by g_0 , we can find a Clifford action on \mathcal{E} for the metric g_t : find $u_t \in \Gamma(M, \text{GL}(\mathcal{V}))$ varying smoothly with t such that $g_t(v, v) = g_0(u_tv, u_tv)$ for $v \in \mathcal{V}_0$ and define the action $\mathbf{c}_t(v)$ by $\mathbf{c}_t(v) = \mathbf{c}(u_tv)$. If M is compact, then, in the \mathbf{K} -theory of \mathcal{V} , the class of the corresponding symbol σ_t is independent of t . So we will often speak of a Clifford bundle \mathcal{E} on \mathcal{V} without specifying the metric on \mathcal{V} .

In particular if $\pi : \mathcal{H} \rightarrow B$ is a Hermitian vector bundle on a Riemannian K -manifold B , using a K -invariant connection, we can choose a metric on $T\mathcal{H}$ such that $T\mathcal{H}$ is isomorphic to the orthogonal direct sum $\pi^*TB \oplus \pi^*\mathcal{H}$ of Euclidean vector bundles. Then any Clifford module for $T\mathcal{H}$ is of the form

$$\pi^*\mathcal{E}|_B \otimes \pi^* \bigwedge \overline{\mathcal{H}},$$

where $\mathcal{E}|_B$ is a Clifford module for TB , and, for $\pi(n) = x$, the Clifford map \mathbf{c}_n is equal to $\mathbf{c}_x^1 \boxtimes \mathbf{c}_x^2$, where

$$\begin{aligned} \mathbf{c}_x^1 : T_x B &\longrightarrow \text{End}(\mathcal{E}|_{B|x}), \\ \mathbf{c}_x^2 : \mathcal{H}|_x &\longrightarrow \text{End}\left(\bigwedge \overline{\mathcal{H}}|_x\right). \end{aligned}$$

5.4 The Bott symbol and direct image

Let N be an Euclidean vector space. The dual space N^* inherits an Euclidean structure through the identification $N^* \rightarrow N, \eta \rightarrow \tilde{\eta}$ given by the relation $\langle \eta, n \rangle = \langle \tilde{\eta}, n \rangle$. The vector space $N \oplus N^*$ is an Euclidean space of even dimension that we identify to $N_{\mathbb{C}}$ by $n \oplus \eta \rightarrow n \oplus i\tilde{\eta}$. The group $O(N)$ acting diagonally on $N \oplus N^*$ becomes a subgroup of $U(N_{\mathbb{C}})$, and acts on $S_N := \bigwedge N_{\mathbb{C}}$ via the natural representation of $U(N_{\mathbb{C}})$ in $\bigwedge N_{\mathbb{C}}$.

Recall the definition of the Bott symbol. We denote by $\mathbf{c}_{N_{\mathbb{C}}}$ the Clifford representation of $N_{\mathbb{C}}$ in S_N . Then $\mathbf{c}_{N_{\mathbb{C}}}(n \oplus in')$ satisfies $\mathbf{c}_{N_{\mathbb{C}}}(n \oplus in')^2 = -(\|n\|^2 + \|n'\|^2)\text{Id}$.

Consider the trivial bundle $\mathcal{S} = N \times S_N$. We define the symbol $\text{Bott}(N_{\mathbb{C}}) \in \Gamma(T^*N, \text{hom}(\mathcal{S}^+, \mathcal{S}^-))$ by

$$\text{Bott}(N_{\mathbb{C}})(n, \eta) = \mathbf{c}_{N_{\mathbb{C}}}(n \oplus i\tilde{\eta}) : S_N^+ \rightarrow S_N^-.$$

Here $n \in N, \eta \in N^*$. The characteristic support of $\text{Bott}(N_{\mathbb{C}})$ is $\{0\}$, so $\text{Bott}(N_{\mathbb{C}})$ is an elliptic symbol on N called the Bott symbol. It is equivariant for the action of $O(N)$ and $\text{Index}_{O(N)}^N(\text{Bott}(N_{\mathbb{C}}))$ is the trivial representation of $O(N)$ [4].

Let $\mathcal{N} \rightarrow \mathcal{X}$ be an Euclidean vector bundle. The orthogonal frame bundle P of \mathcal{N} is a $O(n)$ -principal bundle such that $\mathcal{N} = P \times_{O(n)} N$, with $N = \mathbb{R}^n$. Using the notion of fiber product introduced in Section 4.4 with the group $S = \{1\}$, we define the pushforward morphism $i_! : \mathbf{K}_K^0(T_K^*\mathcal{X}) \rightarrow \mathbf{K}_K^0(T_K^*\mathcal{N})$ by the relation

$$i_!(\sigma) := \sigma \diamond \text{Bott}(N_{\mathbb{C}}),$$

if σ is a transversally elliptic symbol on \mathcal{X} .

The following theorem is a consequence of Lemma 4.6.

Theorem 5.9 (Invariance of the index by push-forward)

$$\text{Index}_K^{\mathcal{X}}(\sigma) = \text{Index}_K^{\mathcal{N}}(i_!(\sigma)).$$

A symbol representing $i_!(\sigma)$ is as follows. We write an element $n \in \mathcal{N}$ as (x, v) with $x \in \mathcal{X}$ and $v \in \mathcal{N}|_x$. We choose a K -invariant connection on the bundle $\mathcal{N} \rightarrow \mathcal{X}$, which allows us to parameterize $\text{T}^*\mathcal{N} = \{(n, (\xi, \eta))\}$, with $\xi \in \text{T}_x^*\mathcal{X}$, $\eta \in \mathcal{N}^*|_x$: here ξ is lifted to an horizontal element of $\text{T}_n^*\mathcal{N}$ using the connection, so (ξ, η) parameterize any element in $\text{T}_n^*\mathcal{N}$. Let $\sigma \in \Gamma(\text{T}^*\mathcal{X}, \text{hom}(\mathcal{E}^+, \mathcal{E}^-))$ be a symbol on \mathcal{X} . Then the symbol

$$(5.11) \quad ((x, v), (\xi, \eta)) \mapsto \sigma(x, \xi) \boxtimes \mathbf{c}_x(v \oplus i\tilde{\eta})$$

represents $i_!(\sigma)$. Here \mathbf{c}_x is the Clifford representation of $\mathcal{N}|_x \otimes \mathbb{C}$ on $\bigwedge(\mathcal{N}|_x \otimes \mathbb{C})$, so $(\mathbf{c}_x(v \oplus i\tilde{\eta}))^2 = -(\|v\|^2 + \|\eta\|^2)\text{Id}$ and $\text{Char}(i_!(\sigma)) = \text{Char}(\sigma)$.

5.5 Dirac operators and equivariant indices

Let $\mathcal{E} \rightarrow M$ be a K -equivariant graded Clifford bundle on a compact Riemannian K -manifold M of even dimension.

Definition 5.10 *We denote by $\mathcal{Q}_K(M, \mathcal{E}) \in R(K)$ the index of the symbol $\sigma(M, \mathcal{E})$.*

If K is the trivial group, we denote $\mathcal{Q}_K(M, \mathcal{E})$ simply by $\mathcal{Q}(M, \mathcal{E})$. This is an integer.

We can realize $\mathcal{Q}_K(M, \mathcal{E})$ as the index of a Dirac operator. We choose K -invariant connections on \mathcal{E}^\pm such that the Dirac operator $\mathcal{D}_\mathcal{E}^\pm : \Gamma(M, \mathcal{E}^\pm) \rightarrow \Gamma(M, \mathcal{E}^\mp)$ [8] verifies $(\mathcal{D}_\mathcal{E}^+)^* = \mathcal{D}_\mathcal{E}^-$. The principal symbol of $\mathcal{D}_\mathcal{E}^+$ is $\sigma(M, \mathcal{E})$. Thus $\mathcal{Q}_K(M, \mathcal{E})$ can be realized as the difference of the two finite dimensional representation spaces $[\ker \mathcal{D}_\mathcal{E}^+] - [\ker \mathcal{D}_\mathcal{E}^-]$ and we also say that $\mathcal{Q}_K(M, \mathcal{E})$ is the equivariant index of the Dirac operator $\mathcal{D}_\mathcal{E}^+$.

For example, if M is any K -manifold, and $\mathcal{E} = \bigwedge(\text{TM} \otimes_{\mathbb{R}} \mathbb{C})$, a Dirac operator with principal symbol $\sigma(M, \mathcal{E})$ can be taken as $d + d^*$ where d is the de Rham differential. If K is connected, $\mathcal{Q}_K(M, \mathcal{E})$ is a multiple of the trivial representation of K , the multiplicity being the Euler characteristic of M .

Example 5.11 *An important example is when M is a complex manifold with complex structure $J \in \Gamma(M, \text{TM})$. Then $\text{TM} \otimes_{\mathbb{R}} \mathbb{C}$ breaks into two pieces, the holomorphic and anti-holomorphic tangent spaces*

$$\text{TM} \otimes_{\mathbb{R}} \mathbb{C} = \text{T}^{1,0}M \oplus \text{T}^{0,1}M$$

on which J acts by i , and $-i$, respectively. We consider a Riemannian metric on TM such that $g(Jv, Jv) = g(v, v)$. Let $\mathcal{W} \rightarrow M$ be a holomorphic vector bundle on M . Then the vector bundle of \mathcal{W} -valued anti-holomorphic differential forms

$$\mathcal{E} = \bigwedge (\mathbb{T}^{0,1}M)^* \otimes \mathcal{W}$$

is a graded Clifford bundle. This bundle is isomorphic to $\bigwedge_J TM \otimes \mathcal{W}$, with Clifford action described by the formula (5.9). A Dirac operator with principal symbol $\sigma(M, \mathcal{E})$ is $\bar{\partial}_{\mathcal{W}} + (\bar{\partial}_{\mathcal{W}})^*$ where $\bar{\partial}_{\mathcal{W}}$ is the Dolbeaut operator. Assume that K acts by holomorphic transformations on M, \mathcal{W} . Then $\mathcal{Q}_K(M, \mathcal{E}) = \sum_j (-1)^j H^j(M, \mathcal{O}(\mathcal{W}))$ is the alternate sum of the cohomology groups of the sheaf of holomorphic sections of \mathcal{W} .

Example 5.12 Let K be a compact connected Lie group, and $T \subset K$ be a maximal torus. We consider the flag manifold $\mathbb{F} := K/T$ with base point $\bar{e} \in K/T$. The choice of a Weyl chamber $\mathfrak{t}_{\geq 0}^* \subset \mathfrak{t}^*$ determines a T -invariant complex structure J on $T_{\bar{e}}\mathbb{F} \simeq \mathfrak{k}/\mathfrak{t}$ such that the complex T -module $\mathbb{T}^{1,0}\mathbb{F}$ is equal to $\bigoplus_{\alpha > 0} \mathbb{C}_{\alpha}$, where $\{\alpha > 0\}$ denotes the set of positive roots. The complex structure J defines an integrable complex structure on the flag manifold \mathbb{F} (still denoted J). Let $\Lambda \subset \mathfrak{t}^*$ be the weight lattice. Any weight $\mu \in \Lambda$ determines an holomorphic line bundle $[\mathbb{C}_{\mu}] := K \times_T \mathbb{C}_{\mu}$ on \mathbb{F} . If μ is dominant, the Borel-Weil Theorem tells us that $\mathcal{Q}_K(\mathbb{F}, \bigwedge_J T\mathbb{F} \otimes [\mathbb{C}_{\mu}])$ is equal to the irreducible representation of K with highest weight μ .

5.6 Horizontal Clifford bundle

Let G and K be two compact Lie groups. Let P be a manifold provided with an action of $G \times K$. Assume that the action of K is free and let $M = P/K$. We are in the situation described in Subsection 4.2, and we use the same notations.

For the tangent bundle, we have $TP = \pi^*TM \oplus P \times \mathfrak{k}$; if a metric is chosen on M , we define the metric on P by declaring that this decomposition is orthogonal.

Let \mathcal{E} be a G -equivariant graded Clifford bundle on M . We obtain from \mathcal{E} a symbol

$$\sigma_{hor}(P, \mathcal{E}) \in \Gamma(T^*P, \text{hom}(\pi^*\mathcal{E}^+, \pi^*\mathcal{E}^-))$$

by defining $\sigma_{hor}(P, \mathcal{E})(p, \eta) = \sigma(M, \mathcal{E})(x, \tilde{\theta})$ where $(x, \theta) := \bar{\pi}(p, \eta)$.

Assume P compact. It is clear that $\sigma_{hor}(P, \mathcal{E})$ is a K -transversally elliptic symbol on P . It will be called the horizontal Dirac symbol. Furthermore,

this symbol is G -equivariant, and for any irreducible representation V_μ of K , the finite dimensional space

$$[\text{Index}_{G \times K}^P(\sigma_{hor}(P, \mathcal{E})) \otimes V_\mu^*]^K$$

carries a representation of G .

We form the G -equivariant complex vector bundle $\mathcal{V}_\mu := P \times_K V_\mu$ on M , and the twisted Clifford bundle $\mathcal{E} \otimes \mathcal{V}_\mu$. We consider the elliptic symbol $\sigma(M, \mathcal{E} \otimes \mathcal{V}_\mu)$ on M .

The following result follows easily from the Peter-Weyl decomposition of $\Gamma(P, \pi^* \mathcal{E})$ under the free action of K . It is a particular case of the free action property of transversally elliptic symbols.

Theorem 5.13 *In $\hat{R}(G \times K)$ we have*

$$\text{Index}_{G \times K}^P(\sigma_{hor}(P, \mathcal{E})) = \sum_{\mu \in \hat{K}} \mathcal{Q}_G(M, \mathcal{E} \otimes \mathcal{V}_\mu^*) \otimes V_\mu.$$

$$\text{In particular } [\text{Index}_{G \times K}^P(\sigma_{hor}(P, \mathcal{E}))]^K = \mathcal{Q}_G(M, \mathcal{E}).$$

Consider now the case of a $G \times K$ Riemannian manifold P , but assume now that K acts only infinitesimally freely: for any $p \in P$, the stabilizer K_p is a finite subgroup. So $M = P/K$ is an orbifold, but maybe not a manifold. Consider \mathcal{H} , the subbundle of TP defined by the orthogonal decomposition $TP = \mathcal{H} \oplus P \times \mathfrak{k}$ and call \mathcal{H} the horizontal tangent bundle.

Definition 5.14 • *If \mathcal{E} is a graded Clifford bundle for the horizontal tangent bundle \mathcal{H} , equivariant for the $G \times K$ -action, we define $\sigma_{hor}(P, \mathcal{E})(p, \nu) = \mathbf{c}_{\mathcal{E}_p}(\tilde{\nu}_{\mathcal{H}})$ where $\tilde{\nu}_{\mathcal{H}}$ is the horizontal component of $\tilde{\nu} \in T_p P$.*

- *We define $\mathcal{Q}_G(M, \mathcal{E})$ by the formula*

$$\mathcal{Q}_G(M, \mathcal{E}) := [\text{Index}_{G \times K}^P(\sigma_{hor}(P, \mathcal{E}))]^K.$$

It can be shown, see [39], that this is indeed the equivariant index of a Dirac operator on the G -orbifold M in the sense of Kawasaki [18].

6 Deformation à la Witten of Dirac operators

We start to discuss the real purpose of this paper. We will study the deformation à la Witten of elliptic symbols $\sigma(M, \mathcal{E})$ by vector fields associated to K -equivariant maps $\Phi : M \rightarrow \mathfrak{k}^*$.

We recall that we chose an invariant scalar product in \mathfrak{k} providing an identification $\mathfrak{k} \simeq \mathfrak{k}^*$.

6.1 Kirwan vector field

Let M be a K -manifold and let $\Phi : M \rightarrow \mathfrak{k}^*$ be an equivariant map.

Definition 6.1 • *The Kirwan vector field associated to Φ is defined by*

$$(6.12) \quad \kappa_\Phi(m) = -\Phi(m) \cdot m, \quad m \in M.$$

• *We denote by Z_Φ the set of zeroes of κ_Φ . Thus Z_Φ is a K -invariant closed subset of M .*

The Z_Φ is not necessarily smooth. If $0 \in \Phi(M)$, then clearly Z_Φ contains $\Phi^{-1}(0)$.

In order to describe more precisely Z_Φ , we consider the set $\{(\mathfrak{k}_i), i \in I\}$ of conjugacy classes of subalgebras of \mathfrak{k} representing the infinitesimal stabilizers of the K -action on M : if M is compact, the set I is finite. We denote by $M_{\mathfrak{k}_i}$ the submanifold of points $m \in M$ such that $\mathfrak{k}_i = \mathfrak{k}_m$. We see that $Z_\Phi \cap M_{\mathfrak{k}_i} = M_{\mathfrak{k}_i} \cap \Phi^{-1}(\mathfrak{k}_i^*)$, hence

$$(6.13) \quad Z_\Phi = \bigsqcup_{i \in I} K \left(M_{\mathfrak{k}_i} \cap \Phi^{-1}(\mathfrak{k}_i^*) \right),$$

where \mathfrak{k}_i^* is viewed as a subspace of \mathfrak{k}^* thanks to the scalar product.

Definition 6.2 *An equivariant map $\Phi : M \rightarrow \mathfrak{k}^*$ is called a moment map if there exists an invariant (real) 2-form Ω on M satisfying the relations*

$$(6.14) \quad \iota(X_M)\Omega + d\langle\Phi, X\rangle = 0 \quad \text{and} \quad d\Omega = 0$$

for $X \in \mathfrak{k}$. The data (Ω, Φ) on M is called a weak Hamiltonian structure.

Classically, a moment map in the sense of Hamiltonian geometry requires that Ω is non degenerate. In this text, we will see that some localization results can be proven without this strong hypothesis.

A particularly important example of moment map is provided by the following construction.

Definition 6.3 *Let L be a K -equivariant Hermitian line bundle with Hermitian connection ∇ with curvature R_L . Let $\Omega_L = \frac{1}{i}R_L$. Then Ω_L is a closed real two form. This data determines a moment map*

$$\Phi_L : M \rightarrow \mathfrak{k}^*$$

by the relation $\mathcal{L}(X) - \nabla_{X_M} = i\langle\Phi_L, X\rangle$, for all $X \in \mathfrak{k}$. Here $\mathcal{L}(X)$ is the infinitesimal action of $X \in \mathfrak{k}$ on the sections of L . The relation $\iota(X_M)\Omega_L + d\langle\Phi_L, X\rangle = 0$ follows.

Remark 6.4 *The subset $\Phi_L^{-1}(0)$ of the zeroes of the associated Kirwan vector field has some particular importance when L is an ample bundle on a projective manifold provided with an action of K . In this case, if Φ_L is the moment map associated to the projective embedding, $\Phi_L^{-1}(0)/K$ is isomorphic to the geometric invariant theory (GIT) quotient $M//K_{\mathbb{C}}$ of M by $K_{\mathbb{C}}$, and is provided with structure of projective variety with corresponding line bundle $L//K_{\mathbb{C}}$. When $M//K_{\mathbb{C}}$ is smooth, Guillemin and Sternberg [15] have proved the first $[Q, R] = 0$ identity :*

$$[H^0(M, L)]^K \simeq H^0(M//K_{\mathbb{C}}; L//K_{\mathbb{C}}).$$

Remark 6.5 *If Φ is a moment map, by Equation (6.14), we have*

$$d\|\Phi\|^2 = 2\iota(\kappa_{\Phi})\Omega.$$

Thus the set Z_{Φ} is contained in the critical set of the function $\|\Phi\|^2 : \beta \in \Phi(Z_{\Phi})$, then $\|\beta\|^2$ is a critical value of the square of the moment map Φ .

One of the properties that a moment map shares with the moment map of a Hamiltonian action is the following.

Proposition 6.6 • *For any $X \in \mathfrak{k}$, the function $\langle \Phi, X \rangle$ is constant on each connected component of the manifold M^X .*

• *If 0 is a regular value of a moment map Φ , then the action of K is infinitesimally free on the submanifold $\Phi^{-1}(0)$. The reduced space $M_{red} = \Phi^{-1}(0)/K$ is an orbifold.*

• *Conversely, if Φ is the moment map of a Hamiltonian action, and the action of K is infinitesimally free on the submanifold $\Phi^{-1}(0)$, then 0 is a regular value of Φ .*

Proof. Equation (6.14) tells us that for a point $m \in M$ where $X_M(m) = 0$, the differential $T_m\Phi$ sends T_mM to $(\mathbb{R}X)^{\perp} \subset \mathfrak{k}^*$. Our three points follow from this simple fact.

We note that any invariant 1-form α defines a moment map $\varphi_{\alpha} : M \rightarrow \mathfrak{k}^*$ by the relation : $\langle \varphi_{\alpha}, X \rangle = \iota(X_M)\alpha$, $X \in \mathfrak{k}$: the corresponding 2-form is $d\alpha$.

We will use the following equivalence relation between moment maps on M :

$$(6.15) \quad \Phi \sim \Phi' \iff \exists \text{ an invariant 1-form } \alpha \text{ such that } \Phi - \Phi' = \varphi_{\alpha}.$$

Let us describe $\Phi(Z_\Phi)$ when Φ is a moment map. As before, let $\{\mathfrak{k}_i, i \in I\}$ be a set of representatives of the infinitesimal stabilizers of the K -action on M , and let $M_{\mathfrak{k}_i}$ be the submanifold of points $m \in M$ such that $\mathfrak{k}_i = \mathfrak{k}_m$. If \mathcal{X} is a connected component of $M_{\mathfrak{k}_i}$, Relations (6.14) imply that the image $\Phi(\mathcal{X})$ is contained in an affine space $A(\mathcal{X}, \Phi) \subset \mathfrak{k}^*$ with direction \mathfrak{k}_i^\perp . The orthogonal projection $\beta(\mathcal{X}, \Phi)$ of 0 on the affine space of $A(\mathcal{X}, \Phi)$ belongs to \mathfrak{k}_i^* , viewed as a subspace of \mathfrak{k}^* thanks to the scalar product.

Lemma 6.7 *The set $\Phi(Z_\Phi)$ is contained in*

$$C_\Phi := \bigcup_{i \in I} \bigcup_{\mathcal{X} \subset M_{\mathfrak{k}_i}} K \cdot \beta(\mathcal{X}, \Phi).$$

More precisely, the orbit $K \cdot \beta(\mathcal{X}, \Phi)$ is contained in $\Phi(Z_\Phi)$ if and only if $\beta(\mathcal{X}, \Phi)$ belongs to $\Phi(\mathcal{X})$.

Proof. The proof is left to the reader.

Concerning the set C_Φ , we have the useful remark.

Proposition 6.8 *Let Φ, Φ' be two moment maps on the K -manifold M .*

- *If $\Phi \sim \Phi'$ then $C_\Phi = C_{\Phi'}$.*
- *If M is compact, C_Φ is a finite union of coadjoint orbits.*
- *If Z_Φ is compact, then $\Phi(Z_\Phi)$ is a finite union of coadjoint orbits.*

Proof. The first point is due to the fact that $A(\mathcal{X}, \Phi) = A(\mathcal{X}, \Phi + \varphi_\alpha)$ for any invariant 1-form α and any connected component \mathcal{X} of $M_{\mathfrak{k}_i}$. If M is compact, the set I is finite and each submanifold $M_{\mathfrak{k}_i}$ has a finite number of connected components, so there is a finite number of terms $\beta(\mathcal{X}, \Phi)$. If Z_Φ is compact, we consider an invariant neighborhood \mathcal{U} of Z_Φ such that $\overline{\mathcal{U}}$ is compact. The subset $I_\mathcal{U} \subset I$ formed by the element $i \in I$ such that $M_{\mathfrak{k}_i} \cap \mathcal{U} \neq \emptyset$ is finite, and we see that $\Phi(Z_\Phi)$ is contained in

$$C_\Phi^\mathcal{U} := \bigcup_{i \in I_\mathcal{U}} \bigcup_{\mathcal{X} \subset M_{\mathfrak{k}_i} \cap \mathcal{U}} K \cdot \beta(\mathcal{X}, \Phi).$$

Since $C_\Phi^\mathcal{U}$ is a finite union of coadjoint orbits, the last assertion is proved.

For any moment map Φ , if Z_Φ is compact, it admits a finite decomposition $Z_\Phi = \coprod_{\mathcal{O}} Z_{\mathcal{O}}$ where $Z_{\mathcal{O}} = Z_\Phi \cap \Phi^{-1}(\mathcal{O})$.

Definition 6.9 We denote by $\mathcal{B}(\Phi)$ the set of coadjoint orbits contained in $\Phi(Z_\Phi)$.

In practice, we parameterize the set $\mathcal{B}(\Phi)$ by a finite set \mathcal{B} contained in \mathfrak{k}^* : $\Phi(Z_\Phi) = \coprod_{\beta \in \mathcal{B}} K\beta$ and then

$$(6.16) \quad Z_\Phi = \coprod_{\beta \in \mathcal{B}} Z_\beta$$

where $Z_\beta = K(M^\beta \cap \Phi^{-1}(\beta))$.

6.2 Deforming a symbol

Let M be a K -manifold. We do not necessarily assume M to be compact, and the compact Lie group K nor M are necessarily connected. Let $\Phi : M \rightarrow \mathfrak{k}^*$ be an equivariant map, and let \mathcal{E} be a graded Clifford bundle on M .

Definition 6.10 The symbol $\sigma(M, \mathcal{E}, \Phi)$ pushed by the vector field κ_Φ is the symbol on M defined by

$$\sigma(M, \mathcal{E}, \Phi)(m, \nu) = \sigma(M, \mathcal{E})(m, \tilde{\nu} - \kappa_\Phi(m)) = \mathbf{c}_{\mathcal{E}_m}(\tilde{\nu} - \kappa_\Phi(m))$$

for any $(m, \nu) \in T^*M$.

Note that $\sigma(M, \mathcal{E}, \Phi)(m, \nu)$ is invertible except if $\tilde{\nu} = \kappa_\Phi(m)$. If furthermore (m, ν) belongs to the subset T_K^*M of cotangent vectors orthogonal to the K -orbits, then $\nu = 0$ and $\kappa_\Phi(m) = 0$. Indeed, as $\kappa_\Phi(m)$ is tangent to $K \cdot m$, the two equations $\tilde{\nu} = \kappa_\Phi(m)$ and $\langle \nu, \kappa_\Phi(m) \rangle = 0$ imply $\tilde{\nu} = \kappa_\Phi(m) = 0$. So we note that $(m, \nu) \in \text{Char}(\sigma(M, \mathcal{E}, \Phi)) \cap T_K^*M$ if and only if $\nu = 0$ and $\kappa_\Phi(m) = 0$. Hence $\sigma(M, \mathcal{E}, \Phi)$ is transversally elliptic whenever Z_Φ is compact.

Definition 6.11 If Z_Φ is compact, we define $\mathcal{Q}_K(M, \mathcal{E}, \Phi) \in \hat{R}(K)$ as the equivariant index of the transversally elliptic symbol $\sigma(M, \mathcal{E}, \Phi) \in \mathbf{K}_K^0(T_K^*M)$.

When M is compact, it is clear that the classes of the symbols $\sigma(M, \mathcal{E}, \Phi)$ and $\sigma(M, \mathcal{E})$ are equal in $\mathbf{K}_K^0(T_K^*M)$, hence the equivariant indices $\mathcal{Q}_K(M, \mathcal{E})$ and $\mathcal{Q}_K(M, \mathcal{E}, \Phi)$ are equal.

For any K -invariant, relatively compact, open subset $\mathcal{U} \subset M$ such that $\mathcal{U} \cap Z_\Phi$ is closed⁴ in M , we see that the restriction $\sigma(M, \mathcal{E}, \Phi)|_{\mathcal{U}}$ is a transversally elliptic symbol on \mathcal{U} , and so its equivariant index is a well defined element in $\hat{R}(K)$, independent of the choice of such a \mathcal{U} .

⁴ $\mathcal{U} \cap Z_\Phi$ is then compact.

Definition 6.12 • A closed K invariant subset $Z \subset Z_\Phi$ is called a **component** if it is a union of connected components of Z_Φ .

- For a compact component Z of Z_Φ , we denote by

$$\sigma(M, \mathcal{E}, Z, \Phi) \in \mathbf{K}_K^0(\mathbb{T}_K^* M)$$

the image of $\sigma(M, \mathcal{E}, \Phi)|_{\mathcal{U}} \in \mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{U})$ by the pushforward morphism $\mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{U}) \rightarrow \mathbf{K}_K^0(\mathbb{T}_K^* M)$ attached to a relatively compact invariant neighborhood \mathcal{U} of Z satisfying $\mathcal{U} \cap Z_\Phi = Z$.

- For a compact component Z of Z_Φ , we denote by

$$\mathcal{Q}_K(M, \mathcal{E}, Z, \Phi) \in \hat{R}(K)$$

the equivariant index of $\sigma(M, \mathcal{E}, Z, \Phi)$. By definition, if $Z = \emptyset$, then $\mathcal{Q}_K(M, \mathcal{E}, \emptyset, \Phi) = 0$.

In particular, if $\Phi : M \rightarrow \mathfrak{k}^*$ is a proper moment map, we can define

$$\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi) \in \hat{R}(K).$$

By taking its invariant part $[\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)]^K \in \mathbb{Z}$, we will define in a natural way invariants on the reduced “manifold” $M_{red} = \Phi^{-1}(0)/K$.

The simplest example of a pushed symbol $\sigma(M, \mathcal{E}, \Phi)$ is when the map Φ is constant, equal to a K -invariant element $\beta \in \mathfrak{k}^* \simeq \mathfrak{k}$. The vector field $m \mapsto \beta \cdot m$ is then K -equivariant, and we define

$$\sigma(M, \mathcal{E}, \beta)(m, \nu) = \sigma(M, \mathcal{E})(m, \tilde{\nu} + \beta \cdot m).$$

The first example of pushed symbol $\sigma(M, \mathcal{E}, \beta)$ is the Atiyah symbol that we will study in the next section.

We end this section by considering the case where Φ is a moment map such that Z_Φ is compact. For any $\mathcal{O} \subset C_\Phi$, the set $Z_{\mathcal{O}} = Z_\Phi \cap \Phi^{-1}(\mathcal{O})$ is a compact component of Z_Φ , hence we may consider the localized index $\mathcal{Q}_K(M, \mathcal{E}, Z_{\mathcal{O}}, \Phi)$. This generalized character is non-zero only if $Z_{\mathcal{O}} \neq \emptyset$, and this happens only for a finite number of orbits (those included in $\Phi(Z_\Phi)$).

Thanks to the excision property, we get the first form of the localization theorem :

$$(6.17) \quad \mathcal{Q}_K(M, \mathcal{E}, \Phi) = \sum_{\mathcal{O} \subset C_\Phi} \mathcal{Q}_K(M, \mathcal{E}, Z_{\mathcal{O}}, \Phi).$$

The aim of Section 7.2 is the description by induction of the representations $\mathcal{Q}_K(M, \mathcal{E}, Z_{\mathcal{O}}, \Phi)$.

Let us show that, when M is compact, the character $\mathcal{Q}_K(M, \mathcal{E}, Z_{\mathcal{O}}, \Phi)$ depends only of the class of Φ relative to the equivalence relation (6.15).

Proposition 6.13 *Let $\Phi \sim \Phi'$ be two moment maps on a compact K -manifold. Let $Z_{\mathcal{O}} \subset Z_{\Phi}$ and $Z'_{\mathcal{O}} = Z_{\Phi'} \cap (\Phi')^{-1}(\mathcal{O})$ be the components attached to $\mathcal{O} \in C_{\Phi} = C_{\Phi'}$. Then we have*

$$\mathcal{Q}_K(M, \mathcal{E}, Z_{\mathcal{O}}, \Phi) = \mathcal{Q}_K(M, \mathcal{E}, Z'_{\mathcal{O}}, \Phi').$$

Proof. Let α be the invariant 1-form such that $\Phi' - \Phi = \varphi_{\alpha}$. We consider the family of moment maps $\Phi^t = \Phi + t\varphi_{\alpha}$, $t \in \mathbb{R}$. We denote simply by σ^t the symbol $\sigma(M, \mathcal{E}, \Phi^t)$. For any t , consider the component $Z^t_{\mathcal{O}} = Z_{\Phi^t} \cap (\Phi^t)^{-1}(\mathcal{O})$ and the generalized character

$$\mathcal{Q}^t := \mathcal{Q}_K(M, \mathcal{E}, Z^t_{\mathcal{O}}, \Phi^t)$$

which is the equivariant index of $\sigma^t|_{U_t}$ where U_t is a neighborhood of $(\Phi^t)^{-1}(\mathcal{O})$ such that $U_t \cap Z_{\Phi^t} = Z^t_{\mathcal{O}}$.

Let us prove that the multiplicity \mathcal{Q}^t is independent of t . It is sufficient to prove that $t \rightarrow \mathcal{Q}^t$ is locally constant : let us show that it is constant in a neighborhood of 0.

Note that the set C_{Φ^t} does not depend of t : let us simply denote it by C . We remark that for any invariant subset $U \subset M$, we have

$$U \cap Z_{\Phi^t} \subset \bigcup_{\mathcal{P} \subset \Phi^t(U) \cap C} Z^t_{\mathcal{P}}$$

By an obvious continuity argument, we see that if U_0 is a neighborhood of $(\Phi^0)^{-1}(\mathcal{O})$ such that $\Phi^0(U_0) \cap C = \mathcal{O}$, then we have

$$\Phi^t(U_0) \cap C = \mathcal{O} \quad \text{and} \quad U_0 \cap Z_{\Phi^t} = Z^t_{\mathcal{O}}$$

for small t (says $|t| \leq \epsilon$). As the family $\sigma^t|_{U_0}$, $t \in [0, \epsilon]$ defines an homotopy of transversally elliptic symbols, they have the same equivariant index. Finally, for small t , we have

$$\begin{aligned} \mathcal{Q}^0 &= \text{Index}_K(\sigma^0|_{U_0}) &= \text{Index}_K(\sigma^t|_{U_0}) \\ & &= \mathcal{Q}^t. \end{aligned}$$

6.3 Atiyah symbol

Let N be a **Hermitian vector space**. We denote by U the unitary group of transformations of N , and by $U(1) \subset U$ the subgroup formed the homotheties $x \mapsto e^{i\theta}x$. Consider $S = \bigwedge \bar{N}$, the irreducible Clifford module for N

with Clifford action $\mathbf{c}(n)$. If we identify $\mathbb{T}^*N = N \oplus N^*$ with $N \oplus N$, the U -equivariant symbol $\sigma(N, S)$ is defined by

$$\sigma(N, S)(x, \nu) = \mathbf{c}(\tilde{\nu}) : \bigwedge^{\text{even}} \overline{N} \longrightarrow \bigwedge^{\text{odd}} \overline{N}, \quad (x, \nu) \in \mathbb{T}^*N.$$

The matrix $\beta = i\text{Id}$ is an invariant element of $\text{Lie}(U)$. We have $\beta \cdot x = Jx$ where J is the multiplication by i on N . We can then define the pushed symbol $\sigma(N, S, \beta)$.

Definition 6.14 *The Atiyah symbol $\text{At}(N) \in \Gamma(\mathbb{T}^*N, \text{hom}(\mathcal{S}^+, \mathcal{S}^-))$ is defined by*

$$\text{At}(N)(x, \nu) = \mathbf{c}(\tilde{\nu} + Jx) : \bigwedge^{\text{even}} \overline{N} \longrightarrow \bigwedge^{\text{odd}} \overline{N}, \quad (x, \nu) \in \mathbb{T}^*N.$$

The symbol $\text{At}(N)$ is not elliptic since $\text{supp}(\text{At}(N)) = \{(x, \nu), \tilde{\nu} + Jx = 0\}$ is not compact. But remark that $x \rightarrow Jx$ is the tangent vector produced by the action of $U(1)$ on N . Thus At is a $U(1)$ -transversally elliptic symbol. Indeed the two equations $\tilde{\nu} + Jx = 0$ and $\langle \nu, Jx \rangle = 0$ imply $\tilde{\nu} = x = 0$. So a fortiori the symbol $\text{At}(N)$ is U -transversally elliptic. Let us consider $\text{Sym}(N) = \bigoplus_{k=0}^{\infty} \text{Sym}^k(N)$ where $\text{Sym}^k(N)$ is the subspace of $\otimes^k N$ formed of symmetric tensors.

The following proposition is proved in [1] (see also [9]).

Proposition 6.15 *We have $\text{Index}_U^N(\text{At}(N)) = \text{Sym}(N)$ in $\hat{R}(U)$.*

More generally, consider the case of a real vector space N equipped with a linear action of a compact Lie group K . Let $\beta \in \mathfrak{k}$ be a K -invariant element such that $\mathcal{L}(\beta) : N \rightarrow N$ is invertible. We denote $\mathbb{T}_\beta \subset K$ the closed torus contained in the center equal to the closure of $\exp(\mathbb{R}\beta)$.

We denote by N_{J_β} the Hermitian vector space N equipped with the invariant complex structure J_β . We consider the K -equivariant \mathbb{T}_β -transversally elliptic symbol $\text{At}_\beta(N)$ defined by

$$\text{At}_\beta(N)(v, \xi) = \mathbf{c}(\tilde{\xi} + \beta \cdot v) : \bigwedge^{\text{even}} \overline{N_{J_\beta}} \longrightarrow \bigwedge^{\text{odd}} \overline{N_{J_\beta}}.$$

Its equivariant index satisfies

$$\text{Index}_K^N(\text{At}_\beta(N)) = \text{Sym}(N_{J_\beta}).$$

The meaning of this equation is as follows. We write $\text{Sym}(N_{J_\beta})$ as the sum of the finite dimensional spaces $\text{Sym}^k(N_{J_\beta})$. We consider $\text{Sym}^k(N_{J_\beta})$ as a representation space for K , write $\text{Sym}^k(N_{J_\beta}) = \sum_{\mu \in \hat{K}} m(\mu, k) V_\mu$. For each irreducible representation $V_\mu \in \hat{K}$, the infinitesimal action of β on V_μ is by a scalar $iz_\mu \text{Id}_{V_\mu}$, as β is fixed by K . On the other hand, the action of β in $\text{Sym}^k(N_{J_\beta})$ is diagonalizable with eigenvalues $\sum_J ia_j$ with J a subset of cardinal k of $[1, 2, \dots, \dim_{\mathbb{C}} N]$: here $a_j > 0$ are the eigenvalue of $\frac{\mathcal{L}(\beta)}{i}$ on N_{J_β} . As $\sum_J a_j$ is larger than $k \min(a_j)$, we see that k has to be small enough in order that $m(\mu, k) \neq 0$. Thus the sum $\sum_k \sum_{\mu \in \hat{K}} m(\mu, k) V_\mu$ is a well defined element of $\hat{R}(K)$.

We now consider a K -equivariant Euclidean vector bundle $\mathcal{N} \rightarrow \mathcal{X}$ on a connected K -manifold \mathcal{X} . Let us assume that $\beta \in \mathfrak{k}$ is an element fixed by K and acting fiberwise on \mathcal{N} by an invertible transformation : in other words $\mathcal{N}^\beta = \mathcal{X}$. The eigenvalues of β^2 on the fibers are strictly negative and constant over each connected component of \mathcal{X} . Using Definition 5.3 of J_β , we define on \mathcal{N} a Hermitian structure invariant by K . We denote by \mathcal{N}_{J_β} the vector bundle \mathcal{N} considered as a complex vector bundle with complex structure J_β . By definition of J_β , the list of eigenvalues of β on \mathcal{N}_m for the complex structure J_β are $[ia_1, ia_2, \dots, ia_R]$ where the a_j are strictly positive.

We denote by $\bigwedge \mathcal{N}_{J_\beta}$ the complex exterior bundle of the complex vector bundle \mathcal{N}_{J_β} . We denote by $\text{Sym}(\mathcal{N}_{J_\beta})$ the \mathbb{Z} -graded symmetric vector bundle of \mathcal{N}_{J_β} .

Let N be the fiber $\mathcal{N}|_{x_o}$ at some point $x_o \in \mathcal{X}$, equipped with the linear action of β , the complex structure J_β and its Euclidean structure. Let U be the subgroup of $U(N)$ of elements that commutes with $\mathcal{L}(\beta)$. Let $P_U \rightarrow \mathcal{X}$ be the U -unitary framed bundle of \mathcal{N} : for $x \in \mathcal{X}$, we take

$$P_U|_x := \{f : N \rightarrow \mathcal{N}|_x, \text{ unitary and } \mathcal{L}(\beta) - \text{equivariant}\}.$$

Hence the vector bundle \mathcal{N} is equal to $P \times_U N$. The Atiyah symbol on N defines a class

$$\text{At}_\beta(N) \in \mathbf{K}_{U \times \mathbb{T}_\beta}^0(\mathbb{T}_{\mathbb{T}_\beta}^* N).$$

Let us use the notion of fiber product introduced in Section 4.4: here we work with the group $S := \mathbb{T}_\beta$. We can define the morphism $\text{At}_\beta : \mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{X}) \rightarrow \mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{N})$ by the relation

$$(6.18) \quad \text{At}_\beta(\sigma) := \sigma \diamond \text{At}_\beta(N)$$

where σ is a K -transversally elliptic symbol on \mathcal{X} .

Lemma 4.6 gives

$$(6.19) \quad \text{Index}_K^{\mathcal{N}}(\text{At}_\beta(\sigma)) = \text{Index}_K^{\mathcal{X}}(\sigma \otimes \text{Sym}(\mathcal{N}_{J_\beta}))$$

$$(6.20) \quad = \sum_{k=0}^{\infty} \text{Index}_K^{\mathcal{X}}(\sigma \otimes \text{Sym}^k(\mathcal{N}_{J_\beta})).$$

A symbol representing $\text{At}_\beta(\sigma)$ is as follows. We write an element $n \in \mathcal{N}$ as (x, v) with $x \in \mathcal{X}$ and $v \in \mathcal{N}|_x$. We choose a K -invariant connection on the bundle $\mathcal{N} \rightarrow \mathcal{X}$, which allows us to parameterize $\text{T}^*\mathcal{N} = \{(n, (\xi, \nu))\}$, with $\xi \in \text{T}_x^*\mathcal{X}$, $\nu \in \mathcal{N}^*|_x$ (here ξ is lifted to an element of $\text{T}_n^*\mathcal{N}$ using the connection). Let $\sigma(x, \xi) \in \Gamma(\text{T}^*\mathcal{X}, \text{hom}(\mathcal{E}^+, \mathcal{E}^-))$ be a symbol on \mathcal{X} . Then

$$(6.21) \quad \text{At}_\beta(\sigma)((x, v), (\xi, \nu)) = \sigma(x, \xi) \boxtimes \mathbf{c}_x(\tilde{\nu} + \beta v).$$

Here \mathbf{c}_x is the Clifford action on $\bigwedge \overline{\mathcal{N}}_{J_\beta}|_x$, so $\mathbf{c}_x(\tilde{\nu} + \beta v)^2 = -\|\tilde{\nu} + \beta v\|^2 \text{Id}$. We see that $\text{Char}(\text{At}_\beta(\sigma)) \cap \text{T}_K^*\mathcal{N} = \text{Char}(\sigma) \cap \text{T}_K^*\mathcal{X}$.

7 Abelian Localization formula

7.1 Atiyah-Segal-Singer localization formula

Let M be a compact K -manifold. Let $\beta \in \mathfrak{k}$ be a central element, and let M^β be the submanifold of points fixed by the infinitesimal action of β . Let \mathcal{N} be the normal bundle of M^β in M . Then β acts by a fiberwise invertible transformation on \mathcal{N} . Denote by \mathcal{N}_{J_β} the (complex) vector bundle \mathcal{N} polarized by β .

Let \mathcal{U} be a tubular neighborhood of M^β in M that is diffeomorphic with the normal bundle \mathcal{N} . We denote by $\varphi : \mathcal{U} \rightarrow \mathcal{N}$ the K -diffeomorphism and by $i : \mathcal{U} \hookrightarrow M$ the inclusion. Let $\varphi^* : \mathbf{K}_K^0(\text{T}_K^*\mathcal{N}) \rightarrow \mathbf{K}_K^0(\text{T}_K^*\mathcal{U})$ and $i_* : \mathbf{K}_K^0(\text{T}_K^*\mathcal{U}) \rightarrow \mathbf{K}_K^0(\text{T}_K^*M)$ the corresponding maps in \mathbf{K} -theory.

We denote by

$$\mathbf{At}_\beta : \mathbf{K}_K^0(\text{T}_K^*M^\beta) \longrightarrow \mathbf{K}_K^0(\text{T}_K^*M)$$

the map $i_* \circ \varphi^* \circ \mathbf{At}_\beta$. For the equivariant index, we have the equality

$$\text{Index}_K^M(\mathbf{At}_\beta(\sigma)) = \text{Index}_K^{M^\beta}(\sigma \otimes \text{Sym}(\mathcal{N}_{J_\beta}))$$

for any $\sigma \in \mathbf{K}_K^0(\text{T}_K^*M^\beta)$.

Let $\mathcal{E} \rightarrow M$ be a (graded) Clifford module on M . The bundle $\bigwedge \overline{\mathcal{N}}_{J_\beta}$ is a Spin^c bundle for the Euclidean bundle \mathcal{N} .

Definition 7.1 *There exists a unique graded Clifford bundle $\mathfrak{d}_\beta(\mathcal{E})$ over the tangent bundle TM^β , such that, when we restrict \mathcal{E} to M^β , we get the isomorphism*

$$(7.22) \quad \mathcal{E}|_{M^\beta} \simeq \mathfrak{d}_\beta(\mathcal{E}) \otimes \bigwedge \overline{\mathcal{N}_{J_\beta}}$$

of graded Clifford bundles relative to the decomposition $\mathrm{TM}|_{M^\beta} := \mathrm{TM}^\beta \oplus \mathcal{N}$.

We have two elliptic symbols : $\sigma(M, \mathcal{E}) \in \mathbf{K}_K^0(\mathrm{T}^*M)$ and $\sigma(M^\beta, \mathfrak{d}_\beta(\mathcal{E})) \in \mathbf{K}_K^0(\mathrm{T}^*M^\beta)$.

Proposition 7.2 (Atiyah-Segal-Singer localization formula) *The following relation holds in $\mathbf{K}_K^0(\mathrm{T}_K^*M)$:*

$$\mathrm{At}_\beta \left(\sigma(M^\beta, \mathfrak{d}_\beta(\mathcal{E})) \right) = \sigma(M, \mathcal{E}).$$

In particular $\mathcal{Q}_K(M, \mathcal{E}) = \mathcal{Q}_K(M^\beta, \mathfrak{d}_\beta(\mathcal{E}) \otimes \mathrm{Sym}(\mathcal{N}_{J_\beta}))$.

Remark 7.3 *In Atiyah-Segal, this equation is proved in the tensor product of $\mathbf{K}_K^0(\mathrm{T}^*M)$ with the field of fractions of $R(K)$. In [1], this more precise relation is proved in $\mathbf{K}_K^0(\mathrm{T}_K^*M)$.*

Proof. We deform the symbol $\sigma(M, \mathcal{E})$ by the vector field β_M . So we apply the deformation introduced in Definition 6.10 with the constant map $\Phi = \beta$. The pushed-symbol $\sigma(M, \mathcal{E}, \beta)$ is

$$\sigma(M, \mathcal{E}, \beta)(m, \nu) := \mathbf{c}_{\mathcal{E}_m}(\tilde{\nu} + \beta \cdot m) : \mathcal{E}_m^+ \rightarrow \mathcal{E}_m^-.$$

The excision property tells us that $\sigma(M, \mathcal{E}) = i_*(\sigma(M, \mathcal{E}, \beta)|_{\mathcal{U}})$ in $\mathbf{K}_K^0(\mathrm{T}_K^*M)$. We now analyze the symbol $\tilde{\sigma} := (\varphi^{-1})^*(\sigma(M, \mathcal{E}, \beta)|_{\mathcal{U}})$ on \mathcal{N} . Let $\pi : \mathcal{N} \rightarrow M^\beta$ be the projection. Over \mathcal{N} , the Clifford bundle $\tilde{\mathcal{E}} := (\varphi^{-1})^*(\mathcal{E}|_{\mathcal{U}})$ is isomorphic to $\pi^*(\mathfrak{d}_\beta(\mathcal{E})) \otimes \pi^*(\bigwedge \overline{\mathcal{N}_{J_\beta}})$. With notations of Subsection 5.3, we may assume that for $(n, \nu) \in \mathrm{T}^*\mathcal{N}$, and $n = (x, v)$, the map $\tilde{\sigma}(n, (\xi, \nu))$ is the exterior product of the map

$$\mathbf{c}_x^1(\tilde{\xi}) : \mathfrak{d}_\beta(\mathcal{E})|_x^+ \mapsto \mathfrak{d}_\beta(\mathcal{E})|_x^-$$

with the map

$$\mathbf{c}_x^2(\tilde{\nu} + \beta \cdot v) : \bigwedge^{\text{even}} \overline{\mathcal{N}_{J_\beta}}|_x \mapsto \bigwedge^{\text{odd}} \overline{\mathcal{N}_{J_\beta}}|_x.$$

Here ξ is viewed as a horizontal vector in $\mathrm{T}_n^*\mathcal{N}$ and $\tilde{\nu}, \beta \cdot v$ are vertical vectors in $\mathrm{T}_n\mathcal{N}$. By definition, this means that $\tilde{\sigma}$ is equal to $\sigma(M^\beta, \mathfrak{d}_\beta(\mathcal{E})) \diamond \mathrm{At}_\beta(N)$.

7.2 Deformed Atiyah-Segal-Singer localization formula

We now prove a slightly more general version of the preceding Atiyah-Segal-Singer localization formula.

Let $\mathcal{N} \rightarrow \mathcal{X}$ be a K -equivariant vector bundle over \mathcal{X} , and let β be a central element in \mathfrak{k} such that $\mathcal{X} = \mathcal{N}^\beta$. Thus β acts invertibly on the vector bundle \mathcal{N} . Let $\Phi : \mathcal{N} \rightarrow \mathfrak{k}^*$ be a K -invariant map and $Z_\Phi \subset \mathcal{N}$ be the set of zeroes of the associated Kirwan vector field κ_Φ . We do not assume the basis \mathcal{X} compact, but we assume that $\Phi_\mathcal{X} := \Phi|_\mathcal{X} : \mathcal{X} \rightarrow \mathfrak{k}^*$ is a proper map.

The intersection of the set Z_Φ with \mathcal{X} is equal to $Z_{\Phi_\mathcal{X}}$.

Lemma 7.4 *Assume that $Z_\Phi \cap \mathcal{X} = \Phi_\mathcal{X}^{-1}(\beta)$. Then $\Phi_\mathcal{X}^{-1}(\beta)$ is a compact component of Z_Φ : there exists a K -invariant neighborhood \mathcal{U} of $\Phi_\mathcal{X}^{-1}(\beta)$ in \mathcal{N} such that $Z_\Phi \cap \mathcal{U} = \Phi_\mathcal{X}^{-1}(\beta)$.*

Proof. Consider a K -invariant decomposition of the tangent bundle $T\mathcal{N} = H \oplus V$ in vertical and horizontal tangent bundles. Here $V = \pi^*\mathcal{N}$. As \mathcal{N} is a K -equivariant vector bundle, for $X \in \mathfrak{k}$, and $n = (x, v) \in \mathcal{N}$, the vertical projection $[X_\mathcal{N}]^V(x, v) \in \mathcal{N}|_x$ of the vector field $X_\mathcal{N}$ depends linearly of v . In other words, there exists $\mu(X) \in \Gamma(\mathcal{X}, \text{End}(\mathcal{N}))$ such that $[X_\mathcal{N}]^V(x, v) = \mu(X)_x \cdot v \in \mathcal{N}|_x$, where $x \in \mathcal{X}$, $v \in \mathcal{N}_x$.

For $X = \beta$, $-\mu(\beta)$ is equal to the linear action $\mathcal{L}(\beta)$ of β on \mathcal{N}_x . If $\mathcal{K} \subset \mathcal{X}$ is a compact subset, there exists $c_\mathcal{K} > 0$ such that $\|[X_\mathcal{N}]^V(x, v)\| \leq c_\mathcal{K} \|X\| \|v\|$ for all $(x, v) \in \pi^{-1}(\mathcal{K})$.

The Kirwan vector field $\kappa_\Phi(n) = -\Phi(n) \cdot n$ admits then the decomposition $\kappa_\Phi = [\kappa_\Phi]^V \oplus [\kappa_\Phi]^H$ with $[\kappa_\Phi]^V(n) = -\mu(\Phi(n))_x \cdot v \in \mathcal{N}|_x$.

Choose a relatively compact open neighborhood \mathcal{V} of $\Phi_\mathcal{X}^{-1}(\beta)$ in \mathcal{X} , and consider $\mathcal{U}_\epsilon := \{n \in \pi^{-1}(\mathcal{V}); \|\Phi(n) - \beta\| < \epsilon\}$ which is an open neighborhood of $\Phi_\mathcal{X}^{-1}(\beta)$ in \mathcal{N} .

If $R(n) = \Phi(n) - \beta$, we have

$$(7.23) \quad \mu(\Phi(n))_x \cdot v = \beta \cdot v + \mu(R(n))_x \cdot v.$$

We choose ϵ sufficiently small so that $\|\mu(R(n))_x \cdot v\| < \|\beta \cdot v\|$ for any $n = (x, v) \in \mathcal{U}_\epsilon$. If $\kappa_\Phi(n) = 0$, taking the scalar product with the vertical vector $\beta \cdot v$, we obtain

$$0 = -\langle \kappa_\Phi(n), \beta \cdot v \rangle = -\langle [\kappa_\Phi]^V(n), \beta \cdot v \rangle = \|\beta \cdot v\|^2 + \langle \mu(R(n))_x \cdot v, \beta \cdot v \rangle$$

and this implies $v = 0$. Thus the set of zeroes of κ_Φ on \mathcal{U}_ϵ is the set of zeroes of κ_Φ on \mathcal{X} .

Let $\mathcal{E} \rightarrow \mathcal{N}$ be a graded Clifford bundle on the manifold \mathcal{N} . We can then define the K -transversally elliptic class $\sigma(\mathcal{N}, \mathcal{E}, Z, \Phi) \in \mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{N})$ attached to the compact component $Z = \Phi_{\mathcal{X}}^{-1}(\beta)$.

Let $\mathfrak{d}_\beta(\mathcal{E})$ be the graded Clifford bundle on \mathcal{X} induced by \mathcal{E} (see Definition 7.1). We can also define the K -transversally elliptic class $\sigma(\mathcal{X}, \mathfrak{d}_\beta(\mathcal{E}), Z, \Phi_{\mathcal{X}}) \in \mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{X})$.

Proposition 7.5 (Variation on Atiyah-Segal localization formula) *The following relation holds in $\mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{N})$:*

$$(7.24) \quad \mathbf{At}_\beta(\sigma(\mathcal{X}, \mathfrak{d}_\beta(\mathcal{E}), Z, \Phi_{\mathcal{X}})) = \sigma(\mathcal{N}, \mathcal{E}, Z, \Phi).$$

In particular $\mathcal{Q}_K(\mathcal{N}, \mathcal{E}, Z, \Phi) = \mathcal{Q}_K(\mathcal{X}, \mathfrak{d}_\beta(\mathcal{E}) \otimes \text{Sym}(\mathcal{N}_{J_\beta}), Z, \Phi_{\mathcal{X}})$.

Proof. The proof is a variation of the proof of Proposition 7.2. We consider the vector field $\kappa^0(n) = -\beta \cdot n \oplus \kappa_{\mathcal{X}}(x)$ where $\kappa_{\mathcal{X}}(x) \in T_x \mathcal{X}$ is lifted horizontally in $\mathbb{T}\mathcal{N} = \pi^* \mathbb{T}\mathcal{X} \oplus \pi^* \mathcal{N}$. Then $\kappa^t = t\kappa^0 + (1-t)\kappa_\Phi$ has vertical component $-\beta \cdot v + (1-t)\mu(R(n))_x \cdot v$. The restriction to \mathcal{X} of κ^t is the vector field $\kappa_{\Phi_{\mathcal{X}}}$ for any $t \in [0, 1]$. We then consider

$$\sigma^t(n, \nu) = \mathbf{c}_n(\tilde{\nu} - \kappa^t(n))$$

for $t \in [0, 1]$. We see, as before, that $\text{Char}(\sigma^t)$ intersected with $\mathbb{T}_K^*(\mathcal{N})$ remains equal to Z . Thus the class of σ^t is constant in $\mathbf{K}_K^0(\mathbb{T}_K^* \mathcal{N})$, and σ^0 is equal to $\mathbf{At}_\beta(\sigma(\mathcal{X}, \mathfrak{d}_\beta(\mathcal{E}), Z, \Phi_{\mathcal{X}}))$.

8 Non abelian localization formula

In this section, given an equivariant map $\Phi : M \rightarrow \mathfrak{k}^*$, we establish a general localization formula for the equivariant index of a Dirac operator.

8.1 Localization on $\Phi^{-1}(0)$

Let $\Phi : M \rightarrow \mathfrak{k}^*$ be an equivariant proper map. We describe what is the representation $\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)$, in the special case where 0 is a regular value of Φ , and where K acts infinitesimally freely on the compact submanifold $P = \Phi^{-1}(0)$. Let $M_{red} := P/K$ be the corresponding ‘‘reduced’’ space, and let $\pi : P \rightarrow M_{red}$ be the projection map.

Remark 8.1 *By Proposition 6.6, the hypothesis that 0 is a regular value implies that K acts infinitesimally freely on P in the case where Φ is a moment map.*

On P , we obtain an exact sequence $0 \rightarrow TP \rightarrow TM|_P \xrightarrow{T\Phi} [\mathfrak{k}^*] \rightarrow 0$, where $[\mathfrak{k}^*]$ is the trivial bundle $P \times \mathfrak{k}^*$. We have also an orthogonal decomposition $TP = T_K P \oplus [\mathfrak{k}]$ where $[\mathfrak{k}]$ is the subbundle identified to $P \times \mathfrak{k}$ through the map $(p, X) \mapsto X \cdot p$. So $TM|_P$ admits the orthogonal decomposition $TM|_P \simeq T_K P \oplus [\mathfrak{k}] \oplus [\mathfrak{k}^*]$. We rewrite this as

$$(8.25) \quad TM|_P \simeq T_K P \oplus [\mathfrak{k}_{\mathbb{C}}]$$

with the convention $[\mathfrak{k}] = P \times (\mathfrak{k} \otimes \mathbb{R})$ and $[\mathfrak{k}^*] = P \times (\mathfrak{k} \otimes i\mathbb{R})$. Note that the bundle $T_K P$ is naturally identified with $\pi^*(TM_{red})$.

Following Definition 7.1, we can divide the graded Clifford bundle $\mathcal{E}|_P$ by the Spin^c -bundle $\bigwedge \mathfrak{k}_{\mathbb{C}}$ for the vector space $\mathfrak{k}_{\mathbb{C}}$.

Definition 8.2 *Let \mathcal{E}_{red} be the graded Clifford bundle on the vector bundle $T_K P \rightarrow P$ such that*

$$\mathcal{E}|_P \simeq \mathcal{E}_{red} \otimes [\bigwedge \mathfrak{k}_{\mathbb{C}}]$$

is an isomorphism of graded Clifford bundles on $TM|_P$. We still denote by \mathcal{E}_{red} the induced Clifford (orbi-)bundle on TM_{red} .

For any representation space V_{μ} of K , we can form the twisted Clifford bundle $\mathcal{E}_{red} \otimes \mathcal{V}_{\mu}^*$ of the Clifford bundle \mathcal{E}_{red} by the equivariant (orbi-)bundle $\mathcal{V}_{\mu}^* = P \times_K V_{\mu}^*$.

Consider the horizontal Dirac symbol, $\sigma_{hor}(P, \mathcal{E}_{red})$ on P defined in Section 5.6. This is a K transversally elliptic symbol on P . We now prove

Theorem 8.3 *If 0 is a regular value of Φ and K acts locally freely on $\Phi^{-1}(0)$, then $\Phi^{-1}(0)$ is a component of Z_{Φ} , and we have*

$$\begin{aligned} \mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi) &= \text{Index}_K^P(\sigma_{hor}(P, \mathcal{E}_{red})) \\ &= \sum_{\mu \in \hat{K}} \mathcal{Q}(M_{red}, \mathcal{E}_{red} \otimes \mathcal{V}_{\mu}^*) V_{\mu}. \end{aligned}$$

In particular we have $[\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)]^K = \mathcal{Q}(M_{red}, \mathcal{E}_{red})$.

Proof. Let us study the pushed symbol $\sigma(M, \mathcal{E}, \Phi)(m, \nu) = \sigma_{\mathcal{E}_m}(\tilde{\nu} - \kappa_{\Phi}(m))$ when m is in a neighborhood of $P = \Phi^{-1}(0)$. So we can assume that $M = P \times \mathfrak{k}^*$. For a point $m = (x, \xi) \in P \times \mathfrak{k}^*$, the tangent vector $\tilde{\nu} \in T_m M$ is represented by $\eta \oplus a \oplus b$: here $\eta \in T_K P$, $a \in \mathfrak{k}$ represents the tangent vector $a \cdot x \in T_x P$, and $b = T_m \Phi(\tilde{\nu}) \in \mathfrak{k}^*$. For $X \in \mathfrak{k}$, the vector field $X_M(x, \xi)$ associated to the action of K on $P \times \mathfrak{k}^*$ is represented by $0 \oplus -X \oplus [\xi, X]$:

in particular the Kirwan vector field $\kappa_\Phi(x, \xi)$ is equal to $0 \oplus -\xi \oplus 0$. For the pushed symbol we obtain

$$\sigma(M, \mathcal{E}, \Phi)(m, \nu) = \mathbf{c}_{\mathcal{E}_{red}}(\eta) \boxtimes \mathbf{c}_{\mathfrak{k}_\mathbb{C}}((a + \xi) \oplus ib).$$

Let us see that the support of the pushed symbol intersected with T_K^*M is just equal to P . Indeed if $(x, \xi; \eta \oplus a \oplus b) \in T_K^*M$, we have $([\xi, X], b) - (X, a) = 0$ for all $X \in \mathfrak{k}$, e.g. $a = [b, \xi]$. Then, if $\mathbf{c}_{\mathcal{E}_x}(\eta) \boxtimes \mathbf{c}_{\mathfrak{k}_\mathbb{C}}((a + \xi) \oplus ib)$ is not invertible, we obtain $\eta = 0$, $b = 0$ and $a + \xi = 0$. If furthermore $(x, \xi; \eta \oplus a \oplus b)$ is in T_K^*M , we obtain $a = \xi = 0$. Thus we see that $\Phi^{-1}(0) = P$ is a component of Z_Φ , and $\sigma(M, \mathcal{E}, \Phi)$ restricted to a neighborhood of P defines a transversally elliptic symbol on M . We can then define its index $\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)$.

Consider now the injection $i : P \rightarrow P \times \mathfrak{k}^* = M$ and the horizontal symbol $\sigma_{hor}(P, \mathcal{E}_{red})$ on P . By definition (see Section 5.4), the direct image $i_!\sigma_{hor}(P, \mathcal{E}_{red})$ is defined to be the restriction to T_K^*M of the symbol

$$\mathbf{c}_{\mathcal{E}_{red}}(\eta) \boxtimes \mathbf{c}_{\mathfrak{k}_\mathbb{C}}(\xi \oplus ib).$$

We see that for $t \in [0, 1]$, the support of the symbol

$$\sigma^t(x, \xi, \nu) = \mathbf{c}_{\mathcal{E}_{red}}(\eta) \boxtimes \mathbf{c}_{\mathfrak{k}_\mathbb{C}}((ta + \xi) \oplus ib)$$

intersected with T_K^*M remains equal to P .

Thus $\sigma(M, \mathcal{E}, \Phi)$ defines the same class in K -theory than the symbol $i_!\sigma_{hor}(P, \mathcal{E}_{red})$. Thus by the direct image property of the symbol, we obtain our result.

If furthermore we had an action of $G \times K$ on M , we would obtain an action of G in each space $\mathcal{Q}(M_{red}, \mathcal{E}_{red} \otimes \mathcal{V}_\mu^*)$ and the $G \times K$ decomposition

$$(8.26) \quad \mathcal{Q}_{G \times K}(M, \mathcal{E}, \Phi^{-1}(0), \Phi) = \sum_{\mu \in \hat{K}} \mathcal{Q}_G(M_{red}, \mathcal{E}_{red} \otimes \mathcal{V}_\mu^*) \otimes V_\mu.$$

8.2 Coadjoint orbits and slices

Let $\Phi : M \rightarrow \mathfrak{k}^*$ be our K -equivariant map (with K not necessarily connected).

Consider the coadjoint action of K on \mathfrak{k}^* . Let $\xi \in \mathfrak{k}^*$. Let us consider the K_ξ -invariant decomposition $\mathfrak{k} = \mathfrak{k}_\xi \oplus \mathfrak{q}$, with dual decomposition $\mathfrak{k}^* = \mathfrak{k}_\xi^* \oplus \mathfrak{q}^*$.

Proposition 8.4 *Let $\xi \in \mathfrak{k}^*$, and let B be a sufficiently small K_ξ -invariant neighborhood of ξ in \mathfrak{k}_ξ^* . Then*

- $Y = \Phi^{-1}(B)$ is a K_ξ -invariant submanifold of M (perhaps empty).
- $\Phi^{-1}(KB)$ is an open neighborhood of $\Phi^{-1}(K\xi)$, and is diffeomorphic to $K \times_{K_\xi} Y$.

We will call the manifold Y a slice of M at ξ .

Proof. Consider the coadjoint orbit $K\xi$. The map $X \rightarrow X \cdot \xi$ is a surjective map from \mathfrak{k} to \mathfrak{q}^* and \mathfrak{k}_ξ^* identifies to the normal space to the coadjoint orbit $K\xi$ at ξ . The tubular neighborhood theorem implies that if B is sufficiently small, then KB is a neighborhood of $K\xi$ isomorphic to $K \times_{K_\xi} B$.

Write $\Phi_{\mathfrak{q}}$ for the component of Φ in \mathfrak{q}^* . The differential map $T_y\Phi_{\mathfrak{q}} : T_yM \rightarrow \mathfrak{q}^*$ is surjective, if $\Phi(y) = \xi$. Thus it is surjective at any point $y \in Y$, if B is sufficiently small. The rest of the proof is standard.

We also consider ξ as an element of \mathfrak{k}_ξ , using our K -invariant identification $\mathfrak{k} = \mathfrak{k}^*$. So ξ induces an invertible skew-symmetric transformation on \mathfrak{q} . Denote by \mathfrak{q}_{J_ξ} the vector space $\mathfrak{q} = \mathfrak{k}/\mathfrak{k}_\xi$ equipped with the complex structure J_ξ ,

Suppose now that M carries a graded Clifford bundle \mathcal{E} . The tangent bundle TM , when restricted to the slice Y , decomposes as $TM|_Y \simeq TY \oplus [\mathfrak{k}/\mathfrak{k}_\xi]$, where $[\mathfrak{k}/\mathfrak{k}_\xi]$ denotes the trivial bundle $\mathfrak{k}/\mathfrak{k}_\xi \times Y$.

As in Definition 7.1, we can divide the graded Clifford bundle $\mathcal{E}|_Y$ by the Spin^c -bundle $\bigwedge \mathfrak{q}_{J_\xi}$ for the Euclidean vector space \mathfrak{q} . There exists a unique graded Clifford bundle \mathcal{E}_{dY} over the tangent bundle TY , such that, when we restrict \mathcal{E} to Y , we get the isomorphism

$$(8.27) \quad \mathcal{E}|_Y \simeq \mathcal{E}_{dY} \otimes \left[\bigwedge \mathfrak{q}_{J_\xi} \right]$$

of graded Clifford bundles relative to the decomposition $TM|_Y \simeq TY \oplus [\mathfrak{k}/\mathfrak{k}_\xi]$.

8.3 Localization on Z_β

We consider our equivariant map $\Phi : M \rightarrow \mathfrak{k}^*$, and we assume that

$$Z_\beta = K(M^\beta \cap \Phi^{-1}(\beta))$$

is a compact component of Z_Φ . Here β is a non zero element in \mathfrak{k}^* .

Let $\mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi) \in \hat{R}(K)$ be the localized equivariant index attached to an equivariant graded Clifford bundle \mathcal{E} . Let $\mathfrak{d}_\beta(\mathcal{E})$ be the induced graded Clifford bundle on M^β (see Section 7.1). The restriction of Φ to M^β is a

map that takes value in \mathfrak{k}_β^* : for simplicity we still denote $\Phi : M^\beta \rightarrow \mathfrak{k}_\beta^*$. The set $M^\beta \cap \Phi^{-1}(\beta)$ is a compact component of $Z_\Phi \cap M^\beta$. Let \mathcal{N}_{J_β} be the (complex) normal bundle of M^β in M polarized by β . The aim of this subsection is to prove the following localization formula.

Theorem 8.5 *We have the following relation in $\hat{R}(K)$:*

$$\mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi) = \text{Ind}_{K_\beta}^K \left(\mathcal{Q}_{K_\beta}(M^\beta, \mathfrak{d}_\beta(\mathcal{E}) \otimes \text{Sym}(\mathcal{N}_{J_\beta}), M^\beta \cap \Phi^{-1}(\beta), \Phi) \otimes \bigwedge (\mathfrak{k}/\mathfrak{k}_\beta \otimes \mathbb{C}) \right).$$

Proof. The proof will have two steps.

First Step : Let $B \subset \mathfrak{k}^*$ be a small K_β -invariant ball around β in \mathfrak{k}_β^* and let $Y = \Phi^{-1}(B)$ be the corresponding slice. A neighborhood of Z_β in M is then diffeomorphic with $K \times_{K_\beta} Y$. We denote by $\Phi_Y : Y \rightarrow \mathfrak{k}_\beta^*$ the restriction of Φ to the submanifold Y . Let \mathcal{E}_{dY} be the induced Clifford bundle on TY (see (8.27)).

The restriction of the symbol $\sigma(M, \mathcal{E}, \Phi)$ to T^*Y is equal to $\sigma(Y, \mathcal{E}_{dY}, \Phi_Y) \otimes \bigwedge \mathfrak{q}_{J_\beta}$, hence the induction formula of Proposition 4.7 gives that

$$\begin{aligned} \mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi) &= \text{Ind}_{K_\beta}^K \left(\mathcal{Q}_{K_\beta}(Y, \mathcal{E}_{dY}, M^\beta \cap \Phi^{-1}(\beta), \Phi_Y) \otimes \bigwedge \mathfrak{q}_{J_\beta} \right) \\ &\text{(as } \Phi^{-1}(\beta) \text{ is contained in } Y, M^\beta \cap \Phi^{-1}(\beta) = Y^\beta \cap \Phi_Y^{-1}(\beta)). \end{aligned}$$

Second Step : Now we use the variation of the Atiyah-Segal-Singer localization formula for computing $\mathcal{Q}_{K_\beta}(Y, \mathcal{E}_{dY}, M^\beta \cap \Phi^{-1}(\beta), \Phi_Y)$. As we are working on a neighborhood of $Y^\beta \cap \Phi_Y^{-1}(\beta)$, we may assume that Y is the normal bundle \mathcal{N}_Y of Y^β in Y .

Let $\mathfrak{d}_\beta(\mathcal{E}_{dY})$ be the Clifford bundle on the submanifold Y^β determined by \mathcal{E}_{dY} (see Section 7.1). Let $(\mathcal{N}_Y)_{J_\beta}$ be the normal bundle of Y^β in Y polarized by β . We apply the variation of Atiyah-Segal-Singer localisation formula to $\mathcal{N}_Y \rightarrow Y^\beta$ (the group K being now K_β) and we obtain the following formula:

$$\begin{aligned} \mathcal{Q}_{K_\beta}(Y, \mathcal{E}_{dY}, M^\beta \cap \Phi^{-1}(\beta), \Phi_Y) &= \\ &\mathcal{Q}_{K_\beta}(Y^\beta, \mathfrak{d}_\beta(\mathcal{E}_{dY}) \otimes \text{Sym}((\mathcal{N}_Y)_{J_\beta}), M^\beta \cap \Phi^{-1}(\beta), \Phi_Y). \end{aligned}$$

The manifold Y^β is an open subset of M^β , and for the normal bundles \mathcal{N} of M^β in M versus \mathcal{N}_Y (the normal bundle of Y^β in Y), we have $\mathcal{N} = \mathcal{N}_Y \oplus [\mathfrak{k}/\mathfrak{k}_\beta]$. For the polarized ones, we get $\mathcal{N}_{J_\beta} = (\mathcal{N}_Y)_{J_\beta} \oplus [\mathfrak{q}_{J_\beta}]$. If we use the relation $\text{Sym}(\mathfrak{q}_{J_\beta}) \otimes \bigwedge \mathfrak{q}_{J_\beta} = 1$ in $\hat{R}(K_\beta)$, we see that

$$\begin{aligned} \text{Sym}(\mathcal{N}_{J_\beta}) \otimes \left[\bigwedge (\mathfrak{k}/\mathfrak{k}_\beta \otimes \mathbb{C}) \right] &= \text{Sym}((\mathcal{N}_Y)_{J_\beta}) \otimes \text{Sym}(\mathfrak{q}_{J_\beta}) \otimes \left[\bigwedge \mathfrak{q}_{J_\beta} \right] \otimes \left[\bigwedge \overline{\mathfrak{q}_{J_\beta}} \right] \\ &= \text{Sym}((\mathcal{N}_Y)_{J_\beta}) \otimes \left[\bigwedge \overline{\mathfrak{q}_{J_\beta}} \right]. \end{aligned}$$

So we are left to check that $\mathfrak{d}_\beta(\mathcal{E}_{dY}) \otimes [\bigwedge \mathfrak{q}_{J_\beta}]$ and $\mathfrak{d}_\beta(\mathcal{E}) \otimes [\bigwedge \overline{\mathfrak{q}_{J_\beta}}]$ are equal⁵ as Clifford module over TY^β .

We have the identity $\mathcal{E}|_Y = [\bigwedge \mathfrak{q}_{J_\beta}] \otimes \mathcal{E}_{dY}$ as Clifford modules over $\mathfrak{k}/\mathfrak{k}_\beta \oplus \mathrm{TY}$. We have also $(\mathcal{E}_{dY})|_{Y^\beta} = \bigwedge (\overline{\mathcal{N}_Y})_{J_\beta} \otimes \mathfrak{d}_\beta(\mathcal{E}_{dY})$ as an equality of Clifford modules over $\mathcal{N}_Y \oplus \mathrm{TY}^\beta$. Finally we get

$$(8.28) \quad \mathcal{E}|_{Y^\beta} = \bigwedge (\overline{\mathcal{N}_Y})_{J_\beta} \otimes \left[\bigwedge \mathfrak{q}_{J_\beta} \right] \otimes \mathfrak{d}_\beta(\mathcal{E}_{dY})$$

as an equality of Clifford modules over $\mathcal{N}_Y \oplus [\mathfrak{k}/\mathfrak{k}_\beta] \oplus \mathrm{TY}^\beta$. By definition, we have also $\mathcal{E}|_{M^\beta} = \bigwedge \overline{\mathcal{N}_{J_\beta}} \otimes \mathfrak{d}_\beta(\mathcal{E})$ as an equality of modules over $\mathcal{N} \oplus \mathrm{TM}^\beta$. If we restrict the previous identity to the open subset Y^β , we get

$$(8.29) \quad \mathcal{E}|_{Y^\beta} = \bigwedge (\overline{\mathcal{N}_Y})_{J_\beta} \otimes \left[\bigwedge \overline{\mathfrak{q}_{J_\beta}} \right] \otimes \mathfrak{d}_\beta(\mathcal{E}).$$

If we use (8.28) and (8.29), we get the desired identity.

8.4 The non abelian localization theorem

Let $\mathcal{E} \rightarrow M$ be a equivariant graded Clifford module over an even dimensional K -manifold M (M is not necessarily compact, the group K is compact but not necessarily connected). Let Φ be a moment map, such that Z_Φ is compact. We can then construct $\mathcal{Q}_K(M, \mathcal{E}, \Phi) \in \hat{R}(K)$. If M is compact, $\mathcal{Q}_K(M, \mathcal{E}, \Phi)$ is just equal to $\mathcal{Q}_K(M, \mathcal{E})$.

The set of zeroes Z_Φ is a disjoint union of its components $Z_\beta, \beta \in \mathcal{B}$, with \mathcal{B} finite (see (6.16)). The following decomposition of $\mathcal{Q}_K(M, \mathcal{E}, \Phi)$ is the \mathbf{K} -theoretical analogue of Witten non abelian localization theorem in equivariant cohomology. This reduces the study of $\mathcal{Q}_K(M, \mathcal{E}, \Phi)$ by induction to the neighborhood of the zeroes of the moment map.

Theorem 8.6 (Non abelian localization theorem) *We have*

$$\mathcal{Q}_K(M, \mathcal{E}, \Phi) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi).$$

Furthermore, for $\beta \neq 0$, the generalized character $\mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi)$ is equal to

$$\mathrm{Ind}_{K_\beta}^K \left(\mathcal{Q}_{K_\beta}(M^\beta, \mathfrak{d}_\beta(\mathcal{E}) \otimes \mathrm{Sym}(\mathcal{N}_{J_\beta}), \Phi^{-1}(\beta) \cap M^\beta, \Phi) \otimes \bigwedge \mathfrak{q}_{J_\beta} \otimes \bigwedge \overline{\mathfrak{q}_{J_\beta}} \right).$$

⁵In fact we can check that $\mathfrak{d}_\beta(\mathcal{E}_{dY}) = (-1)^{\dim \mathfrak{q}_{J_\beta}} \det(\mathfrak{q}_{J_\beta})^{-1} \mathfrak{d}_\beta(\mathcal{E})$.

Here, \mathcal{B} is a set of representatives of the orbits of K in $\Phi(Z_\Phi)$. For $\beta \in \mathcal{B}$, M^β is the submanifold of M stable by $\beta \in \mathfrak{k}$, Z_β is $K(\Phi^{-1}(\beta) \cap M^\beta)$, \mathcal{N}_{J_β} is the normal bundle on M^β in M polarized by β , $\mathfrak{d}_\beta(\mathcal{E})$ is the K_β equivariant Clifford bundle on M^β defined by

$$\mathcal{E}|_{M^\beta} = \mathfrak{d}_\beta(\mathcal{E}) \otimes \bigwedge \overline{\mathcal{N}}_{J_\beta},$$

and \mathfrak{q}_{J_β} denotes the vector space $\mathfrak{q} = \mathfrak{k}/\mathfrak{k}_\beta$ equipped with the complex structure J_β .

As β is a central element in \mathfrak{k}_β , we are reduced by induction to the study of a localized character at a central element for the lower dimensional manifold M^β acted on by the subgroup K_β .

When 0 is a regular value of the moment map, the term $\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)$ is given explicitly:

$$\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi) = \sum_{\lambda \in \hat{K}} \mathcal{Q}(M_{red}, \mathcal{E}_{red} \otimes \mathcal{V}_\lambda^*) V_\lambda.$$

It follows that the multiplicities $m(\lambda)$ of the irreducible representations V_λ in $\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)$ are given explicitly as the equivariant index of a orbi-bundle over the orbifold $M_{red} = \Phi^{-1}(0)/K$.

When K is connected, we may parameterize \hat{K} by the lattice of weights intersected with a positive Weyl chamber. In this case Kawasaki formula implies that $\lambda \rightarrow m(\lambda)$ is a quasi-polynomial function of λ . We will return to this theme in Subsection 12.5.

8.5 Example of $T^*\tilde{K}$

We consider in this subsection the paradigmatic example of the cotangent space $T^*\tilde{K}$ where \tilde{K} is a compact Lie group. This is a \tilde{K} -Hamiltonian manifold with moment map Φ , and in the philosophy of quantization, its quantization is $L^2(\tilde{K})$. We here study the space $\mathcal{Q}_{\tilde{K}}(T^*\tilde{K}, \mathcal{E}, \Phi)$, and verify that indeed this is the space of \tilde{K} -finite functions on \tilde{K} for a particular Clifford module \mathcal{E} . This example will not be used in the rest of this article.

Let \tilde{K} be a compact Lie group and let K be a closed subgroup. Let i denote the inclusion of K into \tilde{K} , $i : \mathfrak{k} \rightarrow \tilde{\mathfrak{k}}$ the induced embedding of Lie algebras, and $\pi : \tilde{\mathfrak{k}}^* \rightarrow \mathfrak{k}^*$ the dual projection.

We consider the following action of $\tilde{K} \times K$ on \tilde{K} : $(\tilde{k}, k) \cdot a = \tilde{k}ak^{-1}$.

The tangent bundle $T\tilde{K}$ is identified with $\tilde{K} \times \tilde{\mathfrak{k}}$ through the left translations: to $(a, X) \in \tilde{K} \times \tilde{\mathfrak{k}}$ we associate $\frac{d}{dt}e^{tX}a|_{t=0} \in T_a\tilde{K}$. The action of

$\tilde{K} \times K$ on the cotangent bundle $T^*\tilde{K} \simeq \tilde{K} \times \tilde{\mathfrak{k}}^*$ is then

$$(\tilde{k}, k) \cdot (a, \xi) = (\tilde{k}ak^{-1}, \tilde{k}\xi).$$

The Liouville form λ of $T^*\tilde{K}$ is thus given by $\lambda_{a,\xi}(X, \eta) = \langle \xi, X \rangle$ for $(a, \xi) \in T^*\tilde{K} \simeq \tilde{K} \times \tilde{\mathfrak{k}}^*$ and $(X, \eta) \in T_a\tilde{K} \times T_\xi\tilde{\mathfrak{k}}$. The symplectic form on $T^*\tilde{K}$ is $\Omega := d\lambda$.

The corresponding moment map relative to the $\tilde{K} \times K$ -action is the map $\Phi = \Phi_{\tilde{K}} \oplus \Phi_K : T^*\tilde{K} \rightarrow \tilde{\mathfrak{k}}^* \oplus \mathfrak{k}^*$ defined by

$$(8.30) \quad \Phi_{\tilde{K}}(a, \xi) = -\xi, \quad \Phi_K(a, \xi) = \pi(a^{-1} \cdot \xi).$$

Lemma 8.7 *The set $Z_\Phi \subset T^*\tilde{K}$ is equal to $\tilde{K} \times \{0\}$. More generally, if we consider the equivariant map $\Phi_t = \Phi_{\tilde{K}} \oplus t\Phi_K$ for any $t \geq 0$, the set $Z_{\Phi_t} \subset T^*\tilde{K}$ is still equal to $\tilde{K} \times \{0\}$.*

Proof. Let κ_t be the Kirwan vector field on $T^*\tilde{K}$ attached to the map Φ_t . A direct computation gives that for $(a, \xi) \in T^*\tilde{K}$, the vector $\kappa_t(a, \xi) \in T_{(a,\xi)}(\tilde{K} \times \tilde{\mathfrak{k}}^*) = \tilde{\mathfrak{k}} \times \tilde{\mathfrak{k}}^*$ is equal to $(\kappa_t^1(a, \xi), 0)$ with

$$\kappa_t^1(a, \xi) = \xi + t a \pi(a^{-1}\xi).$$

For $t \geq 0$, we see that $\kappa_t^1(a, \xi) = 0$ if and only if $\xi = 0$.

Let \mathcal{E} be a $\tilde{K} \times K$ -equivariant graded Clifford bundle on the manifold $T^*\tilde{K}$. Thanks to Lemma 8.7, we know that Z_Φ is compact. Hence we can consider the generalized index

$$\mathcal{Q}_{\tilde{K} \times K}(T^*\tilde{K}, \mathcal{E}, \Phi) \in \hat{R}(\tilde{K} \times K).$$

The restriction of \mathcal{E} at $(1, 0) \in T^*\tilde{K}$ defines a graded Clifford module $\mathcal{E}|_{(1,0)}$ for the Euclidean vector space $T(T^*\tilde{K})|_{(1,0)} \simeq \tilde{\mathfrak{k}} \times \tilde{\mathfrak{k}}^*$. We use the identification $\tilde{\mathfrak{k}} \times \tilde{\mathfrak{k}}^* \simeq \tilde{\mathfrak{k}}_{\mathbb{C}}, (X, \xi) \mapsto X \oplus i\xi$. Thus $\bigwedge \tilde{\mathfrak{k}}_{\mathbb{C}}$ is an irreducible Clifford module for $\tilde{\mathfrak{k}} \times \tilde{\mathfrak{k}}^*$.

Following Definition 7.1, we can “divide” the graded Clifford bundle $\mathcal{E}|_{(1,0)}$ by the irreducible Clifford module $\bigwedge \tilde{\mathfrak{k}}_{\mathbb{C}}$.

Definition 8.8 *Let $E_{\tilde{K} \times K}$ be the graded $\tilde{K} \times K$ -representation space such that*

$$\mathcal{E}|_{(1,0)} \simeq E_{\tilde{K} \times K} \otimes \bigwedge \tilde{\mathfrak{k}}_{\mathbb{C}}.$$

is an isomorphism of graded Clifford modules on $\tilde{\mathfrak{k}} \times \tilde{\mathfrak{k}}^$.*

Proposition 8.9 *We have the following relation in $\hat{R}(\tilde{K} \times K)$:*

$$\mathcal{Q}_{\tilde{K} \times K}(\mathbb{T}^* \tilde{K}, \mathcal{E}, \Phi) = \sum_{V \in \hat{K}} (V^* \otimes V|_K) \otimes E_{\tilde{K} \times K}.$$

Remark 8.10 *If $E_{\tilde{K} \times K} = \mathbb{C}$, we obtain that $\mathcal{Q}_{\tilde{K} \times K}(\mathbb{T}^* \tilde{K}, \mathcal{E}, \Phi)$ is $L^2(\tilde{K})$ after completion, as expected.*

Proof. Thanks to Lemma 8.7, we know that $t \in [0, 1] \mapsto \sigma(\mathbb{T}^* \tilde{K}, \mathcal{E}, \Phi_t)$ is an homotopy of transversally elliptic symbols. It follows that $\mathcal{Q}_{\tilde{K} \times K}(\mathbb{T}^* \tilde{K}, \mathcal{E}, \Phi) = \mathcal{Q}_{\tilde{K} \times K}(\mathbb{T}^* \tilde{K}, \mathcal{E}, \Phi_{\tilde{K}})$. We note that the computation of $\mathcal{Q}_{\tilde{K} \times K}(\mathbb{T}^* K, \mathcal{E}, \Phi_{\tilde{K}})$ follows from (8.26), since $Z_{\Phi_{\tilde{K}}} = \Phi_{\tilde{K}}^{-1}(0)$, 0 is a regular value of $\Phi_{\tilde{K}}$ and $\Phi_{\tilde{K}}^{-1}(0)/\tilde{K}$ is reduced to a point.

9 $[Q, R] = 0$ Theorem

Let M be an even dimensional K -manifold equipped with an equivariant graded Clifford bundle \mathcal{E} .

When M is compact, we consider the representation $\mathcal{Q}_K(M, \mathcal{E})$. In this section, we study $[\mathcal{Q}_K(M, \mathcal{E})]^K$, with the help of a moment map Φ , and the non abelian localization formula, in particular, we will find sufficient conditions on the couple (\mathcal{E}, Φ) in order that the only component of Z_Φ contributing to $[\mathcal{Q}_K(M, \mathcal{E})]^K$ is $\Phi^{-1}(0)$. We will be considering in more detail the case where M is an almost complex manifold.

Our discussion extends to the case where M is non-compact. We will study the generalized character $\mathcal{Q}_K(M, \mathcal{E}, \Phi)$ attached to a moment map Φ such that Z_Φ is compact.

9.1 A preliminary formula

Let M be an even dimensional K -manifold equipped with an equivariant graded Clifford bundle \mathcal{E} and a moment map $\Phi : M \rightarrow \mathfrak{k}^*$ such that Z_Φ is compact. We study $[\mathcal{Q}_K(M, \mathcal{E}, \Phi)]^K$ via the non abelian localization formula. The notations in this subsection are as in Subsection 8.4. From Theorem 8.6, we obtain

$$[\mathcal{Q}_K(M, \mathcal{E}, \Phi)]^K = \sum_{\beta \in \mathcal{B}} [\mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi)]^K$$

and if $\beta \neq 0$, the integer $[\mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi)]^K$ is equal to

$$\left[\mathcal{Q}_{K_\beta}(M^\beta, \mathfrak{d}_\beta(\mathcal{E}) \otimes \text{Sym}(\mathcal{N}_{J_\beta}), \Phi^{-1}(\beta) \cap M^\beta, \Phi) \otimes \bigwedge \mathfrak{q}_{J_\beta} \otimes \bigwedge \bar{\mathfrak{q}}_{J_\beta} \right]^{K_\beta}.$$

Using Lemma 4.1, the vanishing of this last expression is automatic if the linear action (denoted $\mathcal{L}(\beta)$) of β on the fibers of $\mathfrak{d}_\beta(\mathcal{E}) \otimes \text{Sym}(\mathcal{N}_{J_\beta}) \otimes \wedge \mathfrak{q}_{J_\beta} \otimes \wedge \overline{\mathfrak{q}}_{J_\beta}$ has only non-zero eigenvalues. We thus introduce some definitions.

Consider a K_β -equivariant vector bundle $\mathcal{F} \rightarrow M^\beta$. Let \mathcal{X} be a connected component of M^β . Let $c \in \mathbb{R}$. If all the eigenvalues of $\mathcal{L}(\beta)$ acting on $\mathcal{F}|_{\mathcal{X}}$ are of the form $i\theta$ with $\theta > c$ (respectively $\theta \geq c$), we say that $\frac{1}{i}\mathcal{L}(\beta) > c$ on $\mathcal{F}|_{\mathcal{X}}$ (respectively $\frac{1}{i}\mathcal{L}(\beta) \geq c$).

Proposition 9.1 *Let $\beta \neq 0$.*

• *If $\frac{1}{i}\mathcal{L}(\beta) \geq 0$ on $\mathfrak{d}_\beta(\mathcal{E})|_{\mathcal{X}} \otimes \det(\mathfrak{q}_{J_\beta})^{-1}$ for any connected component $\mathcal{X} \subset M^\beta$ such that $\mathcal{X} \cap \Phi^{-1}(\beta) \neq \emptyset$. Then*

$$[\mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi)]^K = \left[\mathcal{Q}_{K_\beta}(M^\beta, \mathcal{E}_{[\beta]}, \Phi^{-1}(\beta) \cap M^\beta, \Phi) \right]^{K_\beta}$$

where

$$(9.31) \quad \mathcal{E}_{[\beta]} := (-1)^{\dim \mathfrak{q}_{J_\beta}} [\mathfrak{d}_\beta(\mathcal{E}) \otimes \det(\mathfrak{q}_{J_\beta})^{-1}]^\beta$$

is a graded K_β -equivariant Clifford bundle on M^β .

• *If $\frac{1}{i}\mathcal{L}(\beta) > 0$ on $\mathfrak{d}_\beta(\mathcal{E})|_{\mathcal{X}} \otimes \det(\mathfrak{q}_{J_\beta})^{-1}$ for any connected component $\mathcal{X} \subset M^\beta$ such that $\mathcal{X} \cap \Phi^{-1}(\beta) \neq \emptyset$, we have $\mathcal{E}_{[\beta]} = [0]$ in a neighborhood of $M^\beta \cap \Phi^{-1}(\beta)$ and then $[\mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi)]^K = 0$.*

Remark that $[\mathfrak{d}_\beta(\mathcal{E}) \otimes \det(\mathfrak{q}_{J_\beta})^{-1}]^\beta$ is isomorphic to the subbundle of $\mathfrak{d}_\beta(\mathcal{E})$ formed by the vectors satisfying $\mathcal{L}(\beta)v = \mathbf{Tr}_{\mathfrak{q}_{J_\beta}}(\beta)v$.

Proof.

Introduce the union $M_c^\beta = \cup \mathcal{X}$ of the connected components \mathcal{X} of M^β such that \mathcal{X} intersects $\Phi^{-1}(\beta)$. Then M_c^β is K_β invariant and clearly, we can replace M^β by M_c^β in Theorem 8.6. Using Lemma 4.1, we obtain $[\mathcal{Q}_K(M, \mathcal{E}, Z_\beta, \Phi)]^K$ is equal to

$$\left[\mathcal{Q}_{K_\beta}(M_c^\beta, [\mathcal{F}_{M_c^\beta}]^\beta, M_c^\beta \cap \Phi^{-1}(\beta), \Phi) \right]^{K_\beta}$$

where $\mathcal{F}_{M_c^\beta} = \mathfrak{d}_\beta(\mathcal{E})|_{M_c^\beta} \otimes (\det(\mathfrak{q}_{J_\beta})^{-1} \otimes \det(\mathfrak{q}_{J_\beta})) \otimes \text{Sym}(\mathcal{N}_{J_\beta})|_{M_c^\beta} \otimes \wedge \mathfrak{q}_{J_\beta} \otimes \wedge \overline{\mathfrak{q}}_{J_\beta}$.

Let \mathcal{X} be a connected component of M_c^β , thus \mathcal{X} intersects $\Phi^{-1}(\beta)$. On $\text{Sym}(\mathcal{N}_{J_\beta})|_{\mathcal{X}} \otimes \wedge \mathfrak{q}_{J_\beta}$, we have $\frac{1}{i}\mathcal{L}(\beta) \geq 0$ by definition of the complex structures associated to β , and $[\text{Sym}(\mathcal{N}_{J_\beta})|_{\mathcal{X}} \otimes \wedge \mathfrak{q}_{J_\beta}]^\beta = [\mathbb{C}]$. Similarly

$\frac{1}{i}\mathcal{L}(\beta) \geq 0$ on $[\det(\mathfrak{q}_{J_\beta}) \otimes \wedge \overline{\mathfrak{q}_{J_\beta}}]$ and $[\det(\mathfrak{q}_{J_\beta}) \otimes \wedge \overline{\mathfrak{q}_{J_\beta}}]^\beta = (-1)^{\dim \mathfrak{q}_{J_\beta}}$. Then the Clifford bundle $[\mathcal{F}_\mathcal{X}]^\beta$ is reduced to $\mathcal{E}_{[\beta]}$ if $\frac{1}{i}\mathcal{L}(\beta) \geq 0$ on $\mathfrak{d}_\beta(\mathcal{E})|_\mathcal{X} \otimes \det(\mathfrak{q}_{J_\beta})^{-1}$. This proves the first point. The second point is an obvious consequence of the first.

9.2 Φ -positivity

Definition 9.2 Let $\Phi : M \rightarrow \mathfrak{k}^*$ be an equivariant map.

- A K -equivariant bundle $\mathcal{F} \rightarrow M$ is weakly Φ -positive (resp. strictly Φ -positive) if

$$\frac{1}{i}\mathcal{L}(\beta) \geq 0 \quad (\text{resp. } > 0) \quad \text{on } \mathcal{F}|_\mathcal{X}$$

for any couple (\mathcal{X}, β) where $\mathcal{X} \subset M^\beta$ is a connected component such that $\mathcal{X} \cap \Phi^{-1}(\beta) \neq \emptyset$, and $\beta \neq 0$.

- A K -equivariant Clifford bundle $\mathcal{E} \rightarrow M$ is weakly Φ -positive (resp. strictly Φ -positive) if

$$\frac{1}{i}\mathcal{L}(\beta) \geq 0 \quad (\text{resp. } > 0) \quad \text{on } \mathfrak{d}_\beta(\mathcal{E})|_\mathcal{X} \otimes \det(\mathfrak{q}_{J_\beta})^{-1}$$

for any couple (\mathcal{X}, β) where $\mathcal{X} \subset M^\beta$ is a connected component such that $\mathcal{X} \cap \Phi^{-1}(\beta) \neq \emptyset$ and $\beta \neq 0$.

An obvious example of weakly Φ -positive bundle \mathcal{F} on M is the trivial bundle $M \times \mathbb{C}$.

If \mathcal{E} is a weakly Φ -positive Clifford bundle and \mathcal{F} a weakly Φ -positive bundle, then the Clifford module $\mathcal{E} \otimes \mathcal{F}$ is weakly Φ -positive.

Another example arise when we have an equivariant map $\varphi : M' \rightarrow M$. Suppose that we start with a weakly Φ -positive bundle \mathcal{F} on M . We consider the pull-backs $\mathcal{F}' = \varphi^*\mathcal{F}$ and $\Phi' = \varphi^*\Phi$ on M' . It is obvious that $\varphi(\{\kappa_{\Phi'} = 0\}) \subset \{\kappa_\Phi = 0\}$. It is a small exercise to check the following lemma.

Lemma 9.3 If $\varphi(\{\kappa_{\Phi'} = 0\}) = \{\kappa_\Phi = 0\}$, the bundle \mathcal{F}' is weakly Φ' -positive

A particularly important example of strictly Φ -positive bundle is the following.

Definition 9.4 A K -equivariant line bundle L over M is called a Φ -moment line bundle if, for any $m \in M$ and any $X \in \mathfrak{k}_m$, the action of $\mathcal{L}(X)$ on the fiber $L|_m$ is equal to $i\langle \Phi(m), X \rangle \text{Id}$.

Such a line bundle L is strictly Φ -positive, since for any $m \in \Phi^{-1}(\beta) \cap M^\beta$, the action of $\mathcal{L}(\beta)$ on $L|_m$ is equal to $i\|\beta\|^2\text{Id}$. In particular, let Φ_L be the moment map associated to an Hermitian connection (Definition 6.3). Clearly L is a Φ_L -line bundle, and the line bundle L is strictly Φ_L -positive.

Example 9.5 *Let us consider the case studied in Section 8.5 : the Hamiltonian action of $\tilde{K} \times K$ on $\mathbb{T}^*\tilde{K}$. The moment map is $\Phi(a, \xi) = -\xi \oplus \pi(a^{-1}\xi)$. In this situation, the trivial line bundle over $\mathbb{T}^*\tilde{K}$ is a Φ -moment line bundle.*

The following theorem follows from Proposition 9.1.

Theorem 9.6 *Let M be a K -manifold equipped with a moment map Φ such that Z_Φ is compact. Let \mathcal{E} be an equivariant graded Clifford bundle on M .*

- *If \mathcal{E} is weakly Φ -positive, then we have*

$$\begin{aligned} [\mathcal{Q}_K(M, \mathcal{E}, \Phi)]^K &= [\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)]^K \\ &+ \sum_{\beta \in \mathcal{B}, \beta \neq 0} \left[\mathcal{Q}_{K_\beta}(M^\beta, \mathcal{E}_{[\beta]}, M^\beta \cap \Phi^{-1}(\beta), \Phi) \right]^{K_\beta} \end{aligned}$$

where the Clifford bundles $\mathcal{E}_{[\beta]}$ are defined by (9.31).

- *When the Clifford bundle \mathcal{E} is strictly Φ -positive, we have*

$$[\mathcal{Q}_K(M, \mathcal{E}, \Phi)]^K = [\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)]^K.$$

If furthermore 0 is a regular value of Φ , then K acts locally freely on $\Phi^{-1}(0)$, and we have $[\mathcal{Q}_K(M, \mathcal{E})]^K = \mathcal{Q}(M_{red}, \mathcal{E}_{red})$.

The equality $[\mathcal{Q}_K(M, \mathcal{E}, \Phi^{-1}(0), \Phi)]^K = \mathcal{Q}(M_{red}, \mathcal{E}_{red})$ is proved in Theorem 8.3.

Finally let us remark the following.

Proposition 9.7 *Let Φ be a moment map, and assume that L is a Φ -moment line bundle. Let $\Phi_t, t \in [0, 1]$ be a smooth family of moment maps such that $\Phi_0 = \Phi$. We assume that there exists a relatively compact open subset $\mathcal{U} \subset M$ such that $Z_{\Phi_t} \subset \mathcal{U}$ for all $t \in [0, 1]$. Then there exists $\epsilon > 0$ such that L is weakly Φ_t -positive for any $t \in [0, \epsilon]$.*

Proof. Take the notations of Lemma 6.7. Consider a finite set $\{\mathfrak{k}_i\}$ of representatives of the infinitesimal stabilizers of the action of K on \mathcal{U} , and let \mathcal{C}_i be the set of connected components \mathcal{X} of $M_{\mathfrak{k}_i} \cap \mathcal{U}$. The action of $\frac{1}{i}\beta(\mathcal{X}, \Phi) \in \mathfrak{k}_i$ on $L|_m$, for any $m \in \mathcal{X}$, is given by $\|\beta(\mathcal{X}, \Phi)\|^2\text{Id}$ as $\beta(\mathcal{X}, \Phi) \in \mathfrak{k}_i$ and $\Phi(\mathcal{X}) \subset \beta(\mathcal{X}, \Phi) + \mathfrak{k}_i^\perp$.

Let $m \in Z_{\Phi_t} \subset \mathcal{U}$. Then, up to conjugacy, there exists an i such that $m \in \Phi_t^{-1}(\mathfrak{k}_i) \cap M_{\mathfrak{k}_i} \cap \mathcal{U}$. Let \mathcal{X} be the connected component of $M_{\mathfrak{k}_i} \cap \mathcal{U}$ containing m . Then $\beta_t = \Phi_t(m) = \beta(\mathcal{X}, \Phi_t) \in \mathfrak{k}_i$ and is close to the point $\beta(\mathcal{X}, \Phi) \in \mathfrak{k}_i$. Thus if $\beta(\mathcal{X}, \Phi) \neq 0$, and t small enough, the action of β_t is positive on $L|_m$. If $\beta(\mathcal{X}, \Phi) = 0$, then $\Phi(\mathcal{X})$ is contained in \mathfrak{k}_i^\perp , and so the action of $\beta_t \in \mathfrak{k}_i$ on $L|_m$ is equal to $i\langle \Phi(m), \beta_t \rangle = 0$.

9.3 $[Q, R] = 0$ in the asymptotic sense

Let M be a compact even dimensional K -manifold equipped with an equivariant graded Clifford bundle \mathcal{E} . Let L be a K -equivariant Hermitian line bundle with Hermitian connection ∇ , and consider its moment map Φ_L .

Theorem 9.8 *When k is sufficiently large, we have*

$$(9.32) \quad \left[\mathcal{Q}_K(M, \mathcal{E} \otimes L^{\otimes k}) \right]^K = \left[\mathcal{Q}_K(M, \mathcal{E} \otimes L^{\otimes k}, \Phi_L^{-1}(0), \Phi_L) \right]^K.$$

If furthermore 0 is a regular value of Φ_L , then K acts locally freely on $\Phi_L^{-1}(0)$, and we have $\left[\mathcal{Q}_K(M, \mathcal{E} \otimes L^{\otimes k}) \right]^K = \mathcal{Q}(M_{red}, \mathcal{E}_{red} \otimes L_{red}^{\otimes k})$.

This theorem follows right away from Theorem 9.6. Indeed, when $\beta \neq 0$, for any couple (\mathcal{X}, β) where $\mathcal{X} \subset M^\beta$ is a connected component such that $\mathcal{X} \cap \Phi_L^{-1}(\beta) \neq \emptyset$, the action of β/i on $L^{\otimes k}|_{\mathcal{X}}$ is given by $k\|\beta\|^2$, so when k is large, the Clifford bundle $\mathcal{E} \otimes L^{\otimes k}$ is strictly Φ_L -positive.

So, although the map Φ_L varies with the choice of connection on L and the topological space M_{red} may vary dramatically, then for k large, the quantity $\mathcal{Q}(M_{red}, \mathcal{E}_{red} \otimes L_{red}^{\otimes k})$ is independent of the choice of the connection on L .

10 $[Q, R] = 0$ for almost complex manifolds

Let us assume that our K -manifold M is provided with a K -invariant almost complex structure J . Then, as in the complex case (see Example 5.11), a natural Clifford bundle on M is $\bigwedge_J TM$ and any Clifford module on M is a twisted Clifford bundle $\bigwedge_J TM \otimes \mathcal{F}$ where \mathcal{F} is a K -equivariant complex vector bundle.

Definition 10.1 • *When M is compact, we define the Riemann-Roch character*

$$RR_K^J(M, \mathcal{F}) := \mathcal{Q}_K(M, \bigwedge_J TM \otimes \mathcal{F}).$$

- When M is not necessarily compact, we can attach the localized Riemann-Roch character

$$RR_K^J(M, \mathcal{F}, \Phi) := \mathcal{Q}_K(M, \bigwedge_J TM \otimes \mathcal{F}, \Phi)$$

to any equivariant map $\Phi : M \rightarrow \mathfrak{k}^*$ such that Z_Φ is compact.

- When Z is a compact component of Z_Φ , we define the localized Riemann-Roch character by

$$RR_K^J(M, \mathcal{F}, Z, \Phi) := \mathcal{Q}_K(M, \bigwedge_J TM \otimes \mathcal{F}, Z, \Phi).$$

An important example of almost complex manifold is the case of symplectic manifold (M, Ω) . In this case, we construct J as follows: choose a K -invariant Riemannian metric g on M , and write $\Omega_m(v, w) = g(v, A_m w)$ where A_m is a invertible antisymmetric matrix, and choose $J_m = \frac{A_m}{\sqrt{-A_m A_m^*}}$.

Our aim is to compute $[RR_K^J(M, \mathcal{F}, \Phi)]^K$ for a well chosen moment map Φ .

10.1 Statement of the results

We start with a definition.

Definition 10.2 Consider the data (Ω, J) where Ω is a two form and J is an almost complex structure on M . We say that (Ω, J) is adapted if, for any $m \in M$, $\Omega(Jv, Jw) = \Omega(v, w)$ and the quadratic form $v \in T_m M \mapsto \Omega(v, Jv)$ is semi-positive.

We have the fundamental fact that we will prove in next subsection.

Proposition 10.3 If (Ω, Φ) is a weakly Hamiltonian structure on M such that (Ω, J) is adapted, then the Clifford bundle $\bigwedge_J TM$ is weakly Φ -positive.

For the rest of this section, we work with a weakly Hamiltonian structure (Ω, Φ) on M such that (Ω, J) is adapted.

Consider $\beta \in \mathfrak{k}$. For the tangent bundle, we have the decomposition $TM|_{M^\beta} = TM^\beta \oplus \mathcal{N}$, where $TM^\beta = \ker(\mathcal{L}(\beta))$ and $\mathcal{N} = \text{Image}(\mathcal{L}(\beta))$. Since the complex structure J on M commutes with $\mathcal{L}(\beta)$, the vector subbundles TM^β and \mathcal{N} are stable under J . On \mathcal{N} , as we have another complex structure J_β that commutes with J , we get the decomposition $\mathcal{N} = \mathcal{N}_{J_\beta, +} \oplus \mathcal{N}_{J_\beta, -}$ where $\mathcal{N}_{J_\beta, \pm} = \{v \in \mathcal{N} \mid J_\beta v = \pm Jv\}$.

Also this lemma will be proved in the next subsection.

Lemma 10.4 *For any non-zero element $\beta \in \mathfrak{k}$, and any $m \in M^\beta \cap \Phi^{-1}(\beta)$, the real dimension of $\mathcal{N}_{J,\beta,+}|_m$ is greater or equal than $\dim_{\mathbb{R}}(\mathfrak{k}/\mathfrak{k}_\beta)$.*

Recall that the critical set Z_Φ is parameterized by a finite set \mathcal{B} . Let us fix a non-zero element $\beta \in \mathcal{B}$.

Definition 10.5 *For any non-zero element $\beta \in \mathcal{B}$, define $M_o^\beta \subset M^\beta$ as the union $\cup \mathcal{X}$ of the connected components \mathcal{X} of M^β intersecting $\Phi^{-1}(\beta)$ and such that the real rank of the vector bundle $\mathcal{N}_{J,\beta,+}|_{\mathcal{X}}$ is equal to $\dim_{\mathbb{R}}(\mathfrak{k}/\mathfrak{k}_\beta)$.*

We notice that M_o^β is a submanifold of M^β stable under the action of the group K_β .

The main result of this subsection is the following theorem.

Theorem 10.6 *Let \mathcal{F} be a K -equivariant bundle on an almost complex manifold (M, J) . Let (Ω, Φ) be a weakly Hamiltonian structure such that (Ω, J) is adapted and Z_Φ is compact.*

- *If the vector bundle \mathcal{F} is weakly Φ -positive, we have*

$$\begin{aligned} \left[RR_K^J(M, \mathcal{F}, \Phi) \right]^K &= \left[RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi) \right]^K + \\ &\sum_{\beta \in \mathcal{B} \setminus \{0\}} \left[RR_{K_\beta}(M_o^\beta, [\mathcal{F}|_{M_o^\beta}]^\beta, M_o^\beta \cap \Phi^{-1}(\beta), \Phi) \right]^{K_\beta}. \end{aligned}$$

- *If the vector bundle \mathcal{F} is strictly Φ -positive, then*

$$\left[RR_K^J(M, \mathcal{F}, \Phi) \right]^K = \left[RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi) \right]^K.$$

- *If 0 is a regular value of Φ , we have*

$$\left[RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi) \right]^K = \mathcal{Q}(M_{red}, \mathcal{S}_{red} \otimes \mathcal{F}_{red}).$$

Here $\mathcal{F}_{red} = \mathcal{F}|_{\Phi^{-1}(0)}/K$ is a orbibundle over $M_{red} = \Phi^{-1}(0)/K$, and \mathcal{S}_{red} is the Spin^c -bundle on M_{red} induced by $\wedge_J TM$.

Suppose now that (Ω, Φ) is an Hamiltonian structure : the closed 2-form Ω is non-degenerate. When M is non-compact, we assume furthermore that Φ is proper and that the set Z_Φ is compact. Theorem 10.6 can be improved as follows.

Theorem 10.7 *Let (M, J) be an almost complex manifold, and let \mathcal{F} be a K -equivariant vector bundle. Let (Ω, Φ) be a Hamiltonian structure on M such that (Ω, J) is adapted, the moment map Φ is proper and Z_Φ is compact.*

If the vector bundle \mathcal{F} is weakly Φ -positive, and 0 is in the image of the moment map, then:

$$\left[RR_K^J(M, \mathcal{F}, \Phi)\right]^K = \left[RR_K^J(M, \mathcal{F}, \Phi^{-1}(0), \Phi)\right]^K.$$

Our manifold M is not necessarily connected, but, considering the action of K on connected components of M , our study reduces easily to the case where $M = KM^0$, where M^0 is connected. Then M^0 is connected and stable by the connected component K^0 of K . We now assume that we are in this situation. In the Hamiltonian setting, we then have the following lemma.

Lemma 10.8 *The set of points in $\Phi(M)$ at minimum distance of the origin is a K -orbit $K\beta_{\min}$. Furthermore $\Phi^{-1}(K\beta_{\min}) \subset M^{\beta_{\min}}$.*

We denote by $M' = M^{\beta_{\min}}$, $K' = K_{\beta_{\min}}$, $\Phi' = \Phi|_{M'} - \beta_{\min}$, J' the restriction of J to M' , and \mathcal{F}' the restriction of \mathcal{F} to M' . We have then the following theorem, which allows us to always reduce the study of $\left[RR_K^J(M, \mathcal{F}, \Phi)\right]^K$ when \mathcal{F} is weakly Φ -positive, to the study of a Hamiltonian manifold with 0 in the image of its moment map.

Theorem 10.9 *Let (M, J) be an almost complex manifold, and let \mathcal{F} be a K -equivariant vector bundle. Let (Ω, Φ) be a Hamiltonian structure on M such that (Ω, J) is adapted, the moment map Φ is proper and Z_Φ is compact. If the vector bundle \mathcal{F} is weakly Φ -positive, then \mathcal{F}' is weakly Φ' -positive, $Z'_{\Phi'}$ is compact and*

$$\left[RR_K^J(M, \mathcal{F}, \Phi)\right]^K = \left[RR_{K'}^{J'}(M', \mathcal{F}', \Phi')\right]^{K'}.$$

Let us summarize what we have proved in this section for a line bundle $L \rightarrow M$ on an almost complex manifold (M, J) . Let (Ω, Φ) be a weak Hamiltonian structure on M such that (Ω, J) is adapted and Z_Φ is compact. Then the relation $\left[RR_K^J(M, L, \Phi)\right]^K = \left[RR_K^J(M, L, \Phi^{-1}(0), \Phi)\right]^K$ holds in two cases:

1. L is a Φ -moment line bundle,
2. L is weakly Φ -positive, Ω is non-degenerate and $\Phi^{-1}(0) \neq \emptyset$.

10.2 Proofs

We now concentrate on the proofs of Lemma 10.4, Proposition 10.3 and Theorems 10.6 – 10.9.

Let $\beta \in \mathfrak{k} \setminus \{0\}$. If $m \in M^\beta \cap \Phi^{-1}(\beta)$, we have $\mathfrak{k}_m \subset \mathfrak{k}_\beta$ and then the map $X \mapsto X \cdot m$, $\mathfrak{q}_\beta \rightarrow \mathcal{N}|_m \subset \mathbb{T}_m M$, is injective, \mathbb{R} -linear and equivariant relatively to the action of $\mathcal{L}(\beta)$. We consider the map

$$p_m : \mathfrak{q}_\beta \rightarrow \mathcal{N}|_m^{\beta,+}$$

which is the composition of the injective map $\mathfrak{q}_\beta \hookrightarrow \mathcal{N}|_m$ with the projection $\mathcal{N}|_m \rightarrow \mathcal{N}|_m^{\beta,+}$. The map p_m is complex linear if both vector spaces \mathfrak{q}_β and $\mathcal{N}|_m$ are endowed with the complex structure J_β .

Lemma 10.4 is a consequence of

Lemma 10.10 *The map p_m is injective for any $m \in \Phi^{-1}(\beta) \cap M^\beta$.*

Proof. Consider the symmetric \mathbb{R} -bilinear form

$$(10.33) \quad H_m(v, w) = \Omega_m(v, \mathcal{L}(\beta)w)$$

on $\mathcal{N}|_m$. By our condition on Ω , $H_m(v, v) \leq 0$ on $\mathcal{N}_{J_\beta, -}|_m$. In contrast, on the subspace $\mathfrak{q} \cdot m \subset \mathcal{N}|_m$, the quadratic form H_m is positive definite. Indeed $H_m(X \cdot m, X \cdot m) = (\Phi(m), [X, [\Phi(m), X]]) = \|[\beta, X]\|^2$. Thus $\mathfrak{q} \cdot m$ does not intersect $\mathcal{N}_{J_\beta, -}|_m$, and then p_m is injective.

Remark 10.11 *The proof above shows that if p_m is bijective, then $\mathfrak{q} \cdot m$ is a maximal subspace of $\mathcal{N}|_m$ where the quadratic form associate to H_m is definite positive.*

The injectivity of the complex linear map p_m implies that $\frac{1}{i}\mathcal{L}(\beta) \geq 0$ on $\det(\mathcal{N}_{J_\beta, +}|_m) \otimes \det(\mathfrak{q}_{J_\beta})^{-1}$, and for a connected component \mathcal{X} of M^β intersecting $\Phi^{-1}(\beta)$ we have

$$(10.34) \quad \mathcal{X} \subset M_o^\beta \iff \mathcal{L}(\beta) = 0 \text{ on } \det(\mathcal{N}_{J_\beta, +}|_\mathcal{X}) \otimes \det(\mathfrak{q}_{J_\beta})^{-1}.$$

Moreover, when m is on a neighborhood $\Phi^{-1}(\beta) \cap M_o^\beta$ of $\Phi^{-1}(\beta) \cap M_o^\beta$, the map p_m stay injective and realize a trivialization of the bundle $\det(\mathcal{N}_{J_\beta, +}) \otimes \det(\mathfrak{q}_{J_\beta})^{-1}$.

Let us determine the Clifford bundle $\mathfrak{d}_\beta(\mathcal{E})$ associated to the Spin^c -bundle $\mathcal{E} = \bigwedge_J \mathbb{T}M$ on M . By definition, we have $\mathcal{E}|_{M^\beta} = \bigwedge \overline{\mathcal{N}_{J_\beta}} \otimes \mathfrak{d}_\beta(\mathcal{E})$.

Since $\mathcal{N}_{J_\beta} \simeq \mathcal{N}_{J_\beta,+} \oplus \overline{\mathcal{N}_{J_\beta,-}}$ as complex vector bundles, we have

$$\begin{aligned} \bigwedge \overline{\mathcal{N}_{J_\beta}} &\simeq \bigwedge \overline{\mathcal{N}_{J_\beta,+}} \otimes \bigwedge \mathcal{N}_{J_\beta,-} \\ &\simeq (-1)^{\text{rank} \mathcal{N}_{J_\beta,+}} \det(\mathcal{N}_{J_\beta,+})^{-1} \otimes \bigwedge \mathcal{N}_{J_\beta,+} \otimes \bigwedge \mathcal{N}_{J_\beta,-} \\ &\simeq (-1)^{\text{rank} \mathcal{N}_{J_\beta,+}} \det(\mathcal{N}_{J_\beta,+})^{-1} \otimes \bigwedge_J \mathcal{N} \end{aligned}$$

as graded Spin^c -bundles on $\mathcal{N} \rightarrow M^\beta$. On the other hand we have also $\bigwedge_J \text{TM}|_{M^\beta} \simeq \bigwedge_J \text{TM}^\beta \otimes \bigwedge_J \mathcal{N}$. This shows that

$$(10.35) \quad \mathfrak{d}_\beta(\mathcal{E}) := (-1)^{\text{rank} \mathcal{N}_{J_\beta,+}} \bigwedge_J \text{TM}^\beta \otimes \det(\mathcal{N}_{J_\beta,+}).$$

At this stage we have proved Proposition 10.3: The Clifford bundle $\mathcal{E} = \bigwedge_J \text{TM}$ is weakly Φ -positive since for any connected component \mathcal{X} of M^β intersecting $\Phi^{-1}(\beta)$ we have $\frac{1}{i} \mathcal{L}(\beta) \geq 0$ on $\mathfrak{d}_\beta(\mathcal{E})|_{\mathcal{X}} \otimes \det(\mathfrak{q}_{J_\beta})^{-1}$ as $\frac{1}{i} \mathcal{L}(\beta) \geq 0$ on $\det(\mathcal{N}_{J_\beta,+}) \otimes \det(\mathfrak{q}_{J_\beta})^{-1}$ restricted to \mathcal{X} .

Moreover we compute that

$$\begin{aligned} \mathcal{E}_{[\beta]} &:= (-1)^{\dim \mathfrak{q}_{J_\beta}} [\mathfrak{d}_\beta(\mathcal{E}) \otimes \det(\mathfrak{q}_{J_\beta})^{-1}]^\beta \\ &= (-1)^{n_\beta} \mathbb{D}_\beta \otimes \bigwedge_J \text{TM}^\beta \end{aligned}$$

where $n_\beta = \dim_{\mathbb{C}} \mathfrak{q}_{J_\beta} - \text{rank}_{\mathbb{C}} \mathcal{N}_{J_\beta,+}$ and

$$\mathbb{D}_\beta = [\det(\mathcal{N}_{J_\beta,+}) \otimes \det(\mathfrak{q}_{J_\beta})^{-1}]^\beta.$$

Two cases hold for a connected component \mathcal{X} of M^β intersecting $\Phi^{-1}(\beta)$. Thanks to the discussion above (see (10.34)), we see that either $\mathcal{X} \not\subset M_o^\beta$ and then $\mathbb{D}_\beta|_{\mathcal{X}} = [0]$, or $\mathcal{X} \subset M_o^\beta$ and then $n_\beta = 0$ on \mathcal{X} and $\mathbb{D}_\beta|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{C}$ on a neighborhood \mathcal{U} of $\mathcal{X} \cap \Phi^{-1}(\beta)$ in \mathcal{X} .

Finally we see that Theorem 10.6 is a consequence of Theorem 9.6. If we work with the Clifford bundle $\mathcal{E}' = \bigwedge_J \text{TM} \otimes \mathcal{F}$, we have $\mathcal{E}'_{[\beta]}|_{\mathcal{X}} = [0]$ unless \mathcal{X} is contained in M_o^β , and we have $\mathcal{E}'_{[\beta]}|_{M_o^\beta} \simeq \bigwedge_J \text{TM}_o^\beta \otimes [\mathcal{F}|_{M_o^\beta}]^\beta$ in a neighborhood of $M_o^\beta \cap \Phi^{-1}(\beta)$. So we can conclude.

We now concentrate ourselves on the proof of Theorem 10.7 and Theorem 10.9. First we prove the preliminary lemma, Lemma 10.8. Let K^0 be the connected component of K . We assume $M = KM^0$ where M^0 is connected

and stable by K^0 . Let T be the maximal torus of K^0 with Lie algebra \mathfrak{t} . We make the choice of a Weyl chamber $\mathfrak{t}_{\geq 0}^*$. The Convexity Theorem [2, 14, 20, 22] is telling us that the set $\Delta(M) = \Phi(M) \cap \mathfrak{t}_{\geq 0}^*$ is a convex polytope, called the Kirwan polytope. So there is a unique point $\beta_{\min} \in \Delta(M)$ at minimum distance of the origin. So $K\beta_{\min}$ is the unique orbit at minimal distance in $\Phi(M)$.

Let $\beta_{\min} \in \Phi(M)$ at minimum distance of the origin. Equation $d\|\Phi\|^2 = 2\iota(\kappa_\Phi)\Omega$ shows that $\Phi^{-1}(\beta_{\min})$ is contained in $M^{\beta_{\min}} = M'$. Let K' be $K_{\beta_{\min}}$. We notice that M' is a K' -Hamiltonian manifold with moment map $\Phi_{M'}$, and J induces an invariant almost complex structure J' on M' . We have the following basic properties whose proof is left to the reader.

Lemma 10.12 • *The data (Ω', J') is adapted.* • *The map $\Phi_{M'}$ is proper and $Z_{\Phi_{M'}}$ is compact.* • *If \mathcal{F} is weakly Φ -positive, then $\mathcal{F}_{M'}$ is weakly $\Phi_{M'}$ -positive.*

Let us go back to the proofs of Theorem 10.7 and Theorem 10.9.

Using Theorem 10.6, and Lemma 10.8 we are left to prove the following

Lemma 10.13 *Let $\beta \in \mathcal{B} \setminus \{0\}$. The submanifold M_o^β is non-empty if and only if $\beta = \beta_{\min}$.*

Thus, let $\beta \in \mathcal{B}$ and consider $m \in \Phi^{-1}(\beta) \cap M^\beta$ such that the real rank of $\mathcal{N}_{J_{\beta,+}}$ is equal to $\dim_{\mathbb{R}} \mathfrak{q}$. We want to prove that $\beta = \beta_{\min}$. For that, we can only consider the action of the connected group K^0 on a connected component M^0 of M . Changing locally of notations for the aim of this proof, we now assume K and M connected. Let us use the notations of the proof of Lemma 10.10. As $m \in M^\beta$, m is a critical point of the map $(\Phi, \beta) : M \rightarrow \mathbb{R}$. Its Hessian H_m at m is the symmetric bilinear form defined in (10.33). The fact that the 2-form Ω is symplectic implies that H_m is non-degenerate on $\mathcal{N}|_m$.

Let U be the connected component of the set of elements ξ in \mathfrak{k}_β such that $\mathfrak{k}_\xi \subset \mathfrak{k}_\beta$. This is an open set in \mathfrak{k}_β containing β . The subset $Y = \Phi^{-1}(U)$ is connected, and is a K_β -symplectic submanifold of M such that $KY \subset M$ is a dense open subset of M .

Let \mathcal{N}^Y be the normal bundle of Y^β in Y . At the level of tangent spaces we have the decompositions

$$\begin{aligned} \mathrm{T}_m M &= \mathrm{T}_m M^\beta \oplus \mathcal{N}|_m \\ &= \mathrm{T}_m Y^\beta \oplus \mathfrak{q} \cdot m \oplus \mathcal{N}^Y|_m \end{aligned}$$

that are orthogonal with respect to Ω_m (and also under H_m).

Now, if the map p_m is an isomorphism, $\mathfrak{q}\cdot m$ is a maximal subspace of $\mathcal{N}|_m$ where the quadratic form associate to H_m is definite positive (see Remark 10.11). As H_m is non-degenerate on $\mathcal{N}^Y|_m$, necessarily $H_m(v, v) < 0$ for any non-zero element $v \in \mathcal{N}^Y|_m$. So, we have $(\Phi, \beta) \geq \|\beta\|^2$ on a neighborhood of m in Y . In the symplectic setting, any function of the type (Φ, β) has a unique local minimum, thus $(\Phi, \beta) \geq \|\beta\|^2$ on Y , and consequently $\|\Phi\|^2 \geq \|\beta\|^2$ on Y . By K -invariance it implies that $\|\Phi\|^2 \geq \|\beta\|^2$ on M . So $\beta = \beta_{\min}$ as desired.

10.3 $[Q, R] = 0$ and semi-ample line bundles

We define the notion of semi-ample line bundles in the almost complex setting.

Definition 10.14 *Let L be a K -equivariant line bundle on an almost complex manifold (M, J) . We say that L is **semi-ample**, if there exists a K -invariant Hermitian connection with curvature two form $R_L = i\Omega_L$ such that $\Omega_L(v, Jv) \geq 0$ for $v \in \text{TM}$. The corresponding moment map Φ_L will be called adapted.*

The following corollary is clear.

Corollary 10.15 *Let L be a K -equivariant semi-ample line bundle on a compact almost complex manifold M , with adapted moment map Φ_L . Then*

$$\left[RR_K^J(M, L^{\otimes k})\right]^K = \left[RR_K^J(M, L^{\otimes k}, \Phi_L^{-1}(0), \Phi_L)\right]^K$$

for any $k \geq 1$. In particular, $\left[RR_K^J(M, L^{\otimes k})\right]^K \neq 0$ for some $k \geq 1$ only if $0 \in \Phi_L(M)$.

This theorem is an analogue of the GIT construction. We will obtain more precise results on the description of $[RR_K(M, \mathcal{F})]^K$ via a moment map Φ if we can choose (Ω, Φ) such that Ω is non degenerate, and \mathcal{F} weakly Φ -positive. This is the object of Section 12.4.

Example 10.16 *We follow the notations of Example 5.12. Let $\mathbb{F} := K/T$ be the flag manifold equipped with an integrable complex structure. The line bundle $[\mathbb{C}_\mu] := K \times_T \mathbb{C}_\mu$ associated to a weight $\mu \in \Lambda$ is semi-ample if and only if μ is dominant, and $[\mathbb{C}_\mu]$ is ample if and only if μ is regular dominant.*

Example 10.17 *The cotangent bundle $T^*\tilde{K}$ of a compact Lie group \tilde{K} is equipped with the symplectic form $\Omega = d\lambda$ where λ is the Liouville 1-form (see section 8.5). Let J be a almost complex structure on $T^*\tilde{K}$ that is compatible with Ω . Let us consider the trivial line bundle $[\mathbb{C}] = T^*\tilde{K} \times \mathbb{C}$ with the connection $\nabla := d + i\lambda$. We see then that the two form $\Omega_{[\mathbb{C}]} := \frac{1}{i}\nabla^2$ is equal to Ω .*

11 A slice theorem for deformed symbol

11.1 An induction formula

Let H be a closed subgroup of K , and consider a H -invariant decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{q}$. Let N' be a H -manifold, and consider the K -manifold

$$N = K \times_H (B_{\mathfrak{q}^*} \times N')$$

where $B_{\mathfrak{q}^*} \subset \mathfrak{q}^*$ is a H -invariant ball centered at 0. Then N' is a submanifold of N , and the normal bundle of N' in N is isomorphic to the trivial bundle with fiber $\mathfrak{q} \oplus \mathfrak{q}^*$: for $n' \in N'$, the element $X \in \mathfrak{q}$ defines the vector $\frac{d}{dt}|_{t=0} e^{tX} \cdot [1, 0, n'] \in T_{[1, n', 0]}N$.

The vector space $\mathfrak{q} \oplus \mathfrak{q}^*$ is identified to $\mathfrak{q}_{\mathbb{C}}$ by $X \oplus \xi \rightarrow X \oplus i\tilde{\xi}$: here $\xi \in \mathfrak{q}^* \rightarrow \tilde{\xi} \in \mathfrak{q}$ is the isomorphism associated to an invariant scalar product on \mathfrak{k} . Let $S_{\mathfrak{q}} = \bigwedge \mathfrak{q}_{\mathbb{C}}$ be the spinor space for $\mathfrak{q}_{\mathbb{C}}$. Thus if \mathcal{E} is a K -equivariant graded Clifford bundle on N , there exists a H -equivariant graded Clifford bundle \mathcal{E}' on N' such that

$$\mathcal{E}|_{N'} = S_{\mathfrak{q}} \otimes \mathcal{E}'.$$

Let $\Phi' : N' \rightarrow \mathfrak{h}^*$ be a H -equivariant map, and let $\Phi : N \rightarrow \mathfrak{k}^*$ be a K -equivariant map. We assume that these maps are linked by the following relations :

$$(11.36) \quad \begin{cases} \Phi|_{N'} = \Phi', \\ \Phi([1; \xi, n']) \in \mathfrak{h}^* \iff \xi = 0, \\ (\Phi([1; \xi, n']), \xi) \geq 0, \end{cases}$$

for $(\xi, n') \in B_{\mathfrak{q}^*} \times N'$.

The following induction result will be useful.

Proposition 11.1 • *The critical sets $Z_{\Phi} \subset N$ and $Z_{\Phi'} \subset N'$ are related by $Z_{\Phi} = K \times_H (\{0\} \times Z_{\Phi'})$.*

• Let Z be a compact component of Z_Φ and Z' its intersection with N' . Then Z' is a compact component of $Z_{\Phi'}$ and

$$\mathcal{Q}_K(N, \mathcal{E}, Z, \Phi) = \text{Ind}_H^K (\mathcal{Q}_H(N', \mathcal{E}', Z', \Phi')).$$

This leads to the relation $[\mathcal{Q}_K(N, \mathcal{E}, Z, \Phi)]^K = [\mathcal{Q}_H(N', \mathcal{E}', Z', \Phi')]^H$.

Proof. Let $(\xi, n') \in B_{\mathfrak{q}^*} \times N'$. We identify $\mathbb{T}_{[1; \xi, n']} N$ to $\mathfrak{q} \oplus \mathfrak{q}^* \oplus \mathbb{T}_{n'} N'$. Let us write $\Phi([1; \xi, n']) = \Phi_{\mathfrak{q}}(\xi, n') \oplus \Phi_{\mathfrak{h}}(\xi, n')$ in the decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{q}$ (and we have identified \mathfrak{k}^* and \mathfrak{k}). We have

$$(11.37) \quad \kappa_\Phi([1; \xi, n']) = -\Phi_{\mathfrak{q}}(\xi, n') \oplus [\xi, \Phi_{\mathfrak{h}}(\xi, n')] \oplus -\Phi_{\mathfrak{h}}(\xi, n') \cdot n'.$$

From (11.36), we have that $\Phi_{\mathfrak{q}}(\xi, n') = 0$ only if $\xi = 0$. Hence the vector field κ_Φ vanishes at $[1; X, n']$ if and only if $\xi = 0$ and $\kappa_{\Phi'}(n') = -\Phi_{\mathfrak{h}}(0, n') \cdot n' = 0$. So if Z a component of Z_Φ , its intersection Z' with $\mathfrak{q} \times N'$ is a component of $Z_{\Phi'}$ such that $Z = K \times_H (\{0\} \times Z')$.

Consider a compact component Z of Z_Φ . Let \mathcal{U}' be a relatively compact neighborhood of Z' in N' , so that $\mathcal{V} = B_{\mathfrak{q}} \times \mathcal{U}'$ is a relatively compact neighborhood of Z' in $\mathfrak{q} \times M'$. Let τ be the restriction of $\sigma(M, \mathcal{E}, \Phi)$ to $\mathbb{T}^* \mathcal{V}$. We are working on $K \times_H \mathcal{V}$, a small neighborhood of $Z = KZ'$ in M , and by the induction formula of Proposition 4.7, we have

$$\mathcal{Q}_K(M, \mathcal{E}, Z, \Phi) = \text{Ind}_H^K (\text{Index}_H^{\mathcal{V}}(\tau)).$$

Let us consider the injection $i : \mathcal{U}' \rightarrow \mathcal{V}$. The symbol $\sigma' = \sigma(\mathcal{U}', \mathcal{E}', \Phi')$ is a H -equivariant transversally elliptic symbol on \mathcal{U}' . Its index is by definition $\mathcal{Q}_H(N', \mathcal{E}', Z', \Phi')$. We can construct the push-forward $i_!(\sigma')$ of the symbol $\sigma' \in \mathbf{K}_H^0(\mathbb{T}_H^* \mathcal{U}')$ in $\mathbf{K}_H^0(\mathbb{T}_H^* \mathcal{V})$ as defined in Section 5.4. Our proposition 11.1 will follow from the following formula and Theorem 5.9.

Lemma 11.2

$$\tau = i_!(\sigma').$$

Proof. By definition the symbol $i_!(\sigma')$ is equal to $\sigma' \boxtimes \text{Bott}(\mathfrak{q}_{\mathbb{C}})$. For $(\xi, n') \in \mathfrak{q}^* \times N'$, and $(q, \nu) \in \mathfrak{q} \times \mathbb{T}_{n'}^* N'$, we have

$$i_!(\sigma')_{(\xi, n')}(q, \nu) = \mathbf{c}_{\mathcal{E}'_{n'}}(\tilde{\nu} - \kappa_{\Phi'}(n')) \otimes \mathbf{c}_{S_{\mathfrak{q}}}(\tilde{\xi} \oplus iq).$$

We compute $\tau_{(X, n')}(q, \nu)$. Using Formula 11.37 we have

$$\tau_{(\xi, n')}(q, \nu) = \mathbf{c}_{\mathcal{E}'_{n'}}(\tilde{\nu} + \Phi_{\mathfrak{h}}(\xi, n') \cdot n') \boxtimes \mathbf{c}_{S_{\mathfrak{q}}}(\Phi_{\mathfrak{q}}(\xi, n') \oplus i(q - [\xi, \Phi_{\mathfrak{h}}(\xi, n')])).$$

For any $t \in [0, 1]$, we define

$$\begin{aligned} A^t(\xi, n', q) &= t\Phi_{\mathfrak{q}}(\xi, n') + (1-t)\xi \oplus i(q - t[\xi, \Phi_{\mathfrak{h}}(\xi, n')]) \in \mathfrak{q}_{\mathbb{C}} \\ B^t(\xi, n') &= -t\Phi_{\mathfrak{h}}(\xi, n') \cdot n' + (1-t)\kappa_{\Phi'}(n') \in \mathbb{T}_{n'}N', \end{aligned}$$

and the symbol

$$\tau_{(\xi, n')}^t(q, \nu) = \mathbf{c}_{\mathcal{E}'_{n'}}(\tilde{\nu} - B^t(\xi, n')) \boxtimes \mathbf{c}_{S_{\mathfrak{q}}}(A^t(\xi, n', q)).$$

We have $\|A^t(\xi, n', q)\|^2 = \|t\Phi_{\mathfrak{q}}(\xi, n') + (1-t)\xi\|^2 + \|\tilde{q} - t[\xi, \Phi_{\mathfrak{h}}(\xi, n')]\|^2$.

If $((\xi, n'), (q, \nu))$ is in the support of τ^t , necessarily $A^t(\xi, n', q) = 0$. We use the conditions (11.36). As $(\xi, \Phi_{\mathfrak{q}}(\xi, n')) = (\Phi([1; \xi, n']), \xi) \geq 0$, we see that $\|t\Phi_{\mathfrak{q}}(\xi, n') + (1-t)\xi\|^2 \geq t^2\|\xi\|^2 + (1-t)^2\|\Phi_{\mathfrak{q}}(\xi, n')\|^2$ and this last expression vanishes only when $\xi = 0$. Thus, we see that, when $(\xi, n') \in B_{\mathfrak{q}} \times N'$, the vector $A^t(\xi, n', q)$ vanishes if and only if $\xi = 0$ and $q = 0$. And for any $t \in [0, 1]$, $B^t(0, n') = \kappa_{\Phi'}(n')$.

Thus τ^t is an homotopy of transversally elliptic symbol on $B_{\mathfrak{q}} \times N'$. Hence $\tau^1 = \tau$ defines the same \mathbf{K} -theory class than $\tau^0 = i_!(\sigma')$.

In the next subsection, we give a first application of this formula.

11.2 The shifting formula

We consider a K -manifold M , oriented and of even dimension, that is equipped with a K -equivariant Clifford bundle \mathcal{G} . Let $\Phi_M : M \rightarrow \mathfrak{k}^*$ be an equivariant map. Let $a \in \mathfrak{k}^*$, and let $H = K_a$ be the stabilizer subgroup of a . On the coadjoint orbit Ka , consider the inclusion $\Phi_a : Ka \hookrightarrow \mathfrak{k}^*$. We assume that Ka is equipped with a K -equivariant Clifford bundle \mathcal{F} . On the product $M \times Ka$, consider the map $\Phi = \Phi_M - \Phi_a$.

In this section we assume that $\Phi^{-1}(0)$ is a compact component of Z_{Φ} : this property holds when Φ_M is a moment map and M is compact.

Consider

$$\mathcal{Q}_K(M \times Ka, \mathcal{G} \otimes \mathcal{F}, \Phi^{-1}(0), \Phi)$$

the equivariant index localized near the component $\Phi^{-1}(0) \subset M \times Ka$.

We look at a small H -invariant open neighborhood B of a in \mathfrak{h}^* such that $Y = \Phi_M^{-1}(B) \subset M$ is a slice at $a \in \mathfrak{k}^*$ for the map Φ_M . Then Y is a H -invariant manifold of M such that $\Phi_M(Y) \subset \mathfrak{h}^*$ and such that $K \times_H Y$ is diffeomorphic to an invariant open neighborhood of $\Phi_M^{-1}(Ka)$. We denote by Φ_Y the restriction of the map Φ_M to Y .

The element $a \in \mathfrak{k}$ acts by an invertible skew-symmetric transformation on $\mathfrak{k}/\mathfrak{k}_a$. We denote by \mathfrak{q}_{J_a} the vector space $\mathfrak{k}/\mathfrak{k}_a$ equipped with the complex structure J_a . We write $\mathcal{G}|_Y = \mathcal{G}_{dY} \otimes \bigwedge \mathfrak{q}_{J_a}$ where \mathcal{G}_{dY} is a H -equivariant Clifford bundle on Y , and $\mathcal{F}|_{\{a\}} = \bigwedge \overline{\mathfrak{q}_{J_a}} \otimes F$ where $F \in R(H)$ (here $\{a\}$ is the base point of Ka).

Proposition 11.3 *For any equivariant Clifford bundles $\mathcal{G} \rightarrow M$ and $\mathcal{F} \rightarrow Ka$, and any equivariant map $\Phi(k, \xi) = \Phi_M(m) - \xi$ such that $\Phi^{-1}(0)$ is a compact component of Z_Φ , we have*

$$\mathcal{Q}_K(M \times Ka, \mathcal{G} \otimes \mathcal{F}, \Phi^{-1}(0), \Phi) = \text{Ind}_H^K(\mathcal{Q}_H(Y, \mathcal{G}_{dY}, \Phi_Y^{-1}(a), (\Phi_Y - a)) \otimes F).$$

Proof. We show that we are in the situation of Subsection 11.1. A neighborhood of $\Phi^{-1}(Ka)$ in M is of the form $K \times_H Y$. Let us consider the diffeomorphism $\varphi : K \times_H (Ka \times Y) \rightarrow Ka \times (K \times_H Y)$ defined by $\varphi([k; \xi, y]) = (k\xi, ky)$. Through φ , the map Φ becomes on $K \times_H (Ka \times Y)$ the map $\tilde{\Phi}([k; \xi, y]) = k(\Phi_Y(y) - \xi)$, and the set $\tilde{\Phi}^{-1}(0)$ is equal to $K \times (\{a\} \times \phi^{-1}(a))$.

We are then working on a H -invariant neighborhood of $\{a\}$ in Ka . This neighborhood can indeed be parameterized by \mathfrak{q}^* through the map $\xi \rightarrow e^{\tilde{\xi}}a$. So we work with the K -manifold $N = K \times_H (B_{\mathfrak{q}^*} \times Y)$ and the H -manifold $N' = Y$. The equivariant maps are $\Phi([k; \xi, y]) := k(\Phi_Y(y) - e^{\tilde{\xi}}a)$ on N and $\Phi' = \Phi_Y - a$ on Y . We check easily that relations (11.36) hold if the ball $B_{\mathfrak{q}^*}$ is small enough. Indeed, $(e^{\tilde{\xi}}a, \tilde{\xi}) = 0$ for any $\xi \in \mathfrak{q}^*$, and for small ξ the term $e^{\tilde{\xi}}a = a + [\tilde{\xi}, a] + O(\|\tilde{\xi}\|^2)$ belongs to \mathfrak{h} only if $\xi = 0$.

At the level of Clifford bundles, if one considers $\mathcal{E} := \mathcal{G} \otimes \mathcal{F}$, the corresponding \mathcal{E}' on Y is $\mathcal{E}' := \mathcal{G}_{dY} \otimes F$ since the Clifford bundle $S_{\mathfrak{q}_a}$ is equal to $\bigwedge \mathfrak{q}_{J_a} \otimes \bigwedge \overline{\mathfrak{q}_{J_a}}$. So we are in the setting of Proposition 11.1 and we can conclude.

12 The Hamiltonian setting

Let us assume that our K -manifold M is provided with a Hamiltonian structure $(\Phi, \Omega) : \Omega$ is a non degenerate closed 2-form on M and the moment map $\Phi : M \rightarrow \mathfrak{k}^*$ and Ω are related by the Kostant relations (6.14). Then we can always choose an invariant almost complex structure J on M such that $\Omega(v, Jw), v, w \in TM$ is a Riemannian metric on M : such almost complex structure J is called compatible with Ω .

Consider the case where M compact. In this context, the representation $\mathcal{Q}_K(M, \bigwedge_J TM \otimes \mathcal{F})$ does not depend on the choice of J since the set of Ω -compatible almost complex structures is contractible, so, given Ω , we denote it by $RR_K(M, \mathcal{F})$ whenever the almost complex structure J is compatible with the given choice of Ω .

We can make use of the result of Theorem 10.7, since our hypothesis “ J compatible with Ω ” is stronger than the hypothesis “ (Ω, J) is adapted”. In fact we will gain much more, as we will be able to compute geometrically the multiplicities of $RR_K(M, L)$ when L is a Φ -moment bundle : it is the heart of the $[Q, R] = 0$ theorem of Meinrenken-Sjamaar that is explained in the next section.

Our proof of the Meinrenken-Sjamaar Theorem relies on the Witten deformation argument presented in the previous section. We attach a localized Riemann-Roch character

$$RR_K(M, \mathcal{F}, Z_\beta, \Phi) = \mathcal{Q}_K(M, \bigwedge_J TM \otimes \mathcal{F}, Z_\beta, \Phi)$$

to each component Z_β of Z_Φ [29, 30, 23], where J is an almost complex structure compatible with Ω . In this way, we decompose the character $RR_K(M, \mathcal{F})$ in sum of (infinite) dimensional representations, while Meinrenken-Sjamaar decomposes $RR_K(M, \mathcal{F})$ in sums of (finite) dimensional representations by cutting the Kirwan polytope in small polytopes. In both of these methods, when \mathcal{F} is weakly Φ -positive, the study of $[RR_K(M, \mathcal{F})]^K$ is reduced to the study of M around $\Phi^{-1}(0)$, and this can be done explicitly, at least in the case where 0 is a quasi-regular value. The use of the compact connected component $\Phi^{-1}(0)$ of the zeroes of the Kirwan vector field κ_Φ replaces the compactification of a neighborhood of $\Phi^{-1}(0)$ by cutting used in Meinrenken-Sjamaar.

When M is non-compact, we can also study the localized Riemann-Roch character $RR_K(M, \mathcal{F}, \Phi)$ when the set Z_Φ is proper [30, 23]. In Section 13 we will concentrate ourselves to the case of an Hamiltonian action of $\tilde{K} \times K$ on the cotangent bundle $T^*\tilde{K}$.

12.1 $[Q, R] = 0$ Theorem of Meinrenken-Sjamaar

Assume in this subsection that K is a compact connected Lie group, and let T be its maximal torus. Let $\Lambda \subset \mathfrak{t}^*$ be the weight lattice of T , and consider the set of dominant weights $\Lambda_{\geq 0} = \Lambda \cap \mathfrak{t}_{\geq 0}^*$ that parameterizes the irreducible representations of K : for $\mu \in \Lambda_{\geq 0}$, we denote by V_μ^K , the irreducible representation with highest weight μ .

For the remaining part of this section we assume that M is compact and connected, and we denote $\Delta(M) := \Phi(M) \cap \mathfrak{k}_{\geq 0}^*$ the Kirwan polytope.

Let $L \rightarrow M$ be a Φ -moment bundle. Write

$$RR_K(M, L) = \sum_{\mu \in \Lambda_{\geq 0}} m_{\mu}(L) V_{\mu}^K.$$

We will explain how to compute geometrically the multiplicity $m_{\mu}(L)$ of the representation V_{μ}^K in $RR_K(M, L)$ in terms of the fibers of the moment map.

Definition 12.1 *For any $a \in \mathfrak{k}^*$, we define the topological space $M_a := \Phi^{-1}(Ka)/K$.*

We say that M_a is the reduced space of M at a . In particular, the reduced space $M_{red} = \Phi^{-1}(0)/K$ is denoted M_0 .

Let us recall the notion of quasi-regular value of Φ .

Definition 12.2 *An element $a \in \mathfrak{k}^*$ is a quasi-regular value of Φ if $a \in \Phi(M)$ and the K -orbits in $\Phi^{-1}(Ka)$ all have the same dimension.*

Equivalently, a is a quasi-regular value if there exists a subalgebra $\mathfrak{s} \subset \mathfrak{k}$ such that $\Phi^{-1}(Ka)$ is contained in the infinitesimal orbit type stratum $M_{(\mathfrak{s})} = \{m \in M, (\mathfrak{k}_m) = (\mathfrak{s})\}$. As we will see, this implies that $\Phi^{-1}(Ka)$ is a smooth K -manifold. Furthermore, if a is a quasi-regular value, then the orbifold stratification of $M_a := \Phi^{-1}(Ka)/K$ consists of one piece only, and M_a is therefore a symplectic orbifold.

Assume that a dominant weight μ is a quasi-regular value of Φ . We can then consider the orbifold line bundle over M_{μ} ,

$$L_{\mu} := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu}) / K_{\mu},$$

and we may define the Riemann-Roch number $RR(M_{\mu}, L_{\mu})$.

In general, we consider quasi-regular values a of Φ close to μ , and we can, as we will see, define the orbifold line bundle

$$L_{a,\mu} := (L|_{\Phi^{-1}(a)} \otimes \mathbb{C}_{-\mu}) / K_a,$$

over the symplectic orbifold $M_a = \Phi^{-1}(a)/K_a$. The line orbi-bundle $L_{a,\mu}$ is well defined since for a close enough to μ , the action of \mathfrak{k}_m on $L|_m \otimes \mathbb{C}_{-\mu}$ is trivial for any $m \in \Phi^{-1}(a)$ (see Lemma 12.20).

We can now state the Meinrenken-Sjamaar Theorem [27].

Theorem 12.3 *Let $\mu \in \Lambda_{\geq 0}$.*

- *If $\mu \notin \Delta(M)$, then $m_\mu(L) = 0$.*
- *If $\mu \in \Delta(M)$, then $m_\mu(L) = RR(M_a, L_{a,\mu})$ for any quasi-regular value a of Φ that is close enough to μ .*

We can restate this result by defining Riemann-Roch numbers on (singular) symplectic reduced spaces as follows.

Definition 12.4 *Let L be a Φ -moment bundle. For any dominant weight $\mu \in \Lambda_{\geq 0}$, the Riemann-Roch number $RR(M_\mu, L_\mu)$ is defined by the following dichotomy*

$$RR(M_\mu, L_\mu) = \begin{cases} 0 & \text{if } \mu \notin \Delta(M), \\ RR(M_a, L_{a,\mu}) & \text{if } \mu \in \Delta(M) \text{ and } a \text{ is a quasi regular} \\ \text{value close enough to } \mu. & \end{cases}$$

The $[Q, R] = 0$ Theorem of Meinrenken-Sjamaar says that for a Φ -moment bundle L , the multiplicity $m_\mu(L)$ of the representation V_μ^K in $RR_K(M, L)$ is equal to $RR(M_\mu, L_\mu)$.

12.2 Localization on $\Phi^{-1}(0)$

In this section, we consider a K -Hamiltonian manifold (M, Ω, Φ) not necessarily compact nor connected, and the group K is not supposed connected. We study the character $RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi)$, in the case where 0 is a quasi-regular value of Φ and the manifold $\Phi^{-1}(0)$ is compact.

12.2.1 Regular case

We suppose here that 0 is a regular value of Φ . The group K acts infinitesimally freely on the compact submanifold $P = \Phi^{-1}(0)$, and $M_0 := P/K$ is a compact orbifold equipped with an induced symplectic form Ω_0 . We denote $\pi : P \rightarrow P/K = M_0$ the projection.

Proposition 12.5 *For any equivariant vector bundle $\mathcal{F} \rightarrow M$ we have*

$$(12.38) \quad [RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi)]^K = RR(M_0, \mathcal{F}_0)$$

where $\mathcal{F}_0 = \mathcal{F}|_{P/K}$.

Proof. Let $\mathcal{E} := \bigwedge_J TM$ be the Clifford bundle associated to a compatible almost complex structure J . We consider the restriction of the tangent bundle TM on the submanifold P . Let $[\mathfrak{k}] \subset TM|_P$ be the subbundle which is the image of the map $(m, X) \in P \times \mathfrak{k} \mapsto X \cdot m \in T_m P \subset T_m M$. We consider the subbundle $[\mathfrak{k}] \oplus J[\mathfrak{k}]$ of $TM|_P$ that is canonically isomorphic to $[\mathfrak{k}_{\mathbb{C}}]$. Let $E \subset TM|_P$ be the orthogonal of $[\mathfrak{k}] \oplus J[\mathfrak{k}]$ relatively to the symplectic form. We see that $E \rightarrow P$ is a symplectic bundle isomorphic to the pull-back $\pi^* TM_0$, and that J induces an almost complex structure J_0 on E compatible with $\pi^*(\Omega_0)$. We see then that $\mathcal{E}|_P \simeq \bigwedge \mathfrak{k}_{\mathbb{C}} \otimes \mathcal{E}_0$ with $\mathcal{E}_0 = \pi^*(\bigwedge_{J_0} TM_0)$. Our result follows then from Theorem 8.3.

12.2.2 Reduction in stage

In this section, we explore the case of reduction in stage. Suppose that we have a Hamiltonian action of the compact Lie group $G \times K$ on the manifold (M, Ω) . Let $\Phi = \Phi_G \oplus \Phi_K : M \rightarrow \mathfrak{g}^* \oplus \mathfrak{k}^*$ be the corresponding moment map. We suppose that

- 0 is a regular value of Φ_K ,
- K acts freely on $P := \Phi_K^{-1}(0)$,
- $Z := \Phi^{-1}(0)$ is compact.

We denote by $\pi : P \rightarrow M_0 := P/K$ the corresponding G -equivariant principal fibration. We denote by $\phi : M_0 \rightarrow \mathfrak{g}^*$ the equivariant map induced by Φ_G : it is the moment map relative to the Hamiltonian action of G on the symplectic manifold (M_0, Ω_0) .

Proposition 12.6 *For any equivariant vector bundle $\mathcal{F} \rightarrow M$, we have the following relation*

$$[RR_{K \times G}(M, \mathcal{F}, \Phi^{-1}(0), \Phi)]^K = RR_G(M_0, \mathcal{F}_0, \phi^{-1}(0), \phi) \quad \text{in } \hat{R}(G).$$

Here $\mathcal{F}_0 = (\mathcal{F}|_P)/K$.

Proof. We follow the proof of Proposition 12.5. We analyze the symbol $\sigma(M, \Phi)(m, \nu) := \mathbf{c}_m(\tilde{\nu} - \kappa_{\Phi}(m)) : \bigwedge_J^{even} TM \rightarrow \bigwedge_J^{odd} TM$ on the model $\hat{M} = P \times \mathfrak{k}^*$ in a neighborhood of $Z \times \{0\}$.

The choice of a $G \times K$ -invariant connection 1-form θ on the K -principal bundle $P \rightarrow M_0$ defines a projection map $T_p P \rightarrow \mathfrak{k}$ which associates to a tangent vector its vertical part. This projection induces a map $j : P \rightarrow$

$\text{hom}(\mathfrak{g}, \mathfrak{k})$, where the vertical part of the vector $X \cdot p \in T_p P$ is equal to $j_p(X) \in \mathfrak{k}$ for any $(X, p) \in \mathfrak{g} \times P$.

The symplectic form can be taken as $\hat{\Omega} = \pi^* \Omega_0 + d\langle \theta, \xi \rangle$, and the moment maps have the following expressions : for $(p, \xi) \in P \times \mathfrak{k}^*$

- $\Phi_K(p, \xi) = \xi$,
- $\Phi_G(p, \xi) = \phi(\bar{p}) + j_p^*(\xi)$, where $\bar{p} = \pi(p)$.

We denote by κ_G (resp. κ_ϕ) the Kirwan vector field on P (resp. M_0) associated to the equivariant map $\Phi_G : P \rightarrow \mathfrak{g}^*$ (resp. ϕ). The Kirwan vector field κ_Φ has the following expression $\kappa_\Phi = \kappa_\Phi^{vert} + \kappa_\Phi^{hor}$, where

- $-\kappa_\Phi^{vert}(p, \xi) = (Id + j_p j_p^*)\xi + j_p \phi(\bar{p})$,
- $-\kappa_\Phi^{hor}(p, \xi) \simeq -\kappa_\phi(\bar{p}) + j_p^*(\xi) \cdot \bar{p}$ through the tangent map $T_p \pi$.

Let $V \subset U$ be two invariant relatively compact neighborhoods of $Z = \Phi_G^{-1}(0) \cap P$ in P such that $\pi(\bar{U}) \cap \{\kappa_\phi = 0\} = \phi^{-1}(0)$, and $\bar{V} \subset U$. The vector field κ_ϕ does not vanish on the compact set $\pi(\bar{U} \setminus V)$, hence we can choose the connection 1-form θ such that the vector field κ_G is horizontal on $\bar{U} \setminus V$. In other words, the term $j_p \phi(\bar{p})$ vanishes for $p \in \bar{U} \setminus V$. Let $c > 0$ such that $\|j_p \phi(\bar{p})\| \leq c$ on V . We deform Φ in $\Phi_t(p, \xi) = \xi \oplus (\phi(\bar{p}) + t j_p(\xi))$ for $t \in [0, 1]$, and we denote κ_t the corresponding vector field. The following lemma is an easy computation

Lemma 12.7 *For any $t \in [0, 1]$, the set $\{(p, \xi) \in U \times \mathfrak{k}^*, \kappa_t(p, \xi) = 0\}$ is contained in the compact set $\bar{V} \times \{\|\xi\| \leq c\}$.*

Thanks to the previous lemma, we see that the symbols $\sigma(M, \Phi)$ and $\sigma(M, \Phi_0)$ define the same class in $\mathbf{K}_{G \times K}^0(T_{G \times K}(U \times \mathfrak{k}^*))$. Let $\sigma_{hor} \in \mathbf{K}_{G \times K}^0(T_{G \times K} U)$ be the pullback of $\sigma(M_0, \phi)|_{\pi(U)} \in \mathbf{K}_G^0(T_G \pi(U))$.

For $m = (p, \xi) \in U \times \mathfrak{k}^*$ and $\nu := \eta \oplus a \oplus b \in T^*(U \times \mathfrak{k}^*)$, we have

$$\sigma(M, \Phi_0)(m, \nu) = \sigma_{hor}(p, \eta) \boxtimes \mathbf{c}_{\wedge \mathfrak{k}_\mathbb{C}}((a + \gamma_p + \xi) \oplus ib)$$

where $\gamma_p = j_p \phi(\bar{p})$. We see that the support of the symbol $\sigma(M, \Phi_0)$ intersected with $T_{G \times K}^*(U \times \mathfrak{k}^*)$ is just equal to $Z \times \{0\}$.

We see also that for $t \in [0, 1]$, the support of the symbol

$$\sigma^t(m, \nu) = \sigma_{hor}(p, \eta) \boxtimes \mathbf{c}_{\wedge \mathfrak{k}_\mathbb{C}}((t(a + \gamma_p) + \xi) \oplus ib)$$

intersected with $T_{G \times K}^*(U \times \mathfrak{k}^*)$ remains equal to $Z \times \{0\}$. Thus we have proved that $\sigma(M, \Phi)$ is equal to $i_!(\sigma_{hor})$ in $\mathbf{K}_{G \times K}^0(T_{G \times K}(U \times \mathfrak{k}^*))$. Our result follows from this fact.

12.2.3 Quasi regular case

In this subsection, we study the character $RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi)$, in the case where 0 is a quasi-regular value of Φ .

In particular, we prove the first form of the $[Q, R] = 0$ theorem.

Theorem 12.8 *If 0 is a quasi-regular value of the moment map, and L a Φ -moment bundle, then*

$$[RR_K(M, L)]^K = RR(M_0, L_0).$$

Here $L_0 = L|_{\Phi^{-1}(0)}/K$.

Consider first the basic example of a Hermitian vector space W with complex structure J and Hermitian inner product $h(v, w)$. Consider the symplectic form on W given by $\Omega = -\text{Im}(h)$. Thus J is a complex structure adapted to Ω . Let K be a compact group acting by unitary transformations on W . Then W is a K -Hamiltonian manifold with moment map $\langle \Phi(x), X \rangle = \frac{i}{2}h(X \cdot x, x)$, for $X \in \mathfrak{k}$, $x \in W$.

Assume that $\Phi^{-1}(0) = \{0\}$: in this case 0 is a quasi-regular value. Consider a complex representation space F of K , and the K -equivariant vector bundle $[F] = W \times F$.

Proposition 12.9 *Assume that $\Phi^{-1}(0) = 0$. Then*

$$RR_K(W, [F], \Phi^{-1}(0), \Phi) = \text{Sym}(W^*) \otimes F.$$

Furthermore, $[\text{Sym}(W^*)]^K = \mathbb{C}$.

Remark 12.10 *The second claim follows from Mumford GIT theory. In spirit, this is the first case of the $[Q, R] = 0$ theorem.*

Proof. Let $x \in W$, and $\kappa_\Phi(x) = -\Phi(x) \cdot x$ be the Kirwan vector field associated to Φ . Then κ_Φ vanishes only at 0: indeed we have $h(Jx, \kappa_\Phi(x)) = 2\|\Phi(x)\|^2$.

Our deformed symbol is $\sigma(W, \Phi)(x, \nu) = \mathbf{c}(\tilde{\nu} - \kappa_\Phi(x)) : \bigwedge^{\text{even}} W \rightarrow \bigwedge^{\text{odd}} W$, while Atiyah symbol $\text{At}(\overline{W})$ is $\text{At}(\overline{W})(x, \nu) = \mathbf{c}(\tilde{\nu} - Jx) : \bigwedge^{\text{even}} W \rightarrow \bigwedge^{\text{odd}} W$.

Let $t \in [0, 1]$ and $\sigma^t(x, \nu) = \mathbf{c}(\tilde{\nu} - (1-t)Jx - t\kappa_\Phi(x))$. Let us see that σ^t is a transversally elliptic symbol with support $\{(0, 0)\}$ in T_K^*W . Indeed if $\tilde{\nu} = (1-t)Jx + t\kappa_\Phi(x)$ and is orthogonal to $\mathfrak{k} \cdot x$, we obtain

$$0 = \text{Re}(h(\tilde{\nu}, \kappa_\Phi(x))) = 2(1-t)\|\Phi(x)\|^2 + t\|\kappa_\Phi(x)\|^2,$$

and this implies $x = 0$. We obtain our first claim since the equivariant index of $\text{At}(\overline{W})$ is $\text{Sym}(W^*)$ (see Proposition 6.15).

Let us now prove that $[\text{Sym}(W^*)]^K = \mathbb{C}$. If $P \in \text{Sym}(W^*)$ is K -invariant, P is constant on the trajectory of the vector field κ_Φ which is tangent to the K -orbits. So the value of P coincide with the value of P on the unique stationary point of κ_Φ . So $P(x) = P(0)$ and P is constant.

Remark 12.11 *In the proof above we have checked that the transversally elliptic symbols $\sigma(W, \Phi)$ and $\text{At}(\overline{W})$ define the same class in $\mathbf{K}_K^0(\mathbb{T}_K^*W)$. If \mathbb{T} is the 1-dimension center of $U(V)$, the same proof shows that $\sigma(W, \Phi)$ and $\text{At}(\overline{W})$ define the same class in $\mathbf{K}_{K \times \mathbb{T}}^0(\mathbb{T}_\mathbb{T}^*W)$*

We now consider the general case where 0 is a quasi-regular value, and we study the character $RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi)$ for any vector bundle $\mathcal{F} \rightarrow M$.

Let \mathfrak{s} be the subalgebra of \mathfrak{k} such that $Z := \Phi^{-1}(0)$ is contained in the submanifold $M_{(\mathfrak{s})}$. Let $M_{\mathfrak{s}} = \{m \in M, \mathfrak{k}_m = \mathfrak{s}\}$. Let H be the normalizer subgroup of \mathfrak{s} in K . Thus $M_{(\mathfrak{s})} = K \times_H M_{\mathfrak{s}}$ and $Z = K \times_H Z_{\mathfrak{s}}$ where $Z_{\mathfrak{s}} := \Phi^{-1}(0) \cap M_{\mathfrak{s}}$.

We write a H -invariant decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{q}$, with corresponding decomposition $\mathfrak{k}^* = \mathfrak{h}^* \oplus \mathfrak{q}^*$, and write $\Phi = \Phi_{\mathfrak{h}} \oplus \Phi_{\mathfrak{q}}$. Here $\Phi_{\mathfrak{h}} : M \rightarrow \mathfrak{h}^*$ is the moment map relative to the H -action on M , and the map $\Phi_{\mathfrak{q}} : M \rightarrow \mathfrak{q}^*$ is H -equivariant. We start with the basic lemma.

Lemma 12.12 *The map $\Phi_{\mathfrak{q}}$ is equal to zero on $M_{\mathfrak{s}}$.*

Proof. Let $m \in M_{\mathfrak{s}}$. Since $\mathfrak{k}_m = \mathfrak{s}$, the term $\Phi(m) \in \mathfrak{k}^* \simeq \mathfrak{k}$ satisfies $[\mathfrak{s}, \Phi(m)] = 0$ and a fortiori $\Phi(m) \in \mathfrak{h} = N_{\mathfrak{k}}(\mathfrak{s})$.

Thus the restriction ϕ of Φ to $M_{\mathfrak{s}}$ takes values in \mathfrak{h}^* : the manifold $M_{\mathfrak{s}}$ is a H -Hamiltonian manifold, with moment map ϕ . Let $M_{\mathfrak{s}}^0$ be the union of connected components of $M_{\mathfrak{s}}$ that intersect $\phi^{-1}(0)$. Since S acts trivially on $M_{\mathfrak{s}}^0$, the map ϕ take value in $(\mathfrak{h}/\mathfrak{s})^*$ on $M_{\mathfrak{s}}^0$. Since the group H/S acts locally freely on $M_{\mathfrak{s}}^0$, the map $\phi : M_{\mathfrak{s}}^0 \rightarrow (\mathfrak{h}/\mathfrak{s})^*$ is a submersion. In particular, $Z_{\mathfrak{s}} = \phi^{-1}(0) \cap M_{\mathfrak{s}}^0$ is a compact submanifold, so $Z = K \times_H Z_{\mathfrak{s}}$ is a submanifold of M , as asserted previously, and the reduced space

$$M_0 := \Phi^{-1}(0)/K = \phi^{-1}(0) \cap M_{\mathfrak{s}}^0/(H/S)$$

is a compact symplectic orbifold.

Let $\mathcal{N} \rightarrow Z$ be the symplectic normal bundle of the submanifold Z in M : for $x \in Z$,

$$\mathcal{N}|_x = (\mathbb{T}_x Z)^\perp / (\mathbb{T}_x Z)^\perp \cap \mathbb{T}_x Z.$$

Here we have denoted by $(\mathbb{T}_x Z)^\perp$ the orthogonal with respect to the symplectic form.

We can equip \mathcal{N} with an H -invariant Hermitian structure h such that the symplectic structure on the fibers of $\mathcal{N} \rightarrow Z$ is equal to $-\text{Im}(h)$.

The subalgebra \mathfrak{s} acts fiberwise on the complex vector bundle $\mathcal{N}|_{Z_\mathfrak{s}}$. We consider the action of \mathfrak{s} on the fibers of the complex bundle $\text{Sym}(\mathcal{N}^*|_{Z_\mathfrak{s}})$. We will use the following.

Lemma 12.13 *The subbundle $[\text{Sym}(\mathcal{N}^*|_{Z_\mathfrak{s}})]^\mathfrak{s}$ is reduced to the trivial bundle $[\mathbb{C}] \rightarrow Z_\mathfrak{s}$.*

Proof. Let $x \in Z_\mathfrak{s}$. The vector space $\mathfrak{k} \cdot x \subset \mathbb{T}_x M$ is totally isotropic, since $\Omega_x(X \cdot x, Y \cdot x) = \langle \Phi(x), [X, Y] \rangle = 0$, hence we can consider the vector space $E_x := (\mathfrak{k} \cdot x)^\perp / \mathfrak{k} \cdot x$ that is equipped with a K_x -equivariant symplectic structure Ω_x : we denote by $\Phi_{E_x} : E_x \rightarrow \mathfrak{s}^*$ the corresponding moment map. A local model for a symplectic neighborhood of Kx is $U := K \times_{K_x} ((\mathfrak{k}_x/\mathfrak{s})^* \times E_x)$ where the moment map is $\Phi_x[k; \xi, v] = k(\xi + \Phi_{E_x}(v))$. Our hypothesis that $\Phi^{-1}(0)$ is contained in $M_{(\mathfrak{s})}$ implies that $\Phi_{E_x}^{-1}(0) = (E_x)^\mathfrak{s}$. We get then the decompositions

$$\begin{aligned} \mathbb{T}_x Z &= (\mathfrak{k}/\mathfrak{s}) \cdot x \oplus (E_x)^\mathfrak{s}, \\ \mathbb{T}_x M &= (\mathfrak{k}/\mathfrak{s}) \cdot x \oplus (\mathfrak{k}/\mathfrak{s})^* \oplus E_x. \end{aligned}$$

Let S be the connected component of K_x . We see then that $\mathcal{N}|_x$ is equal to the symplectic orthogonal of $(E_x)^\mathfrak{s}$ in E_x . Hence $\mathcal{N}|_x$ is a symplectic vector space with a Hamiltonian action of the compact group S such that $\Phi_{\mathcal{N}|_x}^{-1}(0) = \Phi_{E_x}^{-1}(0) \cap \mathcal{N}|_x = 0$. As we have seen before, it implies that $[\text{Sym}(\mathcal{N}^*|_x)]^\mathfrak{s} = \mathbb{C}$.

Definition 12.14 *If $\mathcal{F} \rightarrow M$ is a K -equivariant complex vector bundle, we define on M_0 the (finite dimensional) orbi-bundle*

$$\mathcal{F}_0 := [\mathcal{F}|_{Z_\mathfrak{s}} \otimes \text{Sym}(\mathcal{N}^*|_{Z_\mathfrak{s}})]^\mathfrak{s} / (H/S).$$

If \mathfrak{s} acts trivially on the fibers of $\mathcal{F}|_{Z_\mathfrak{s}}$, the bundle \mathcal{F}_0 is equal to $\mathcal{F}|_{Z_\mathfrak{s}} / (H/S)$.

The aim of this subsection is the following theorem.

Theorem 12.15 *Assume that $\Phi^{-1}(0) \subset M_{(\mathfrak{s})}$. For any K -equivariant complex vector bundle $\mathcal{F} \rightarrow M$, we have*

$$[RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi)]^K = RR(M_0, \mathcal{F}_0).$$

The proof of this theorem will require two steps, inducing and restricting, that are considered in the next subsections.

Inducing from H to K .

Let us show that there is a K -invariant neighborhood N of the compact submanifold $Z_{\mathfrak{s}} := \Phi^{-1}(0) \cap M_{\mathfrak{s}}$ of M of the form $N = K \times_H (N' \times B_{\mathfrak{q}^*})$, where $B_{\mathfrak{q}^*}$ is a small H -invariant ball in \mathfrak{q}^* , and N' a H -invariant symplectic submanifold of M .

Let $\mathcal{V} \subset M$ be a small K -invariant tubular neighborhood of the stratum $M_{(\mathfrak{s})}$ with fibration $p : \mathcal{V} \rightarrow M_{(\mathfrak{s})}$. Let $V = p^{-1}(M_{\mathfrak{s}})$. Thus V is a H -invariant submanifold of M containing $M_{\mathfrak{s}}$, fibered over $M_{\mathfrak{s}}$ and $\mathcal{V} = K \times_H V$ is an open subset of M . We consider the map $\Phi_{\mathfrak{q}} : V \rightarrow \mathfrak{q}^*$.

Lemma 12.16 *Let $z \in Z_{\mathfrak{s}}$:*

- (a) *the differential $T_z \Phi_{\mathfrak{q}} : T_z V \rightarrow \mathfrak{q}^*$ is surjective,*
- (b) *$\ker(T_z \Phi_{\mathfrak{q}}) \subset T_z V$ is a symplectic subspace of $T_z M$.*

Proof. For $z \in Z_{\mathfrak{s}}$, we have $T_z M = \mathfrak{q} \cdot z \oplus T_z V$ with $\mathfrak{q} \cdot z \simeq \mathfrak{q}$ since $\mathfrak{k}_z = \mathfrak{s}$. The subspace $\mathfrak{q} \cdot z$ is totally isotropic since $\Phi(z) = 0$, and by definition the subspace $\ker(T_z \Phi_{\mathfrak{q}}) \subset T_z V$ is equal to $(\mathfrak{q} \cdot z)^{\perp} \cap T_z V$. So we have $(\mathfrak{q} \cdot z)^{\perp} = \mathfrak{q} \cdot z \oplus \ker(T_z \Phi_{\mathfrak{q}})$. This implies that $\ker(T_z \Phi_{\mathfrak{q}}) \subset T_z V$ is a symplectic subspace of $T_z M$ of dimension equal to $\dim(T_z V) - \dim(\mathfrak{q})$: the surjectivity of $T_z \Phi_{\mathfrak{q}} : T_z V \rightarrow \mathfrak{q}^*$ follows.

Let \mathcal{X} be a small H -invariant neighborhood of $Z_{\mathfrak{s}}$ in $M_{\mathfrak{s}}$, and let $V_{\mathcal{X}}$ be a small neighborhood of \mathcal{X} in $p^{-1}(\mathcal{X}) \subset V$ such that for any $z \in V_{\mathcal{X}}$ the statements (a) and (b) hold.

Consider $N' := \{z \in V_{\mathcal{X}}; \Phi_{\mathfrak{q}}(z) = 0\}$. Then N' is a H -Hamiltonian submanifold of $V_{\mathcal{X}}$ containing \mathcal{X} , with symplectic form $\Omega' := \Omega|_{N'}$ and moment map $\Phi' : N' \rightarrow \mathfrak{h}^*$ equal to the restriction of Φ to N' . If \mathcal{X} is small enough, there exists an invariant neighborhood U' of \mathcal{X} in $V_{\mathcal{X}}$ and a diffeomorphism $j : N' \times B_{\mathfrak{q}^*} \rightarrow U'$ such that $\Phi \circ j = \Phi'(n') + \xi$, for any $(n', \xi) \in N' \times B_{\mathfrak{q}^*}$.

Finally we have proved that a neighborhood of $Z_{\mathfrak{s}}$ in M is diffeomorphic to $N = K \times_H (N' \times B_{\mathfrak{q}^*})$. Here (N', Ω', Φ') is a H -Hamiltonian submanifold of (M, Ω, Φ) and the moment map $\Phi_N : N \rightarrow \mathfrak{k}^*$ is defined by the relation

$$\Phi_N([k; n', \xi]) = k \cdot (\Phi'(n') + \xi).$$

We return to the study of $RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi)$ under the hypothesis that $\Phi^{-1}(0)$ is contained in $M_{(\mathfrak{s})}$. We may do the computation in the neighborhood N of $\Phi^{-1}(0)$. Note that $\Phi^{-1}(0) \subset M_{(\mathfrak{s})}$ implies that $Z' := (\Phi')^{-1}(0)$ is equal to $Z_{\mathfrak{s}}$. Recall that $RR_K(N, \mathcal{F}, \Phi^{-1}(0), \Phi) = \mathcal{Q}_K(N, \bigwedge_J \mathrm{TN} \otimes \mathcal{F}, \Phi^{-1}(0), \Phi)$. Here we have chosen J_N , an almost complex structure that is compatible with the symplectic form Ω . We may then further deform the almost complex structure J_N and we will keep the same index.

Let $J_{N'}$ be an almost complex structure on N' that is compatible with Ω' . If we restrict the Clifford bundle $\mathcal{E} := \bigwedge_{J_N} \mathrm{TN}$ to the submanifold N' , we get

$$\mathcal{E}|_{N'} = S_{\mathfrak{q}} \otimes \mathcal{E}'$$

where $S_{\mathfrak{q}} = \bigwedge_{\mathbb{C}} \mathfrak{q}_{\mathbb{C}}$. We need to see that $\mathcal{E}' = \bigwedge_{J_{N'}} \mathrm{TN}'$.

At each point y of N' , the tangent bundle $\mathrm{T}_y N$ admits a canonical identification with $\mathfrak{q} \oplus \mathfrak{q}^* \oplus \mathrm{T}_y N'$. Through the identification $X \oplus \xi \in \mathfrak{q} \oplus \mathfrak{q}^* \mapsto X \oplus i\tilde{\xi} \in \mathfrak{q}_{\mathbb{C}}$, the vector space $\mathfrak{q} \oplus \mathfrak{q}^*$ inherits a complex structure $J_{\mathfrak{q}}$ and a Hermitian form. Take a Riemannian metric g on TN , orthogonal direct sum of the metric $\Omega'(\cdot, J_{N'}\cdot)$ and of the underlying Riemannian metric of $\mathfrak{q}_{\mathbb{C}}$. The symplectic form Ω_y can be written $g(\cdot, A_y\cdot)$ where A_y is a non degenerate antisymmetric matrix. Let $J_y = \frac{A_y}{\sqrt{-A_y A_y^*}}$ be the almost complex structure determined by A_y . When $y \in Z_{\mathfrak{s}}$, A_y is equal to $J_{N'} \oplus J_{\mathfrak{q}}$. As $Z_{\mathfrak{s}}$ is compact, we see that if N' is a sufficiently small neighborhood, J_y is homotopic to $J_{N'} \oplus J_{\mathfrak{q}}$, and we obtain our result.

Proposition 11.1 tells us then that

$$[RR_K(M, \mathcal{F}, \Phi^{-1}(0), \Phi)]^K = [RR_H(N', \mathcal{F}, Z', \Phi')]^H.$$

Here $Z' = Z_{\mathfrak{s}}$, and we have still denoted by \mathcal{F} the restriction of \mathcal{F} to N' .

Localization on $M_{\mathfrak{s}}$.

We now analyze $[RR_H(N', \mathcal{F}, Z', \Phi')]^H$, where $Z' = Z_{\mathfrak{s}}$.

Let us summarize the situation. We consider a H -equivariant decomposition $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{r}$. Here \mathcal{X} is a small neighborhood of $Z_{\mathfrak{s}}$ in $M_{\mathfrak{s}}$, N' is a H -Hamiltonian manifold, with symplectic structure Ω' and moment map

$\Phi' : N' \rightarrow \mathfrak{h}^*$. We have $N'_\mathfrak{s} = \mathcal{X}$. The restriction $\Phi_{\mathcal{X}}$ of Φ' to \mathcal{X} takes its values in \mathfrak{t}^* , and the zero set Z' of the vector field $\kappa_{\Phi'}$ is the compact submanifold $\Phi_{\mathcal{X}}^{-1}(0) = Z_\mathfrak{s}$ of \mathcal{X} .

We denote by $\kappa_{\mathcal{X}}$ the Kirwan vector field on $\mathcal{X} \subset M_\mathfrak{s}$ associated to $\Phi_{\mathcal{X}}$.

As we work on a neighborhood of Z' , we may assume that N' is diffeomorphic to \mathcal{N}' , the total space of the normal bundle of \mathcal{X} in N' . The restriction of Ω' on \mathcal{X} induces a symplectic structure on each fibers : $\Omega_x \in \bigwedge^2 \mathcal{N}'|_x$ for $x \in \mathcal{X}$.

We choose an H -invariant connection on \mathcal{N}' . The subspace $\mathfrak{t} \cdot w$ of $T_w \mathcal{N}'$ projects injectively on $\mathfrak{t} \cdot x$, and keep the same dimension $\dim \mathfrak{t}$. Thus we can assume that the horizontal space contains $\mathfrak{t} \cdot w$.

If $\eta \in T_w \mathcal{N}'$, we denote η^{vert} its vertical component. Let λ be the H -invariant one-form on \mathcal{N}' such that, at a point $w = (x, v) \in \mathcal{N}'$, $\lambda_w(\eta) = \Omega_x(v, \eta^{\text{vert}})$. From the symplectic embedding theorem, we may assume that $\Omega' = d\lambda + p^* \Omega_{\mathcal{X}}$. Here p is the projection $\mathcal{N}' \rightarrow \mathcal{X}$.

We can equip the vector bundle \mathcal{N}' with a H -equivariant Hermitian structure such that the section $x \in \mathcal{X} \mapsto \Omega_x$ is minus the imaginary part of the Hermitian metric. Thus the complex structure J on the fiber $\mathcal{N}'|_x$ is compatible with Ω_x . Let us choose a H -invariant metric on \mathcal{X} , and let $J_{\mathcal{X}}$ be the corresponding almost complex structure on \mathcal{X} compatible with $\Omega_{\mathcal{X}}$. Thanks to the connection, we can consider on the manifold \mathcal{N}' a H -invariant metric orthogonal direct sum of the metric on \mathcal{X} (its horizontal lift) and the Hermitian metric on \mathcal{N}' , let $J_{\mathcal{N}'}$ be the corresponding compatible almost complex structure associated to Ω' on \mathcal{N}' . We consider also with the help of the connection $J' = J \oplus J_{\mathcal{X}}$. On \mathcal{X} , we have $J' = J_{\mathcal{N}'}$. Thus $J_{\mathcal{N}'}$ is a deformation of J' and $RR_H(N', \mathcal{F}, Z', \Phi')$ can be computed using the direct sum J' .

Remark that we have the additional action of the group $U(1)$ acting by homotheties $v \mapsto e^{i\theta} v$ on the fibers of the Hermitian bundle $\mathcal{N}' \rightarrow \mathcal{X}$ and commuting with the H -action, with infinitesimal generator J . Thus we have a morphism $\text{At}_J : \mathbf{K}_H^0(T_H^* \mathcal{X}) \rightarrow \mathbf{K}_H^0(T_H^* \mathcal{N}')$ defined by Equation (6.18).

We compute now the moment map Φ' associated to the action of H on (\mathcal{N}', Ω') . We write $\Phi' = \Phi_\mathfrak{s} \oplus \Phi_\mathfrak{t}$. We obtain for $w = (x, v) \in \mathcal{N}'$:

- $\langle \Phi_\mathfrak{s}(x, v), X \rangle = \frac{1}{2} \Omega_x(Xv, v)$ if $X \in \mathfrak{s}$,
- $\Phi_\mathfrak{t}(x, v) = \Phi_{\mathcal{X}}(x)$.

Let $\mathcal{N} \rightarrow Z$ be the symplectic normal bundle of the submanifold Z in M , and let $\mathcal{N}|_{Z_\mathfrak{s}}$ be its restriction on $Z_\mathfrak{s}$. We will need the following

Lemma 12.17 • *The restriction of symplectic vector bundle $\mathcal{N}' \rightarrow \mathcal{X}$ to $Z_{\mathfrak{s}}$ is equal to $\mathcal{N}|_{Z_{\mathfrak{s}}}$.*

- *The subbundle $[\text{Sym}((\mathcal{N}')^*)]^{\mathfrak{s}}$ is reduced to the trivial bundle $[\mathbb{C}] \rightarrow \mathcal{X}$.*

Proof. Let $x \in Z_{\mathfrak{s}}$. Let \mathfrak{q}_x^* be a subspace of $(\text{T}_x \mathcal{N}')^{\perp} \subset \text{T}_x M$ that is in bijection with \mathfrak{q}^* through the tangent map $\text{T}_x \Phi_{\mathfrak{q}}$. Let $(\mathfrak{h}/\mathfrak{s})_x^*$ be a subspace of $\text{T}_x \mathcal{X}$ that is in bijection with $(\mathfrak{h}/\mathfrak{s})^*$ through the tangent map $\text{T}_x \Phi_{\mathfrak{h}}$. We have the decompositions

$$\begin{aligned} \text{T}_x M &= (\mathfrak{q} \cdot x \oplus \mathfrak{q}_x^*) \overset{\perp}{\oplus} \text{T}_x \mathcal{N}' \\ \text{T}_x \mathcal{N}' &= \text{T}_x \mathcal{X} \overset{\perp}{\oplus} \mathcal{N}'|_x, \\ \text{T}_x \mathcal{X} &= (\mathfrak{h}/\mathfrak{s})_x^* \oplus \text{T}_x Z_{\mathfrak{s}}. \end{aligned}$$

that gives $\text{T}_x M = (\mathfrak{q} \cdot x \oplus \mathfrak{q}_x^*) \overset{\perp}{\oplus} ((\mathfrak{h}/\mathfrak{s})_x^* \oplus \text{T}_x Z_{\mathfrak{s}}) \overset{\perp}{\oplus} \mathcal{N}'|_x$. Now, if we use that $\text{T}_x Z = \mathfrak{q} \cdot x \oplus \text{T}_x Z_{\mathfrak{s}}$, we see that $(\text{T}_x Z)^{\perp} = (\text{T}_x Z)^{\perp} \cap \text{T}_x Z \overset{\perp}{\oplus} \mathcal{N}'|_x$, and then $\mathcal{N}|_x = \mathcal{N}'|_x$. The first point is proved.

The restriction of the bundle $[\text{Sym}((\mathcal{N}')^*)]^{\mathfrak{s}}$ to $Z_{\mathfrak{s}}$ is equal to $[\text{Sym}(\mathcal{N}|_{Z_{\mathfrak{s}}}^*)]^{\mathfrak{s}}$ which is the trivial bundle $[\mathbb{C}] \rightarrow Z_{\mathfrak{s}}$. The second point follows.

We are in a very similar situation than Proposition 7.5 : instead of β acting fiberwise, we have \mathfrak{s} (an H -stable ideal in \mathfrak{h}) acting fiberwise.

Let $w = (x, v) \in \mathcal{N}'$, with $x \in \mathcal{X}$, and $v \in \mathcal{N}'|_x$. The horizontal lift of $\kappa_{\mathcal{X}}$ (still denoted by $\kappa_{\mathcal{X}}$) is the vector field $\Phi_{\mathfrak{r}}(w) \cdot w$. As in the proof of Proposition 12.9, we introduce the deformation $\kappa^t(w) = -\Phi_{\mathfrak{r}}(w) \cdot w - t\Phi_{\mathfrak{s}}(w) \cdot w + (1-t)Jw$ and

$$\sigma^t(w, \nu) = \mathbf{c}_w(\tilde{\nu} - \kappa^t(w)), \quad (w, \nu) \in \text{T}^* \mathcal{N}'$$

acting on $\bigwedge_{J'} \text{T} \mathcal{N}' \otimes \mathcal{F}$. Note that the vector $\Phi_{\mathfrak{r}}(w) \cdot w$ is horizontal, while $\Phi_{\mathfrak{s}}(w) \cdot w$ and Jw are verticals.

By definition, the character $RR_H(N', \mathcal{F}, Z', \Phi')$ is equal to the equivariant index of $\sigma^1 \otimes \mathcal{F}$. Let us prove that the characteristic set of σ^t intersected with $\text{T}_H^* \mathcal{N}'$ stays equal to $Z_{\mathfrak{s}}$. Consider $\eta = \kappa^t(w)$ and orthogonal to $\mathfrak{h} \cdot w$. Thus we obtain $\langle \Phi_{\mathfrak{r}}(w) \cdot w + t\Phi_{\mathfrak{s}}(w) \cdot w - (1-t)Jw, \Phi_{\mathfrak{r}}(w) \cdot w \rangle = 0$ and $\langle \Phi_{\mathfrak{r}}(w) \cdot w + t\Phi_{\mathfrak{s}}(w) \cdot w - (1-t)Jw, \Phi_{\mathfrak{s}}(w) \cdot w \rangle = 0$. Thus considering vertical and orthogonal components, we obtain $\Phi_{\mathfrak{r}}(w) = 0$. Then we have

$$\begin{aligned} 0 &= \langle t\Phi_{\mathfrak{s}}(w) \cdot w - (1-t)Jw, \Phi_{\mathfrak{s}}(w) \cdot w \rangle \\ &= t\|\Phi_{\mathfrak{s}}(w) \cdot w\|^2 + 2(1-t)\|\Phi_{\mathfrak{s}}(w)\|^2 \end{aligned}$$

as $\langle Jw, \Phi_{\mathfrak{s}}(w) \cdot w \rangle = -2\|\Phi_{\mathfrak{s}}(w)\|^2$.

Thus we have proved that if $\kappa^t(w)$ is orthogonal to $\mathfrak{h} \cdot w$, then $\Phi'(w) = 0$: this implies $w = (x, 0)$ with $x \in Z_{\mathfrak{s}}$.

We know then that $RR_H(N', \mathcal{F}, Z', \Phi')$ is equal to the equivariant index of the transversally elliptic symbol $\sigma^0 \otimes \mathcal{F}$. Since the symbol σ^0 is equal to $\text{At}_J(\sigma(\mathcal{X}, Z_{\mathfrak{s}}, \Phi_{\mathcal{X}}))$, we have then by Equation (6.19)

$$[RR_H(N', \mathcal{F}, Z', \Phi')]^H = [RR_H(\mathcal{X}, \mathcal{F} \otimes \text{Sym}((\mathcal{N}')^*), Z_{\mathfrak{s}}, \Phi_{\mathcal{X}})]^H.$$

We obtain

$$\begin{aligned} & [RR_H(\mathcal{X}, \mathcal{F} \otimes \text{Sym}((\mathcal{N}')^*), Z_{\mathfrak{s}}, \Phi_{\mathcal{X}})]^H \\ &= [RR_{H/S}(\mathcal{X}, [\mathcal{F} \otimes \text{Sym}((\mathcal{N}')^*)]_{\mathfrak{s}}, Z_{\mathfrak{s}}, \Phi_{\mathcal{X}})]^{H/S}, \\ &= RR(M_0, \mathcal{F}_0) \quad (1) \end{aligned}$$

where $\mathcal{F}_0 := [\mathcal{F}|_{Z_{\mathfrak{s}}} \otimes \text{Sym}(\mathcal{N}^*|_{Z_{\mathfrak{s}}})]_{\mathfrak{s}} / (H/S)$. In (1), we used the regular case considered in Proposition 12.5.

12.3 Riemann-Roch number on symplectic reduction

Let (M, Ω, Φ) be a Hamiltonian K -manifold not necessarily compact. Suppose that a is a quasi-regular value of Φ such that the fiber $Z := \Phi^{-1}(a)$ is a *compact* submanifold. The quotient $M_a := \Phi^{-1}(a)/K_a$ is a compact symplectic orbifold.

Let $\mathfrak{s} \subset \mathfrak{k}_a$ be a subalgebra such that $\Phi^{-1}(Ka)$ is contained in the submanifold $M_{(\mathfrak{s})}$. We denote by S the connected subgroup of K_a with Lie algebra \mathfrak{s} , and H the normalizer subgroup of \mathfrak{s} in \mathfrak{k}_a . The orbifold $M_a = \Phi^{-1}(a)/K_a$ is also the quotient of the submanifold $Z_{\mathfrak{s}} := \Phi^{-1}(a) \cap M_{\mathfrak{s}}$ by the group H/S .

We consider now the case of an equivariant vector bundle $\mathcal{F} \rightarrow M$ and a character $\chi_{\lambda} : K_a \rightarrow \text{U}(1)$: here $\lambda = \frac{1}{i}d\chi_{\lambda}$ is a weight, and we denote \mathbb{C}_{λ} the 1-dimensional representation of K_a defined by χ_{λ} .

We suppose that the action of S on the fibers of the bundle $\mathcal{F}|_{Z_{\mathfrak{s}}}$ is $t \cdot v = \chi_{\lambda}(t)v$. Then S acts trivially on $\mathcal{F}|_{Z_{\mathfrak{s}}} \otimes \mathbb{C}_{-\lambda}$ and we may form the orbibundle

$$\mathcal{F}_{a,\lambda} := (\mathcal{F}|_{Z_{\mathfrak{s}}} \otimes \mathbb{C}_{-\lambda}) / (H/S)$$

on M_a . The aim of this section is to show that the Riemann-Roch number $RR(M_a, \mathcal{F}_{a,\lambda})$ can be computed by a localization procedure.

Let $(Ka)^{-}$ be the coadjoint orbit Ka with the opposite symplectic structure. Let $[\mathbb{C}_{-\lambda}] = K \times_{K_a} \mathbb{C}_{-\lambda}$ be the line bundle on $(Ka)^{-}$. We consider

the symplectic manifold $M \times (Ka)^-$ with the complex bundle $\mathcal{F} \boxtimes [\mathbb{C}_{-\lambda}]$ on it. The corresponding moment map $\Phi_a : M \times (Ka)^- \rightarrow \mathfrak{k}^*$ is $\Phi_a(m, \xi) = \Phi(m) - \xi$. As the set $\Phi_a^{-1}(0)$ is compact, we may define the localized Riemann Roch character

$$RR_K(M \times (Ka)^-, \mathcal{F} \boxtimes [\mathbb{C}_{-\lambda}], \Phi_a^{-1}(0), \Phi_a) \in \hat{R}(K).$$

The main result of this section is the

Theorem 12.18 *If a is a quasi-regular value of Φ we have*

$$(12.39) \quad [RR_K(M \times (Ka)^-, \mathcal{F} \boxtimes [\mathbb{C}_{-\lambda}], \Phi_a^{-1}(0), \Phi_a)]^K = RR(M_a, \mathcal{F}_{a,\lambda}).$$

Proof. Let $B \subset \mathfrak{k}_a^*$ be a small K_a -invariant ball around a , and consider the slice $Y := \Phi^{-1}(B)$ which is a symplectic submanifold of M : the action of K_a on $(Y, \Omega|_Y)$ is Hamiltonian with moment map $\Phi_Y := \Phi|_Y$.

The computation of the left hand side of (12.39) is done in the manifold $\tilde{M} \times (Ka)^-$ where $\tilde{M} = K \times_{K_a} Y$ is viewed as an open subset of M . Let J_Y be a K_a -invariant almost complex structure on Y compatible with the symplectic form $\Omega|_Y$. Let $J_{\tilde{M}}$ be the K -invariant complex structure on \tilde{M} that is equal to $J_a \oplus J_Y$ at the points $[1, y]$, $y \in Y$. Notice that $J_{\tilde{M}}$ is compatible with the symplectic structure $\Omega|_{\tilde{M}}$.

If we apply the induction formula of Proposition 11.3 to the manifold $\tilde{M} \times (Ka)^-$ equipped with the Clifford bundles $\mathcal{G} := \bigwedge_J TM \otimes \mathcal{F}$ on \tilde{M} and $\mathcal{R} = \bigwedge_{-J_a} TKa \otimes [\mathbb{C}_{-\lambda}]$ on Ka , we get

- $\mathcal{G}_{dY} = \bigwedge_{J_Y} TY \otimes \mathcal{F}|_Y$ for the corresponding Clifford bundle on Y ,
- $\mathcal{R} = \mathbb{C}_{-\lambda}$ for the corresponding K_a -module.

Hence we have

$$(12.40) \quad \begin{aligned} & [RR_K(M \times (Ka)^-, \mathcal{F} \boxtimes [\mathbb{C}_{-\lambda}], \Phi_a^{-1}(0), \Phi_a)]^K \\ &= [RR_{K_a}(Y, \mathcal{F}_\lambda, \phi^{-1}(0), \phi)]^{K_a}. \end{aligned}$$

where $\phi := \Phi_Y - a$ is a moment map on the K_a -Hamiltonian manifold (Y, Ω_Y) , 0 is a quasi regular value of ϕ and $\mathcal{F}_\lambda = \mathcal{F}|_Y \otimes \mathbb{C}_{-\lambda}$.

We are in the setting of Section 12.2.3. Since \mathfrak{s} acts trivially on the fibers of the vector bundle $\mathcal{F}_\lambda|_{Z_s}$, Theorem 12.15 says that

$$[RR_{K_a}(Y, \mathcal{F}_\lambda, \phi^{-1}(0), \phi)]^{K_a} = RR(M_a, \mathcal{F}_{a,\lambda})$$

with $\mathcal{F}_{a,\lambda} = \mathcal{F}_\lambda|_{Z_s}/(H/S)$. The proof is completed.

12.4 Multiplicities as Riemann-Roch numbers

In this section, we give a proof of the $[Q, R] = 0$ Theorem of Meinrenken-Sjamaar.

Let L be a Φ -moment line bundle on the compact Hamiltonian K -manifold (M, Ω, Φ) . We fix a dominant weight μ , and we consider the multiplicity $m_\mu(L) = [RR_K(M, L) \otimes V_\mu^*]^K$. We consider elements $a \in \mathfrak{t}^*$ close enough to μ , so $K_a \subset K_\mu$. Let $(Ka)^-$ be the coadjoint orbit equipped with the opposite symplectic structure and the line bundle $[\mathbb{C}_{-\mu}] \simeq K \times_{K_a} \mathbb{C}_{-\mu}$. We have $RR_K((Ka)^-, [\mathbb{C}_{-\mu}]) = V_\mu^*$, hence the shifting trick gives

$$m_\mu(L) = [RR_K(M \times (Ka)^-, \mathbb{L}(a))]^K.$$

Here $\mathbb{L}(a) = L \boxtimes [\mathbb{C}_{-\mu}]$ and the moment map is $\Phi_a : M \times (Ka)^- \rightarrow \mathfrak{t}^*$, $(m, \xi) \mapsto \Phi(m) - \xi$. We start with the

Lemma 12.19 *There exist $\epsilon > 0$ such that the line bundle $\mathbb{L}(a)$ is weakly Φ_a -positive if $\|a - \mu\| \leq \epsilon$.*

Proof. We consider the maps $\varphi_a : M \times K/T \rightarrow M \times (Ka)^-$ defined by $\varphi(m, [k]) = (m, ka)$. The line bundle $\varphi_a^* \mathbb{L}(a)$ does not depend on a and is denoted \mathbb{L}' . Let $\Phi'_a = \varphi_a^* \Phi_a$.

One checks that $\varphi_a(\{\kappa_{\Phi'_a} = 0\}) = \{\kappa_{\Phi_a} = 0\}$, hence we can use Lemma 9.3 : the line bundle $\mathbb{L}(a)$ is weakly Φ_a -positive if and only if the line bundle \mathbb{L}' is weakly Φ'_a -positive.

Now we see that for $a = \mu$, the line bundle \mathbb{L}' is a Φ'_μ -moment line bundle. Thanks to Proposition 9.7, we know that \mathbb{L}' is weakly Φ'_a -positive if a is close enough to μ .

Now we take $a \in \Delta(M)$ such that $\|a - \mu\| \leq \epsilon$. Since $\mathbb{L}(a)$ is a weakly Φ_a -positive line bundle on the Hamiltonian K -manifold $M \times (Ka)^-$ and $\Phi_a^{-1}(0) \neq \emptyset$, Theorem 10.7 says that

$$m_\mu(L) = [RR_K(M \times (Ka)^-, \mathbb{L}(a), \Phi_a^{-1}(0), \Phi_a)]^K.$$

We suppose furthermore that a is a quasi-regular value of Φ . Let \mathfrak{s} be the subalgebra of \mathfrak{k}_a such that $\Phi^{-1}(Ka)$ is contained in $M_{(\mathfrak{s})}$: we denote $Z_{\mathfrak{s}}$ the submanifold $\Phi^{-1}(a) \cap M_{\mathfrak{s}}$. In order to make use of Theorem 12.18 we need to compute the action of S on the fibers of the line bundle $L|_{Z_{\mathfrak{s}}} \otimes \mathbb{C}_{-\mu}$.

Lemma 12.20 *If a is a quasi-regular value close enough to μ , the group S acts trivially on the fibers of the line bundle $L|_{Z_{\mathfrak{s}}} \otimes \mathbb{C}_{-\mu}$.*

Proof. We consider the K_μ -invariant subset $U \subset M$ formed by the points $m \in M$ such that the algebra $\mathfrak{k}_m \cap \mathfrak{k}_\mu$ acts trivially on $L|_m \otimes \mathbb{C}_{-\mu}$. The Kostant formula shows that $\Phi^{-1}(\mu) \subset U$, and if we use local symplectic models of orbits $K_\mu \cdot m$, one sees that U is an open subset. Hence $\Phi^{-1}(a) \subset U$ for a close enough to μ . The proof is completed.

Finally, we can use Theorem 12.18 when a is a quasi-regular value close enough to μ : relation (12.39) gives

$$m_\mu(L) = [RR_K(M \times (Ka)^-, \mathbb{L}(a), \Phi_a^{-1}(0), \Phi_a)]^K = RR(M_a, L_{\mu,a}).$$

The proof of Theorem 12.3 is then completed.

12.5 Quasi polynomial behavior

We work in the same setting than in Subsection 12.4. We fix a Φ -moment line bundle L on M . For any dominant weight $\mu \in \Lambda_{\geq 0}$ and any integer $k \in \mathbb{N}$, we define $m(k, \mu) \in \mathbb{N}$ as the multiplicity of the representation V_μ^K in $RR_K(M, L^{\otimes k})$.

We introduce

1. the vector subspace $V_M \subset \mathbb{R} \oplus \mathfrak{t}^*$ generated by $\{1\} \times \Delta(M)$,
2. the cone $C_M = \mathbb{R}_{\geq 0}(\{1\} \times \Delta(M))$ contained in V_M ,
3. the lattice $\Lambda_M = \{\mathbb{Z} \oplus \Lambda\} \cap V_M$,
4. the principal face σ for the K -manifold: σ is the relative interior of a face of the Weyl chamber and $\sigma \cap \Delta(M)$ is dense in $\Delta(M)$.

Theorem 12.3 tells us that $m(k, \mu) = 0$ if $(k, \mu) \notin C_M$. We will now give some qualitative properties on the behavior of the multiplicities $m(k, \mu)$ on the cone C_M .

If ξ varies in σ , the subgroup K_ξ is independent of ξ . We denote it by K_σ . Let $Y = \Phi^{-1}(\sigma)$ be the slice attached to the principal face. It is a K_σ -invariant symplectic submanifold of M , and the subgroup $[K_\sigma, K_\sigma]$ acts trivially on $L|_Y \rightarrow Y$ [22]. The vector space $\mathbb{R}\sigma$ is naturally identified with the dual of the Lie algebra of the torus $A_\sigma := K_\sigma/[K_\sigma, K_\sigma]$. Hence the restriction $\Phi_Y = \Phi|_Y : Y \rightarrow \sigma$ corresponds to the moment map relative to the action of A_σ on $(Y, \Omega|_Y)$.

Let $I_M \subset \mathbb{R}\sigma$ be the affine subspace generated by $\Delta(M)$: the orthogonal, denoted \mathfrak{t}_M , of the direction of I_M in $\text{Lie}(A_\sigma)$ corresponds to the generic infinitesimal stabilizer for the A_σ -action on Y . Let $\Delta(M)^0 \subset \Delta(M) \cap \sigma$ be

the open subset formed by the regular values of $\Phi_Y : Y \rightarrow I_M$. Note that any element $\xi \in \Delta(M)^0$ is a quasi-regular value of Φ . The reduced space M_ξ stays of constant dimension when $\xi \in \Delta(M)^0$, and this dimension is the maximal dimension of the reduced spaces M_ξ , where ξ varies over the set of quasi-regular values of Φ .

If $\mathfrak{a} \subset \Delta(M)^0$ is a connected component, we choose a point $\xi \in \mathfrak{a}$, and we consider

- the closed subcone $C_{\mathfrak{a}} = \mathbb{R}_{\geq 0}(\{1\} \times \bar{\mathfrak{a}})$ of C_M ,
- the symplectic orbifold $M_{\mathfrak{a}} := \Phi^{-1}(\xi)/A_\sigma$,
- the family of line orbi-bundle $L_{\mathfrak{a}}^{k,\mu} \rightarrow M_{\mathfrak{a}}$ parameterized by $(k, \mu) \in \Lambda_M$:

$$L_{\mathfrak{a}}^{k,\mu} := \left(L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu} \right) / A_\sigma$$

The line orbi-bundles $L_{\mathfrak{a}}^{k,\mu}$ are well-defined for any $(k, \mu) \in \Lambda_M$ since $\mathcal{L}(X) = 0$ on $L^{\otimes k}|_{\Phi^{-1}(\xi)} \otimes \mathbb{C}_{-\mu}$ for any $X \in \mathfrak{t}_M$.

Definition 12.21 *Let \mathfrak{a} be a connected component of $\Delta(M)^0$. We consider the function $p_{\mathfrak{a}} : \Lambda_M \rightarrow \mathbb{Z}$ defined by the relation*

$$p_{\mathfrak{a}}(k, \mu) := RR(M_{\mathfrak{a}}, L_{\mathfrak{a}}^{k,\mu})$$

The Kawasaki-Riemann-Roch Theorem [18] implies that the function $p_{\mathfrak{a}}$ have a quasi polynomial behavior on the lattice Λ_M .

The function $p_{\mathfrak{a}}$ does not depend on the choice of the choice of ξ in \mathfrak{a} . It is due to the fact that $p_{\mathfrak{a}}(k, \mu)$ is equal to

$$\left[RR_{A_\sigma} \left(Y, L^{\otimes k}|_Y, \Phi_Y^{-1}(\xi), \Phi_Y - \xi \right) \otimes \mathbb{C}_{-\mu} \right]^{A_\sigma}.$$

See (12.39). Hence if we take another $\xi' \in \mathfrak{a}$, we consider a continuous family $\xi_t \in \mathfrak{a}, t \in [0, 1]$ joining ξ and ξ' . By an easy homotopy argument, we see that $\left[RR_{A_\sigma} \left(Y, L^{\otimes k}|_Y, \Phi_Y^{-1}(\xi_t), \Phi_Y - \xi_t \right) \otimes \mathbb{C}_{-\mu} \right]^{A_\sigma}$ is independent of $t \in [0, 1]$.

We can now state the main result of this section. Notice that $C_M = \bigcup_{\mathfrak{a}} C_{\mathfrak{a}}$.

Theorem 12.22 *If $(k, \mu) \in C_{\mathfrak{a}}$, we have $m(k, \mu) = p_{\mathfrak{a}}(k, \mu)$.*

Proof. Let us fix $(k, \mu) \in C_{\mathfrak{a}}$. We apply Theorem 12.3 to the K -manifold $\mathbb{X} := (M, k\Omega, k\Phi)$ equipped with the moment line bundle $\mathbb{L} := L^{\otimes k}$. The multiplicity of V_{μ}^K in $RR_K(\mathbb{X}, \mathbb{L})$ is equal to $RR(\mathbb{X}_a, \mathbb{L}_{a, \mu})$ where a is any quasi-regular value of $k\Phi$ close enough to μ . We can take $a = k\xi$ with $\xi \in \mathfrak{a}$: the reduced space \mathbb{X}_a is equal to M_{ξ} and the line bundle $\mathbb{L}_{a, \mu}$ coincides with $L_a^{k, \mu}$. We have proved that $m(k, \mu) = RR(\mathbb{X}_a, \mathbb{L}_{a, \mu}) = p_{\mathfrak{a}}(k, \mu)$.

12.6 Multiplicities on a face

Let F be a (closed) face of the polytope $\Delta(M)$ that intersects the principal face σ . We say that F is a general face of $\Delta(M)$.

We assume in this section that μ is a dominant weight that belongs to F . Let $T_{\sigma} \subset T$ be the connected component of the center of the subgroup K_{σ} . The dual of the Lie algebra of T_{σ} is the subspace $\mathbb{R}\sigma \subset \mathfrak{t}^*$. Let $T_F \subset T_{\sigma}$ be the subtorus with Lie algebra equal to the orthogonal of the direction $\vec{F} \subset \mathbb{R}\sigma$ of the face F . We denote K_F the subgroup of K that centralizes T_F : we see that $K_{\sigma} \subset K_F$.

We start with the basic fact.

Lemma 12.23 *The pullback $\Phi^{-1}(F \cap \sigma)$ is connected and belongs to the submanifold M^{T_F} .*

Proof. The pullback $\Phi^{-1}(F \cap \sigma)$ belongs to the slice $Y = \Phi^{-1}(\sigma)$, and it is connected as each fiber $\Phi^{-1}(\xi)$ is connected. Since F is a face of the polytope $\Delta(M) \subset \mathbb{R}\sigma$, we see that $\Delta(M)$ is contained in a cone of the form

$$\bigcap_{j=1}^p \{\xi \in \mathbb{R}\sigma \mid \langle \xi, X_j \rangle \geq \langle \mu, X_j \rangle\}$$

where X_1, \dots, X_p is a basis of $\text{Lie}(T_F)$. Since $\Delta(M) \cap \sigma = \text{Image}(\Phi|_Y)$, the functions $y \in Y \mapsto \langle \Phi(y), X_j \rangle$ reach their minimum at any point of $\Phi^{-1}(F \cap \sigma)$. So the differential of the function $y \in Y \mapsto \langle \Phi(y), X_j \rangle$ vanishes at any point of $\Phi^{-1}(F \cap \sigma)$. This implies that $\Phi^{-1}(F \cap \sigma) \subset \bigcap_{j=1}^p M^{X_j} = M^{T_F}$.

Let M_F be the connected component of M^{T_F} that contains $\Phi^{-1}(F \cap \sigma)$. It is a K_F -Hamiltonian manifold equipped with the two form $\Omega_F := \Omega|_{M_F}$ and the moment map $\Phi_F = \Phi|_{M_F} : M_F \rightarrow \mathfrak{k}_F^*$. We denote $L|_F$ the restriction of the Φ -moment line bundle L on M_F . It is easy to see that $L|_F$ is a Φ_F -moment line bundle.

We parameterize the irreducible representations of K_F thanks to the choice of a Weyl chamber $\mathcal{C}_F \subset \mathfrak{t}^*$ that contains the Weyl chamber $\mathfrak{t}_{\geq 0}^*$ of K . So, we may consider the irreducible representation $V_\mu^{K_F}$ of the group K_F with highest weight μ .

A direct consequence of the $[Q, R] = 0$ Theorem is the following proposition.

Proposition 12.24 *Let L be a Φ -moment bundle on M , and let F be a face of $\Delta(M)$ that intersects the principal face σ . For any dominant weight μ that belongs to F , the multiplicity $m_\mu(L)$ of the representation V_μ^K in $RR_K(M, L)$ is equal to the multiplicity of the representation $V_\mu^{K_F}$ in $RR_{K_F}(M_F, L|_F)$.*

Proof. We have $m_\mu(L) = RR(M_a, L_{a,\mu})$ for any quasi-regular value of Φ close enough to μ . Since $\mu \in F$, we can choose a in $F \cap \sigma$, and then $\Phi^{-1}(a) = \Phi_F^{-1}(a) \subset M_F$. The proposition follows then from the fact that the datas $L_{a,\mu} \rightarrow M_a$ and $(L|_F)_{a,\mu} \rightarrow (M_F)_a$ are equal (the last one is a reduction relatively to the action of the group K_F on $L|_F \rightarrow M_F$).

13 Branching laws

Let \tilde{K} be a compact connected Lie group and let K be a closed connected subgroup. Let i denote the inclusion of K into \tilde{K} , $i : \mathfrak{k} \rightarrow \tilde{\mathfrak{k}}$ the induced embedding of Lie algebras, and $\pi : \tilde{\mathfrak{k}}^* \rightarrow \mathfrak{k}^*$ the dual projection.

We consider the following action of $\tilde{K} \times K$ on $\tilde{K} : (\tilde{k}, k) \cdot g = \tilde{k}gk^{-1}$. Then $N = T^*\tilde{K}$, the cotangent bundle of \tilde{K} is a $\tilde{K} \times K$ Hamiltonian manifold. Recall from Subsection 8.5 (Equation 8.30) that the corresponding moment map relative to the $\tilde{K} \times K$ -action is the map $\Phi = \Phi_{\tilde{K}} \oplus \Phi_K : T^*\tilde{K} \rightarrow \tilde{\mathfrak{k}}^* \oplus \mathfrak{k}^*$ defined by

$$\Phi_{\tilde{K}}(g, \xi) = -\xi, \quad \Phi_K(g, \xi) = \pi(g^{-1}\xi).$$

Select maximal tori T in K and \tilde{T} in \tilde{K} such that $T \subset \tilde{T}$, and Weyl chambers $\tilde{\mathfrak{t}}_{\geq 0}^*$ in $\tilde{\mathfrak{k}}^*$ and $\mathfrak{t}_{\geq 0}^*$ in \mathfrak{k}^* , where \mathfrak{t} and $\tilde{\mathfrak{t}}$ denote the Lie algebras of T , resp. \tilde{T} .

Let $\Delta(T^*\tilde{K}) \subset \tilde{\mathfrak{t}}_{\geq 0}^* \times \mathfrak{t}_{\geq 0}^*$ be the Kirwan polytope associated to Φ :

$$\Delta(T^*\tilde{K}) = \left\{ (\alpha, \beta) \in \tilde{\mathfrak{t}}_{\geq 0}^* \times \mathfrak{t}_{\geq 0}^* \mid -\beta \in \pi(\tilde{K}\alpha) \right\}.$$

For $(\tilde{\lambda}, \lambda) \in \tilde{\mathfrak{t}}_{\geq 0}^* \oplus \mathfrak{t}_{\geq 0}^*$, we denote by $N_{\tilde{\lambda}, \lambda}$ the reduced space of $N := T^*\tilde{K}$ at $(\tilde{\lambda}, \lambda)$. Consider the coadjoint orbit $\tilde{K}\tilde{\lambda}^* := -\tilde{K}\tilde{\lambda}$, equipped with the K -action and moment map $\pi|_{\tilde{K}\tilde{\lambda}^*}$. Let $(\tilde{K}\tilde{\lambda}^*)_\lambda$ the reduced manifold of $\tilde{K}\tilde{\lambda}^*$ at λ . The following lemma is obvious.

Lemma 13.1 *As sets, $(\tilde{K}\tilde{\lambda}^*)_\lambda$ is isomorphic to $N_{\tilde{\lambda},\lambda}$.*

Let $\tilde{\Lambda}_{\geq 0} \subset \tilde{\mathfrak{t}}_{\geq 0}^*$, $\Lambda_{\geq 0} \subset \mathfrak{t}_{\geq 0}^*$ be the set of dominant weights. For any $(\lambda, \tilde{\lambda}) \in \Lambda_{\geq 0} \times \tilde{\Lambda}_{\geq 0}$, we denote $V_\lambda, \tilde{V}_{\tilde{\lambda}}$ the corresponding irreducible representations of K and \tilde{K} , and we define

$$(13.41) \quad m(\tilde{\lambda}, \lambda) \in \mathbb{N}$$

as the dimension of the vector space $[\tilde{V}_{\tilde{\lambda}}|_K \otimes V_\lambda]^K$. In other words $m(\tilde{\lambda}, \lambda)$ is the multiplicity of the representation V_λ in $(\tilde{V}_{\tilde{\lambda}})^*|_K$.

We consider the coadjoint orbit $\tilde{K}\tilde{\lambda}^*$, equipped with the prequantum line bundle $L(\tilde{\lambda}^*) := [\mathbb{C}_{\tilde{\lambda}^*}]$. The $[Q, R] = 0$ theorem 12.3 says that $m(\tilde{\lambda}, \lambda) = RR((\tilde{K}\tilde{\lambda}^*)_\lambda, L(\tilde{\lambda}^*)_\lambda)$, in particular $m(\tilde{\lambda}, \lambda) = 0$ if $(\tilde{K}\tilde{\lambda}^*)_\lambda = \emptyset$.

If we use Lemma 13.1, we see that the cone $\Delta(\mathrm{T}^*\tilde{K})$ is related to the multiplicity function (13.41) through the following basic result.

Lemma 13.2 *We have $m(\tilde{\lambda}, \lambda) = 0$ if $(\tilde{\lambda}, \lambda) \notin \Delta(\mathrm{T}^*\tilde{K})$.*

Let (\tilde{X}, X) be an element of $\tilde{\mathfrak{t}} \times \mathfrak{t}$. In the following lemma, which follows by direct checking, we describe the manifold $N^{(\tilde{X}, X)}$ of zeroes of the vector field (\tilde{X}, X) on $N = \mathrm{T}^*\tilde{K}$. We denote $\tilde{K}_X \subset \tilde{K}$ the subgroup that fixes X . The cotangent bundle $\mathrm{T}^*\tilde{K}_X$ is then a submanifold of $\mathrm{T}^*\tilde{K}$.

Lemma 13.3

- *If \tilde{X} is not conjugated in $\tilde{\mathfrak{k}}$ to X , then $N^{(\tilde{X}, X)} = \emptyset$*

- *If $\tilde{X} = aX$ with $a \in \tilde{K}$, then $N^{(\tilde{X}, X)} = a \cdot \mathrm{T}^*\tilde{K}_X$.*

We recall from general principle that the Kirwan polyhedron has non zero interior in $\tilde{\mathfrak{t}}^* \times \mathfrak{t}^*$ if and only if the generic stabilizer of the $\tilde{K} \times K$ -action on $\mathrm{T}^*\tilde{K}$ is finite.

To simplify the exposition, we assume from now on that no nonzero ideal of \mathfrak{k} is an ideal of $\tilde{\mathfrak{k}}$. It implies the following

Lemma 13.4 *The intersection of $\Delta(\mathrm{T}^*\tilde{K})$ with $\tilde{\mathfrak{t}}_{>0}^* \times \mathfrak{t}_{>0}^*$ has a non empty interior, and the generic stabilizer of the $\tilde{K} \times K$ -action on $\mathrm{T}^*\tilde{K}$ is finite.*

Proof. By the preceding computation, we see that the set N^1 of elements of N having a non zero stabilizer is $N^1 = \tilde{K} \cdot \cup_{X \in \mathfrak{t}, X \neq 0} \tilde{\mathrm{T}}^*\tilde{K}_X$.

Using conjugation by K , we may assume that X is in \mathfrak{t} . Thus we obtain $N^1 = (\tilde{K} \times K) \cdot \cup_{X \in \mathfrak{t}, X \neq 0} \mathrm{T}^*\tilde{K}_X$.

The set of subgroups \tilde{K}_X when X varies in \mathfrak{t} is finite. Consider the fiber product $F(X) = (\tilde{K} \times K) \times_{\tilde{K}_X \times K_X} \mathrm{T}^* \tilde{K}_X$. It surjects on $(\tilde{K} \times K) \cdot \mathrm{T}^* \tilde{K}_X$. Thus it is sufficient to prove that for $X \neq 0$ in \mathfrak{t} , and under our hypothesis, the dimension of $F(X)$ is strictly less than $\mathrm{T}^* \tilde{K}$.

Computing dimensions, if the dimensions are equal, then $\dim(K/K_X) = \dim(\tilde{K}/\tilde{K}_X)$. So this implies that $K \cdot X = \tilde{K} \cdot X$. The vector space spanned by $K \cdot X$ is then an ideal of $\tilde{\mathfrak{k}}$ contained in \mathfrak{k} , contradicting our hypothesis.

In the next section, we will give some qualitative information concerning the behavior of the multiplicity function $m(\tilde{\lambda}, \lambda)$ on the cone $\Delta(\mathrm{T}^* \tilde{K})$.

13.1 Quasi polynomial behavior

We consider the slice

$$Y = \Phi^{-1}(\tilde{\mathfrak{t}}_{>0}^* \times \mathfrak{t}_{>0}^*)$$

which is a (non-empty) symplectic submanifold of $\mathrm{T}^* \tilde{K}$ equipped with the action of $\tilde{T} \times T$.

We denote $\Phi_Y : Y \rightarrow \tilde{\mathfrak{t}}^* \times \mathfrak{t}^*$ the restriction of the moment map on Y . Let $\Delta^0 \subset \Delta(\mathrm{T}^* \tilde{K})$ be the open cone formed by the regular values of $\Phi_Y : Y \rightarrow \tilde{\mathfrak{t}}^* \times \mathfrak{t}^*$. If $\mathfrak{a} \subset \Delta^0$ is a connected component of Δ^0 , we choose a point $(\tilde{\xi}, \xi) \in \mathfrak{a}$, and we consider:

- the compact connected symplectic orbifold $N_{\mathfrak{a}} := \Phi_Y^{-1}(\tilde{\xi}, \xi) / \tilde{T} \times T$,
- the family of line orbi-bundle $L_{\mathfrak{a}}^{\tilde{\alpha}, \alpha} \rightarrow N_{\mathfrak{a}}$ parametrized by $(\tilde{\alpha}, \alpha) \in \tilde{\Lambda} \times \Lambda$:

$$L_{\mathfrak{a}}^{\tilde{\alpha}, \alpha} := \Phi_Y^{-1}(\tilde{\xi}, \xi) \times_{\tilde{T} \times T} \mathbb{C}_{-\tilde{\alpha}, -\alpha}.$$

Here $\mathbb{C}_{-\tilde{\alpha}, -\alpha}$ denotes the one dimensional representation of $\tilde{T} \times T$ where the group acts through the character $\chi_{-\tilde{\alpha}, -\alpha}(t_1, t_2) = t_1^{-\tilde{\alpha}} t_2^{-\alpha}$.

Definition 13.5 *Let \mathfrak{a} be a connected component of $\Delta(\mathrm{T}^* K)^0$. We consider the function $p_{\mathfrak{a}} : \tilde{\Lambda} \times \Lambda \rightarrow \mathbb{Z}$ defined by the relation*

$$p_{\mathfrak{a}}(\tilde{\alpha}, \alpha) := RR(N_{\mathfrak{a}}, L_{\mathfrak{a}}^{\tilde{\alpha}, \alpha}).$$

The function $p_{\mathfrak{a}}$ does not depend on the choice of the choice of $\tilde{\xi}, \xi$ in \mathfrak{a} , since the $\tilde{T} \times T$ -manifolds $\Phi_Y^{-1}(\tilde{\xi}_1, \xi_1)$ and $\Phi_Y^{-1}(\tilde{\xi}_2, \xi_2)$ are diffeomorphic for two different choices. The Kawasaki-Riemann-Roch Theorem [18] implies that the function $p_{\mathfrak{a}}$ have a quasi polynomial behavior on the lattice $\tilde{\Lambda} \times \Lambda$.

We can now state the main result of this section. Note that $\Delta(\mathrm{T}^* K) = \bigcup_{\mathfrak{a}} \mathfrak{a}$.

Theorem 13.6 *If $(\tilde{\lambda}, \lambda) \in \bar{\mathfrak{a}}$, we have $\mathfrak{m}(\tilde{\lambda}, \lambda) = p_{\mathfrak{a}}(\tilde{\lambda}, \lambda)$.*

Proof. Let $b := (\tilde{\xi}, \xi) \in \mathfrak{a}$ and $(\tilde{\lambda}, \lambda) \in \bar{\mathfrak{a}}$. By (12.38), the quantity $p_{\mathfrak{a}}(\tilde{\lambda}, \lambda)$ is equal to

$$\left[RR_{\tilde{T} \times T}(Y, \mathbb{C}, \Phi_Y^{-1}(b), \Phi_Y - b) \otimes \mathbb{C}_{-\tilde{\lambda}, -\lambda} \right]^{\tilde{T} \times T}.$$

We write $\Phi_Y = \Phi_{\tilde{T}} \oplus \Phi_T$. An element (g, η) belongs to Y if and only if $-\eta \in \tilde{\mathfrak{t}}_{>0}^*$ and $\pi(g^{-1}\eta) \in \mathfrak{t}_{>0}^*$. The group \tilde{T} acts freely on Y , and the smooth reduced space $Y_{\tilde{\xi}} = \Phi_{\tilde{T}}^{-1}(\tilde{\xi})/\tilde{T}$ is symplectomorphic to the slice

$$\{\eta \in \tilde{K}\tilde{\xi}^* \mid \pi(\eta) \in \mathfrak{t}_{>0}^*\} \subset \tilde{K}\tilde{\xi}^*$$

through the map $(g, \eta) \rightarrow g^{-1}\eta$. We have

$$\begin{aligned} & \left[RR_{\tilde{T} \times T}(Y, \mathbb{C}, \Phi_Y^{-1}(b), \Phi_Y - b) \otimes \mathbb{C}_{-\tilde{\lambda}, -\lambda} \right]^{\tilde{T} \times T} \\ &= \left[RR_T(Y_{\tilde{\xi}}, [\mathbb{C}_{-\tilde{\lambda}}], \Phi_T^{-1}(\xi), \Phi_T - \xi) \otimes \mathbb{C}_{-\lambda} \right]^T \quad (1) \\ &= \left[RR_K(\tilde{K}\tilde{\xi}^* \times (K\xi)^-, [\mathbb{C}_{-\tilde{\lambda}}] \boxtimes [\mathbb{C}_{-\lambda}], \Phi_b^{-1}(0), \Phi_b) \right]^K \quad (2) \end{aligned}$$

where $\Phi_b : \tilde{K}\tilde{\xi}^* \times \overline{K\xi} \rightarrow \mathfrak{k}^*$ is the moment map defined by $\Phi_b(\tilde{\eta}, \eta) = \pi(\tilde{\eta}) - \eta$. Equality (1) follows from the reduction in stage property (see Proposition 12.6), and (2) is a particular case of (12.40).

If $b = (\tilde{\xi}, \xi) \in \mathfrak{a}$ is close enough to $(\tilde{\lambda}, \lambda)$, the line bundle $[\mathbb{C}_{-\tilde{\lambda}}] \boxtimes [\mathbb{C}_{-\lambda}]$ is weakly Φ_b -positive (see Lemma 12.19). Since $\Phi_b^{-1}(0) \neq \emptyset$ we can conclude that

$$\begin{aligned} & \left[RR_K(\tilde{K}\tilde{\xi}^* \times (K\xi)^-, [\mathbb{C}_{-\tilde{\lambda}}] \boxtimes [\mathbb{C}_{-\lambda}], \Phi_b^{-1}(0), \Phi_b) \right]^K \\ &= \left[RR_K(\tilde{K}\tilde{\xi}^* \times (K\xi)^-, [\mathbb{C}_{-\tilde{\lambda}}] \boxtimes [\mathbb{C}_{-\lambda}]) \right]^K \\ &= \left[(\tilde{V}_{\tilde{\lambda}})^*|_K \otimes V_{\lambda}^* \right]^K \\ &= \mathfrak{m}(\tilde{\lambda}, \lambda). \end{aligned}$$

Remark 13.7 *In the proof above, we obtain the following identity*

$$\mathfrak{m}(\tilde{\lambda}, \lambda) = \left[RR_{\tilde{T} \times T}(Y, \mathbb{C}, \Phi_Y^{-1}(b), \Phi_Y - b) \otimes \mathbb{C}_{-\tilde{\lambda}, -\lambda} \right]^{\tilde{T} \times T},$$

that holds for any $b = (\tilde{\xi}, \xi) \in \tilde{\mathfrak{t}}_{>0}^* \times \mathfrak{t}_{>0}^*$ that is close enough to $(\tilde{\lambda}, \lambda)$. The identity $\mathfrak{m}(\tilde{\lambda}, \lambda) = p_{\mathfrak{a}}(\tilde{\lambda}, \lambda)$ follows by taking $(\tilde{\xi}, \xi)$ a regular value of Φ . But we could have taken a quasi-regular value of Φ (see Section 13.2).

We can also view this theorem as a consequence of $[Q, R] = 0$ for the non compact manifold $N = T^* \tilde{K}$. Philosophically, we have used here reduction in stages to reduce to the $\tilde{K} \times K$ reduced space of N to the reduced space with respect to K of the reduced space with respect to \tilde{K} of N .

We could however, at the price of proving $[Q, R] = 0$ directly for the non compact manifold N , prove directly our theorem. Let us sketch the proof. If we use the result of Section 8.5, knowing that the critical set Z_Φ is compact, we see that the localized index $RR_{\tilde{K} \times K}(N, [\mathbb{C}], Z_\Phi)$ is equal to $\sum_{\tilde{\eta}} \tilde{V}_{\tilde{\eta}}^* \otimes \tilde{V}_{\tilde{\eta}}|_K$ in $\hat{R}(\tilde{K} \times K)$.

The quantity $m(\tilde{\lambda}, \lambda)$ is then the multiplicity of $\tilde{V}_{\tilde{\lambda}} \otimes V_\lambda$ in the character $RR_{\tilde{K} \times K}(N, [\mathbb{C}], Z_\Phi)$. Note that here the trivial line bundle $[\mathbb{C}]$ is a Φ -moment line bundle on N . By an adaptation of the shifting trick to this non-compact setting one can prove directly that $m(\tilde{\lambda}, \lambda) = RR(N_{\tilde{\lambda}, \lambda}, [\mathbb{C}]_{\tilde{\lambda}, \lambda})$. This is the content of Theorem 13.6 since the term $RR(N_{\tilde{\lambda}, \lambda}, [\mathbb{C}]_{\tilde{\lambda}, \lambda})$ is equal to $p_{\mathfrak{a}}(\tilde{\lambda}, \lambda)$ for any chamber \mathfrak{a} containing $(\tilde{\lambda}, \lambda)$ in its closure.

13.2 Multiplicities on a face

Let us give the structure of the general faces of the cone $\Delta(T^* \tilde{K}) \subset \tilde{\mathfrak{t}}_{\geq 0}^* \times \mathfrak{t}_{\geq 0}^*$. Let F be a closed face of $\Delta(T^* \tilde{K})$ that intersects $\tilde{\mathfrak{t}}_{> 0}^* \times \mathfrak{t}_{> 0}^*$. We say that F is a general face. We denote $\langle F \rangle$ the vector subspace generated by F , and \mathfrak{g}_F its orthogonal in $\tilde{\mathfrak{t}} \times \mathfrak{t}$. Note that $F = \langle F \rangle \cap \Delta(T^* \tilde{K})$.

We denote \tilde{W} the Weyl group of \tilde{K} relatively to the maximal torus \tilde{T} .

Lemma 13.8 • *There exists a vector subspace $\mathfrak{t}_F \subset \mathfrak{t}$ and $\tilde{w} \in \tilde{W}$ such that $\mathfrak{g}_F = \{(\tilde{w}X, X), X \in \mathfrak{t}_F\}$.*

• *For any weights $(\tilde{\lambda}, \lambda)$ contained in F , the weight $\tilde{w}^{-1}\tilde{\lambda} + \lambda$ vanishes on \mathfrak{t}_F .*

• *The connected component of $(T^* \tilde{K})^{\mathfrak{g}_F}$ that contains $\Phi^{-1}(F \cap \tilde{\mathfrak{t}}_{> 0}^* \times \mathfrak{t}_{> 0}^*)$ is $\tilde{w} \cdot T^* \tilde{K}_F$. Here \tilde{K}_F is the subgroup of \tilde{K} that centralizes \mathfrak{t}_F .*

Proof. We know that any element $(\tilde{X}, X) \in \mathfrak{g}_F$ fixes the points of $\Phi^{-1}(F \cap \tilde{\mathfrak{t}}_{> 0}^* \times \mathfrak{t}_{> 0}^*)$ (see Lemma 12.23), and that forces to have $\tilde{X} = \tau X$ for some $\tau \in \tilde{W}$ (see Lemma 13.3). The first point follows. The second point is obvious and the third follows directly from Lemma 13.3.

Berenstein-Sjamaar [7] and Ressayre [33, 34] have determined, in terms of Schubert calculus, what are the data $(\tilde{w}, \mathfrak{t}_F)$ that comes from a face F . It is usually difficult to list them explicitly.

We denote by K_F the subgroup of K that centralizes \mathfrak{t}_F . The element \tilde{w} is defined modulo the Weyl group of K_F . We thus can choose \tilde{w} such that for any root α of \tilde{K}_F , we have $\alpha > 0 \Leftrightarrow \tilde{w}\alpha > 0$.

The subgroup of $\tilde{K} \times K$ that centralizes \mathfrak{g}_F is of the form $\tilde{w}\tilde{K}_F\tilde{w}^{-1} \times K_F$. So for weights $(\tilde{\lambda}, \lambda)$ contained in F , the weight

$$\hat{\lambda} := \tilde{w}^{-1}\tilde{\lambda}$$

is dominant for \tilde{K}_F : we denote $\tilde{V}_{\tilde{\lambda}}^F$ the corresponding irreducible representation of \tilde{K}_F . Similarly the weight λ is dominant for K_F and we denote V_{λ}^F the corresponding irreducible representation of K_F .

We end this section with the following result obtained by Ressayre in [35], which is a non-compact analogue of Proposition 12.24.

Proposition 13.9 *For any dominant weights $(\tilde{\lambda}, \lambda)$ contained in F , the dimension of $[\tilde{V}_{\tilde{\lambda}}|_K \otimes V_{\lambda}]^K$ is equal to the dimension of $[\tilde{V}_{\tilde{\lambda}}^F|_{K_F} \otimes V_{\lambda}^F]^{K_F}$.*

Proof. We denote G_F the subtorus of $\tilde{T} \times T$ with Lie algebra \mathfrak{g}_F . We start with the identity

$$(13.42) \quad [\tilde{V}_{\tilde{\lambda}}|_K \otimes V_{\lambda}]^K = \left[RR_{\tilde{T} \times T}(Y, \mathbb{C}, \Phi_Y^{-1}(b), \Phi_Y - b) \otimes \mathbb{C}_{-\tilde{\lambda}, -\lambda} \right]^{\tilde{T} \times T},$$

that holds for any $b = (\tilde{\xi}, \xi) \in \tilde{\mathfrak{t}}_{>0}^* \times \mathfrak{t}_{>0}^*$ that is close enough to $(\tilde{\lambda}, \lambda)$. See Remark 13.7. Let us use it with $(\tilde{\xi}, \xi)$ a generic element contained in $F \cap (\tilde{\mathfrak{t}}_{>0}^* \times \mathfrak{t}_{>0}^*)$ and close enough to $(\tilde{\lambda}, \lambda)$: it is a quasi-regular value of Φ such that the fiber $\Phi^{-1}(\tilde{\xi}, \xi) \subset \tilde{w} \cdot \mathbb{T}^* \tilde{K}_F$ is a smooth submanifold with a locally free action of the group $(\tilde{T} \times T)/G_F$,

Thanks to Theorem 12.18, we know that the right hand side of (13.42) is equal to $RR(N_{\tilde{\xi}, \xi}, \mathbb{L}_{\tilde{\lambda}, \lambda})$ where

$$\mathbb{L}_{\tilde{\lambda}, \lambda} = \Phi^{-1}(\tilde{\xi}, \xi) \times_{(\tilde{T} \times T)/G_F} \mathbb{C}_{-\tilde{\lambda}, -\lambda}$$

is an line orbifold on the symplectic orbifold $N_{\tilde{\xi}, \xi} = \Phi^{-1}(\tilde{\xi}, \xi)/((\tilde{T} \times T)/G_F)$

We consider now the Hamiltonian action of $\tilde{K}_F \times K_F$ on $N_F := \mathbb{T}^* \tilde{K}_F$. Let us denote $\Phi_F : N_F \rightarrow \tilde{\mathfrak{t}}_F^* \times \mathfrak{t}_F^*$ the corresponding moment map. The torus T_F with Lie algebra \mathfrak{t}_F is embedded diagonally in $\tilde{T} \times T$. For a generic element $(\alpha, \beta) \in \Delta(N_F)$, the fiber $\Phi_F^{-1}(\alpha, \beta)$ is a smooth submanifold with a locally free action of $(\tilde{T} \times T)/T_F$.

As in the previous setting, we know that the multiplicity $\left[\tilde{V}_\lambda^F|_{K_F} \otimes V_\lambda^F\right]^{K_F}$ is equal to $RR((N_F)_{\hat{\xi}, \xi}, \mathbb{L}_{\hat{\lambda}, \lambda})$ when $(\hat{\xi}, \xi) \in \Delta(N_F)$ is a generic element close enough to $(\hat{\lambda}, \lambda)$. Here

$$\mathbb{L}_{\hat{\lambda}, \lambda} = \Phi_F^{-1}(\hat{\xi}, \xi) \times_{(\tilde{T} \times T)/T_F} \mathbb{C}_{-\hat{\lambda}, -\lambda}$$

is an line orbibundle on the symplectic orbifold $(N_F)_{\hat{\xi}, \xi} = \Phi_F^{-1}(\hat{\xi}, \xi)/((\tilde{T} \times T)/T_F)$.

Now our proposition follows from the fact that the Riemann-Roch numbers $RR(N_{\hat{\xi}, \xi}, \mathbb{L}_{\hat{\lambda}, \lambda})$ and $RR((N_F)_{\hat{\xi}, \xi}, \mathbb{L}_{\hat{\lambda}, \lambda})$ are equal. It is due to the fact that the multiplication by \tilde{w}^{-1} induces a symplectomorphism $N_{\hat{\xi}, \xi} \simeq (N_F)_{\hat{\xi}, \xi}$ and an isomorphism of line bundles $\mathbb{L}_{\hat{\lambda}, \lambda} \simeq \mathbb{L}_{\hat{\lambda}, \lambda}$.

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