Doubly-resonant saddle-nodes in C^3 and the fixed singularity at infinity in the Painlevé equations: formal classification.

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Doubly-resonant saddle-nodes in $\mathbb{C}^3$ and the fixed singularity at infinity in the Painlevé equations: formal classification.

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May 3, 2016

Abstract

In this work we consider formal singular vector fields in $\mathbb{C}^3$ with an isolated and doubly-resonant singularity of saddle-node type at the origin. Such vector fields come from irregular two-dimensional systems with two opposite non-zero eigenvalues, and appear for instance when studying the irregular singularity at infinity in Painlevé equations $(\mathtt{P}_j)$, $j \in \{I, II, III, IV, V\}$, for generic values of the parameters. Under generic assumptions we give a complete formal classification for the action of formal diffeomorphisms (by changes of coordinates) fixing the origin and fibered in the independent variable x. We also identify all formal isotropies (self-conjugacies) of the normal forms. In the particular case where the flow preserves a transverse symplectic structure, *e.g.* for Painlevé equations, we prove that the normalizing map can be chosen to preserve the transverse symplectic form.

Keywords: Painlevé equations, singular vector fields, irregular singularity, resonant singularity, normal form

1 Introduction

1.1 Definition and main result

We consider singular vector fields Y in $\mathbb{C}^3$ which can be written in appropriate coordinates $(x, \mathbf{y}) := (x, y_1, y_2)$ as

$$Y \quad = \quad x^2\frac{\partial}{\partial x} + \Big(-\lambda y_1 + F_1(x,\mathbf{y})\Big)\frac{\partial}{\partial y_1} + \Big(\lambda y_2 + F_2(x,\mathbf{y})\Big)\frac{\partial}{\partial y_2} \quad , (1.1)$$

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where $\lambda \in \mathbb{C}^*$ and F_1, F_2 are formal power series of order at least two. They represent singular irregular 2-dimensional systems having two opposite non-zero eigenvalues and a vanishing third eigenvalue.

Our main motivation is the study of the irregular singularity at infinity in Painlevé equations $(\mathtt{P}_j)$, $j \in \{I, II, III, IV, V\}$, for generic values of the parameters [22]. These equations, discovered (mainly) by Paul Painlevé [18], share the property that the only movable singularities of their solutions are poles (the so-called Painlevé property); this is the complete list of all such equations up to changes of variables. They have been intensively studied since the important work of Okamoto [16]. The study of fixed singularities, and more particularly those at infinity, started to be investigated by Boutroux with his famous *tritronquées* solutions [18]. Recently, several authors provided more complete information about such singularities, studying the so-called *quasi-linear Stokes phenomena* and also giving connection formulas ([10], [12] and [11]). However, to the best of our knowledge there are no general analytic classification for this kind of doubly-resonant saddle-nodes yet (using normal form theory).

More precisely, we would like to understand the action of germs of analytic diffeomorphisms on such vector fields by changes of coordinates. If one tries to do this, a first step would be to provide a formal classification, that is to study the action of formal changes of coordinates on these vector fields. This is the aim of this paper. Based on the usual strategy employed for the classification of resonant vector fields [15] in dimension 2, we give in a forthcoming paper a complete analytic classification for a more specific class of vector fields, by studying the *non-linear Stokes phenomena*.

To state our main results we need to introduce some notations and nomenclature.

- $\mathbb{C}[\![\mathbf{x}]\!]$ is the $\mathbb{C}$-algebra of formal power series in the (multi)variable $\mathbf{x} = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{C}$. We denote by $\mathfrak{m}$ its unique maximal ideal: it is formed by formal series with null constant term. For any formal series $f_1, \ldots, f_m$ in $\mathbb{C}[\![\mathbf{x}]\!]$, we denote by $\langle f_1, \ldots, f_m \rangle$ the ideal generated by these elements.

- $\mathcal{D}^{(1)}$ is the Lie algebra of formal vector fields at the origin of $\mathbb{C}^3$ which are singular (*i.e.* vanish at the origin). Any formal vector field in $\mathcal{D}^{(1)}$ can be written
$$Y = b\frac{\partial}{\partial x} + b_1\frac{\partial}{\partial y_1} + b_2\frac{\partial}{\partial y_2}$$
with $b, b_1, b_2 \in \mathfrak{m}$.

- $\widehat{\mathrm{Diff}}$ is the group of formal diffeomorphisms fixing the origin of $\mathbb{C}^3$. It acts on $\mathcal{D}^{(1)}$ by conjugacy: if $(\Phi, Y) \in \widehat{\mathrm{Diff}} \times \mathcal{D}^{(1)}$,
$$\Phi_* (Y) := (\mathrm{D}\Phi \cdot Y) \circ \Phi^{-1} \qquad , \tag{1.2}$$
where $\mathrm{D}\Phi$ is the Jacobian matrix of Φ.

- $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$ is the subgroup of $\widehat{\mathrm{Diff}}$ of diffeomorphisms fibered in the x-coordinate, *i.e.* of the form $(x, \mathbf{y}) \mapsto (x, \phi(x, \mathbf{y}))$.

Definition 1.1. A **doubly-resonant saddle-node** is a vector field $Y \in \mathcal{D}^{(1)}$ which is $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$ -conjugate to one of the form

$$Y = x^2 \frac{\partial}{\partial x} + \Big(-\lambda y_1 + F_1(x, \mathbf{y}) \Big) \frac{\partial}{\partial y_1} + \Big(\lambda y_2 + F_2(x, \mathbf{y}) \Big) \frac{\partial}{\partial y_2} \quad ,$$

with $\lambda \in \mathbb{C}^*$ and $F_1, F_2 \in \mathfrak{m}^2$. We will denote by $\widehat{\mathcal{SN}}$ the set of all such formal vector fields.

By Taylor expansion up to order 1 with respect to $\mathbf{y}$, given a vector field $Y \in \widehat{\mathcal{SN}}$ written as above we can consider the associated 2-dimensional differential system:

$$x^2 \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \alpha(x) + \mathbf{A}(x) \mathbf{y}(x) + \mathbf{f}(x, \mathbf{y}(x)) \quad , \tag{1.3}$$

where $\mathbf{y} = (y_1, y_2)$, such that the following conditions hold:

- $\alpha(x) = \begin{pmatrix} \alpha_1(x) \\ \alpha_2(x) \end{pmatrix}$, with $\alpha_1, \alpha_2 \in \langle x \rangle^2 \subset \mathbb{C}[\![x]\!]$
- $\mathbf{A}(x) \in \mathrm{Mat}_{2,2}(\mathbb{C}[\![x]\!])$ with $\mathbf{A}(0) = \mathrm{Diag}(-\lambda, \lambda)$, $\lambda \in \mathbb{C}^*$
- $\mathbf{f}(x, \mathbf{y}) = \begin{pmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \end{pmatrix}$, with $f_1, f_2 \in \langle y_1, y_2 \rangle^2 \subset \mathbb{C}[\![x, \mathbf{y}]\!]$.

Based on this expression, we state:

Definition 1.2. The **residue** of $Y \in \widehat{\mathcal{SN}}$ is the complex number

$$\mathrm{res}(Y) := \left(\frac{\mathrm{Tr}(\mathbf{A}(x))}{x} \right)_{|x=0} .$$

We say that Y is **non-degenerate** if $\mathrm{res}(Y) \in \mathbb{C} \backslash \mathbb{Q}_{\leq 0}$, and we denote by $\widehat{\mathcal{SN}}_{\mathrm{nd}} \subset \widehat{\mathcal{SN}}$ the subset of non-degenerate vector fields.

We will prove in subsection 3.1 that the residue of $Y \in \widehat{\mathcal{SN}}$ is invariant under the action of $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$ by conjugacy. We can state now our first main result.

Theorem 1.3. *Let $Y \in \widehat{\mathcal{SN}}_{\mathrm{nd}}$ be a non-degenerate doubly-resonant saddle-node. Then there exists a fibered diffeomorphism $\Phi \in \widehat{\mathrm{Diff}}_{\mathrm{fib}}$ such that:*

$$\begin{aligned} \Phi_*(Y) = {} & x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(v)) y_1 \frac{\partial}{\partial y_1} \\ & + (\lambda + a_2 x + c_2(v)) y_2 \frac{\partial}{\partial y_2} \quad , \end{aligned} \tag{1.4}$$

where we put $v := y_1 y_2$. Here, c_1, c_2 in $\langle v \rangle = v\mathbb{C}[\![v]\!]$ are formal power series with null constant term and $a_1, a_2 \in \mathbb{C}$ are such that $a_1 + a_2 = \mathrm{res}(Y)$.

Remark 1.4. We will see in Corollary 3.4 and Proposition 3.2 that Φ as above is essentially unique (that is, unique up to pre-composition by linear transforms).

Definition 1.5. The **parameter space** for $\widehat{\mathcal{SN}}_{\mathrm{nd}}$ is the set

$$\mathcal{P} \quad := \quad \left\{ \mathbf{p} = (\lambda, a_1, a_2, c_1, c_2) \in \mathbb{C}^* \times \left(\mathbb{C}^2 \backslash \Delta\right) \times \left(v\mathbb{C}\left[\!\left[v\right]\!\right]\right)^2 \right\}$$

where

$$\Delta \quad = \quad \left\{ (a_1, a_2) \in \mathbb{C}^2 \mid a_1 + a_2 \in \mathbb{Q}_{\leq 0} \right\}$$

is the locus of degeneracy. A vector field in the form (1.4) will be called a **normal form** of $\widehat{\mathcal{SN}}_{\mathrm{nd}}$ with parameters $(\lambda, a_1, a_2, c_1, c_2)$ in $\mathcal{P}$.

Let us consider the quotient space

$$\mathcal{P} \Big/ (\mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z})$$

where the group $(\mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z})$ acts on $\mathcal{P}$ as follows. Given $\mathbf{p} = (\lambda, a_1, a_2, c_1, c_2) \in \mathcal{P}$, $\theta \in \mathbb{C}^*$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ we define

$$\begin{aligned} \theta \cdot (\lambda, a_1, a_2, c_1, c_2) &= (\lambda, a_1, a_2, c_1 \circ \varphi_\theta, c_2 \circ \varphi_\theta) \\ \epsilon \cdot (\lambda, a_1, a_2, c_1, c_2) &= \begin{cases} (\lambda, a_1, a_2, c_1, c_2) & , \text{if } \epsilon = 0 \\ (-\lambda, a_2, a_1, c_2, c_1) & , \text{if } \epsilon = 1 \end{cases} \end{aligned} \quad ,$$

where φ_θ is the homothecy $v \mapsto \theta v$. If two parameters $\mathbf{p}, \mathbf{p}' \in \mathcal{P}$ are in the same orbit for this action we write $\mathbf{p} \sim \mathbf{p}'$. Our second main result shows the uniqueness of the normal forms up to this action.

Theorem 1.6. *Suppose Z and Z' are two normal forms of $\widehat{\mathcal{SN}}_{\mathrm{nd}}$ with respective parameters $\mathbf{p} = (\lambda, a_1, a_2, c_1, c_2) \in \mathcal{P}$ and $\mathbf{p}' = (\lambda', a_1', a_2', c_1', c_2') \in \mathcal{P}$. Then Z and Z' are $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$-conjugate if and only $\mathbf{p} \sim \mathbf{p}'$.*

One can rephrase the above results in terms of group actions as follows.

Corollary 1.7. *There exists a bijection*

$$\widehat{\mathcal{SN}}_{\mathrm{nd}} \Big/ \widehat{\mathrm{Diff}}_{\mathrm{fib}} \quad \simeq \quad \mathcal{P} \Big/ (\mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z}) \quad ,$$

where $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$ acts on $\widehat{\mathcal{SN}}_{\mathrm{nd}}$ by conjugacy.

Let us make some remarks.

Remark 1.8.

1. The condition of non-degeneracy is necessary to obtain such normal forms. For instance for any $a_1, a_2 \in \mathbb{C}$ such that $a_1 + a_2 = -\frac{p}{q} \in \mathbb{Q}_{\leq 0}$, with $(p, q) \in \mathbb{N} \times \mathbb{N}^*$, the vector field

$$Y = x^2 \frac{\partial}{\partial x} + \left(-\lambda + a_1 x + x^{p+1} (y_1 y_2)^q\right) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x) y_2 \frac{\partial}{\partial y_2} \ ,$$

with residue $\mathrm{res}\,(Y) = -\dfrac{p}{q}$ is not $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$-conjugate to a normal form as in Theorem 1.3. Indeed, the resonant term $x^{p+1}\,(y_1y_2)^q$ cannot be eliminated by the action of $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$.

2. Notice that the above two results are not immediate consequences of Poincaré-Dulac normal form theory. In fact, the usual Poincaré-Dulac normal form possibly contains several additional resonant terms of the form $\left(x^k\,(y_1y_2)^l\right)_{k,l\in\mathbb{N}}$, and is far from being unique.

1.2 Painlevé equations and the transversally symplectic case

In [22] Yoshida shows that a vector field in the class $\widehat{\mathcal{SN}}_{\mathrm{nd}}$ naturally appears after a suitable compactification (given by the so-called Boutroux coordinates [2]) of the phase-space of Painlevé equations $(\mathtt{P}_j)$, $j \in \{I, II, III, IV, V\}$ (for generic values of the parameters). In these cases the vector field presents an additional Hamiltonian structure that will interest us.

Let us illustrate these computations in the case of the first Painlevé equation:

$$(P_I) \qquad \frac{\mathrm{d}^2 z_1}{\mathrm{d}t^2} = 6z_1^2 + t \qquad .$$

As is well known since Okamoto [17], (P_I) can be seen as a non-autonomous Hamiltonian system

$$\begin{cases} \frac{\partial z_1}{\partial t} = -\frac{\partial H}{\partial z_2} \\ \frac{\partial z_2}{\partial t} = \frac{\partial H}{\partial z_1} \end{cases}$$

with Hamiltonian

$$H\,(t, z_1, z_2) \; := \; 2z_1^3 + tz_1 - \frac{z_2^2}{2}.$$

More precisely, if we consider the standard symplectic form $\omega_{st} := dz_1 \wedge dz_2$ and the vector field

$$Z \; := \; \frac{\partial}{\partial t} - \frac{\partial H}{\partial z_2}\frac{\partial}{\partial z_1} + \frac{\partial H}{\partial z_1}\frac{\partial}{\partial z_2}$$

induced by (P_I), then the Lie derivative

$$\mathcal{L}_Z\,(\omega_{st}) = \left(\frac{\partial^2 H}{\partial t \partial z_1}\mathrm{d}z_1 + \frac{\partial^2 H}{\partial t \partial z_2}\mathrm{d}z_2\right) \wedge \mathrm{d}t = \mathrm{d}z_1 \wedge \mathrm{d}t$$

belongs to the ideal $\langle \mathrm{d}t \rangle$ generated by $\mathrm{d}t$ in the exterior algebra $\Omega^*\left(\mathbb{C}^3\right)$ of differential forms in variables (t, z_1, z_2). Equivalently, for any $t_1, t_2 \in \mathbb{C}$ the flow of Z at time $(t_2 - t_1)$ acts as a symplectomorphism between fibers $\{t = t_1\}$ and $\{t = t_2\}$.

The weighted compactification given by the Boutroux coordinates [3] (see also [6]) defines a chart near $\{t = \infty\}$ as follows:

$$\begin{cases} z_2 = y_2 x^{-\frac{3}{5}} \\ z_1 = y_1 x^{-\frac{2}{5}} \\ t = x^{-\frac{4}{5}} \end{cases} .$$

In the coordinates (x, y_1, y_2), the vector field Z is transformed, up to a translation $y_1 \leftarrow y_1 + \zeta$ with $\zeta = \frac{i}{\sqrt{6}}$, into the vector field

$$\tilde{Z} \quad = \quad -\frac{5}{4x^{\frac{1}{5}}} Y \tag{1.5}$$

where

$$\begin{aligned} Y \quad = \quad & x^2 \frac{\partial}{\partial x} + \left(-\frac{4}{5} y_2 + \frac{2}{5} x y_1 + \frac{2\zeta}{5} x \right) \frac{\partial}{\partial y_1} \\ & + \left(-\frac{24}{5} y_1^2 - \frac{48\zeta}{5} y_1 + \frac{3}{5} x y_2 \right) \frac{\partial}{\partial y_2} \quad . \end{aligned} \tag{1.6}$$

We observe that Y is a *non-degenerate doubly-resonant saddle-node* Y as in Definitions 1.1 and 1.2 with residue $\operatorname{res}(Y) = 1$. Furthermore we have:

$$\begin{cases} \mathrm{d}t & = -\frac{4}{5} 5^{\frac{4}{5}} x^{-\frac{9}{5}} \mathrm{d}x \\ \mathrm{d}z_1 \wedge \mathrm{d}z_2 & = \frac{1}{x} (\mathrm{d}y_1 \wedge \mathrm{d}y_2) + \frac{1}{5x^2} (2y_1 \mathrm{d}y_2 - 3y_2 \mathrm{d}y_1) \wedge \mathrm{d}x \\ & \in \frac{1}{x} (\mathrm{d}y_1 \wedge \mathrm{d}y_2) + \langle \mathrm{d}x \rangle \end{cases} ,$$

where $\langle \mathrm{d}x \rangle$ denotes the ideal generated by $\mathrm{d}x$. We finally obtain

$$\begin{cases} \mathcal{L}_Y \left(\dfrac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{x} \right) = \frac{1}{5x} (3y_2 \mathrm{d}y_1 - (2\zeta + 2y_1) \mathrm{d}y_2) \wedge \mathrm{d}x \\ \mathcal{L}_Y (\mathrm{d}x) = 2x \mathrm{d}x \end{cases} .$$

Therefore, both $x\mathcal{L}_Y \left(\dfrac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{x} \right)$ and $\mathcal{L}_Y (\mathrm{d}x)$ are differential forms which lie in the ideal $\langle \mathrm{d}x \rangle$. This motivates the following definition.

Definition 1.9. Consider the rational 1-form

$$\omega \quad := \quad \frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{x} \quad .$$

We say that a formal vector field $Y \in \mathcal{D}^{(1)}$ is **transversally Hamiltonian** (with respect to ω and $\mathrm{d}x$) if

$$\mathcal{L}_Y (\mathrm{d}x) \in \langle \mathrm{d}x \rangle \quad \text{and} \quad x\mathcal{L}_Y (\omega) \in \langle \mathrm{d}x \rangle \quad .$$

We say a formal diffeomorphism $\Phi \in \widehat{\mathrm{Diff}}$ is **transversally symplectic** (with respect to ω and $\mathrm{d}x$) if

$$\Phi^* (x) = x \text{ and } x\Phi^* (\omega) \in x\omega + \langle \mathrm{d}x \rangle \quad .$$

(Here $\Phi^*(\omega)$ denotes the pull-back of ω by Φ.)

We denote respectively by $\mathcal{D}_\omega$ and $\widehat{\mathrm{Diff}}_\omega$ the sets of transversally Hamiltonian vector fields and transversally symplectic diffeomorphisms.

Remark 1.10.

- The flow of a transversally Hamiltonian X defines a map between fibers $\{x = x_1\}$ and $\{x = x_2\}$ which sends $\omega_{|x=x_1}$ onto $\omega_{|x=x_2}$, since

$$(\exp(X))^*(\omega) \in \omega + \langle \mathrm{d}x \rangle \ .$$

- By our definition, a transversally symplectic diffeomorphism $\Phi \in \widehat{\mathrm{Diff}}_\omega$ is necessarily a fibered diffeomorphism. In other words: $\widehat{\mathrm{Diff}}_\omega \subset \widehat{\mathrm{Diff}}_{\mathrm{fib}}$.

Definition 1.11. A **transversally Hamiltonian doubly-resonant saddle-node** is a vector field $Y \in \mathcal{D}_\omega$ which is $\widehat{\mathrm{Diff}}_\omega$ -conjugate to one of the form

$$Y \quad = \quad x^2\frac{\partial}{\partial x} + \Big(-\lambda y_1 + F_1(x,\mathbf{y})\Big)\frac{\partial}{\partial y_1} + \Big(\lambda y_2 + F_2(x,\mathbf{y})\Big)\frac{\partial}{\partial y_2} \quad ,$$

with $\lambda \in \mathbb{C}^*$ and $F_1, F_2 \in \mathfrak{m}^2$. We will denote by $\widehat{\mathcal{SN}}_\omega$ the set of all such formal vector fields.

Notice that a transversally Hamiltonian doubly-resonant saddle-node $Y \in \widehat{\mathcal{SN}}_\omega$ is necessarily non-degenerate since its residue is always equal to 1. In other words: $\widehat{\mathcal{SN}}_\omega \subset \widehat{\mathcal{SN}}_{\mathrm{nd}}$.

Theorem 1.12. *Let $Y \in \widehat{\mathcal{SN}}_\omega$ be a transversally Hamiltonian doubly-resonant saddle-node. Then, there exists a transversally symplectic diffeomorphism $\Phi \in \widehat{\mathrm{Diff}}_\omega$ such that:*

$$\Phi_*(Y) = \quad x^2\tfrac{\partial}{\partial x} + (-\lambda + a_1 x - c(v))\, y_1 \tfrac{\partial}{\partial y_1} + (\lambda + a_2 x + c(v))\, y_2 \tfrac{\partial}{\partial y_2} \quad . \qquad (1.7)$$

where we put $v := y_1 y_2$. Here, $c(v)$ in $v\mathbb{C}[\![v]\!]$ is a formal power series with null constant term and $a_1, a_2 \in \mathbb{C}$ are such that $a_1 + a_2 = \mathrm{res}(Y) = 1$. Furthermore this normal form is unique with respect to the action of $\widehat{\mathrm{Diff}}_\omega$.

One can rephrase the theorem above in terms of group action.

Corollary 1.13. *There exists a bijection*

$$\widehat{\mathcal{SN}}_\omega \Big/ \widehat{\mathrm{Diff}}_\omega \quad \simeq \quad \mathbb{C}^* \times \Big\{(a_1,a_2) \in \mathbb{C}^2 \mid a_1 + a_2 = 1\Big\} \times v\mathbb{C}[\![v]\!] \qquad .$$

Remark 1.14.

1. As for Theorem 1.3, Φ is essentially unique (Corollary 3.4). This is an immediate consequence of Theorem 1.6. However, the fact that the normalizing diffeomorphism Φ in Theorem 1.12 is transversally symplectic is not an immediate consequence of Theorem 1.6.

2. The above normalization theorem can be interpreted as defining local *action-angle coordinates* for vector fields in $\widehat{\mathcal{SN}}_{\omega}$. More precisely, if we consider the successive symplectic changes of coordinates

$$\begin{cases} y_1 = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\left(u_1 + iu_2\right) \\ y_2 = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\left(u_1 - iu_2\right) \end{cases}$$

and

$$\begin{cases} u_1 = \sqrt{2\rho}\cos\varphi \\ u_2 = \sqrt{2\rho}\sin\varphi \end{cases},$$

then the vector field (1.7) becomes:

$$x^2\frac{\partial}{\partial x} + e^{-i\frac{\pi}{4}}x\sqrt{\rho}\frac{\partial}{\partial\rho} + i\left(\lambda + c\left(i\rho\right) + \frac{\left(a_2 - a_1\right)}{2}x\right)\frac{\partial}{\partial\varphi} \quad .$$

Notice that the corresponding differential equation can be explicitly integrated by quadratures in terms of an anti-derivative of c.

We will explain in section 4 how to compute inductively any finite jet of $c\left(v\right)$ in the case of the Painlevé equations (for which $c\left(v\right)$ is a germ of an analytic function at the origin).

Corollary 1.15. *Let Y be as in (1.6). Then $a_1 = a_2 = \frac{1}{2}$, $\lambda = \frac{8\sqrt{3\zeta}}{5} = \frac{4\cdot 2^{\frac{3}{4}}\cdot 3^{\frac{1}{4}}}{5}e^{\frac{i\pi}{4}}$ and*

$$\begin{aligned} c\left(v\right) &= 3v + \left(9 + \frac{167\cdot 2^{\frac{1}{4}}\cdot 3^{\frac{3}{4}}}{96}e^{\frac{3i\pi}{4}}\right)v^2 \\ &\quad + \left(16 + \frac{31837\sqrt{6}}{6912}i + \frac{5}{2}\cdot 2^{\frac{1}{4}}\cdot 3^{\frac{1}{4}}\cdot e^{\frac{3i\pi}{4}}\right)v^3 + O\left(v^4\right) \quad . \end{aligned}$$

1.3 Known results

In [22], [21] Yoshida shows that the doubly-resonant saddle-nodes arising from the compactification of Painlevé equations $(\mathtt{P}_j)$, $j \in \{I, II, III, IV, V\}$ (for generic values for the parameters) is conjugate to polynomial vector fields of the form

$$\begin{aligned} Z = \quad & x^2\frac{\partial}{\partial x} + \Big(-\left(1 + \gamma y_1 y_2\right) + a_1 x\Big)y_1\frac{\partial}{\partial y_1} \\ & + \Big(1 + \gamma y_1 y_2 + a_2 x\Big)y_2\frac{\partial}{\partial y_2} \quad , \end{aligned} \tag{1.8}$$

with $\gamma \in \mathbb{C}^*$ and $(a_1, a_2) \in \mathbb{C}^2$ such that $a_1 + a_2 = 1$. One drawback of this result is that Yoshida admits fibered transformations $\Psi\left(x, \mathbf{y}\right) = \left(x, \psi_1\left(x, \mathbf{y}\right), \psi_2\left(x, \mathbf{y}\right)\right)$

of a more general form:

$$\psi_i(x,\mathbf{y}) = y_i\left(1+\sum_{\substack{(k_0,k_1,k_2)\in\mathbb{N}^3\\ k_1+k_2\geq 1}}\frac{q_{i,\mathbf{k}}(x)}{x^{k_0}}y_1^{k_1+k_0}y_2^{k_1+k_0}\right) \quad , \tag{1.9}$$

where each $q_{i,\mathbf{k}}$ is a formal power series. Notice that x can appear with negative exponents and therefore $\Psi \notin \widehat{\mathrm{Diff}}$. As we will see in the next subsection, the problem (when seen from the viewpoint of analytic classification) is that the transformations used by Yoshida have "small" regions of convergence, in the sense that one cannot cover an entire neighborhood of the origin in $\mathbb{C}^3$ by taking the union of these regions. On the contrary, we prove in an upcoming work that the formal normalizations presented here can be embodied by diffeomorphisms analytic on finitely many sectors whose union is a neighborhood of the origin. This entails the classical theory of summability of formal power series.

1.4 Analytic results

Several authors studied the problem of convergence of the conjugating transformations described above. Some results (that we recall soon) will hold not only in the class of formal objects, but also for Gevrey (and even summable) ones, or more generally for holomorphic functions with asymptotic expansions in sectorial domains. We refer to [13] and [15] for details on asymptotic expansions, Gevrey and summability theory.

Assuming that the initial vector field is analytic, Yoshida proves in [21] that he can chose a conjugacy of the form (1.9) which is the asymptotic expansion of an analytic function in a domain

$$\left\{(x,\mathbf{z})\in S\times\mathbf{D}(0,\mathbf{r}) \mid |z_1z_2|<\nu|x|\right\}$$

for some small $\nu>0$, where S is a sector of opening less than π with vertex at the origin and $\mathbf{D}(0,\mathbf{r})$ is a polydisc of small poly-radius $\mathbf{r}=(r_1,r_2)$. Moreover, the $(q_{i,\mathbf{k}}(x))_{i,\mathbf{k}}$ are in fact Gevrey-1 series.

Under more restrictive conditions (which correspond to $c_1=c_2=0$ and $\mathrm{Re}(a_1+a_2)>0$ in Theorem 1.3), Shimomura, improving on a result by Iwano [9], shows in [19] that analytic doubly-resonant saddle-nodes satisfying these conditions are conjugate to:

$$x^2\frac{\partial}{\partial x}+(-\lambda+a_1x)\,y_1\frac{\partial}{\partial y_1}+(\lambda+a_2x)\,y_2\frac{\partial}{\partial y_2}$$

via a diffeomorphism whose coefficients have asymptotic expansions as $x\to 0$ in sectors of opening greater than π. Stolovitch generalized this result for any dimension in [20]. Unfortunately, as shown by Yoshida in [22], the hypothesis $c_1=c_2=0$ is not met in the case of Painlevé equations.

In a forthcoming series of papers we will prove an analytic version of Theorem 1.12, valid in sectorial domains with sufficiently large opening, which in turn will help us to provide an analytic classification. Let us insist once more on the key fact that the union of these sectorial domains forms a whole neighborhood of the origin.

1.5 Outline of the paper

- In section 2 we recall some basic concepts and results from the theory of formal vector fields and differential forms.
- In section 3 we prove Theorems 1.3, 1.6 and 1.12, and compute the isotropies of the associated normal forms.
- In section 4 we explain how to compute any finite jet of the formal invariant c in Theorem 1.12 in the case of the Painlevé equations.

Contents

2 Background

We refer the reader to [8], [14] and [4] for a detailed introduction to formal vector fields and formal diffeomorphisms. Although these concepts are well-known by specialists, we will recall briefly the needed results and nomenclature.

2.1 Formal power series, vector fields and diffeomorphisms

As usual, we will denote a formal power series as $f(\mathbf{x}) = \sum_{\mathbf{k}} f_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ where, for all $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}$, $f_{\mathbf{k}} \in \mathbb{C}$ and $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \ldots x_n^{k_n}$. We will also use the notation $|\mathbf{k}| = k_1 + \cdots + k_n$ for the *degree* of a *monomial* $\mathbf{x}^{\mathbf{k}}$ (which is of *homogenous degree* $\mathbf{k} = (k_1, \ldots, k_n)$).

We denote respectively by $\mathbb{C}[\![\mathbf{x}]\!]$, $\mathcal{D}$, $\widehat{\mathrm{Diff}}$ the sets of formal power series (equipped with an algebra structure), vector field (equipped with a Lie algebra structure), diffeomorphisms (equipped with a group structure). The maximal ideal of the algebra $\mathbb{C}[\![\mathbf{x}]\!]$ formed by formal power series with null constant term is denoted by $\mathfrak{m}$.

A vector field will be seen either as a an element of $(\mathbb{C}[\![\mathbf{x}]\!])^n$ or as a derivation on $\mathbb{C}[\![\mathbf{x}]\!]$: for any vector field

$$X = \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n \frac{\partial}{\partial x_n} \in (\mathbb{C}[\![\mathbf{x}]\!])^n , \tag{2.1}$$

its *Lie derivative* is defined as the operator

$$\mathcal{L}_X(f) = \alpha_1 \frac{\partial f}{\partial x_1} + \cdots + \alpha_n \frac{\partial f}{\partial x_n} , \tag{2.2}$$

for any formal power series $f \in \mathbb{C}[\![\mathbf{x}]\!]$. The Lie bracket $[X, Y]$ of two vector fields $X, Y \in \mathcal{D}$ is defined by

$$\mathcal{L}_{[X,Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$$

for all $f \in \mathbb{C}[\![\mathbf{x}]\!]$.

Similarly, a formal diffeomorphism will be seen either as an element of $\Phi(\mathbf{x}) \in (\mathbb{C}[\![\mathbf{x}]\!])^n$ such that $\Phi(\mathbf{0}) = \mathbf{0}$ and $\mathrm{D}_{\mathbf{0}}\Phi = \mathrm{Jac}(\Phi(\mathbf{0})) \in \mathrm{Gl}_n(\mathbb{C})$, or as an algebra automorphism of $\mathbb{C}[\![\mathbf{x}]\!]$: given a formal series $f = \sum_{\mathbf{k} \in \mathbb{N}^n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \in \mathbb{C}[\![\mathbf{x}]\!]$, we denote by

$$\Phi^*(f) = \sum_{\mathbf{k} \in \mathbb{N}^n} a_{\mathbf{k}} \phi_1^{k_1} \cdots \phi_n^{k_n} , \tag{2.3}$$

the *pull-back* of f by $\Phi \in \widehat{\mathrm{Diff}}$, where

$$\phi_1 = \Phi(x_1), \ldots, \phi_n = \Phi(x_n) \ .$$

The Jacobian matrix (or the linear part) of Φ in the basis $(x_1, \ldots, x_n)$ is the matrix $\left(\dfrac{\partial \phi_i}{\partial x_j}(0, \ldots, 0)\right)_{i,j}$.

The *order* $\mathrm{ord}(f)$ (*resp.* $ord(X)$, *resp.* $ord(\Phi)$) of a non-zero formal power series $f \in \mathbb{C}[[\mathbf{x}]]$ (*resp.* vector field $X \in \mathcal{D}$, *resp.* diffeomorphism $\Phi \in \widehat{\mathrm{Diff}}$) is the maximal integer $k \geq 0$ such that $f \in \mathfrak{m}^k$ (*resp.* $\mathcal{L}_X(\mathfrak{m}) \subset \mathfrak{m}^k$, *resp.* $\Phi^*(\mathfrak{m}) \subset \mathfrak{m}^k$). The notion of order allows to define the classical *Krull topology* on $\mathbb{C}[[\mathbf{x}]]$, $\mathcal{D}$ and $\widehat{\mathrm{Diff}}$. The set of formal vector field of order at least k is a submodule denoted by $\mathcal{D}^{(k)} \subset \mathcal{D}$. In particular, $\mathcal{D}^{(1)}$ is the submodule of *singular* formal vector fields. We denote by $\mathcal{A}^{(k)} \subset \widehat{\mathrm{Diff}}$ the normal subgroup formed by those automorphisms Φ such that

$$\Phi(x_i) - x_i \in \mathfrak{m}^{k+1}$$

for each $i = 1, \ldots, n$. Each element of $\mathcal{A}^{(k)}$ will be called a formal diffeomorphism *tangent to the identity* up to order k.

Given a subgroup $\mathcal{G} \subset \widehat{\mathrm{Diff}}$, we say that two vector fields $Y_1,\ Y_2$ in $\mathcal{D}$ are $\mathcal{G}-$*conjugate* if there exits a $\Phi \in \mathcal{G}$ such that:

$$\mathcal{L}_{Y_1} \circ \Phi \quad = \quad \Phi \circ \mathcal{L}_{Y_2} \quad .$$

The following two lemmas will be important in the proof of Theorem 1.3.

Lemma 2.1. *Let $X, Y \in \mathcal{D}^{(1)}$ be two singular formal vector fields. Then:*

$$\mathrm{ord}([X,Y]) \quad \geq \quad \mathrm{ord}(X) + \mathrm{ord}(Y) - 1 \ .$$

Lemma 2.2. *Let $(d_n)_{n\geq 0} \subset \mathbb{N}_{>0}$ be a strictly increasing sequence of positive integers, and $(\Phi_n)_{n\geq 0}$ a sequence of formal diffeomorphisms. Assume that for all $n \geq 0$,*

$$\Phi_n(\mathbf{x}) = \mathbf{x} + P_{d_n}(\mathbf{x}) \ \left(\mathit{mod}\ \mathfrak{m}^{d_n+1} \right) \quad ,$$

where $P_{d_n}(\mathbf{x})$ is a vector homogenous polynomial of degree d_n. Then the sequence $\left(\Phi^{[n]}\right)_{n\geq 0}$, defined by $\Phi^{[n]} = \Phi_n \circ \cdots \circ \Phi_0$, for all $n \geq 0$, is convergent, of limit $\Phi \in \widehat{\mathrm{Diff}}$.

Moreover, if each Φ_n is fibered then Φ is fibered too.

Proof. It suffices to prove by induction that for all $n \geq 0$:

$$\Phi_n \circ \cdots \circ \Phi_0(\mathbf{x}) = \mathbf{x} + P_{d_0}(\mathbf{x}) + \cdots + P_{d_n}(\mathbf{x}) \ \left(\mathrm{mod}\ \mathfrak{m}^{d_n+1} \right) \ ,$$

because the sequence $(d_n)_{n\geq 0} \subset \mathbb{N}_{>0}$ is strictly increasing. ∎

2.2 Exponential map and logarithm

Given formal vector field $X \in \mathcal{D}^{(1)}$ and a formal power series $f \in \mathbb{C}[[\mathbf{x}]]$ and we set

$$\begin{cases} \mathcal{L}_X^{\circ 0}(f) = f \\ \mathcal{L}_X^{\circ(k+1)}(f) := \mathcal{L}_X\left(\mathcal{L}_X^{\circ k}(f)\right) & \text{, for all } k \geq 0 \end{cases}$$

so that we can consider the algebra homomorphism given by:

$$\exp(X)^* : f \mapsto \sum_{k \geq 0} \frac{1}{k!} \mathcal{L}_X^{\circ k}(f) \quad . \tag{2.4}$$

This series is convergent in the Krull topology and defines in fact a formal diffeomorphism, which is called the *time* 1 *formal flow* of $X \in \mathcal{D}^{(1)}$ or the *exponential* of X. (see *e.g.* section 3 in [8]).

For any vector field $X \in \mathcal{D}^{(1)}$, we consider also the *adjoint map*

$$\begin{array}{rccc} \mathrm{ad}_X : & \mathcal{D}^{(1)} & \to & \mathcal{D}^{(1)} \\ & Y & \mapsto & [X, Y] \end{array}$$

and define

$$\begin{cases} (\mathrm{ad}_X)^{\circ 0} := \mathrm{Id} \\ (\mathrm{ad}_X)^{\circ(k+1)} := \mathrm{ad}_X \circ (\mathrm{ad}_X)^{\circ k} & , \forall k \in \mathbb{N} \end{cases} \quad .$$

We will need the following classical formula (see [14]).

Proposition 2.3. *Given* $X, Y \in \mathcal{D}^{(1)}$*:*

$$\exp(X)_*(Y) = \exp(\mathrm{ad}_X)(Y) \quad ,$$

where

$$\exp(\mathrm{ad}_X)(Y) = \sum_{k \geq 0} \frac{1}{k!} (\mathrm{ad}_X)^{\circ k}(Y) = Y + \frac{1}{1!}[X, Y] + \frac{1}{2!}[X, [X, Y]] + \dots \quad .$$

We also recall the existence of a logarithm for all formal diffeomorphisms tangent to the identity (see [8], section 3).

Proposition 2.4. *For any formal diffeomorphism* $\Phi \in \widehat{\mathrm{Diff}}$*, there exists a unique vector field* $F \in \mathcal{D}^{(2)}$ *such that* $\Phi = \varphi \circ \exp(F)$*, where* $\varphi \in \widehat{\mathrm{Diff}}$ *is the linear change of coordinate given by* $D_0\Phi$*. Moreover, for each* $k \geq 2$*, the* exponential map *defines a bijection between* $\mathcal{D}^{(k)}$ *and* $\mathcal{A}^{(k-1)}$*.*

2.3 Jordan decomposition and Dulac-Poincaré normal forms

According to [14], any singular formal vector field $X \in \mathcal{D}^{(1)}$ admits a unique *Jordan decomposition*:

$$X = X_S + X_N, \text{ with } X_S, X_N \in \mathcal{D}^{(1)} \text{ and } [X_S, X_N] = 0 \quad , \tag{2.5}$$

where the restriction of X_S (*resp.* X_N) to each k-jet vector space $\mathrm{J}_k = \mathfrak{m}/\mathfrak{m}^k$ (which is finite dimensional), $k \geq 0$, is semi-simple (*resp.* nilpotent). This decomposition is compatible with truncation and invariant by conjugacy: if $X = X_S + X_N$ is the Jordan decomposition of X then

1. for all $k \geq 0$, $\mathrm{j}_k(X) = \mathrm{j}_k(X_S) + \mathrm{j}_k(X_N)$ is the Jordan decomposition of $\mathrm{j}_k(X)$(here, for any singular vector field $Y \in \mathcal{D}^{(1)}$, $\mathrm{j}_k(Y)$ is the endomorphism $\mathrm{J}_k \to \mathrm{J}_k$ induced by $\mathcal{L}_Y$);

2. for any formal diffeomorphism $\varphi \in \widehat{\mathrm{Diff}}$, $\varphi_*(X) = \varphi_*(X_S) + \varphi_*(X_N)$ is the Jordan decomposition of $\varphi_*(X)$.

Definition 2.5. We say that $X \in \mathcal{D}^{(1)}$ is in *Poincaré-Dulac normal form* if its Jordan decomposition $X = X_S + X_N$ is such that X_S is in diagonal form, *i.e.* $X_S = S(\lambda)$, where $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and

$$S(\lambda) \quad := \quad \lambda_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \lambda_n x_n \frac{\partial}{\partial x_n} \quad . \tag{2.6}$$

As mentioned in the introduction, according to Poincaré-Dulac Theorem, any singular vector field is conjugate to a Poincaré-Dulac normal form, but this normal form is far from being unique: every vector field is conjugate to many distinct Poincaré-Dulac normal forms.

Definition 2.6. A *monomial vector field* is a vector field in $\mathcal{D}$ of the form $\mathbf{x}^{\mathbf{k}} S(\mu)$ for some $\mathbf{k} \in \mathcal{I}$, where $\mathcal{I}$ is the index set

$$\mathcal{I} := \{\mathbf{k} = (k_1, \ldots k_n) \in (\mathbb{Z}_{\geq -1})^n \mid \text{ at most one of the } k_j\text{'s is -1}\} \quad ,$$

and some $\mu \in \mathbb{C}^n$ with the condition that $\mu = (0, \ldots, 0, \underset{\substack{\uparrow \\ j}}{\mu_j}, 0, \ldots 0)$ if $k_j = -1$.

Fixing $\lambda \in \mathbb{C}^n$, each monomial vector field $\mathbf{x}^{\mathbf{k}} S(\mu)$ is an eigenvector for $\mathrm{ad}_{S(\lambda)}$ with eigenvalue

$$\langle \lambda, \mathbf{k} \rangle := \lambda_1 k_1 + \cdots + \lambda_n k_n \quad .$$

This is a consequence of the following elementary lemma.

Lemma 2.7. *For all $\lambda, \mu \in \mathbb{C}^n$, and for all $\mathbf{l}, \mathbf{m} \in \mathbb{Z}^n$:*

$$\left[\mathbf{x}^{\mathbf{l}} S(\lambda), \mathbf{x}^{\mathbf{m}} S(\mu)\right] \quad = \quad \mathbf{x}^{\mathbf{l}+\mathbf{m}} \left(\langle \lambda, \mathbf{m} \rangle S(\mu) - \langle \mu, \mathbf{l} \rangle S(\lambda)\right) \quad .$$

Remark 2.8. Notice that each $X \in \mathcal{D}$ can be uniquely written as an infinite sum of monomial vector fields of the form

$$X = \sum_{\mathbf{k} \in \mathcal{I}} \mathbf{x}^{\mathbf{k}} S(\mu_{\mathbf{k}}) \quad ,$$

which is a Krull-convergent series in $\mathcal{D}$. We will call this expression the *monomial expansion* of X.

Assume now that $X = S(\lambda) + X_N$ is in Poincaré-Dulac normal form and let us consider the monomial expansion of X_N :

$$X_N = \sum_{\mathbf{k} \in \mathcal{I}} \mathbf{x}^{\mathbf{k}} S(\mu_{\mathbf{k}}) \quad .$$

The condition $[X_S, X_N] = 0$ is equivalent to require

$$\forall \mathbf{k} \in \mathcal{I},\ \langle \lambda, \mathbf{k} \rangle \neq 0 \Longrightarrow \mu_{\mathbf{k}} = 0 \quad ;$$

in other words, each $\mathbf{x}^{\mathbf{k}}$ in the monomial expansion of X_N is a so-called *resonant monomial*.

Proposition 2.9. *Let $X, Y \in \mathcal{D}^{(1)}$ be two vector fields in Poincaré-Dulac normal form with the same semi-simple part $S(\mu)$ for some $\mu \in \mathbb{C}^n$, and with nilpotent parts in $\mathcal{D}^{(2)}$:*

$$\begin{cases} X = S(\mu) + X_N & , \text{ with } X_N \in \mathcal{D}^{(2)},\ \text{nilpotent, and } [S(\mu), X_N] = 0 \\ Y = S(\mu) + Y_N & , \text{ with } Y_N \in \mathcal{D}^{(2)},\ \text{nilpotent, and } [S(\mu), Y_N] = 0 \end{cases} .$$

Assume X and Y are conjugate by a formal diffeomorphism Φ such that $D_0\Phi = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$. If we write $\Phi = \varphi \circ \exp(F)$ for some vector field $F \in \mathcal{D}^{(2)}$, where $\varphi \in \widehat{\text{Diff}}$ is the linear diffeomorphism associated to $D_0\Phi = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then necessarily $[S(\mu), F] = 0$.

Remark 2.10. Recall that the condition $[S(\mu), F] = 0$ means that if we write $F = \sum_{\mathbf{k} \in \mathcal{I}} \mathbf{x}^{\mathbf{k}} S(\lambda_{\mathbf{k}})$, then $\langle \mu, \mathbf{k} \rangle \neq 0 \Longrightarrow \lambda_{\mathbf{k}} = 0$.

Proof. We can assume without loss of generality that Φ is tangent to the identity. Indeed by setting $P := (\mathrm{D}_0\Phi)^{-1}$, we obtain that $P \circ \Phi$ is tangent to the identity and conjugates X to $\tilde{Y} = \mathrm{D}P \cdot (Y \circ P^{-1})$. Since $\mathrm{D}P$ is diagonal, the assumptions made on Y are also met by $\tilde{Y}$. Moreover, it is clear that the property we have to prove is true for Φ if and only if it is true for $P \circ \Phi$. Therefore we may suppose that Φ is tangent to the identity. According to Proposition 2.4, there exists $F \in \mathcal{D}^{(2)}$ such that $\exp(F) = \Phi$, while according to Proposition 2.3 we have:

$$\exp(F)_* (S(\mu)) = S(\mu) + [F, S(\mu)] + \frac{1}{2!} [F, [F, S(\mu)]] + \ldots \tag{2.7}$$

Since $\exp(F)_* (S(\mu)) = S(\mu)$ by uniqueness of the Jordan decomposition, we have

$$[F, S(\mu)] + \frac{1}{2!}[F, [F, S(\mu)]] + \dots \quad = \quad 0 \quad . \tag{2.8}$$

This implies that $[F, S(\mu)] = 0$, using Lemma 2.1 and the fact that $\operatorname{ord}(F) \geq 2$. $\square$

Remark 2.11. The assumption that $\mathrm{D}_0\Phi$ is in diagonal form necessarily holds if $\mu_i \neq \mu_j$, for all $i \neq j$.

2.4 Formal differential forms

Definition 2.12. We denote by $\widehat{\Omega^1}(\mathbb{C}[[\mathbf{x}]])$ (or just $\widehat{\Omega^1}$ for simplicity) the set of *formal 1-forms* in $\mathbb{C}^n$. It is the dual of $\operatorname{Der}(\mathbb{C}[[\mathbf{x}]])$ as $\mathbb{C}[[\mathbf{x}]]$-module.

Fixing the dual basis $(\mathrm{d}x_1, \dots \mathrm{d}x_n)$ of $(\mathbb{C}^n)^*$, $\widehat{\Omega^1}(\mathbb{C}[[\mathbf{x}]])$ is a free $\mathbb{C}[[\mathbf{x}]]$-module of rank n, generated by $\mathrm{d}x_1, \dots, \mathrm{d}x_n$.

Definition 2.13. For any $p \in \mathbb{N}$, we denote the p-exterior product of $\widehat{\Omega^1}(\mathbb{C}[[\mathbf{x}]])$ by

$$\widehat{\Omega^p}(\mathbb{C}[[\mathbf{x}]]) := \bigwedge^p \widehat{\Omega^1}(\mathbb{C}[[\mathbf{x}]])$$

(or just $\widehat{\Omega^p}$). Its elements will be called *formal p-forms*.

The set of 0-forms is the set of formal series: $\widehat{\Omega^0}(\mathbb{C}[[\mathbf{x}]]) := \mathbb{C}[[\mathbf{x}]]$.

Definition 2.14. We denote by

$$\widehat{\Omega}(\mathbb{C}[[\mathbf{x}]]) := \bigoplus_{p=0}^{+\infty} \widehat{\Omega^p}(\mathbb{C}[[\mathbf{x}]])$$

(or just $\widehat{\Omega}$ for simplicity) the exterior algebra of the formal forms in $\mathbb{C}^n$, and by d the exterior derivative on it.

One can also extend the Krull topology to $\widehat{\Omega}$.

We can define the action of $\widehat{\mathrm{Diff}}$ by *pull-back* on $\widehat{\Omega}(\mathbb{C}[[\mathbf{x}]])$ thanks to the following properties:

1. $\mathbb{C}$-linearity
2. for all $f \in \mathbb{C}[[\mathbf{x}]]$, $\Phi^*(f)$ is defined as in (2.3)
3. $\forall \alpha, \beta \in \widehat{\Omega}(\mathbb{C}[[\mathbf{x}]])$, $\forall \Phi \in \widehat{\mathrm{Diff}}$, $\Phi^*(\alpha \wedge \beta) = \Phi^*(\alpha) \wedge \Phi^*(\beta)$
4. $\forall \Phi \in \widehat{\mathrm{Diff}}$, $\Phi^* \circ \mathrm{d} = \mathrm{d} \circ \Phi^*$.

For any $X \in \mathcal{D}^{(1)}$ and $\alpha \in \widehat{\Omega}\left(\mathbb{C}\left[\left[\mathbf{x}\right]\right]\right)$, we denote by $\mathcal{L}_X\left(\alpha\right)$ the *Lie derivative* of α with respect to X. We recall that $\mathcal{L}_X$ is uniquely determined by the following properties:

1. for all $k \geq 0$, $\mathcal{L}_X : \widehat{\Omega^k}\left(\mathbb{C}\left[\left[\mathbf{x}\right]\right]\right) \longrightarrow \widehat{\Omega^k}\left(\mathbb{C}\left[\left[\mathbf{x}\right]\right]\right)$ is linear
2. for all $f \in \mathbb{C}\left[\left[\mathbf{x}\right]\right]$ (*i.e.* f is a 0-form), $\mathcal{L}_X\left(f\right)$ is as in Definition (2.2)
3. $\mathcal{L}_X$ is a derivation of $\widehat{\Omega}\left(\mathbb{C}\left[\left[\mathbf{x}\right]\right]\right)$, *i.e.* for all $\alpha, \beta \in \widehat{\Omega}\left(\mathbb{C}\left[\left[\mathbf{x}\right]\right]\right)$:

$$\mathcal{L}_X\left(\alpha \wedge \beta\right) = \mathcal{L}_X\left(\alpha\right) \wedge \beta + \alpha \wedge \mathcal{L}_X\left(\beta\right)$$

   (Leibniz rule)

4. $\mathcal{L}_X \circ \mathrm{d} = \mathrm{d} \circ \mathcal{L}_X$.

We will need the following classical formula, which extends (2.4).

Proposition 2.15. *$\forall \alpha \in \widehat{\Omega}\left(\mathbb{C}\left[\left[\mathbf{x}\right]\right]\right),\ X \in \mathcal{D}^{(1)}$:*

$$\exp\left(X\right)^*\left(\alpha\right) = \exp\left(\mathcal{L}_X\right)\left(\alpha\right) = \sum_{k \geq 0} \frac{1}{k!} \mathcal{L}_X^{\circ k}\left(\alpha\right) \quad .$$

Proof. (Sketch)

This formula is true for 0-forms, and we just has to prove it for 1-forms, because we can then it extend to any p-form using the exterior product and the Leibniz formula. In order to prove the result for 1-forms, one has to use the fact that $\mathcal{L}_X \circ \mathrm{d} = \mathrm{d} \circ \mathcal{L}_X$. □

With the same arguments, and using formulas $\Phi^* \circ \mathrm{d} = \mathrm{d} \circ \Phi^*$ and $\Phi^*\left(\alpha \wedge \beta\right) = \Phi^*\left(\alpha\right) \wedge \Phi^*\left(\beta\right)$, we can prove the following Proposition.

Proposition 2.16. *For all $\Phi \in \widehat{\mathrm{Diff}}$, $X \in \mathcal{D}^{(1)}$ and $\theta \in \widehat{\Omega}\left(\mathbb{C}\left[\left[\mathbf{x}\right]\right]\right)$, we have;*

$$\Phi^*\left(\mathcal{L}_{\Phi_*(X)}\left(\omega\right)\right) = \mathcal{L}_X\left(\Phi^*\left(\omega\right)\right) \quad .$$

In other words, the following diagram is commutative for all $p \geq 0$:

$$\begin{tikzcd}
\widehat{\Omega^p} \arrow[r, "\Phi^*"] \arrow[d, "\mathcal{L}_{\Phi_*(X)}"'] & \widehat{\Omega^p} \arrow[d, "\mathcal{L}_X"] \\
\widehat{\Omega^p} \arrow[r, "\Phi^*"'] & \widehat{\Omega^p}
\end{tikzcd}$$

From now on, we set $n = 3$, we denote by $\mathbf{x} = \left(x, \mathbf{y}\right) = \left(x, y_1, y_2\right)$ the coordinates in $\mathbb{C}^3$.

Definition 2.17. We denote by $\langle \mathrm{d}x \rangle$ the ideal generated by $\mathrm{d}x$ in $\widehat{\Omega} = \widehat{\Omega}\left(\mathbb{C}\left[\left[x, \mathbf{y}\right]\right]\right)$: it is the set of forms $\theta \in \widehat{\Omega}$ such that $\theta = \mathrm{d}x \wedge \eta$, for some $\eta \in \widehat{\Omega}$.

Proposition 2.18. *Let $\theta \in \widehat{\Omega}$, $X \in \mathcal{D}^{(1)}$ and set $\Phi := \exp(X) \in \widehat{\mathrm{Diff}}$. Then the following assertions are equivalent:*

1. $\mathcal{L}_X(x) = 0$ *and* $\mathcal{L}_X(\theta) \in \langle dx \rangle$
2. $\Phi^*(x) = x$ *and* $\Phi^*(\theta) \in \theta + \langle dx \rangle$.

Proof. It is just a consequence of Propositions 2.15 and 2.3. □

The next Lemma is proved by induction, as Lemma 2.2.

Lemma 2.19. *In the situation described in Lemma 2.2, if we further assume that there exists a form $\theta \in \widehat{\Omega}$ such that $\Phi_n^*(\theta) \in \theta + \langle dx \rangle$, for all $n \geq 0$, then $\Phi^*(\theta) \in \theta + \langle dx \rangle$.*

2.5 Transversal Hamiltonian vector fields and transversal symplectomorphisms

We will need in fact to deal with forms with rational coefficients, and more precisely with

$$\omega \quad := \quad \frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{x} \; .$$

Given a formal vector field X such that $\mathcal{L}_X(x) \in \langle x \rangle$ we can easily extend its Lie derivative action to the set $x^{-1}\widehat{\Omega}(\mathbb{C}[\![x,\mathbf{y}]\!])$ by setting:

$$\begin{aligned}\mathcal{L}_X\left(\frac{1}{x}\theta\right) \quad &:= \quad -\frac{\mathcal{L}_X(x)}{x^2}\theta + \frac{1}{x}\mathcal{L}_X(\theta)\ ,\ \theta \in \widehat{\Omega}(\mathbb{C}[\![x,\mathbf{y}]\!]) \\ &\in \quad x^{-1}\widehat{\Omega}(\mathbb{C}[\![x,\mathbf{y}]\!])\ ,\ \text{because } \mathcal{L}_X(x) \in \langle x \rangle \qquad .\end{aligned}$$

In particular we have

$$x\mathcal{L}_X\left(\frac{1}{x}\theta\right) \in \widehat{\Omega}(\mathbb{C}[\![x,\mathbf{y}]\!]) \quad .$$

Notice that if a vector field X satisfy $\mathcal{L}_X(\mathrm{d}x) \in \langle \mathrm{d}x \rangle$, then $\mathcal{L}_X(x) \in \langle x \rangle$. Similarly, we naturally extend the action of fibered diffeomorphisms by pull-back on rational forms in $x^{-1}\widehat{\Omega}(\mathbb{C}[\![x,\mathbf{y}]\!])$ by:

$$\Phi^*\left(\frac{1}{x}\theta\right) = \frac{1}{x}\Phi^*(\theta)\,,\ \text{for }\ (\Phi,\theta) \in \widehat{\mathrm{Diff}}_{\mathrm{fib}} \times \widehat{\Omega}(\mathbb{C}[\![x,\mathbf{y}]\!])$$

so that

$$x\Phi^*\left(\frac{1}{x}\theta\right) = \Phi^*(\theta) \qquad .$$

Recalling Definition 1.9, we can now state a result analogous to Proposition 2.18.

Proposition 2.20. *Let $F \in \mathcal{D}^{(1)}$ be a singular vector field. The following two statements are equivalent:*

1. $\exp(F)$ *is a transversally symplectic diffeomorphisms ,*
2. $\mathcal{L}_F(x) = 0$ *and F is transversally Hamiltonian.*

Proof. It is just a consequence of Proposition 2.18. □

Lemma 2.21. *Let $\Phi \in \widehat{\mathrm{Diff}}_\omega$ and $X \in \mathcal{D}_\omega$. Then, $\Phi_*(X) \in \mathcal{D}_\omega$.*

Proof. This comes from Proposition 2.16: $\Phi^*\left(\mathcal{L}_{\Phi_*(X)}(\omega)\right) = \mathcal{L}_X(\Phi^*(\omega))$, and from the fact that $\widehat{\mathrm{Diff}}_\omega$ is a group, so $\Phi^{-1} \in \widehat{\mathrm{Diff}}_\omega$. Consequently we have:

$$\begin{aligned} x\mathcal{L}_{\Phi_*(X)}(\omega) &= x\left(\Phi^{-1}\right)^* \mathcal{L}_X(\Phi^*(\omega)) \\ &= x\left(\Phi^{-1}\right)^* \mathcal{L}_X(\omega + \langle \mathrm{d}x\rangle) \\ &= x\left(\Phi^{-1}\right)^* (\mathcal{L}_X(\omega)) + x\left(\Phi^{-1}\right)^* (\mathcal{L}_X(\langle \mathrm{d}x\rangle)) \\ &= x\left(\Phi^{-1}\right)^* (\langle \mathrm{d}x\rangle) + x\left(\Phi^{-1}\right)^* (\langle \mathrm{d}x\rangle) \\ &\in \langle \mathrm{d}x\rangle \quad . \end{aligned}$$

□

Remark 2.22. In other words, we have an action of the group $\widehat{\mathrm{Diff}}_\omega$ on $\mathcal{D}_\omega$, and then on $\widehat{\mathcal{SN}}_\omega$.

We would like now to give a characterization of transversally Hamiltonian vector fields in terms of its monomial expansion (see Remark 2.8). Consider a monomial vector field

$$X = x^{k_0} y_1^{k_1} y_2^{k_2} S(\mu) \quad ,$$

with $\mu = (\mu_0, \mu_1, \mu_2) \in \mathbb{C}^3 \setminus \{0\}$, such that $\mathcal{L}_X(x) \in \langle x\rangle$. We necessarily have either $\mu_0 = 0$ or $k_0 \geq 0$. Let us compute its Lie derivative applied to ω:

$$\begin{aligned} \mathcal{L}_X(\omega) &= -\frac{\mathcal{L}_X(x)}{x^2}\mathrm{d}y_1 \wedge \mathrm{d}y_2 + \frac{1}{x}\mathrm{d}(\mathcal{L}_X(y_1)) \wedge \mathrm{d}y_2 + \frac{1}{x}\mathrm{d}y_1 \wedge \mathrm{d}(\mathcal{L}_X(y_2)) \\ &= -\mu_0 x^{k_0-1} y_1^{k_1} y_2^{k_2}\mathrm{d}y_1 \wedge \mathrm{d}y_2 + \frac{\mu_1}{x}\mathrm{d}\left(x^{k_0} y_1^{k_1+1} y_2^{k_2}\right) \wedge \mathrm{d}y_2 \\ &\quad + \frac{\mu_2}{x}\mathrm{d}y_1 \wedge \mathrm{d}\left(x^{k_0} y_1^{k_1} y_2^{k_2+1}\right) \\ &= (\mu_1(k_1+1) + \mu_2(k_2+1) - \mu_0)\, x^{k_0-1} y_1^{k_1} y_2^{k_2}\mathrm{d}y_1 \wedge \mathrm{d}y_2 + \langle \mathrm{d}x\rangle \quad . \end{aligned}$$

Moreover:

$$\begin{aligned} \mathcal{L}_X(\mathrm{d}x) &= \mathrm{d}(\mathcal{L}_X(x)) \\ &= \mathrm{d}\left(\mu_0 x^{k_0+1} y_1^{k_1} y_2^{k_2}\right) \\ &= \mu_0\left((k_0+1)\, x^{k_0} y_1^{k_1} y_2^{k_2}\mathrm{d}x + k_1 x^{k_0+1} y_1^{k_1-1} y_2^{k_2}\mathrm{d}y_1 + k_2 x^{k_0+1} y_1^{k_1} y_2^{k_2-1}\mathrm{d}y_2\right) \quad . \end{aligned}$$

Thus, we see that X is transversally Hamiltonian if and only if the following two conditions hold:

1. $\mu_1(k_1+1)+\mu_2(k_2+1)=\mu_0$

2. either $\mu_0=0$ or $k_1=k_2=0$.

So we have the following:

Proposition 2.23. *Let $X\in\mathcal{D}^{(1)}$ be a singular vector field and let*

$$X=\sum_{\mathbf{k}\in\mathcal{I}}\mathbf{x}^{\mathbf{k}}S(\mu_{\mathbf{k}})$$

be its monomial expansion. X is transversally Hamiltonian if and only if for all $\mathbf{k}\in\mathcal{I}$, $\mathbf{x}^{\mathbf{k}}S(\mu_{\mathbf{k}})$ is transversally Hamiltonian.

Proof. Clearly if $\mathbf{x}^{\mathbf{k}}S(\mu_{\mathbf{k}})$ is transversally Hamiltonian for all $\mathbf{k}\in\mathcal{I}$, then X is transversally Hamiltonian is obvious, by convergence of the above series in the Krull topology. Assume conversely that X is transversally Hamiltonian. First of all, notice that we necessarily have, for all $\mathbf{k}\in\mathcal{I}$, $\mathcal{L}_{\mathbf{x}^{\mathbf{k}}S(\mu_{\mathbf{k}})}(\mathrm{d}x)\in\langle\mathrm{d}x\rangle$. Indeed, if it were not the case, consider $\mathbf{k}$ with $|\mathbf{k}|$ minimum among the set of multi-index $\mathbf{l}$ satisfying

$$\mathcal{L}_{x^{l}S(\mu_{\mathbf{l}})}(\mathrm{d}x)\notin\langle\mathrm{d}x\rangle$$

to obtain a contradiction, by looking at the terms of higher order. Similarly, according to the computation above, for each $\mathbf{k}\in\mathcal{I}$:

$$\mathcal{L}_{\mathbf{x}^{\mathbf{k}}S(\mu_{\mathbf{k}})}(\omega)\quad=\quad(\mu_1(k_1+1)+\mu_2(k_2+1)-\mu_0)\,x^{k_0-1}y_1^{k_1}y_2^{k_2}\mathrm{d}y_1\wedge\mathrm{d}y_2+\langle\mathrm{d}x\rangle$$

If one of the two conditions 1. or 2. above were not satisfied by a couple $(\mathbf{k},\mu_{\mathbf{k}})$ with $|\mathbf{k}|$ minimal, then we could not have $x\mathcal{L}_X(\omega)\in\langle\mathrm{d}x\rangle$ (just consider the terms of higher order). □

3 Formal classification under fibered transformations

3.1 Invariance of the residue by fibered conjugacy

We start this section by proving that the non-degenerate condition defined in the introduction only depends on the conjugacy class of the vector field under the action of fibered diffeomorphisms. More precisely, the following proposition states that the residue is an invariant of a doubly-resonant saddle-node under the action of $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$.

Proposition 3.1. *Let $X,Y\in\widehat{\mathcal{SN}}$. If X and Y are $\widehat{\mathrm{Diff}}_{\mathrm{fib}}-$conjugate, then* $\mathrm{res}(X)=\mathrm{res}(Y)$.

Proof. Consider the system

$$x^2\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}=\alpha(x)+\mathbf{A}(x)\mathbf{y}(x)+\mathbf{f}(x,\mathbf{y}(x))\qquad,\tag{3.1}$$

with $\mathbf{y}=(y_1,y_2)$ and where the following conditions hold:

- $\alpha(x) = \begin{pmatrix} \alpha_1(x) \\ \alpha_2(x) \end{pmatrix}$, with $\alpha_1, \alpha_2 \in \langle x \rangle^2 \subset \mathbb{C}[\![x]\!]$
- $\mathbf{A}(x) \in \mathrm{Mat}_{2,2}(\mathbb{C}[\![x]\!])$ with $\mathbf{A}(0) = \mathrm{Diag}(-\lambda, \lambda)$, $\lambda \in \mathbb{C}^*$
- $\mathbf{f}(x, \mathbf{y}) = \begin{pmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \end{pmatrix}$, with $f_1, f_2 \in \langle y_1, y_2 \rangle^2 \subset \mathbb{C}[\![x, \mathbf{y}]\!]$.

Perform the change of coordinates given by $\mathbf{y} = \beta(x) + \mathbf{P}(x)\mathbf{z} + \mathbf{h}(x, \mathbf{z})$, with $\mathbf{z} = (z_1, z_2)$ and where:

- $\beta(x) = \begin{pmatrix} \beta_1(x) \\ \beta_2(x) \end{pmatrix}$, with $\beta_1, \beta_2 \in \langle x \rangle \subset \mathbb{C}[\![x]\!]$
- $\mathbf{P}(x) \in \mathrm{Mat}_{2,2}(\mathbb{C}[\![x]\!])$ such that $\mathbf{P}(0) \in \mathrm{GL}_2(\mathbb{C})$
- $\mathbf{h}(x, \mathbf{y}) = \begin{pmatrix} h_1(x, \mathbf{y}) \\ h_2(x, \mathbf{y}) \end{pmatrix}$, with $h_1, h_2 \in \langle z_1, z_2 \rangle^2 \subset \mathbb{C}[\![x, \mathbf{z}]\!]$.

Then one obtain the following system satisfied by $\mathbf{z}(x)$:

$$\begin{aligned} x^2 \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x} &= \mathbf{P}(x)^{-1} \left(\alpha(x) + \mathbf{A}(x)\beta(x) + \mathbf{f}(x, \beta(x)) - x^2 \frac{\mathrm{d}\beta}{\mathrm{d}x}(x) \right) \\ &\quad + \mathbf{P}(x)^{-1} \left(\mathbf{A}(x)\mathbf{P}(x) - x^2 \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}x}(x) + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(x, \beta(x))\mathbf{P}(x) \right) \mathbf{z} + \langle z_1, z_2 \rangle^2 \end{aligned} ,$$

Since $\mathbf{A}(0) \in \mathrm{GL}_2(\mathbb{C})$, $\mathbf{f}(x, \mathbf{y}) \in \langle y_1, y_2 \rangle^2$ and $\mathrm{ord}(\beta) \geq 1$, the order of

$$\mathbf{P}(x)^{-1} \left(\alpha(x) + \mathbf{A}(x)\beta(x) + \mathbf{f}(x, \beta(x)) - x^2 \frac{\mathrm{d}\beta}{\mathrm{d}x}(x) \right)$$

is at least 2 if and only if $\mathrm{ord}(\beta) \geq 2$. Then:

$$\mathrm{Tr}\left(\mathbf{P}(x)^{-1} \left(\mathbf{A}(x)\mathbf{P}(x) - x^2 \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}x}(x) + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(x, \beta(x))\mathbf{P}(x) \right) \right) \in \mathrm{Tr}(\mathbf{A}(x)) + \langle x \rangle^2 .$$

So $\left(\frac{\mathrm{Tr}(\mathbf{A}(x))}{x} \right)_{|x=0}$ is invariant by fibered change of coordinates on system of the form (3.1) with $\mathrm{ord}(\alpha) \geq 2$. $\square$

3.2 Proof of Theorems 1.3 and 1.12

We will use the tools described in Section 2.

Proof. Let $Y \in \widehat{\mathcal{SN}}_{\mathrm{nd}}$ (*resp.* in $\widehat{\mathcal{SN}}_{\omega}$) be a non-degenerate (*resp.* transversally Hamiltonian) doubly-resonant saddle-node:

$$Y = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + F_1(x, y_1, y_2)) \frac{\partial}{\partial y_1} + (\lambda y_2 + F_2(x, y_1, y_2)) \frac{\partial}{\partial y_1} ,$$

with $\lambda \in \mathbb{C}^*$, and $F_\nu(x, \mathbf{y}) \in \mathfrak{m}^2$, for $\nu = 1, 2$. As seen in the previous subsection, we can assume that $F_1(x, 0, 0) = F_2(x, 0, 0) = 0$.

The general idea is to apply successive (infinitely many) diffeomorphisms of the form

$$\exp\left(x^{j_0} y_1^{j_1} y_2^{j_2} S(0, \mu_{1,\mathbf{j}}, \mu_{2,\mathbf{j}})\right)$$

for convenient choices of $\mathbf{j}, \mu_{1,\mathbf{j}}, \mu_{2,\mathbf{j}}$, in order to remove all the terms we want to. Let us consider the monomial expansion of Y:

$$Y = \lambda S(0, -1, 1) + xS(1, 0, 0) + \sum_{\mathbf{k} \in \mathcal{I},\, |\mathbf{k}| \geq 1} x^{k_0} y_1^{k_1} y_2^{k_2} S(0, \mu_{1,\mathbf{k}}, \mu_{2,\mathbf{k}}) \quad (3.2)$$

Since Y in non-degenerate we necessarily have

$$\mu_{1,(1,00)} + \mu_{2,(1,0,0)} = \mathrm{res}(Y) \in \mathbb{C} \backslash \mathbb{Q}_{\leq 0} \qquad .$$

In the transversally Hamiltonian case, each term in the sum

$$\sum_{\mathbf{k} \in \mathcal{I},\, |\mathbf{k}| \geq 1} x^{k_0} y_1^{k_1} y_2^{k_2} S(0, \mu_{1,\mathbf{k}}, \mu_{2,\mathbf{k}})$$

must satisfy

$$\mu_{1,\mathbf{k}}(k_1 + 1) + \mu_{2,\mathbf{k}}(k_2 + 1) = 0 \qquad ,$$

if $\mathbf{k} \neq (1, 0, 0)$ and $\mu_{1,(1,0,0)} + \mu_{2,(1,0,0)} = 1$.

The normalizing conjugacy Φ is constructed in two steps.

1. The first step is aimed at removing all **non-resonant** monomial terms, *i.e.* those of the form

$$x^{k_0} y_1^{k_1} y_2^{k_2} S(0, \mu_{1,\mathbf{k}}, \mu_{2,\mathbf{k}}) \text{ , with } \mathbf{k} \in \mathcal{I}, |\mathbf{k}| \geq 1 \text{ and } k_1 \neq k_2 \qquad .$$

2. The second step is aimed at removing certain resonant monomial terms, and more precisely those of the form

$$x^{k_0} (y_1 y_2)^k S(0, \eta_{1,\mathbf{i}}, \eta_{2,\mathbf{i}}) \text{ , except for } (k_0, k) = (1, 0) \text{ and } k_0 = 0 \qquad .$$

We will see that each one of these steps allows us to define a fibered diffeomorphism Φ_j (transversally symplectic in the transversally Hamiltonian case), for $j = 1, 2$. Finally we define $\Phi := \Phi_2 \circ \Phi_1$. The main tool used at each step is Proposition 2.15. Moreover, each Φ_j will be constructed using Corollary 2.2. The fact that each Φ_j is a fibered diffeomorphism (transversally symplectic in the transversally Hamiltonian case) will again come from Lemma 2.2 (and Lemma 2.19 in the transversally symplectic case, and each $Y_j = (\Phi_j)_*(Y_{j-1})$, $j = 1, 2$ with $Y_0 := Y$, will be transversally Hamiltonian according to Lemma 2.21).

1. *First step:* we remove all non-resonant monomial terms, using diffeomorphisms of the form

$$\exp\left(x^{i_0}y_1^{i_1}y_2^{i_2}S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)\right)\quad,$$

with $\mathbf{i}\in\mathcal{I}$, $|\mathbf{i}|\geq 1$, $i_1\neq i_2$ and $\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}$ to be determined. We have, thanks to Proposition 2.15:

$$\left(\exp\left(x^{i_0}y_1^{i_1}y_2^{i_2}S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)\right)\right)_*\left(Y_0\right)=Y_0+\frac{1}{1!}\left[x^{i_0}y_1^{i_1}y_2^{i_2}S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right),Y_0\right]+\dots,$$

where $(\dots)$ are terms computed *via* successive nested brackets, and they are all of order at least $|\mathbf{i}|+1$. Let us compute the first bracket:

$$\begin{aligned}
&\left[x^{i_0}y_1^{i_1}y_2^{i_2}S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right),Y_0\right]\\
=\;&\lambda\left(i_1-i_2\right)x^{i_0}y_1^{i_1}y_2^{i_2}S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)\\
&-i_0x^{i_0+1}y_1^{i_1}y_2^{i_2}S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)\\
&+\sum_{\mathbf{k}\in\mathcal{I},\,|\mathbf{k}|\geq 1}x^{i_0+k_0}y_1^{i_1+k_1}y_2^{i_2+k_2}\left(k_1\eta_{1,i_0}+k_2\eta_{2,i_2}\right)S\left(0,\mu_{1,\mathbf{k}},\mu_{2,\mathbf{k}}\right)\\
&-\sum_{\mathbf{k}\in\mathcal{I},\,|\mathbf{k}|\geq 1}x^{i_0+k_0}y_1^{i_1+k_1}y_2^{i_2+k_2}\left(i_1\mu_{1,\mathbf{k}}+i_2\mu_{2,\mathbf{k}}\right)S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)\quad.
\end{aligned}$$

Then one can remove all terms of the form $x^{i_0}y_1^{i_1}y_2^{i_2}S\left(0,\mu_{1,\mathbf{i}},\mu_{2,\mathbf{i}}\right)$ with $|\mathbf{i}|\geq 1$ and $i_1\neq i_2$ by induction on $|\mathbf{i}|\geq 1$. We then define (using Lemma 2.2) a fibered diffeomorphism Φ_1, such that $Y_1:=\left(\Phi_1\right)_*\left(Y_0\right)$ is still of the form (3.2), but without non-resonant terms:

$$\begin{aligned}
Y_1\;=\;&\lambda S\left(0,-1,1\right)+xS\left(1,a_1,a_2\right)\\
&+\sum_{\substack{k_0+k\geq 1\\(k_0,k)\neq(1,0)}}x^{k_0}y_1^ky_2^kS\left(0,\mu_{1,\mathbf{k}},\mu_{2,\mathbf{k}}\right)
\end{aligned}$$

for maybe different $\mu_{j,\mathbf{k}}$. Notice that a_1,a_2 here are necessarily such that $a_1+a_2\notin\mathbb{Q}_{\leq 0}$ since the vector field is supposed to be non-degenerate, and this condition is invariant under fibered change of coordinates.

Remark. In the transversally Hamiltonian case, the terms $x^{i_0}y_1^{i_1}y_2^{i_2}S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)$ to be removed at this stage satisfy $\eta_{1,\mathbf{i}}\left(i_1+1\right)+\eta_{2,\mathbf{i}}\left(i_2+1\right)=0$, so that Φ_1 is transversally symplectic according to Proposition 2.20 and Lemma 2.19. Moreover, in this case, we necessarily have $a_1+a_2=1$.

2. *Second step:* we finally remove all the terms of the form

$$x^{i_0}\left(y_1y_2\right)^iS\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)\text{ , except for }(i_0,i)=(1,0)\text{ and }i_0=0\qquad,$$

using diffeomorphisms of the form

$$\exp\left(x^{i_0}\left(y_1y_2\right)^iS\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)\right)\qquad,$$

with $i_0 + i \geq 1$, and $\eta_{1,\mathbf{i}}, \eta_{2,\mathbf{i}}$ to be determined. We have, thanks to Proposition 2.15:

$$\left(\exp\left(x^{i_0}\left(y_1y_2\right)^i S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)\right)\right)_*\left(Y_1\right) = Y_1 + \frac{1}{1!}\left[x^{i_0}\left(y_1y_2\right)^i S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right), Y_1\right] + \ldots,$$

where $(\ldots)$ are terms computed *via* successive nested brackets, and they are all of order strictly greater than the order of the first bracket. Let us compute the first bracket:

$$\begin{aligned}
& \left[x^{i_0}\left(y_1y_2\right)^i S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right), Y_3\right] \\
= & -\left(i_0 + i\left(a_1+a_2\right)\right) x^{i_0+1}\left(y_1y_2\right)^i S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right) \\
& + \sum_{\substack{k_0+2k\geq 1 \\ (k_0,k)\neq(1,0)}} x^{i_0+k_0}\left(y_1y_2\right)^{i+k} k\left(\eta_{1,i_0}+\eta_{2,i_2}\right) S\left(0,\mu_{1,\mathbf{k}},\mu_{2,\mathbf{k}}\right) \\
& - \sum_{\substack{k_0+2k\geq 1 \\ (k_0,k)\neq(1,0)}} x^{i_0+k_0}\left(y_1y_2\right)^{i+k} i\left(\mu_{1,\mathbf{k}}+\mu_{2,\mathbf{k}}\right) S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right).
\end{aligned}$$

Then we see that one can remove all terms of the form $x^{i_0}\left(y_1y_2\right)^i S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)$ except for $(i_0, i) = (1,0)$ and for $i_0 = 0$, without creating non-resonant terms, since $(a_1+a_2) \notin \mathbb{Q}_{\leq 0}$. We do this by induction on $I := i_0 + i \geq 1$, and for fixed $I \geq 1$, we remove the terms with i increasing and i_0 decreasing. Notice that at each step we do not create terms already removed earlier in the process.

We then define (using Lemma 2.2) a fibered diffeomorphism Φ_2, such that $Y_2 := \left(\Phi_2\right)_*\left(Y_1\right)$ is of the form

$$Y_2 = \lambda S\left(0,-1,1\right) + xS\left(1,a_1,a_2\right) + \sum_{k\geq 1}\left(y_1y_2\right)^k S\left(0,\mu_{1,\mathbf{k}},\mu_{2,\mathbf{k}}\right) .$$

Remark. In the transversally Hamiltonian case, the terms $x^{i_0}\left(y_1y_2\right)^i S\left(0,\eta_{1,\mathbf{i}},\eta_{2,\mathbf{i}}\right)$ to be removed at this stage satisfy $\left(\eta_{1,\mathbf{i}}+\eta_{2,\mathbf{i}}\right) = 0$, so that Φ_2 is transversally symplectic, according to Proposition 2.20 and and Lemma 2.19.

Finally, we define $\Phi := \Phi_2 \circ \Phi_1$, so that $\Phi_*\left(Y\right) = Y_2$ and Φ is a fibered diffeomorphism (transversally symplectic in the Hamiltonian case). $\square$

3.3 Uniqueness: proof of Theorem 1.6

We now prove Theorem 1.6.

Proof. Let

$$\begin{aligned}
Z &= x^2\frac{\partial}{\partial x} + \left(-\lambda + a_1x + c_1\left(v\right)\right)z_1\frac{\partial}{\partial z_1} + \left(\lambda + a_2x + c_2\left(v\right)\right)z_2\frac{\partial}{\partial z_2} \\
Z' &= x^2\frac{\partial}{\partial x} + \left(-\lambda' + a_1'x + c_1'\left(v\right)\right)z_1\frac{\partial}{\partial z_1} + \left(\lambda' + a_2'x + c_2'\left(v\right)\right)z_2\frac{\partial}{\partial z_2} \quad ,
\end{aligned}$$

where $(\lambda,\lambda',a_1,a_2,a'_1,a'_2)\in(\mathbb{C}^*)^2\times\mathbb{C}^4$, $(a_1+a_2,a'_1+a'_2)\in(\mathbb{C}\backslash\mathbb{Q}_{\leq 0})^2$ and $(c_1,c_2,c'_1,c'_2)\in(v\mathbb{C}[\![v]\!])^4$ are formal power series in $v=z_1z_2$ of order at least one.

- It is clear that if there exists $\varphi:v\mapsto\theta v$ with $\theta\in\mathbb{C}^*$ such that

$$\begin{array}{rcl} (\lambda,a_1,a_2,c_1,c_2) & = & (\lambda',a'_1,a'_2,c'_1\circ\varphi,c'_2\circ\varphi) \\ \left(\textit{resp.}\ \ (\lambda,a_1,a_2,c_1,c_2)\right. & = & \left.(-\lambda',a'_2,a'_1,c'_2\circ\varphi,c'_1\circ\varphi)\right) \end{array}$$

  then Z is $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$-conjugate to Z'.

- Now assume that Z is $\widehat{\mathrm{Diff}}_{\mathrm{fib}}$-conjugate to Z'. First of all, studying the terms of degree 1 with respect to $\mathbf{z}$, we see that we either have $(\lambda,a_1,a_2)=(\lambda',a'_1,a'_2)$ or $(\lambda,a_1,a_2)=(-\lambda',a'_2,a'_1)$. Up to perform a linear change of coordinates beforehand, let us assume that $(\lambda,a_1,a_2)=(\lambda',a'_1,a'_2)$. In the following, and for convenience, we will use the notations:

$$\begin{cases} Z=Z_{(c,r)}:=xS(1,a_1,a_2)+(\lambda+c(v))\,S(0,-1,1)+r(v)\,S(0,a_1,a_2) \\ Z'=Z_{(c',r')}:=xS(1,a_1,a_2)+(\lambda+c'(v))\,S(0,-1,1)+r'(v)\,S(0,a_1,a_2) \end{cases},$$

  where:

$$\begin{cases} c_1=-c+r & ,\ c_2=c+r \\ c'_1=-c'+r' & ,\ c'_2=c'+r' \end{cases},$$

  so that $\mathrm{ord}(c)\geq 1$, $\mathrm{ord}(r)\geq 1$.

  Now we have to prove that if $Z_{(c,r)}$ is $\widehat{\mathrm{Diff}}_{\mathrm{fib}}-$conjugate to $Z_{(c',r')}$, then $(c,r)=(c',r')$. By assumption, there exists $\Phi\in\widehat{\mathrm{Diff}}_{\mathrm{fib}}$ such that

$$\Phi_*\left(Z_{(c,r)}\right)\ =\ Z_{(c',r')}\ .$$

  By Remark 2.11, $\mathrm{D}_0\Phi=\mathrm{diag}(1,\theta_1,\theta_2)$ is diagonal. Now, set $\Psi:=(\mathrm{D}_0\Phi)^{-1}\circ\Phi$, $\varphi:v\mapsto(\theta_1\theta_2)\,v$, and $(\overline{c},\overline{r}):=(c'\circ\varphi,r'\circ\varphi)$, so that:

$$\Psi_*\left(Z_{(c,r)}\right)\ =\ Z_{(\overline{c},\overline{r})}\ .$$

  We are going to prove that $\Psi=\mathrm{Id}$. By Proposition 2.9, there exists $G\in\mathcal{D}^{(1)}$ such that $\Psi=\exp(G)$ and:

$$G\ =\ g_0(x,v)\frac{\partial}{\partial x}+g_1(x,v)\,z_1\frac{\partial}{\partial z_1}+g_2(x,v)\,z_2\frac{\partial}{\partial z_2}\ ,$$

  where $g_i\in\mathfrak{m}\subset\mathbb{C}[\![x,v]\!]$ for $i=1,2$ and $g_0\in\mathfrak{m}^2\subset\mathbb{C}[\![x,v]\!]$ is of order at least two. Since Ψ is fibered in x we deduce that $g_0=0$. Therefore, using the notation (2.6), we can write:

$$G\ =\ A(x,v)\,S(0,-1,1)+B(x,v)\,S(0,a_1,a_2)\ ,$$

where

$$\begin{cases} A = A_{i,j} x^i v^j \\ B = \sum_{\substack{i,j \geq 0 \\ i+j \geq 1}} B_{i,j} x^i v^j \end{cases} .$$

Let us prove that $A = B = 0$ (hence $G = 0$) so that $\Psi = \mathrm{Id}$. We consider the Jordan decompositions of $Z := Z_{(c,r)}$ and $\overline{Z} := Z_{(\overline{c},\overline{r})}$:

$$\begin{cases} Z = Z_S + Z_N & , Z_S \text{ semi-simple, } Z_N \text{ nilpotent, } [Z_S, Z_N] = 0 \\ \overline{Z} = \overline{Z}_S + \overline{Z}_N & , \overline{Z}_S \text{ semi-simple, } \overline{Z}_N \text{ nilpotent, } \left[\overline{Z}_S, \overline{Z}_N\right] = 0 \end{cases} .$$

By uniqueness of this decomposition we clearly have:

$$\begin{cases} Z_S = \overline{Z}_S = S(0, -\lambda, \lambda) \\ Z_N = xS(1, a_1, a_2) + c(v) S(0, -1, 1) + r(v) S(0, a_1, a_2) \\ \overline{Z}_N = xS(1, a_1, a_2) + \overline{c}(v) S(0, -1, 1) + \overline{r}(v) S(0, a_1, a_2) \end{cases} ,$$

and we also know that:

$$\Psi_* (Z) = \overline{Z} \quad \Rightarrow \quad \begin{cases} \Psi_* (Z_S) = \overline{Z}_S \\ \Psi_* (Z_N) = \overline{Z}_N \end{cases} .$$

Let us now consider the associated two-dimensional vector fields in the variables (x, v). In the "chart" (x, v) the vector field G is given by $F = B.S(0, a)$, with $a = a_1 + a_2$. Z and $\overline{Z}$ correspond respectively to:

$$\begin{aligned} Y &:= xS(1, a) + r(v) S(0, a) \quad , \\ \overline{Y} &:= xS(1, a) + \overline{r}(v) S(0, a) \quad . \end{aligned}$$

Thus we have $\exp(F)_* (Y) = \overline{Y}$. By Proposition 2.3 we derive

$$\exp(F)_* (Y) \quad = \quad Y + [F, Y] + \frac{1}{2!} [F, [F, Y]] + \ldots$$

and

$$r(v) S(0, a) + [F, Y] + \frac{1}{2!} [F, [F, Y]] + \ldots \quad = \quad \overline{r}(v) S(0, a) \quad . \tag{3.3}$$

We compute next

$$[F, Y] \quad = \quad \left\{ -x \left(\mathcal{L}_{S(1,a)} (B) \right) + B \left(\mathcal{L}_{S(0,a)} (r) \right) - r \left(\mathcal{L}_{S(0,a)} (B) \right) \right\} S(0, a)$$

and, setting

$$\begin{aligned} C^{(1)} (x, v) \quad := \quad & -x \left(\mathcal{L}_{S(1,a)} (B) \right) + B \left(\mathcal{L}_{S(0,a)} (r) \right) \\ & -r \left(\mathcal{L}_{S(0,a)} (B) \right) \quad , \end{aligned} \tag{3.4}$$

we obtain

$$[F,Y] \quad = \quad C^{(1)}(x,v)\,S(0,a) \quad .$$

Now, it is easy to see that for all $l \in \mathbb{N}$, $\mathrm{ad}_F^{\circ l}(Y)$ ca be written

$$\mathrm{ad}_F^{\circ l}(Y) \quad = \quad C^{(l)}(x,v)\,S(0,a) \quad ,$$

where $C^{(l)}$ is determined by the recursive relation:

$$C^{(l+1)}(x,v) \quad = \quad B(x,v)\left(\mathcal{L}_{S(0,a)}\left(C^{(l)}\right)\right) - C^{(l)}(x,v)\left(\mathcal{L}_{S(0,a)}(B)\right) \quad .$$

In particular, we see that for all $l \geq 2$, $C^{(l)}(x,0) = 0$. Equation (3.3) can now be rewritten:

$$r(v) + C^{(1)}(x,v) + \sum_{l\geq 2} C^{(l)}(x,v) \quad = \quad \overline{r}(v) \quad . \tag{3.5}$$

Let us set $r(v) = \sum_{k\geq 1} r_k v^k$ and $\overline{r}(v) = \sum_{k\geq 1} \overline{r}_k v^k$. Looking at terms independent of v in (3.5) (*i.e.* by taking $v = 0$), we see that $C^{(1)}(x,0) = 0$. Taking (3.4) into account we obtain that $\dfrac{\partial B(x,0)}{\partial x} = 0$. Since $\mathrm{ord}(B) \geq 1$ (by assumption) this means that $B(x,0) = 0$. Let us prove the properties $B_{i,k} = 0$ and $r_k = \overline{r}_k$ for all $i,j \in \mathbb{N}$ and $k \leq j$ by induction on $j \geq 0$.

- $j = 0$. This corresponds to the case described above: for all $i \geq 0$, $B_{i,0} = 0$ (and $r_0 = \overline{r}_0$).
- If the property holds at a rank $j \geq 0$, if we consider for all $i \geq 0$ terms of homogenous degree $(i+1, j+1)$ in (3.5), we obtain:

  $$(i + a(j+1))\,B_{i,j+1} = 0$$

  by induction, and because for all $l \geq 2$ the relation $C^{(l)}(x,0) = 0$ also holds. Since $a \notin \mathbb{Q}_{\leq 0}$ we have $B_{i,j+1} = 0$. On the other hand, if we look at terms of homogeneous degree $(0, j+1)$, we obtain: $r_{j+1} = \overline{r}_{j+1}$.

We conclude that $B = 0$, so that $F = 0$ and $r = \overline{r}$.

Finally, we have $G = A\mathcal{L}(0,-1,1)$. Taking the relation $\exp(G)_*(Z_N) = \overline{Z}_N$ into account, we have:

$$Z_N + [G, Z_N] + \frac{1}{2!}[G,[G,Z_N]] + \ldots \quad = \quad \overline{Z}_N$$

if and only if

$$c(v)\,S(0,-1,1) + [G,Z_N] + \frac{1}{2!}[G,[G,Z_N]] + \ldots \quad = \quad \overline{c}(v)\,S(0,-1,1) \quad .$$

Let us compute $[G, Z_N]$:

$$[G, Z_N] = -\left\{x\mathcal{L}_{S(1,a_1,a_2)}(A) + r(v)\,\mathcal{L}_{S(0,a_1,a_2)}(A)\right\} S(0,-1,1) \quad .$$

All other Lie brackets vanish. There only remains

$$c(v) - x\mathcal{L}_{S(1,a_1,a_2)}(A) - r(v)\,\mathcal{L}_{S(0,a_1,a_2)}(A) = \overline{c}(v) \quad ,$$

which becomes a system of identities between terms of same degree:

$$\begin{cases} c_j - \sum_{k=0}^{j-k_1} akA_{0,k}r_{j-k} = \overline{c}_j & , j \geq 0 \\ (i+aj)\,A_{i,j} + \sum_{k=0}^{j-k_1} akA_{i+1,k}r_{j-k} = 0 & , i \geq 0,\, j \geq 0 \end{cases} \quad .$$

Once again, we prove by induction on $j \geq 0$ that for all $i \geq 0$ and all $0 \leq k \leq j$ the relations $A_{i,k} = 0$ and $c_k = \overline{c}_k$ hold. Thus $A = 0$ and $c = \overline{c}$.

As a conclusion $\Psi = \mathrm{Id}$ and $(c, r) = (\overline{c}, \overline{r})$, so that $\Phi = \mathrm{D}_0\Phi = \mathrm{diag}(1, \theta_1, \theta_2)$ and $(c, r) := (c' \circ \varphi, r' \circ \varphi)$ where $\varphi \ : \ v \mapsto (\theta_1\theta_2)\,v$.

□

3.4 Fibered isotropies of the formal normal form

Looking back at the uniqueness proof in the previous paragraph, we immediately obtain all formal fibered isotropies of the normal form given by Theorems 1.3 and 1.12. We recall that an isotropy of a vector field is a self-conjugacy. For a vector field $X \in \mathcal{D}^{(1)}$, we set:

$$\widehat{\mathrm{Isot}}_{\mathrm{fib}}(X) := \left\{\Phi \in \widehat{\mathrm{Diff}}_{\mathrm{fib}} \mid \Phi_*(X) = X\right\} \quad .$$

Proposition 3.2. *Consider a normal form of* $\widehat{\mathcal{SN}}_{\mathrm{nd}}$

$$Z = x^2\frac{\partial}{\partial x} + (-\lambda + a_1x + c_1(y_1y_2))\,y_1\frac{\partial}{\partial y_1} + (\lambda + a_2x + c_2(y_1y_2))\,y_2\frac{\partial}{\partial y_2} \ ,$$

with parameters $(\lambda, a_1, a_2, c_1, c_2) \in \mathcal{P}$. *Then:*

$$\widehat{\mathrm{Isot}}_{\mathrm{fib}}(Z) = \left\{\mathrm{diag}(1, \theta_1, \theta_2)\,,\, (\theta_1, \theta_2) \in (\mathbb{C}^*)^2 \,\middle|\, (c_1, c_2)(\theta_1\theta_2 v) = (c_1, c_2)(v)\right\} \quad .$$

Remark 3.3. If $(c_1, c_2) \neq (0, 0)$ the condition $c_i(\theta_1\theta_2 v) = c_i(v)$ for each $i \in \{1, 2\}$ is equivalent to requiring that each c_i lie in $\mathbb{C}[\![v^q]\!]$, for some $q \in \mathbb{N}_{>0}$, and that $\theta_1\theta_2$ be a q^{th} root of unity.

This proposition has for immediate consequence the (almost) uniqueness of the normalizing conjugacy $\Phi \in \widehat{\mathrm{Diff}}$ in Theorem 1.3. More precisely:

Corollary 3.4. *Let $Y \in \widehat{\mathcal{SN}}_{\mathrm{nd}}$ be a non-degenerate doubly-resonant saddle-node such that $\mathrm{D}_0 Y = \mathrm{diag}\,(0, -\lambda, \lambda)$, with $\lambda \neq 0$. Then there exists a unique fibered diffeomorphism $\Phi \in \widehat{\mathrm{Diff}}_{\mathrm{fib}}$ tangent to the identity such that:*

$$\begin{aligned}\Phi_*(Y) \quad = \quad & x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(v))\, y_1 \frac{\partial}{\partial y_1} \\ & + (\lambda + a_2 x + c_2(v))\, y_2 \frac{\partial}{\partial y_2} \quad , \end{aligned} \tag{3.6}$$

where we put $v := y_1 y_2$. Here, c_1, c_2 belong to $\langle v \rangle = v\mathbb{C}\,[\![v]\!]$ and $a_1, a_2 \in \mathbb{C}$ are such that $a_1 + a_2 = \mathrm{res}\,(Y)$.

Definition 3.5. Let $Z \in \widehat{\mathcal{SN}}_\omega$. We denote by $\widehat{\mathrm{Isot}}_\omega(Z)$ the subgroup of elements $\Phi \in \widehat{\mathrm{Diff}}_\omega$ such that $\Phi_*(Z) = Z$.

Proposition 3.6. *Let $(\lambda, a_1, a_2) \in \mathbb{C}^* \times \mathbb{C}^2$ such that $a_1 + a_2 = 1$, and $c \in v\mathbb{C}\,[\![v]\!]$ with $v = y_1 y_2$. Consider*

$$Z \quad = \quad x^2 \frac{\partial}{\partial x} + (-(\lambda + c(v)) + a_1 x)\, y_1 \frac{\partial}{\partial y_1} + (\lambda + c(v) + a_2 x)\, y_2 \frac{\partial}{\partial y_2} \quad .$$

Then:

$$\widehat{\mathrm{Isot}}_\omega(Z) \quad = \quad \left\{ \mathrm{diag}\left(1, \alpha, \frac{1}{\alpha}\right),\ \alpha \in \mathbb{C} \backslash \{0\} \right\} \simeq \mathbb{C} \backslash \{0\} .$$

4 Applications to Painlevé equations

In this section we investigate the study of the irregular singularity at infinity in the first Painlevé equation

$$(P_I) \qquad \frac{\mathrm{d}^2 z_1}{\mathrm{d}t^2} \quad = \quad 6 z_1^2 + t$$

in terms of Theorem 1.12. More precisely, we are going to explain that the formal invariant $c \in \mathbb{C}\,[\![v]\!]$ of a doubly-resonant, transversally symplectic saddle-node $Y \in \widehat{\mathcal{SN}}_\omega$ is in fact a germ of an analytic function at the origin, whenever Y is analytic at the origin (and not merely a formal vector field). Moreover, we show how to compute recursively this invariant in some specific cases, including Painlevé equations.

4.1 Asymptotically Hamiltonian vector fields

We deal here with the case of **asymptotically Hamiltonian** vector fields.

Definition 4.1.

- We say that a formal vector field X in $\left(\mathbb{C}^2,0\right)$ is **orbitally linear** if

$$X=U\left(\mathbf{y}\right)\left(\lambda_1 y_1\frac{\partial}{\partial y_1}+\lambda_2 y_2\frac{\partial}{\partial y_2}\right)\ ,$$

  for some unity $U\left(\mathbf{y}\right)\in\mathbb{C}\left[\!\left[\mathbf{y}\right]\!\right]^{\times}$ (*i.e.* $U\left(0,0\right)\neq 0$) and $\left(\lambda_1,\lambda_2\right)\in\mathbb{C}^2$.

- We say that a formal (*resp.* germ of an analytic) vector field X in $\left(\mathbb{C}^2,0\right)$ is formally (*resp.* analytically*)* **orbitally linearizable** if X is formally (*resp.* analytically) conjugate to an orbitally linear vector field.

- We say that a doubly-resonant saddle-node $Y\in\widehat{\mathcal{SN}}$ is formally/analytically **asymptotically orbitally linearizable** if the formal/analytic vector field $Y_{|\{x=0\}}$ in $\left(\mathbb{C}^2,0\right)$ is formally/analytically orbitally linearizable.

Remark 4.2.

1. If a vector field X is analytic at the origin of $\mathbb{C}^2$ and has two opposite eigenvalues, it follows from a classical result of Brjuno (see [14]), that X is analytically orbitally linearizable if and only if it is formally orbitally linearizable.

2. The fact of being orbitally linearizable is naturally invariant under orbital equivalence, and then, by (almost) uniqueness of c_1,c_2 in Theorem 1.6, if $Y\in\widehat{\mathcal{SN}}_{\mathrm{nd}}$ is asymptotically linearizable, then its formal invariants c_1,c_2 satisfy $c_1+c_2=0$. In this case, we write $c:=c_2=-c_1$.

The two remarks above imply the following corollary.

Corollary 4.3. *Let $Y\in\widehat{\mathcal{SN}}_{\mathrm{nd}}$ be a doubly-resonant saddle-node asymptotically orbitally linearizable such that $Y_0:=Y_{|\{x=0\}}$ be a germ of an analytic vector field in $\left(\mathbb{C}^2,0\right)$. Then, there exists $\Phi\in\widehat{\mathrm{Diff}}_{\mathrm{fib}}$ such that $\Phi_{|\{x=0\}}$ be a germ an analytic diffeomorphism in $\left(\mathbb{C}^2,0\right)$ and:*

$$\Phi_*\left(Y\right)\quad=\quad x^2\frac{\partial}{\partial x}+\left(-\lambda+a_1x-c\left(v\right)\right)y_1\frac{\partial}{\partial y_1}+\left(\lambda+a_2x+c\left(v\right)\right)y_2\frac{\partial}{\partial y_2}\quad ,$$

where we put $v:=y_1y_2$. Here, $c\left(v\right)\in v\mathbb{C}\left\{v\right\}$ is a germ of an analytic function vanishing at the origin, and $a_1,a_2\in\mathbb{C}$ are such that $a_1+a_2=\mathrm{res}\left(Y\right)$. Moreover, Φ is unique up to linear transformations.

It is important to notice that the following property holds.

Proposition 4.4. *If $Y\in\widehat{\mathcal{SN}}_{\omega}$ is doubly-resonant transversally Hamiltonian saddle-node, then Y is asymptotically orbitally linearizable.*

Proof. The facts that $\mathcal{L}_Y\left(\omega\right)\in\left\langle\mathrm{d}x\right\rangle$ and $\mathcal{L}_Y\left(x\right)=x^2$ imply that $a_1+a_2=1$ and then:

$$\mathcal{L}_Y\left(\mathrm{d}y_1\wedge\mathrm{d}y_2\right)\quad=\quad x\left(\mathrm{d}y_1\wedge\mathrm{d}y_2\right)+\left\langle\mathrm{d}x\right\rangle\quad .$$

Consequently, if we denote $Y_0 := Y_{|x=0}$ the restriction of Y to the invariant hypersurface $\{x=0\}$, we have:

$$\mathcal{L}_{Y_0}(\mathrm{d}y_1 \wedge \mathrm{d}y_2) = 0 \ .$$

This means that Y_0 is a Hamiltonian vector field, *i.e.* there exists $H(\mathbf{y}) \in \mathbb{C}[\![\mathbf{y}]\!]$ such that:

$$Y_0(y_1,y_2) = -\frac{\partial H}{\partial y_2}(y_1,y_2)\frac{\partial}{\partial y_1} + \frac{\partial H}{\partial y_1}(y_1,y_2)\frac{\partial}{\partial y_2} \ .$$

Possibly by performing a linear change of coordinate, we can assume that $H(\mathbf{y}) \in \lambda y_1 y_2 + \mathfrak{m}^3$, therefore we can write:

$$Y = x^2\frac{\partial}{\partial x} + \left(-\frac{\partial H}{\partial y_2} + xF_1(x,\mathbf{y})\right)\frac{\partial}{\partial y_1} + \left(\frac{\partial H}{\partial y_1} + xF_2(x,\mathbf{y})\right)\frac{\partial}{\partial y_2} \ ,$$

where $F_1, F_2 \in \mathbb{C}[\![x,\mathbf{y}]\!]$ vanish at the origin. If we define $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ and $\nabla H := {}^t(\mathrm{D}H)$, then $Y_{|\{x=0\}} = J\nabla H$. According to the Morse lemma for holomorphic functions, there exists an analytic change of coordinates $\varphi \in \widehat{\mathrm{Diff}}$ in $(\mathbb{C}^2,0)$ tangent to the identity such that $\widetilde{H}(\mathbf{y}) := H(\varphi^{-1}(\mathbf{y})) = y_1y_2$. Let us now recall a trivial result from linear algebra.

Fact. *Let* $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$, *and* $P \in M_2(\mathbb{C})$. *Then,* $PJP^{\mathrm{t}} = det(P)\, J$.

We deduce the next result.

Lemma. *Let* $H \in \mathfrak{m}^2 \subset \mathbb{C}[\![\mathbf{y}]\!]$, $Y_0 := J\nabla H$ *the associated Hamiltonian vector field in* $\mathbb{C}^2$ *(for the standard symplectic form* $dy_1 \wedge dy_2$*), and an analytic diffeomorphism near the origin denoted by* φ. *Then:*

$$\varphi_*(Y_0) \ := \ \left(\mathrm{D}\varphi \circ \varphi^{-1}\right)\cdot\left(Y_0 \circ \varphi^{-1}\right) = \det\left(\mathrm{D}\varphi \circ \varphi^{-1}\right) J\nabla\widetilde{H} \ ,$$

where $\widetilde{H} := H \circ \varphi^{-1}$.

As a conclusion,the previous lemma shows that Y is asymptotically orbitally linearizable. □

The next property is a straightforward consequence of Corollary 4.3, Proposition 4.4 and Theorem 1.12.

Corollary 4.5. *Let* $Y \in \widehat{\mathcal{SN}}_\omega$ *be a transversally Hamiltonian doubly-resonant saddle-node. Then, there exists a transversally symplectic diffeomorphism* $\Phi \in \widehat{\mathrm{Diff}}_\omega$ *such that* $\Phi_{|\{x=0\}}$ *be a germ an analytic diffeomorphism in* $(\mathbb{C}^2,0)$ *and:*

$$\Phi_*(Y) = \ x^2\tfrac{\partial}{\partial x} + (-\lambda + a_1x - c(v))\,y_1\tfrac{\partial}{\partial y_1} + (\lambda + a_2x + c(v))\,y_2\tfrac{\partial}{\partial y_2} \ . \quad (4.1)$$

where we put $v := y_1y_2$. *Here,* $c(v) \in v\mathbb{C}\{v\}$ *is a germ of an analytic function vanishing at the origin, and* $a_1, a_2 \in \mathbb{C}$ *are such that* $a_1 + a_2 = \mathrm{res}(Y) = 1$. *Moreover,* Φ *is unique up to linear symplectic transformations, and:*

$$\left(\Phi_{|\{x=0\}}\right)^*(dy_1 \wedge dy_2) \ = \ dy_1 \wedge dy_2 \ .$$

4.2 Periods of the Hamiltonian on $\{x = 0\}$

From now on, we consider a vector field

$$Y = x^2 \frac{\partial}{\partial x} + \left(\left(-\frac{\partial H}{\partial y_2} + x F_1 (x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\frac{\partial H}{\partial y_1} + x F_2 (x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} \right) ,$$

with $H(\mathbf{y}) = \lambda y_1 y_2 + \underset{\mathbf{y} \to 0}{O} \left(\|\mathbf{y}\|^3 \right)$ analytic at the origin of $\mathbb{C}^2$, and $F_1, F_2 \in \mathbb{C}[\![x, \mathbf{y}]\!]$ vanishing at the origin. Let us consider the restriction $Y_0 := Y_{|\{x=0\}}$: it is an analytic Hamiltonian vector field in $\left(\mathbb{C}^2, 0\right)$:

$$Y_0 = -\frac{\partial H}{\partial y_2} \frac{\partial}{\partial y_1} + \frac{\partial H}{\partial y_1} \frac{\partial}{\partial y_2} .$$

We fix a small polydisc $\mathbf{D}(0, \mathbf{r}) \subset \mathbb{C}^2$ on which H is analytic with $\mathbf{r} = (r_1, r_2)$. The leaves of the foliation defined by Y_0 in $\mathbf{D}(0, \mathbf{r})$ are given by the level curves $L_a := \{H = a\} \cap \mathbf{D}(0, \mathbf{r})$, $a \in \mathrm{D}(0, r)$, with $r > 0$ small enough. The Morse Lemma for holomorphic functions tells us that L_a is topologically a cylinder for $a \neq 0$, and r,r_1, r_2 small enough. Thus we can consider a generator γ_a of the first homology group of L_a. We also consider a time-form for Y_0, which is a meromorphic 1-form τ_{Y_0} in $\mathbf{D}(0, \mathbf{r})$ with a unique pole at the origin and such that $\tau_{Y_0} \cdot (Y_0) = 1$. For instance, take $\tau_{Y_0} = -\frac{\mathrm{d}y_1}{\frac{\partial H}{\partial y_2}}$.

Now we define the associated period map:

$$\begin{array}{rcl} T_H : \mathrm{D}(0, r) \setminus \{0\} & \longrightarrow & \mathbb{C} \\ a & \longmapsto & T_H(a) := \frac{1}{2i\pi} \oint_{\gamma_a} \tau_{Y_0} . \end{array}$$

This mapping is a well-defined meromorphic function of $a \in \mathrm{D}(0, r)$.

Proposition 4.6. *For $r > 0$ small enough, and $a \in D(0, r) \setminus \{0\}$, $T_{Y_0}(a)$ only depends on the class of γ_a in $H_1(L_a, \mathbb{Z})$. In other words, if τ'_{Y_0} is another time-form of Y_0 and γ'_a is any loop in L_a homologous to γ_a, then*

$$\oint_{\gamma_a} \tau_{Y_0} = \oint_{\gamma'_a} \tau'_{Y_0} .$$

Proof. The fact that this quantity does not depend on a specific choice of a representative of γ_a in its homology class comes from Stokes Theorem. The fact that it does not depend on the choice of a specific time-form comes from the fact that γ_a lies in a leaf of the foliation generated by Y_0. If

$$\begin{array}{rcl} \gamma_a : [0, 1] & \rightarrow & L_a \\ t & \mapsto & (\gamma_{a,1}(t), \gamma_{a,2}(t)) , \end{array}$$

then $\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_a)(t) = v_a(t) Y_0(\gamma_a(t))$, where

$$v_a(t) = \frac{1}{\left(-\frac{\partial H}{\partial y_2}(\gamma_a(t))\right)} \frac{\mathrm{d}\gamma_{a,1}(t)}{\mathrm{d}t} = \frac{1}{\left(\frac{\partial H}{\partial y_1}(\gamma_a(t))\right)} \frac{\mathrm{d}\gamma_{a,2}(t)}{\mathrm{d}t} .$$

Then:

$$\begin{aligned}
\oint_{\gamma_a} \tau_{Y_0} &= \int_0^1 \tau_{Y_0}\left(\gamma_a(t)\right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_a\right)(t)\right) \mathrm{d}t \\
&= \int_0^1 \tau_{Y_0}\left(\gamma_a(t)\right) \cdot \left(v_a(t) Y_0\left(\gamma_a(t)\right)\right) \mathrm{d}t \\
&= \int_0^1 v_a(t)\, \mathrm{d}t
\end{aligned}$$

since $\tau_{Y_0} \cdot (Y_0) = 1$. □

Definition 4.7. We call T_H the period map of H near the origin.

Now, consider a germ of an analytic diffeomorphism Ψ fixing the origin of $\mathbb{C}^2$. Then:

$$\begin{aligned}
T_H(a) &= \tfrac{1}{2i\pi} \oint_{\gamma_a} \tau_{Y_0} \\
&= \tfrac{1}{2i\pi} \oint_{\Psi^{-1}(\gamma_a)} \Psi^*\left(\tau_{Y_0}\right) \quad .
\end{aligned}$$

Notice that if we write $X_0 := \left(\Psi^{-1}\right)_*(Y_0)$ and $\tau_{X_0} := \Psi^*\left(\tau_{Y_0}\right)$, then:

$$\tau_{X_0} \cdot (X_0) = \left(\Psi^*\left(\tau_{Y_0}\right)\right) \cdot \left(\left(\Psi^{-1}\right)_*(Y_0)\right) = \tau_{Y_0} \cdot (Y_0) = 1 \; .$$

Now, let us take $\Psi^{-1} = \Phi_{|\{x=0\}}$ as in Corollary 4.5 such that

$$X_0 = (\lambda + c(v))\left(-y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}\right) \; ,$$

with $v = y_1 y_2$, $c \in \mathbb{C}\{v\}$ and $c(0) = 0$. Then $\tilde{\gamma}_a := \Psi^{-1}(\gamma_a) = \Phi_{|\{x=0\}}(\gamma_a)$ is a loop generating the homology of the leaf $\Phi_{|\{x=0\}}(L_a)$.

Consider $h := H \circ \Psi$ near the origin. Then, $\Phi_{|\{x=0\}}(L_a) = \{h = a\}$ in a neighborhood of the origin. Notice that h depends in fact only on $v = y_1 y_2$, and $h(v) = \lambda v + \underset{|v|\to 0}{o}(|v|)$. Since $\lambda \neq 0$, the inverse function theorem ensures the existence of an analytic function $g \in \mathbb{C}\{v\}$ such that $g(0) = 0$ and $h \circ g(v) = g \circ h(v) = v$ in a neighborhood of 0. Thus, $\{h(v) = a\} = \{v = g(a)\}$. Consequently, taking for instance $\tau_{X_0} = -\dfrac{\mathrm{d}y_1}{y_1(\lambda + c(v))}$, we see that:

$$\begin{aligned}
T_H(a) &= \tfrac{1}{2i\pi} \oint_{\tilde{\gamma}_a} \tau_{X_0} \\
&= \frac{1}{2i\pi} \frac{1}{\lambda + c(g(a))} \oint_{\tilde{\gamma}_a} -\frac{\mathrm{d}y_1}{y_1} \\
&= \frac{-1}{\lambda + c(g(a))}
\end{aligned}$$

according to the orientation chosen for γ_a. In particular, we see that T_H is analytic at the origin, and T_H can be extend at 0 by $\frac{-1}{\lambda}$.

The fact that $\Phi_{|\{x=0\}}$ satisfies

$$\left(\Phi_{|\{x=0\}}\right)^* (\mathrm{d}y_1 \wedge \mathrm{d}y_2) = \mathrm{d}y_1 \wedge \mathrm{d}y_2$$

implies that $\det\left(\Phi_{|\{x=0\}}\right) = 1$, so that

$$X_0 = -\frac{\partial h}{\partial y_2}\frac{\partial}{\partial y_1} + \frac{\partial h}{\partial y_1}\frac{\partial}{\partial y_2}$$

and

$$\frac{\mathrm{d}h}{\mathrm{d}v} = \lambda + c(v) \quad .$$

As a consequence, for $a = h(v)$, we have the following relation:

$$\frac{\mathrm{d}h}{\mathrm{d}v}(v) . T_H(h(v)) = -1 \quad .$$

If we consider the antiderivative S_H of T_H such that $S_H(0) = 0$, we have

$$S_H(h(v)) = -v \quad ,$$

and in particular

$$S_H = -g \; .$$

Let us summarize this study in the following proposition.

Proposition 4.8. *Let*

$$Y_0 = -\frac{\partial H}{\partial y_2}\frac{\partial}{\partial y_1} + \frac{\partial H}{\partial y_1}\frac{\partial}{\partial y_2}$$

be the restriction on $\{x=0\}$ of a transversally Hamiltonian doubly-resonant saddle-node $Y \in \widehat{\mathcal{SN}}_\omega$, where $H(\mathbf{y}) = \lambda y_1 y_2 + \underset{\mathbf{z}\to 0}{o}\left(\|\mathbf{y}\|^2\right)$ is analytic at the origin of $\mathbb{C}^2$. Consider its unique transversally Hamiltonian normal form

$$X = x^2\frac{\partial}{\partial x} + \left(-(\lambda + c(v)) + a_1 x\right) y_1 \frac{\partial}{\partial y_1} + \left(\lambda + c(v) + a_2 x\right) y_2 \frac{\partial}{\partial y_2}$$

given by Theorem 1.12. Consider the period map T_H as defined above. Then the following holds:

1. *c is the germ of an analytic function at the origin.*
2. *T_H defines the germ of an analytic function in a neighborhood of $0 \in \mathbb{C}^2$, such that $T_H(0) = \frac{-1}{\lambda}$.*
3. *If S_H is the primitive of T_H such that $S_H(0) = 0$, then $(-S_H)$ is invertible (for the composition), and its inverse h satisfy:*

$$\frac{dh}{dv}(v) = \lambda + c(v) \quad .$$

The conclusion is that if one is able to compute the period map of the original Hamiltonian vector field on $\{x=0\}$, then one can compute the formal invariant c in the normal form given in Theorem 1.12, which is in fact even analytic in this case.

Remark 4.9. The Hamiltonian function $h(y_1y_2)=\lambda y_1y_2+\int^{y_1y_2}c(v)\,\mathrm{d}v$ is in fact the *symplectic normal form* of the original Hamiltonian function $H(y_1,y_2)=\lambda y_1y_2+\underset{\mathbf{z}\to 0}{o}\left(\|\mathbf{y}\|^2\right)$, as described in [7] (section 2.7).

4.3 Example: the case of the first Painlevé equation

In the case of the first Painlevé equation, in appropriate coordinates, we are working with the Hamiltonian

$$H(y_1,y_2)=\frac{1}{5}\left(-2y_2^2+24\zeta y_1^2+8y_1^3\right)$$

where $\zeta=\frac{i}{\sqrt{6}}$, according to equation (1.6). These are not the system of coordinates which diagonalizes the linear part of the vector field, but the value of the period does not changes by symplectic changes of coordinates (those which preserve $\mathrm{d}y_1\wedge\mathrm{d}y_2$). Now, if we fix $a\neq 0$ with $|a|$ small enough and look at the level curve $\{H=a\}$ near the origin in $\mathbb{C}^2$, we can compute the associated period:

$$\begin{aligned}T_H(a) &= \frac{1}{2i\pi}\oint_{\gamma_a}\frac{5\mathrm{d}y_1}{-4y_2}\\ &= \frac{1}{2i\pi}\oint_{\gamma_{a,1}}\frac{5\mathrm{d}y_1}{-4\sqrt{12\zeta y_1^2+4y_1^3-\frac{5}{2}a}}\,,\end{aligned}$$

where $\gamma_{a,1}$ is the component of γ_a with respect to $\frac{\partial}{\partial y_1}$.

Remark 4.10. The period $T_H(a)$ is one of the periods of the Weierstrass function $\wp$ associated to the cubic

$$H(y_1,y_2)\ =\ a$$

(see *e.g.* [5], [1]). To compute it we can chose for instance

$$\begin{aligned}\gamma_{a,1}:[0,2\pi] &\longrightarrow \mathbb{C}\\ t &\longmapsto \rho_a e^{it}\end{aligned}$$

where $\rho_a>0$ is such that $\left|12\zeta y_1^2+4y_1^3\right|>\frac{5|a|}{2}$, for all $y_1=\gamma_{a,1}(t)$, $t\in[0,2\pi]$. Now we write:

$$\begin{aligned}\frac{1}{-4\sqrt{12\zeta y_1^2+4y_1^3-\frac{5}{2}a}} &= \frac{\sqrt{2}}{-4\sqrt{24\zeta y_1^2+8y_1^3}}\frac{1}{\sqrt{1-\frac{5a}{24\zeta y_1^2+8y_1^3}}}\\ &= \frac{\sqrt{2}}{-4\sqrt{24\zeta y_1^2+8y_1^3}}\sum_{k\geq 0}\binom{\frac{-1}{2}}{k}\left(\frac{5a}{24\zeta y_1^2+8y_1^3}\right)^k .\end{aligned}$$

As we have normal convergence, we can swap the order of summation and integration:

$$
\begin{aligned}
T_H(a) &= \frac{5}{2i\pi}\oint_{\gamma_{a,1}} \frac{\sqrt{2}}{-4\sqrt{24\zeta y_1^2+8y_1^3}} \sum_{k\geq 0} \binom{\frac{-1}{2}}{k} \left(\frac{5a}{24\zeta y_1^2+8y_1^3}\right)^k \mathrm{d}y_1 \\
&= \frac{-5\sqrt{2}}{8i\pi} \sum_{k\geq 0} \binom{\frac{-1}{2}}{k} 5^k \left(\oint_{\gamma_{a,1}} \left(24\zeta y_1^2+8y_1^3\right)^{-\left(k+\frac{1}{2}\right)} \mathrm{d}y_1\right) a^k \quad .
\end{aligned}
$$

Notice that $y_1 \mapsto \left(24\zeta y_1^2+8y_1^3\right)^{\left(k+\frac{1}{2}\right)}$ is in fact analytic in a neighborhood of the origin, with a zero of order $2k+1$. Hence we can compute the integral above using the residue theorem. As we have

$$
\left(24\zeta y_1^2+8y_1^3\right)^{-\left(k+\frac{1}{2}\right)} = (24\zeta)^{-\left(k+\frac{1}{2}\right)} y_1^{-(2k+1)} \sum_{j\geq 0} \binom{-\left(k+\frac{1}{2}\right)}{j} \left(\frac{8}{24\zeta}\right)^j y_1^j \quad ,
$$

we see that the associated residue at 0 is equal to $8^{2k}(24\zeta)^{-\left(3k+\frac{1}{2}\right)} \binom{-\left(k+\frac{1}{2}\right)}{2k}$, so that

$$
T_H(a) = \sum_{k\geq 0} T_{H,k} a^k
$$

with:

$$
T_{H,k} = -5^{k+1} \binom{\frac{-1}{2}}{k} \binom{-\left(k+\frac{1}{2}\right)}{2k} 8^{-(k+1)} (3\zeta)^{-\left(3k+\frac{1}{2}\right)} \quad .
$$

Using notations of Proposition 4.8 we have:

$$
S_H(a) = \sum_{k\geq 0} \frac{T_{H,k}}{k+1} a^{k+1} = \sum_{k\geq 1} S_{H,k} a^k
$$

with $S_{H,k} = \frac{T_{H,k-1}}{k}$ for $k \geq 1$. Since $S_H(0) = 0$ and $\frac{\mathrm{d}S_H}{\mathrm{d}a}(0) = T_H(0) \neq 0$, the mapping $(-S_H)$ is invertible for the composition and we can compute recursively its inverse (denoted by h):

$$
h(v) = \sum_{k\geq 1} h_k v^k \quad .
$$

For all $k \geq 1$, the coefficient h_k is uniquely determined by the coefficients $S_{H,j}$, $j \leq k$. Finally, we have

$$
\lambda + c(v) = \frac{\mathrm{d}h}{\mathrm{d}v}(v) = \sum_{k\geq 0} (k+1) h_{k+1} v^k = \lambda + \sum_{k\geq 1} c_k v^k \quad .
$$

As a conclusion, the jet of order k of T_H gives us the jet of order k of c. After computations performed with Maple, we obtain for instance:

$$\begin{aligned}
\lambda &= \frac{8\sqrt{3\zeta}}{5} = \frac{4\cdot 2^{\frac{3}{4}}\cdot 3^{\frac{1}{4}}}{5}e^{\frac{i\pi}{4}} \\
c_1 &= 3 \\
c_2 &= 9+\frac{167\cdot 2^{\frac{1}{4}}\cdot 3^{\frac{3}{4}}}{96}e^{\frac{3i\pi}{4}} \\
c_3 &= 16+\frac{31837\sqrt{6}}{6912}i+\frac{5}{2}\cdot 2^{\frac{1}{4}}\cdot 3^{\frac{1}{4}}\cdot e^{\frac{3i\pi}{4}} \quad .
\end{aligned}$$

One can in fact compute any finite jet of c.

Remark 4.11. Similar computations can be performed for any Hamiltonian of the form $H(y_1,y_2)=\beta y_2^2+\alpha y_1^2+f(y_1)$, where $\alpha,\beta\in\mathbb{C}\backslash\{0\}$ and $f\in\mathbb{C}\{y_1\}$.

5 Acknowledgement

I would like to express my gratitude to my supervisors Daniel Panazzolo and Loïc Teyssier for their useful comments and suggestions. The many discussions we had played an important role in the improvement of this paper.

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