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Consistency of likelihood estimation for Gibbs point processes

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Abstract

Strong consistency of the maximum likelihood estimator (MLE) for parametric Gibbs point process models is established. The setting is very general and includes pairwise pair potentials, finite and infinite multibody interactions and geometrical interactions, where the range can be finite or infinite. The Gibbs interaction may depend linearly or non-linearly on the parameters, a particular case being hardcore parameters and interaction range parameters. As important examples, we deduce the consistency of the MLE for all parameters of the Strauss model, the hardcore Strauss model, the Lennard-Jones model and the area-interaction model.

Keywords. Parametric estimation ; Variational principle ; Strauss model ; Lennard-Jones model ; Area-interaction model

1 Introduction

Gibbs point processes are popular and widely used models in spatial statistics to describe the repartition of points or geometrical structures in space. They initially arose from statistical physics where they are models for interacting continuum particles, see for instance [22]. They are now used in as different domains as astronomy, biology, computer science, ecology, forestry, image analysis and materials science. The main reason is that Gibbs point processes provide a clear interpretation of the interactions between the points, such as attraction or repulsion depending on their interdistance. We refer to [2], [12], [18] and [25] for classical text books on spatial statistics and stochastic geometry, including examples and applications of Gibbs point processes.

Assuming a parametric form of the Gibbs interaction, the natural method to estimate the parameters is likelihood inference. A practical issue however is that the likelihood depends on an intractable normalizing constant, called the partition function in the statistical physics literature, that has to be approximated. Some sparse data approximation were first proposed, e.g. in [20], before simulation-based methods have been developed [10]. To avoid the latter approximation, other estimation methods have been introduced, including pseudolikelihood and moments based methods, see the books cited above. With modern computers, the approximation of the partition function is no longer an important issue and maximum likelihood estimation for spatial data is feasible and widely used in practice.

From a theoretical point of view, very few is known about the asymptotic properties of the maximum likelihood estimator (MLE) for Gibbs point processes. The asymptotic here means that the window containing the point pattern increases to the whole space $\mathbb{R}^d$. It is commonly believed that the MLE is consistent and more efficient, at least asymptotically, than the other estimation methods. In 1992, S. Mase [16] proved that the MLE of regular parameters in a certain class of interactions is uniform locally asymptotic normal, thus confirming the latter belief. However his result, relying on strong cluster estimates, is established for Gibbs measure generating very sparse

point patterns, which implies restrictive conditions on the parameter space. In the same period, J. Jensen [13] proved the asymptotic normality of the MLE under the Dobrushin uniqueness region, where the Gibbs process satisfies mixing properties. Here again, as noticed in [10], this assumption implies strong restrictitions on the parameter space. Without these conditions, phase transition may occur (the Gibbs measure is not unique) and some long-range dependence phenomena can appear. The MLE might then exhibit a non standard asymptotic behavior, in the sense that the rate of convergence might differ from the standard square root of the size of the window and the limiting law might be non-gaussian. This has been already noticed in the discrete case of Gibbs interactions on a lattice, see for instance [21] for the Ising model, but no theoretical studies confirm this behavior in the continuous case of Gibbs point processes on $\mathbb{R}^d$. We do not address this question in the present paper.

Regarding the basic question of consistency, it is conjectured to hold in a very general setting but no proof were so far available in the continuous case. This is in contrast with the discrete case where consistency is established for most standard parametric models, regardless of the occurrence of phase transition, see [15] and [11]. In fact, the continuous case is more challenging in that most parametric Gibbs models involve irregular parameters as hardcore parameters (controlling the support of the measure) or interaction range parameters. This results in a discontinuous likelihood contrast function, even asymptotically, and some further technical difficulties (for instance the true unknown Gibbs measure is not absolutely continuous with respect to the Gibbs measure associated to an estimation of the hardcore parameter). In an unpublished manuscript [17], S. Mase addressed the consistency of the MLE for superstable and regular pairwise interactions (a formal definition will be given later). His main tool was the variational principle for Gibbs processes, following the initial idea developed in the discrete case in [15]. He restricted his study to pairwise interactions that are linear in their parameters, which yields a convex contrast function. His result do not imply any restriction on the parameter space, thus including the possibility of phase transition, but the parametric interactions considered in [17] remain nonetheless rather restrictive and do not include hardcore or interaction range parameters.

We prove in this paper that the MLE is strongly consistent for a wide class of stationary Gibbs interactions, without any major restriction on the parameter space. Our assumptions include finite and infinite-body interactions with finite or infinite range, and we do not assume any continuity with respect to the parameters. Our result covers in particular the consistency of the MLE of all parameters for the Strauss model (including the range of interaction), the hardcore Strauss model (including the hardcore parameter), the Lennard-Jones model (including the exponent parameters), and the area-interaction process (including the radius of balls). Our main tools are the variational principle as in [15] and [17], a minimum contrast result in presence of discontinuities, and the control of physical quantities as the pressure and the mean energy with respect to the parameters.

The remainder of the paper is organized as follows. Parametric Gibbs point processes and the MLE procedure are described in Section 2. Section 3 contains our main result, namely minimal conditions on the Gibbs interaction to ensure strong consistency of the MLE. While the latter result is established in a very general setting, we present in Section 4 standard families of models where our main result applies. Specifically, we deal with finite range pair potentials with or without hardcore (including the Strauss model and the hardcore Strauss model), infinite range pair potentials (including the Lennard-Jones model), and infinite-body interactions (including the area-interaction model). Section 5 contains the proofs of our results.

2 Gibbs point processes and the MLE

2.1 State space and reference measure

We consider the continuous space $\mathbb{R}^d$ of arbitrary dimension $d \geq 1$. The Lebesgue measure on $\mathbb{R}^d$ is denoted by λ^d and the symbol Λ will always refer to a bounded Borel subset of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, $|x|$ denotes the Euclidean norm of x while for $\Lambda \subset \mathbb{R}^d$, $|\Lambda| := \lambda^d(\Lambda)$.

A *configuration* is a subset ω of $\mathbb{R}^d$ which is locally finite, meaning that $\omega \cap \Lambda$ has finite cardinality $N_\Lambda(\omega) := \#(\omega \cap \Lambda)$ for every bounded Borel set Λ. The space Ω of all configurations is equipped with the σ-algebra $\mathcal{F}$ generated by the counting variables N_Λ.

For convenience, we will often write ω_Λ in place of $\omega \cap \Lambda$. Similarly for every ω and every x in ω we abbreviate $\omega \cup \{x\}$ to $\omega \cup x$ and $\omega \backslash \{x\}$ to $\omega \backslash x$

As usual, we take the reference measure on $(\Omega, \mathcal{F})$ to be the distribution π of the Poisson point process with intensity measure λ^d on $\mathbb{R}^d$. Recall that π is the unique probability measure on $(\Omega, \mathcal{F})$ such that the following hold for all subsets Λ: (i) N_Λ is Poisson distributed with parameter $\lambda^d(\Lambda)$, and (ii) given $N_\Lambda = n$, the n points in Λ are independent with uniform distribution on Λ. The Poisson point process restricted to Λ will be denoted by π_Λ.

Translation by a vector $u \in \mathbb{R}^d$ is denoted by τ_u, either acting on $\mathbb{R}^d$ or on Ω. A probability P on Ω is said stationary if $P = P \circ \tau_u^{-1}$ for any u in $\mathbb{R}^d$. In this paper we consider only stationary probability measures P with finite intensity $E_P(N_{[0,1]^d})$, where E_P stands for the expectation with respect to P. We denote by $\mathcal{P}$ the space of such probability measures.

We denote by Δ_0, $\mathcal{I}_n$ and Λ_n the following sets

$$\Delta_0 = [0,1)^d, \quad \mathcal{I}_n = \{-n, -n+1, \ldots, n-1\}^d \quad \text{and } \Lambda_n = \bigcup_{k \in \mathcal{I}_n} \tau_k(\Delta_0) = [-n, n)^d.$$

In the following, some infinite range interaction processes will be considered. To ensure their existence, we must restrict the set of configurations to the so-called set of tempered configurations as in [24]. The definition of the latter may depend on the type of interactions at hand. In case of superstable pairwise interactions, it is simply defined by

$$\Omega_T = \{\omega \in \Omega;\ \exists t > 0, \forall n \geq 1, \sum_{i \in \mathcal{I}_n} N^2_{\tau_i(\Delta_0)}(\omega) \leq t(2n)^d\}.$$

Unless specified otherwise, this is the definition we consider in the sequel. From the ergodic theorem, any second order stationary measure on Ω is supported on Ω_T, so the restriction to Ω_T is a mild assumption in practice.

2.2 Gibbs point processes

From a general point of view, a *family of interaction energies* is a collection $\mathcal{H} = (H_\Lambda)$, indexed by bounded Borel sets Λ, of measurable functions from Ω_T to $\mathbb{R} \cup \{+\infty\}$ such that for every $\Lambda \subset \Lambda'$, there exists a measurable function $\varphi_{\Lambda,\Lambda'}$ from Ω_T to $\mathbb{R} \cup \{+\infty\}$ and for every $\omega \in \Omega_T$

$$H_{\Lambda'}(\omega) = H_\Lambda(\omega) + \varphi_{\Lambda,\Lambda'}(\omega_{\Lambda^c}). \tag{1}$$

This decomposition is equivalent to (6.11) and (6.12) in [22, p. 92]. In physical terms, $H_\Lambda(\omega) = H_\Lambda(\omega_\Lambda \cup \omega_{\Lambda^c})$ represents the potential energy of the configuration ω_Λ inside Λ given the configuration ω_{Λ^c} outside Λ.

As explained in introduction, we aim at including a large class of interactions in our study, to cover for instance infinite-body interactions like the area-interaction process considered in Section 4.4. Nevertheless, most of standard parametric Gibbs models are pairwise interaction point processes. They are introduced below and we will come back to this important class of Gibbs models in Section 4. *Pairwise potential interactions* take the particular form

$$H_\Lambda(\omega) = z\ N_\Lambda(\omega) + \sum_{\{x,y\} \in \omega, \{x,y\} \cap \omega_\Lambda \neq \emptyset} \phi(x-y), \tag{2}$$

where $z > 0$ is the intensity parameter and ϕ is the pair potential, a function from $\mathbb{R}^d$ to $\mathbb{R} \cup \{+\infty\}$ which is symmetric, i.e. $\phi(-x) = \phi(x)$ for all x in $\mathbb{R}^d$. Let us present two well-known examples.

Example 1: The Strauss pair potential, defined for some interacting parameter $\beta \geq 0$ and some range of interaction $R \geq 0$ corresponds to

$$\phi(x) = \begin{cases} \beta & \text{if} \quad |x| < R, \\ 0 & \text{if} \quad |x| \geq R. \end{cases} \tag{3}$$

Example 2: The general Lennard-Jones (n, m) pair potential is defined for some $d < m < n$, and some $A > 0$, $B \in \mathbb{R}$ by

$$\phi(x) = A|x|^{-n} - B|x|^{-m}, \quad x \in \mathbb{R}^d. \tag{4}$$

The standard Lennard-Jones model in dimension $d = 2$ and $d = 3$ corresponds to $n = 12$ and $m = 6$.

For a family of energies $\mathcal{H} = (H_\Lambda)$, we denote by Ω_∞ the space of configurations which have a locally finite energy, i.e. $\omega \in \Omega_\infty$ if and only if, for any bounded Borel set Λ, $H_\Lambda(\omega)$ is finite.

The Gibbs measures P associated to $\mathcal{H}$ are defined through their local conditional specification, as described below. For every Λ and every ω in $\Omega_\infty \cap \Omega_T$, the local conditional density f_Λ of P with respect to π_Λ is defined by

$$f_\Lambda(\omega) = \frac{1}{Z_\Lambda(\omega_{\Lambda^c})} e^{-H_\Lambda(\omega)}, \tag{5}$$

where $Z_\Lambda(\omega_{\Lambda^c})$ is the normalization constant, or partition function, given by

$$Z_\Lambda(\omega_{\Lambda^c}) = \int e^{-H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} \pi_\Lambda(d\omega'_\Lambda).$$

Some regularity assumptions on $\mathcal{H}$ are required to ensure that $0 < Z_\Lambda(\omega_{\Lambda^c}) < +\infty$, implying that the local density is well-defined. They will be part of our general hypothesis in Section 3 and are fulfilled for all our examples in Section 4.

We are now in position to define the Gibbs measures associated to $\mathcal{H}$, see for instance [9].

Definition 1 *A probability measure P on Ω is a Gibbs measure for the family of energies $\mathcal{H}$ if $P(\Omega_\infty \cap \Omega_T) = 1$ and, for every bounded borel set Λ, for any measurable and integrable function g from Ω to $\mathbb{R}$,*

$$\int g(\omega) P(d\omega) = \int \int g(\omega'_\Lambda \cup \omega_{\Lambda^c}) f_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c}) \pi_\Lambda(\mathrm{d}\omega'_\Lambda) P(d\omega). \tag{6}$$

Equivalently, for P-almost every ω the conditional law of P given ω_{Λ^c} is absolutely continuous with respect to π_Λ with the density f_Λ defined in (5).

The equations (6) are called the Dobrushin–Lanford–Ruelle (DLR) equations. Conditions on $\mathcal{H}$ are mandatory to ensure the existence of a measure P satisfying (6). For general interactions, we will later assume that both (5) is well-defined and P exists. On the other hand, the unicity of P does not necessarily hold, leading to phase transition. As explained in introduction, our consistency results are not affected by this phenomenon.

2.3 Parametric Gibbs models and the MLE procedure

We consider a parametric Gibbs model that depends on a hardcore parameter $\delta \in \mathcal{I}$, where $\mathcal{I} = [\delta_{\min}, \delta_{\max}]$ with $0 \leq \delta_{\min} \leq \delta_{\max}$, and on a parameter $\theta \in \Theta$ with $\Theta \subset \mathbb{R}^p$ and $\mathring{\Theta} \neq \emptyset$. Specifically we assume that the conditional density writes for any $\delta \in \mathcal{I}$ and any $\theta \in \Theta$

$$f_\Lambda^{\delta,\theta}(\omega) = \frac{1}{Z_\Lambda^{\delta,\theta}(\omega_{\Lambda^c})} e^{-H_\Lambda^\theta(\omega)} \mathbb{1}_{\Omega_\infty^\delta}(\omega), \quad \omega \in \Omega_T, \tag{7}$$

where

$$\omega \in \Omega_\infty^\delta \Leftrightarrow \inf_{\{x,y\}\in\omega} |x-y| \geq \delta,$$

while for any $\theta \in \Theta$ and any $\omega \in \Omega_T$, $H_\Lambda^\theta(\omega) < \infty$. This specific parametric form clearly indicates that the hardcore parameter δ only rules the support Ω_∞^δ of the measure, and has no effect on the interaction energy H_Λ^θ. This is what happens for most models in spatial statistics, see Section 4. A counter-example is the Diggle-Gratton model [5, Section 6], where the interaction energy depends also on δ. This situation is not covered by our study.

Let $\mathcal{G}^{\delta,\theta}$ be the set of stationary Gibbs measures defined by the conditional density (7) and with finite intensity. A minimal condition is to assume that for any value of the parameters, (7) is well-defined and that the latter set is not empty.

[**Existence**]: For any $\delta \in \mathcal{I}$, any $\theta \in \Theta$, any bounded set Λ in $\mathbb{R}^d$ and any $\omega \in \Omega_\infty^\delta \cap \Omega_T$, $Z_\Lambda^{\delta,\theta}(\omega_{\Lambda^c}) < \infty$ and the set $\mathcal{G}^{\delta,\theta}$ is not empty.

Let P^* be a Gibbs measure in $\mathcal{G}^{\delta^*,\theta^*}$ for unknown parameters $\delta^* \in \mathcal{I}$ and θ^* in Θ. For technical reasons, we assume that θ^* is not on the boundary of Θ, i.e. $\theta \in \mathring{\Theta}$. We denote by ω^* a realization of P^*. The MLE of (δ^*, θ^*) from the observation of ω^* on Λ_n is defined as follows.

Definition 2 *Let $\mathcal{K}$ be a compact subset of $\mathring{\Theta}$ such that $\theta^* \in \mathcal{K}$. The MLE of δ^* and θ^* from the observation of ω^* in Λ_n is defined by*

$$(\hat{\delta}_n, \hat{\theta}_n) = \operatorname*{argmax}_{(\delta,\theta)\in\mathcal{I}\times\mathcal{K}} f_{\Lambda_n}^{\delta,\theta}(\omega_{\Lambda_n}^*). \tag{8}$$

In Lemma 2 below, we give sufficient conditions which ensures the existence of the argmax in (8). Let us note that we consider here the MLE with free boundary condition, meaning that the configuration $\omega_{\Lambda_n^c}^*$ outside Λ_n is not involved. This is the most natural setting given that we observe ω^* only on Λ_n. A MLE procedure that depends on the outside configuration could have been considered as well and similar theoretical results would have been proved. To implement this alternative procedure in practice yet, the interaction has to be finite range so that the outside configuration reduces to boundary effects that can be handled by minus sampling. For infinite range potentials, as considered in Section 4.3, this method is not feasible and the MLE with free boundary condition makes more sense.

We will use in the following the equivalent definition

$$(\hat{\delta}_n, \hat{\theta}_n) = \operatorname*{argmin}_{(\delta,\theta)\in\mathcal{I}\times\mathcal{K}} K_n^{\delta,\theta}(\omega_{\Lambda_n}^*),$$

where $K_n^{\delta,\theta}$ is the contrast function

$$K_n^{\delta,\theta}(\omega_{\Lambda_n}^*) = \frac{\ln(Z_{\Lambda_n}^{\delta,\theta})}{|\Lambda_n|} + \frac{H_{\Lambda_n}^\theta(\omega_{\Lambda_n}^*)}{|\Lambda_n|} + \infty \mathbb{1}_{\tilde{\delta}_n(\omega_{\Lambda_n}^*)<\delta} \tag{9}$$

with

$$\tilde{\delta}_n(\omega_{\Lambda_n}) = \min_{\{x,y\}\in\omega_{\Lambda_n}} |x-y| \tag{10}$$

and

$$Z_\Lambda^{\delta,\theta} = Z_\Lambda^{\delta,\theta}(\emptyset) = \int e^{-H_\Lambda^\theta(\omega_\Lambda)} \mathbb{1}_{\Omega_\infty^\delta}(\omega_\Lambda)\pi_\Lambda(d\omega_\Lambda). \tag{11}$$

In statistical mechanics, the first term in (9) is called the finite volume pressure while the second term corresponds to the specific energy.

In the following lemma we give an explicit expression for $\hat{\delta}_n$.

Lemma 1 *Under* [**Existence**]*, the MLE of the hardcore parameter* δ *is* $\hat{\delta}_n = \tilde{\delta}_n$ *if* $\tilde{\delta}_n \in \mathcal{I}$ *and* $\hat{\delta}_n = \delta_{\max}$ *otherwise, where* $\tilde{\delta}_n$ *is given by* (10)*, consequently*

$$\hat{\theta}_n = \operatorname*{argmin}_{\theta \in \mathcal{K}} K_n^{\hat{\delta}_n, \theta}(\omega^*_{\Lambda_n}). \tag{12}$$

Proof. If $\delta > \tilde{\delta}_n$, then $K_n^{\delta,\theta}(\omega^*_{\Lambda_n}) = \infty$ and so $\hat{\delta}_n \leq \tilde{\delta}_n$. For any $\delta \leq \tilde{\delta}_n$ and any $\theta \in \Theta$, $K_n^{\delta,\theta}(\omega^*_{\Lambda_n}) = |\Lambda_n|^{-1}(\ln(Z^{\delta,\theta}_{\Lambda_n}) + H^{\theta}_{\Lambda_n}(\omega^*_{\Lambda_n}))$. Note that if $\delta < \delta'$, then $\Omega^{\delta}_{\infty} \subset \Omega^{\delta'}_{\infty}$ and from (11) the function $\delta \mapsto Z^{\delta,\theta}_{\Lambda_n}$ is decreasing. Therefore, for any $\theta \in \Theta$ and for any $\delta \leq \tilde{\delta}_n$, $\delta \mapsto K_n^{\delta,\theta}(\omega^*)$ is decreasing, proving that $\hat{\delta}_n \geq \tilde{\delta}_n$. Hence $\hat{\delta}_n = \tilde{\delta}_n$ if $\tilde{\delta}_n \in \mathcal{I}$. The other statements of Lemma 1 are straightforward. ■

This lemma provides a useful result for practical purposes, since it states that the MLE for the hardcore parameter is just the minimal distance between pairs of points observed in the pattern, as it is usually implemented. Let us remark that a crucial assumption is that the space of parameters in [**Existence**] is the cartesian product $\mathcal{I} \times \Theta$, i.e. the parametric model is well-defined for any value of $\theta \in \Theta$, whatever δ is. This is the case for the examples considered in Section 4.

As explained in introduction, the computation of $\hat{\theta}_n$ given by (12) is more difficult because there is in general no closed form expression for $Z^{\delta,\theta}_{\Lambda}$. In practice the optimization (12) is conducted from an approximated contrast function, based on Markov chain Monte Carlo methods where the error of approximation can be controlled, see for instance [10]. Even to ensure the existence of $\hat{\theta}_n$, we need some extra assumptions, as detailed in the following lemma.

[**Argmax**]: For any $\omega \in \Omega_T$ and any $n \geq 1$, the function $\theta \mapsto H^{\theta}_{\Lambda_n}(\omega_{\Lambda_n})$ is lower semicontinuous over $\mathcal{K}$, i.e. for any $\theta \in \mathcal{K}$ $\liminf_{\theta' \mapsto \theta} H^{\theta'}_{\Lambda_n}(\omega_{\Lambda_n}) \geq H^{\theta}_{\Lambda_n}(\omega_{\Lambda_n})$, and there exists an upper semicontinuous version $\theta \mapsto \tilde{H}^{\theta}_{\Lambda_n}(\omega_{\Lambda_n})$ such that for every $\theta \in \mathcal{K}$, $\tilde{H}^{\theta}_{\Lambda_n} = H^{\theta}_{\Lambda_n}$ π_{Λ_n}-almost surely.

Lemma 2 *Under* [**Existence**] *and* [**Argmax**]*, the MLE of* θ *exists, i.e. there exists at least one* $\hat{\theta}_n \in \mathcal{K}$ *such that* $K_n^{\hat{\delta}_n, \hat{\theta}_n}(\omega^*_{\Lambda_n}) = \min_{\theta \in \mathcal{K}} K_n^{\hat{\delta}_n, \theta}(\omega^*_{\Lambda_n})$.

Proof. It is sufficient to show that $\theta \mapsto K_n^{\hat{\delta}_n, \theta}(\omega^*_{\Lambda_n})$ is lower semicontinuous. As a direct consequence of [**Argmax**], $\theta \mapsto H^{\theta}_{\Lambda_n}(\omega^*_{\Lambda_n})$ is lower semicontinuous. For the finite volume pressure, this comes from [**Argmax**] and the Fatou's lemma since

$$\begin{aligned}
\liminf_{\theta' \mapsto \theta} Z^{\hat{\delta}_n, \theta'}_{\Lambda_n} &= \liminf_{\theta' \mapsto \theta} \int e^{-\tilde{H}^{\theta'}_{\Lambda_n}(\omega_{\Lambda_n})} \mathbb{1}_{\Omega^{\hat{\delta}_n}_{\infty}}(\omega_{\Lambda}) \pi_{\Lambda_n}(d\omega_{\Lambda_n}) \\
&\geq \int \liminf_{\theta' \mapsto \theta} e^{-\tilde{H}^{\theta'}_{\Lambda_n}(\omega_{\Lambda_n})} \mathbb{1}_{\Omega^{\hat{\delta}_n}_{\infty}}(\omega_{\Lambda}) \pi_{\Lambda_n}(d\omega_{\Lambda_n}) \\
&\geq \int e^{-\tilde{H}^{\theta}_{\Lambda_n}(\omega_{\Lambda_n})} \mathbb{1}_{\Omega^{\hat{\delta}_n}_{\infty}}(\omega_{\Lambda}) \pi_{\Lambda_n}(d\omega_{\Lambda_n}) = Z^{\hat{\delta}_n, \theta}_{\Lambda_n}.
\end{aligned}$$

■

3 Consistency of the MLE for general Gibbs interactions

We detail in this section the minimal assumptions on the parametric family of Gibbs measure that imply the strong consistency of the MLE. All of them are fulfilled by the examples of Section 4.

The first ones, [**Existence**] and [**Argmax**] introduced in the previous section, imply the existence of a Gibbs measure for any $\delta \in \mathcal{I}$ and $\theta \in \Theta$, and the existence of the MLE.

We assume the following mild assumptions on the family of energies. They gather the stability of the energy function, a standard stationary decomposition in terms of mean energy per unit volume and they deal with boundary effects.

[**Stability**]: For any compact set $\mathcal{K} \subset \Theta$, there exists a constant $\kappa \geq 0$ such that for any Λ, any $\theta \in \mathcal{K}$ and any $\omega \in \Omega_T$

$$H^\theta_\Lambda(\omega_\Lambda) \geq -\kappa N_\Lambda(\omega). \tag{13}$$

[**MeanEnergy**]: There exist measurable functions H^θ_0 and (∂H_n) from Ω_T to $\mathbb{R}$ such that for all $n \geq 1$ and all $\theta \in \Theta$

$$H^\theta_{\Lambda_n} = \sum_{k \in \mathcal{I}_n} H^\theta_0 \circ \tau_{-k} + \partial H^\theta_{\Lambda_n}, \tag{14}$$

where for all $P \in \mathcal{G}$ and all $\theta \in \Theta$ the *mean energy* of P defined by

$$H^\theta(P) = E_P(H^\theta_0) \tag{15}$$

is finite and for any compact set $\mathcal{K} \subset \Theta$, for P-almost every $\omega \in \Omega_T$,

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sup_{\theta \in \mathcal{K}} \left|\partial H^\theta_{\Lambda_n}(\omega)\right| = 0.$$

[**Boundary**]: For all $P \in \mathcal{G}$, for any compact set $\mathcal{K} \subset \Theta$ and for P-almost every $\omega \in \Omega_T$

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sup_{\theta \in \mathcal{K}} \left|H^\theta_{\Lambda_n}(\omega_{\Lambda_n}) - H^\theta_{\Lambda_n}(\omega)\right| = 0.$$

To prove the consistency of the MLE, we need to control the asymptotic behaviour of the finite volume pressure and the specific energy in (9). The following assumption deals with the latter.

[**Regularity**]: For any P in $\mathcal{G}$, for any compact set $\mathcal{K} \subset \Theta$, there exists a function g from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\lim_{r\to 0} g(r) = 0$ such that for any $\theta \in \mathcal{K}$ and any $r > 0$

$$E_P\left(\sup_{\substack{\theta' \in \mathcal{K} \\ |\theta-\theta'| \leq r}} \left|H^\theta_0 - H^{\theta'}_0\right|\right) \leq g(r). \tag{16}$$

Moreover for any $\eta > 0$ and any $\theta_0 \in \mathcal{K}$, there exists a finite subset $\mathcal{N}(\theta_0) \subset B(\theta_0, \eta) \cap \Theta$ and $r(\theta_0) > 0$ such that $r(\theta_0) < \eta$ and for any Λ

$$\max_{\theta \in \mathcal{N}(\theta_0)} \inf_{\theta' \in B(\theta_0, r(\theta_0)) \cap \mathcal{K}} \inf_{\omega_\Lambda \in \Omega^{\delta_{\min}}_\infty} \left(\frac{H^\theta_\Lambda(\omega_\Lambda) - H^{\theta'}_\Lambda(\omega_\Lambda)}{N_\Lambda(\omega_\Lambda)}\right) \geq -g(r(\theta_0)). \tag{17}$$

Assumption (16) implies in particular that the mean energy $H^\theta(P)$ is continuous with respect to θ. Let us note that this does not imply that the specific energy or $\theta \mapsto H^\theta_0$ is continuous with respect to θ. As a counterexample, the Strauss model in Example 1, or more generally any model from Sections 4.1 and 4.2, has a discontinuous specific energy but the mean energy is continuous, which is verified in the proof of Theorem 2. Assumption (17) is related to the regularity of the coefficient κ in [**Stability**] with respect to the parameter θ.

Finally the following assumption relates the pressure and the specific entropy defined below, through the variational principle. The variational principle is conjectured to hold for all models of statistical mechanics even if it is not proved in such a general setting.

For any $\delta \in \mathcal{I}$ and $\theta \in \Theta$, the *specific entropy* of $\mu^{\delta,\theta} \in \mathcal{G}^{\delta,\theta}$ is the limit

$$\mathcal{I}(\mu^{\delta,\theta}) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathcal{I}_{\Lambda_n}(\mu^{\delta,\theta})$$

where $\mathcal{I}_{\Lambda_n}(\mu^{\delta,\theta}) = \int \ln f^{\delta,\theta}_{\Lambda_n}(\omega_{\Lambda_n})\mu^{\delta,\theta}_{\Lambda_n}(d\omega)$ is the finite volume entropy of $\mu^{\delta,\theta}$. Note that this limit always exists, see [9].

[VarPrin]: For any $\delta \in \mathcal{I}$ and any $\theta \in \Theta$, the *pressure* defined as the following limit exists and is finite

$$p(\delta,\theta) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \ln(Z^{\delta,\theta}_{\Lambda_n}). \tag{18}$$

Moreover the map $\delta \mapsto p(\delta,\theta)$ is right continuous on $\mathcal{I}$, i.e. $\lim_{\delta'\to\delta,\delta'\geq\delta} p(\delta',\theta) = p(\delta,\theta)$. In addition, for any δ,δ' in $\mathcal{I}$ with $\delta' \geq \delta$, for any θ,θ' in Θ, for all $\mu^{\delta',\theta'} \in \mathcal{G}^{\delta',\theta'}$ the inequality

$$p(\delta,\theta) \geq -\mathcal{I}(\mu^{\delta',\theta'}) - H^{\theta}(\mu^{\delta',\theta'}) \tag{19}$$

holds. If $(\delta',\theta') = (\delta^*,\theta^*)$, then equality holds in (19) if and only if $(\delta,\theta) = (\delta^*,\theta^*)$.

The formulation of the variational principle presented here is slightly weaker than the classical one in statistical mechanics where the inequality (19) is required for any probability measure P on Ω_T, and not only for $\mu^{\delta',\theta'} \in \mathcal{G}^{\delta',\theta'}$. Note that the right continuity of the pressure holds trivially true when δ^* is known, i.e. $\mathcal{I} = \{\delta^*\}$. Let us also remark that the last part in **[VarPrin]** concerning (δ^*,θ^*) is just an identifiability assumption.

We are now in position to state the consistency of the MLE.

Theorem 1 *Under the assumptions* **[Existence]**, **[Argmax]**, **[Stability]**, **[MeanEnergy]**, **[Boundary]**, **[Regularity]** *and* **[VarPrin]**, *for any* $(\delta^*,\theta^*) \in \mathcal{I}\times\mathcal{K}$ *and any* $P \in \mathcal{G}^{\delta^*,\theta^*}$, *the MLE* $(\hat{\delta}_n,\hat{\theta}_n)$ *is well-defined and converges* P*-almost surely to* (δ^*,θ^*) *when* n *goes to infinity.*

As a noticeable example stated in the following corollary, if the energy H^{θ}_{Λ} depends linearly on the parameter θ, which is called the exponential model, then the assumptions **[Argmax]** and **[Regularity]** are automatically satisfied. Specifically, for any $1 \leq k \leq p$, let (H^k_{Λ}) be a family of interaction energies satisfying (1). For any $\theta \in \mathbb{R}^p$ and any bounded set Λ an exponential model takes the form

$$H^{\theta}_{\Lambda} = \sum_{k=1}^{p} \theta_k H^k_{\Lambda}. \tag{20}$$

Let us introduce the following modification of **[MeanEnergy]**.

[MeanEnergy']: The assumption **[MeanEnergy]** holds with $H^{\theta}_0 = \sum_{k=1}^{p} \theta_k H^k_0$ where H^k_0 comes from the decomposition (14) applied to $H^k_{\Lambda_n}$.

Corollary 1 *Assume that the family of interaction energies satisfy* (20). *Then under the assumptions* **[Existence]**, **[Stability]**, **[MeanEnergy']**, **[Boundary]** *and* **[VarPrin]**, *for any* $(\delta^*,\theta^*) \in \mathcal{I}\times\mathcal{K}$ *and any* $P \in \mathcal{G}^{\delta^*,\theta^*}$, *the MLE* $(\hat{\delta}_n,\hat{\theta}_n)$ *is well-defined and converges* P*-almost surely to* (δ^*,θ^*) *when* n *goes to infinity.*

4 Consistency for pairwise interactions and other examples

4.1 Finite range piecewise continuous pair potentials with hardcore

In this section, we consider conditional densities given by (7) with a non-vanishing hardcore parameter, i.e. $\delta_{\min} > 0$, and a Hamiltonian H^{θ}_{Λ} defined as in (2) by

$$H^{\theta}_{\Lambda}(\omega) = z\ N_{\Lambda}(\omega) + \sum_{\{x,y\}\in\omega,\{x,y\}\cap\omega_{\Lambda}\neq\emptyset} \phi^{\beta,R}(x-y), \tag{21}$$

where $\theta = (z, \beta, R)$ is in the set $\Theta = \mathbb{R}_+ \times \mathcal{B} \times \mathbb{R}_+^q$ with $\mathcal{B} \subset \mathbb{R}^p$ and q is a positive integer. For $\beta \in \mathcal{B}$ and $R = (R_1, \dots, R_q) \in \mathbb{R}_+^q$, we consider a pair potential interaction $\phi^{\beta,R}$ defined as follows. If $x \in \mathbb{R}^d$ and $|x| \neq R_1, R_2, \dots, R_q$,

$$\phi^{\beta,R}(x) = \mathbb{1}_{[0,R_1)}(|x|)\, \varphi_1^{\beta,R}(|x|) + \cdots + \mathbb{1}_{(R_{q-1},R_q)}(|x|)\, \varphi_q^{\beta,R}(|x|), \tag{22}$$

where for any $k = 1, \dots, q$, $(\beta, R, x) \mapsto \varphi_k^{\beta,R}(|x|)$ is assumed to be continuous on $\mathcal{B} \times \mathbb{R}_+^q \times \mathbb{R}^d$, while if $|x| = R_k$ for some $1 \leq k \leq q$,

$$\phi^{\beta,R}(x) = \min(\varphi_k^{\beta,R}(|x|), \varphi_{k+1}^{\beta,R}(|x|))$$

with the convention $\varphi_{q+1}^{\beta,R} = 0$. Note that with the latter choice, the function $(\beta, R) \mapsto \phi^{\beta,R}(x)$ is lower semicontinuous which is crucial for the assumption **[Argmax]**.

This setting ensures the existence of the model for any $\delta \in \mathcal{I}$, $z > 0$, $\beta \in \mathcal{B}$ and $R \in \mathbb{R}_+^q$, as explained in the proof of the following theorem.

If $R_k \geq R_{k+1}$, $\mathbb{1}_{(R_k,R_{k+1})} = 0$ and identifiability issues appear. So we assume that the unknown parameter $(\delta^*, z^*, \beta^*, R^*)$ is such that $\delta^* > \delta_{\min}$, $z^* > 0$, $\beta^* \in \mathring{\mathcal{B}}$, and $R^* \in \mathring{\mathcal{R}}$ with $R_1^* > \delta^*$ and $\mathcal{R} = \{R,\ 0 \leq R_1 \leq \cdots \leq R_q\}$. Moreover, we assume that if $(\beta, R) \neq (\beta^*, R^*)$, then

$$\lambda^d \left(x \in \mathbb{R}^d, \mathbb{1}_{[\delta^*,+\infty)}(|x|) \left(\phi^{\beta^*,R^*}(x) - \phi^{\beta,R}(x) \right) \neq 0 \right) > 0. \tag{23}$$

This implies that if $(\delta, \theta) \neq (\delta^*, \theta^*)$ then $\mathcal{G}^{\delta,\theta} \cap \mathcal{G}^{\delta^*,\theta^*} = \emptyset$.

Notice that the assumption $R_1^* > \delta^*$ is necessary to ensure identifiability of R^* and δ^*, but we do not need to add the constraint $R_1 > \delta$ in the MLE optimisation (8), as confirmed by the choice of $\mathcal{K}$ in the theorem below. This is in particular important to agree with the setting of Lemma 1, where it is necessary that the optimisation (8) is carried out on a cartesian product $\mathcal{I} \times \mathcal{K}$.

This framework gathers all finite-range pairwise models with hardcore described in [2], [12], [18] and [25], including the hardcore Strauss model, the hardcore piecewise constant pairwise interaction model and the Fiksel model.

Theorem 2 *Let $\mathcal{K}$ be a compact subset of $(0, \infty) \times \mathring{\mathcal{B}} \times \mathring{\mathcal{R}}$ such that $(z^*, \beta^*, R^*) \in \mathcal{K}$. Then under the above setting, the MLE of $(\delta^*, z^*, \beta^*, R^*)$ given by (8) is strongly consistent.*

4.2 Finite range piecewise continuous pair potentials without hardcore

We consider the same setting as in the previous section except that there is no hardcore, i.e. $\delta^* = 0$ and δ^* is known, meaning that it has not to be fitted, or equivalently $\delta_{\min} = \delta_{\max} = 0$ implying $\mathcal{I} = \{0\}$ in (8). The Hamiltonian H_Λ^θ is defined as in (21) and (22) with the same continuity assumptions on $\phi^{\beta,R}$. The standard example we have in mind for this section is the Strauss model of Example 1 or any of the examples mentioned in the previous section without assuming the presence of a hardcore part.

Whereas this setting may appear to be simpler than the previous one from a statistical point of view (there is one parameter less to fit), it requires additional assumptions to ensure the existence of the model and to control its regularity. Therefore we assume that the Hamiltonian H_Λ^θ satisfies **[Stability]** and that for any $\theta = (z, \beta, R) \in \Theta$ the pair potential $\phi^{\beta,R}$ is superstable, which means that it is the sum of a stable pair potential plus a non negative potential which is positive around zero (see [24]). The identifiability assumptions remain similar as in the previous section and include (23) with $\delta^* = 0$.

To handle the second part of **[Regularity]**, we need a last hypothesis on the potential function. We assume that for any $(\beta_0, R_0) \in \mathcal{B} \times \mathbb{R}_+^q$ and any $\eta > 0$ there exists $\beta_1 \in B(\beta_0, \eta)$ such that for any $k = 1, \dots, q$

$$\varphi_k^{\beta_1,R_0} > \varphi_k^{\beta_0,R_0}. \tag{24}$$

Theorem 3 *Let $\mathcal{K}$ be a compact subset of $(0,\infty)\times\mathring{\mathcal{B}}\times\mathring{\mathcal{R}}$ such that $(z^*,\beta^*,R^*)\in\mathcal{K}$. Then under the above setting, the MLE of (z^*,β^*,R^*) given by (8) is strongly consistent.*

This theorem can be easily applied to all standard finite range pairwise potentials (without hardcore) described in [2], [12], [18] and [25]. In particular, condition (24) turns out to be non-restrictive. Because of its central role in spatial statistics, we focus in the following corollary on the Strauss model of Example 1. Its proof is straightforward and is not detailed.

Corollary 2 *Consider the Strauss model corresponding in (22) to $q=p=1$ and $\varphi_1^{\beta,R}\equiv\beta$ where $\beta\geq 0$ and $R>0$. Let $\mathcal{K}$ be a compact subset of $(0,\infty)^3$ such that $(z^*,\beta^*,R^*)\in\mathcal{K}$. Then the MLE of (z^*,β^*,R^*) given by (8) is strongly consistent.*

4.3 Infinite range pair potentials with a smooth parametrization

In this section, we consider pairwise models in the spirit of the Lennard-Jones model given in Example 2. Our general setting concerns infinite range pair potentials without hardcore, i.e. $\delta_{\min}=\delta_{\max}=0$, that are uniformly regular and non-integrably divergent at the origin in the sense of Ruelle [24]. Specifically, we assume that there exist two positive decreasing functions ψ and χ from $(0,\infty)$ to $\mathbb{R}$ and $r_0>0$ with

$$\int_{r_0}^{+\infty}\psi(t)t^{d-1}dt<\infty,\qquad \int_0^{r_0}\chi(t)t^{d-1}dt=+\infty$$

such that for any parameter $\beta\in\mathcal{B}$, where $\mathcal{B}$ is an open set in $\mathbb{R}^p$, the function $\phi^\beta:\mathbb{R}^d\to\mathbb{R}$ satisfies for any $x\in\mathbb{R}^d$

$$\phi^\beta(x)\geq\chi(|x|)\text{ whenever }|x|\leq r_0\quad\text{and}\quad|\phi^\beta(x)|\leq\psi(|x|)\text{ whenever }|x|>r_0. \tag{25}$$

The Hamiltonian H_Λ^θ is defined as in (2) by

$$H_\Lambda^\theta(\omega)=z\ N_\Lambda(\omega)+\sum_{\{x,y\}\in\omega,\{x,y\}\cap\omega_\Lambda\neq\emptyset}\phi^\beta(x-y), \tag{26}$$

where $\theta=(z,\beta)$ is in $\Theta=\mathbb{R}_+\times\mathcal{B}$.

We assume that ϕ^β is a symmetric function on $\mathbb{R}^d\backslash\{0\}$ and that for any $x\in\mathbb{R}^d\backslash\{0\}$, the map $\beta\mapsto\phi^\beta(x)$ is differentiable on $\mathcal{B}$. Denoting by $\nabla\phi^\beta(x)$ its gradient, we also assume that for any compact set $\mathcal{K}\subset\mathcal{B}$, for any $\beta\in\mathcal{K}$ and $x\in\mathbb{R}^d$ with $|x|>r_0$

$$|\nabla\phi^\beta(x)|\leq\psi(|x|) \tag{27}$$

and

$$\sup_{\beta\in\mathcal{K}}\sup_{x\in\mathbb{R}^d}1\!\!1_{|x|\leq r_0}\sup_{\beta'\in\mathcal{K}}e^{-\phi^{\beta'}(x)}\max(\phi^\beta(x),|\nabla\phi^\beta(x)|)<\infty. \tag{28}$$

In addition we suppose that for any compact set $\mathcal{K}\subset\mathcal{B}$ there exists an open set U in $\mathbb{R}^p$ and a stable pair potential $\tilde{\phi}$ from $\mathbb{R}^d$ to $\mathbb{R}$ such that for any $u\in U$, any $x\in\mathbb{R}^d$ and any $\beta\in\mathcal{K}$

$$\nabla\phi^\beta(x).u\geq\tilde{\phi}(x), \tag{29}$$

where $v.u$ denotes the scalar product of vectors u,v in $\mathbb{R}^p$.

Finally, for identifiability reasons, we assume that for any $\beta\neq\beta'$ in $\mathcal{B}$,

$$\lambda^d\left(x\in\mathbb{R}^d,\phi^\beta(x)\neq\phi^{\beta'}(x)\right)>0. \tag{30}$$

Theorem 4 *Let $(\phi^\beta)_{\beta\in\mathcal{B}}$ be a family of pair potentials which are uniformly regular, non-integrably divergent at the origin and satisfy assumptions (27)-(30). Let $\mathcal{K}$ be a compact subset of $\mathring{\Theta}$ such that $\theta^*=(z^*,\beta^*)$ belongs to $\mathcal{K}$. Then the MLE of (z^*,β^*) given by (8) is strongly consistent.*

As a fundamental example, the following corollary focuses on the Lennard-Jones model of Example 2. Let the triangle domain $\mathcal{T} = \{(n,m) \in \mathbb{R}^2, d < m < n\}$.

Corollary 3 *Consider the Lennard-Jones model given by* (26) *and* (4) *with parameters* $z > 0$ *and* $\beta = (A, B, n, m) \in \mathcal{B}$ *where* $\mathcal{B} = (0, \infty) \times \mathbb{R} \times \mathcal{T}$. *Let* $\mathcal{K}$ *be a compact subset of* $\mathbb{R}_+ \times \mathcal{B}$ *such that* $(z^*, \beta^*) \in \mathcal{K}$. *Then the MLE of* (z^*, β^*) *given by* (8) *is strongly consistent.*

Proof. Denote by $\mathcal{K}_\mathcal{T}$ the projection of $\mathcal{K}$ onto $\mathcal{T}$, i.e. $(n,m) \in \mathcal{K}_\mathcal{T}$ if and only if there exists $z > 0$, $A > 0$ and B such that $(z, A, B, n, m) \in \mathcal{K}$. It is easy to find $r_0 > 0$ and two positive constants c_0, c_1 such that ϕ^β given by (4) is uniformly regular and non-integrably divergent at the origin for r_0, $\psi(t) = c_0 t^{-m_0}$ and $\chi(t) = t^{-n_0}$ with $d < m_0 < m$ and $d < n_0 < n$ for all $(n,m) \in \mathcal{K}_\mathcal{T}$.

The map $\beta \mapsto \phi^\beta$ is clearly differentiable on $\mathcal{B}$ with

$$\nabla\phi^\beta(x) = \left(|x|^{-n}, |x|^{-m}, -A\ln(|x|)|x|^{-n}, B\ln(|x|)|x|^{-m}\right).$$

In adjusting the constant c_0 in the definition of ψ, we show that assumption (27) holds. Assumptions (28) and (30) are obvious. It remains to show (29). Let u_0 be the vector $(1,1,1,1)$ in $\mathbb{R}^4$ and u any vector in the open ball $B(u_0, 1/2)$ in $\mathbb{R}^4$. Denoting $\mathcal{K}_\mathcal{B}$ the projection of $\mathcal{K}$ onto $\mathcal{B}$, we find that for any $\beta \in \mathcal{K}_\mathcal{B}$ and $x \in \mathbb{R}^d$

$$\nabla\phi^\beta(x).u \geq \frac{1}{2}\mathbb{1}_{[0,1]}(|x|)\Big(|x|^{-n} - |b_0\ln(|x|)||x|^{-m}\Big) - \frac{3}{2}\mathbb{1}_{(1,+\infty)}(|x|)(a_1 + |b_1|)\ln(|x|)|x|^{-m}$$

where a_1, b_0 and b_1 are chosen so that for any A and B, $A < a_1$ and $b_0 < B < b_1$. There exists $\delta > 0$ such that for all $(n,m) \in \mathcal{K}_\mathcal{T}$, $m < n - \delta$, whereby

$$\nabla\phi^\beta(x).u \geq c\mathbb{1}_{[0,1]}(|x|)|x|^{-n_0} - c\mathbb{1}_{(1,+\infty)}(|x|)\ln(|x|)|x|^{-m_0}$$

for some $c > 0$. The right hand term of this inequality is regular non-integrably divergent at the origin. Therefore it is stable and (29) is proved. ■

We can also deduce from Theorem 4 the same kind of result as in [17] concerning exponential models. Note however that the proof in [17] crucially relies on the convexity of $\beta \mapsto \phi^\beta(x)$, whereas the following corollary is obtained by different techniques.

Corollary 4 *Assume that in* (26) *the family of pair potentials* $(\phi^\beta)_{\beta\in\mathcal{B}}$ *is uniformly regular, non-integrably divergent at the origin, satisfies the identifiability condition* (30) *and that*

$$\phi^\beta(x) = \sum_{i=1}^{p} \beta_i \phi_i(x)$$

where $\phi_1(x) \geq \chi(|x|)$ *if* $|x| < r_0$ *while for* $i \geq 2$, $\phi_i(x) = o(\phi_1(x))$ *when* $x \to 0$. *Let* $\mathcal{K}$ *be a compact subset of* $\mathring{\Theta}$ *such that* $\theta^* = (z^*, \beta^*)$ *belongs to* $\mathcal{K}$. *Then the MLE of* (z^*, β^*) *given by* (8) *is strongly consistent.*

Proof. In view of Theorem 4, we just need to check (27)-(29). Let $\beta = (\beta_1, \ldots, \beta_p) \in \mathcal{K}$ and $\beta' = (\beta_1, \ldots, \beta_{p-1}, \beta'_p) \in \mathcal{K}$ with $\beta'_p \neq \beta_p$. Since ϕ^β is uniformly regular, we have $|\phi^\beta(x) - \phi^{\beta'}(x)| \leq 2\psi(|x|)$, meaning that $|\phi_p(x)| \leq c\psi(|x|)$ for some $c > 0$. We obtain likewise $|\phi_i(x)| \leq c\psi(|x|)$ for any $i = 1, \ldots, p$. Consequenlty $|\nabla\phi^\beta(x)| \leq \sum_{i=1}^p |\phi_i(x)| \leq pc\Psi(|x|)$ and (27) is proved.

The relation (28) holds trivially true by choosing r_0 sufficiently small. Finally, letting U be any open set included in $\mathcal{K}$, we have $\nabla\phi^\beta(x).u = \phi^u(x)$ which is stable by assumption. ■

4.4 Examples of infinite-body interactions

Our general result in Theorem 1 is also adapted to non-pairwise potential models. For instance, it is not difficult to generalize the results of the three previous sections to the case of a finite-body interaction of order greater than two, as for instance a triplet or quadruplet interaction. For brevity reasons, we do not include this generalization in the present paper. Instead, we give in this section examples arising from stochastic geometry which involve infinite-body interactions. Specifically, we focus on the area-interaction and the Quermass-interaction processes, but other models could have been considered as well.

The area-interaction process [1], also called Widom-Rowlinson model in the statistical physics community [27], is probably the most popular model of infinite-body interaction. For $R \geq 0$, $\omega \in \Omega$ and Λ a bounded set in $\mathbb{R}^d$, we introduce the notation

$$\mathcal{A}^R(\omega_\Lambda) = \lambda^d \left(\bigcup_{x \in \omega_\Lambda} B(x, R) \right).$$

The Hamiltonian H_Λ^θ of the area-interaction process is defined for any $\omega \in \Omega$ and any bounded set Λ by

$$H_\Lambda^\theta(\omega) = zN_\Lambda(\omega) + \beta \left(\mathcal{A}^R \left(\omega_{\Lambda \oplus B(0,2R)} \right) - \mathcal{A}^R \left(\omega_{\Lambda \oplus B(0,2R) \setminus \Lambda} \right) \right), \tag{31}$$

where the parameter $\theta = (z, R, \beta)$ belongs to $\Theta = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ and the operator $\oplus$ stands for the Minkowski sum acting on the sets in $\mathbb{R}^d$. Note that the unknown radius R is part of the parameters and is estimated consistently as stated in the following proposition.

Proposition 1 *Let $\mathcal{H} = (H^\theta)_\Lambda$ be the family of area-interaction energies defined in* (31) *and $\mathcal{K}$ a compact subset of $\mathring{\Theta}$ such that $\theta^* = (z^*, R^*, \beta^*)$ belongs to $\mathcal{K}$. Then the MLE of θ^* given by* (8) *is strongly consistent.*

The Quermass-interaction model is a generalization of the area-interaction process, where not only the volume of the union of balls is involved in the Hamiltonian but also the other Minkowski functionals. We denote by M_k, $k = 1 \ldots d+1$, the $d+1$ Minkowski functionals in $\mathbb{R}^d$ and for short

$$\mathcal{M}_k^R(\omega_\Lambda) = M_k \left(\bigcup_{x \in \omega_\Lambda} B(x, R) \right),$$

where $R \geq 0$, $\omega \in \Omega$ and Λ is a bounded subset of $\mathbb{R}^d$. Recall that for $d = 2$, M_1 corresponds to the Euler-Poincaré characteristic, M_2 is the perimeter and M_3 is the area. We refer to [2] for more details about Minkowski functionals.

The Hamiltonian H_Λ^θ of the Quermass-interaction process is defined for any $\omega \in \Omega$ and any bounded set Λ by

$$H_\Lambda^\theta(\omega) = zN_\Lambda(\omega) + \sum_{k=1}^{d+1} \beta_k \left(\mathcal{M}_k^R \left(\omega_{\Lambda \oplus B(0,2R)} \right) - \mathcal{M}_k^R \left(\omega_{\Lambda \oplus B(0,2R) \setminus \Lambda} \right) \right) \tag{32}$$

where $R \geq 0$, $z > 0$ and $\beta = (\beta_1, \ldots, \beta_{d+1}) \in \mathbb{R}^{d+1}$.

This model has been introduced in [14]. Its existence on $\mathbb{R}^d$ has been solved so far only when $d \leq 2$ in [3]. Therefore we restrict the following study to the case $d \leq 2$. Moreover, we assume that R is known and we only consider the MLE estimation of $\theta := (z, \beta)$ in (32). The reason is that we did not succeed to prove that (17) in [**Regularity**] holds in presence of the Euler-Poincaré characteristic M_1 when R is part of the unknown parameter. With the assumption that R is known, the Quermass-interaction process becomes an infinite-body interactions exponential model and Corollary 1 applies. In this framework the set of parameters is $\Theta = \mathbb{R}_+ \times \mathbb{R}^3$.

Proposition 2 *Let $\mathcal{H} = (H^\theta)_\Lambda$ be the family of Quermass-interaction energies defined in* (32) *for $d = 2$ and let $\mathcal{K}$ be a compact subset of $\mathring{\Theta}$ such that $\theta^* = (z^*, \beta^*)$ belongs to $\mathcal{K}$. Then the MLE of θ^* given by* (8) *is strongly consistent.*

5 Proofs

5.1 Proof of Theorem 1

Let P be a probability measure in $\mathcal{G}^{\delta^*,\theta^*}$ and let ω^* be a realisation of P. If P is not ergodic, it can be represented as the mixture of ergodic stationary Gibbs measures, see [22]. Therefore the proof of the consistency of the MLE reduces to the case when P is ergodic, which is assumed henceforth.

Let us start by proving the convergence of the hardcore parameter.

Lemma 3 *The MLE $\hat{\delta}_n$ converges P-almost surely to δ^* when n goes to infinity.*

Proof. From Lemma 1, $\hat{\delta}_n = \min(\tilde{\delta}_n, \delta_{max})$ where $\tilde{\delta}_n$ is given by (10). Clearly $\tilde{\delta}_n$ is a decreasing sequence and $\tilde{\delta}_n \geq \delta^*$. So it remains to prove that, for any $\delta' > \delta^*$, $\tilde{\delta}_n$ is smaller than δ' for n large enough. By the DLR equations (6) and the definition of local densities (7), there exists a bounded set Λ such

$$P(\text{ there exist } x, y \in \omega_\Lambda \text{ such that } \delta^* < |x-y| < \delta') > 0.$$

Thanks to the ergodic theorem, it implies that

$$P(\text{there exist } n \geq 1 \text{ and } x, y \in \omega_{\Lambda_n} \text{ such that } \delta^* < |x-y| < \delta') = 1$$

which proves the expected result. ■

According to Lemma 2 and 3, we have now to prove the consistency of $\hat{\theta}_n$ where

$$\hat{\theta}_n = \text{argmin}_{\theta \in \mathcal{K}} K_n^{\hat{\delta}_n, \theta}(\omega^*_{\Lambda_n}).$$

The strong consistency relies on a minimum contrast function result stated in the next lemma, where neither the contrast function nor its limit need to be continuous with respect to θ.

Lemma 4 *For any $\theta \in \Theta$, let $(h_n^\theta)_{n\geq 1}$ be a family of parametric measurable functions from Ω_T to $\mathbb{R}$. Let $\mathcal{K}$ be a compact subset of $\mathring{\Theta}$ and let P be a probability measure on Ω_T. We assume that*

i) For any $\theta \in \Theta$, h_n^θ converges P-almost surely to a finite real number (denoted by h^θ) when n goes to infinity.

ii) The function $\theta \mapsto h^\theta$ admits an unique minimum over Θ:

$$\theta^* = argmin_{\theta \in \Theta} h^\theta,$$

and $\theta^* \in \mathcal{K}$.

iii) For P-almost every $\omega \in \Omega_T$ and for n sufficiently large, $\theta \mapsto h_n^\theta$ admits an infimum over $\mathcal{K}$ attained by at least one element $\hat{\theta}_n(\omega)$.

iv) The function $\theta \mapsto h^\theta$ is lower semicontinuous on Θ (i.e. for any $\theta \in \Theta$, $\liminf_{\theta' \mapsto \theta} h^{\theta'} \geq h^\theta$).

v) There exists a function g_0 from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying

$$\lim_{x \mapsto 0} g_0(x) = 0$$

and such that for any $\varepsilon > 0$ and for any $\theta \in \mathcal{K}$, there exists a finite subset $\mathcal{N}(\theta)$ included in $B(\theta, g_0(\varepsilon)) \cap \Theta$ and $r(\theta) > 0$ such that

$$P\left(\limsup_{n\to\infty}\left(\min_{\theta' \in \mathcal{N}(\theta)} h_n^{\theta'} - \inf_{\theta' \in B(\theta, r(\theta)) \cap \mathcal{K}} h_n^{\theta'}\right) \geq \varepsilon\right) = 0. \tag{33}$$

Then the sequence $\hat{\theta}_n = argmin_{\theta \in \mathcal{K}} h_n(\theta)$ converges P-almost surely to θ^* when n goes to infinity.

Proof. Let $\varepsilon > 0$ and A_ε be the following subset of $\mathcal{K}$

$$A_\varepsilon = \left\{\theta \in \mathcal{K} \oplus B(0, g(\varepsilon)) \text{ such that } h^\theta \leq h^{\theta^*} + 2\varepsilon\right\},$$

where g comes from *(v)*. We add the ball $B(0, g(\varepsilon))$ to the set $\mathcal{K}$ for technical reasons which appear below. By *iv)* the function $\theta \mapsto h^\theta$ is lower semicontinuous and so the set A_ε is a compact set. Moreover by *ii)*,

$$\{\theta^*\} = \bigcap_{\varepsilon > 0} A_\varepsilon. \tag{34}$$

We deduce that the diameter of A_ε goes to zero when ε goes to zero. Let us note that the diameter of $B_\varepsilon := A_\varepsilon \oplus B(0, g(\varepsilon) + \varepsilon)$ goes to zero as well. So to prove that $(\hat{\theta}_n)$ converges P-almost surely to θ^*, it is sufficient to show that for any $\varepsilon > 0$

$$P\left(\limsup_{n\to\infty} \left\{\hat{\theta}_n \in B_\varepsilon^c\right\}\right) = 0, \tag{35}$$

where $B_\varepsilon^c = \mathcal{K} \setminus (B_\varepsilon \cap \mathcal{K})$.

Since $\mathcal{K}$ is compact, there exists a finite sequence $(\theta_i)_{1\leq i\leq N}$ in $\mathcal{K}$ and $\{r_i\}_{1\leq i\leq N}$ in $\mathbb{R}_+$, where $r_i = r(\theta_i)$ comes from *(v)*, such that the union of balls $B(\theta_i, r_i)$ covers $\mathcal{K}$. Note that if (33) holds for some r, then it holds for any $r' \leq r$. So without loss of generality we can assume $r_i \leq \varepsilon$ for all $i = 1, \ldots, N$. Let I_ε be the subset of indexes $i \in \{1, \ldots, N\}$ such that the ball $B(\theta_i, r_i)$ intersects the set B_ε^c. We denote by $\mathcal{N}_i$ the subset $\mathcal{N}(\theta_i)$ in assumption *(v)* and by $\mathcal{N}_\varepsilon$ the union of $\mathcal{N}_i$ for $i \in I_\varepsilon$. By definition and since for all $i = 1, \ldots, N$, $r_i \leq \varepsilon$, $\mathcal{N}_i \subset B(\theta_i, g(\varepsilon))$, we have

$$B_\varepsilon^c \subset \bigcup_{i\in\mathcal{I}_\varepsilon} B(\theta_i, r_i), \quad \mathcal{N}_\varepsilon \subset \mathcal{K} \oplus B(0, g(\varepsilon)) \quad \text{and} \quad \mathcal{N}_\varepsilon \cap A_\varepsilon = \emptyset. \tag{36}$$

Let us finally consider the following sequence of random variables:

$$\delta_n^\varepsilon = \max_{i\in\mathcal{I}_\varepsilon} \left(\min_{\theta'\in\mathcal{N}_i} h_n^{\theta'} - \inf_{\theta'\in B(\theta_i, r_i)\cap\mathcal{K}} h_n^{\theta'}\right).$$

Note that from (33) and since $\mathcal{I}_\varepsilon$ is finite

$$P\left(\limsup_{n\to\infty} \left\{\delta_n^\varepsilon \geq \varepsilon\right\}\right) = 0. \tag{37}$$

To prove (35), we consider the following inclusions.

$$\begin{aligned}
\limsup_{n\to\infty} \left\{\hat{\theta}_n \in B_\varepsilon^c\right\} &\subset \limsup_{n\to\infty} \left\{\inf_{\theta\in B_\varepsilon^c} h_n^\theta \leq h_n^{\theta^*}\right\} \\
&\subset \limsup_{n\to\infty} \left\{\min_{\theta\in\mathcal{N}_\varepsilon} h_n^\theta - h_n^{\theta^*} \leq \delta_n^\varepsilon\right\} \\
&\subset \limsup_{n\to\infty} \left\{\delta_n^\varepsilon \geq \varepsilon\right\} \bigcup \left\{\min_{\theta\in\mathcal{N}_\varepsilon} h_n^\theta - h_n^{\theta^*} < \varepsilon\right\} \\
&= \limsup_{n\to\infty} \left\{\delta_n^\varepsilon \geq \varepsilon\right\} \bigcup \limsup_{n\to\infty} \left\{\min_{\theta\in\mathcal{N}_\varepsilon} h_n^\theta - h_n^{\theta^*} < \varepsilon\right\} \\
&= \limsup_{n\to\infty} \left\{\delta_n^\varepsilon \geq \varepsilon\right\} \bigcup \left\{\liminf_{n\to\infty} \left(\min_{\theta\in\mathcal{N}_\varepsilon} h_n^\theta - h_n^{\theta^*}\right) \leq \varepsilon\right\} \\
&\subset \limsup_{n\to\infty} \left\{\delta_n^\varepsilon \geq \varepsilon\right\} \bigcup \left\{\min_{\theta\in\mathcal{N}_\varepsilon} h^\theta - h^{\theta^*} \leq \varepsilon\right\} \bigcup \Omega_\varepsilon,
\end{aligned}$$

where Ω_ε is the event where h_n^θ does not converge to h^θ, when n goes to infinity, for some θ in $\mathcal{N}_\varepsilon \cup \{\theta^*\}$. Since $\mathcal{N}_\varepsilon \cup \{\theta^*\}$ is countable and by assumption *i)*, $P(\Omega_\varepsilon) = 0$. Moreover, from the

definition of A_ε and (36), the event $\left\{\min_{\theta\in\mathcal{N}_\varepsilon} h^\theta - h^{\theta^*} \le \varepsilon\right\}$ is empty. Recalling (37), we thereby deduce (35) and the lemma is proved. ■

Let us show that the assumptions *i)*, *ii)*, *iii)*, *iv)* and *v)* of Lemma 4 hold for the family of contrast functions

$$h_n^\theta(\omega^*_{\Lambda_n}) = K_n^{\hat\delta_n,\theta}(\omega^*_{\Lambda_n}) = \frac{\ln(Z^{\hat\delta_n,\theta}_{\Lambda_n})}{|\Lambda_n|} + \frac{H^\theta_{\Lambda_n}(\omega^*_{\Lambda_n})}{|\Lambda_n|}.$$

To prove *i)*, first note, from [**MeanEnergy**], [**Boundary**] and the ergodic theorem, that for any $\theta\in\Theta$ and P-almost all ω

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}H^\theta_{\Lambda_n}(\omega_{\Lambda_n}) = E_P(H^\theta_0) = H^\theta(P).$$

Second, recall from assumption [**VarPrin**] that for any $\delta\in\mathcal{I}$ and $\theta\in\Theta$, $\ln(Z^{\delta,\theta}_{\Lambda_n})/|\Lambda_n|$ converges to $p(\delta,\theta)$ and $\delta\mapsto p(\delta,\theta)$ is right continuous. On the other hand, from Lemma 3, $\hat\delta_n\to\delta^*$ almost surely with $\hat\delta_n>\delta^*$. Let $\theta\in\Theta$, $\epsilon>0$ and $\delta^+>\delta^*$ such that $|p(\delta^*,\theta)-p(\delta^+,\theta)|<\epsilon$. Let n be sufficiently large so that $\hat\delta_n<\delta^+$, $|\ln(Z^{\delta^+,\theta}_{\Lambda_n})/|\Lambda_n|-p(\delta^+,\theta)|<\epsilon$ and $|\ln(Z^{\delta^*,\theta}_{\Lambda_n})/|\Lambda_n|-p(\delta^*,\theta)|<\epsilon$. Then, since $\delta\mapsto Z^{\delta,\theta}_{\Lambda_n}$ is decreasing, we have

$$\ln(Z^{\hat\delta_n,\theta}_{\Lambda_n})/|\Lambda_n| - p(\delta^*,\theta) \le \ln(Z^{\delta^*,\theta}_{\Lambda_n})/|\Lambda_n| - p(\delta^*,\theta) < \epsilon$$

and

$$p(\delta^*,\theta) - \ln(Z^{\hat\delta_n,\theta}_{\Lambda_n})/|\Lambda_n| \le p(\delta^*,\theta) - p(\delta^+,\theta) + p(\delta^+,\theta) - \ln(Z^{\delta^+,\theta}_{\Lambda_n})/|\Lambda_n| < 2\epsilon,$$

proving that $\ln(Z^{\hat\delta_n,\theta}_{\Lambda_n})/|\Lambda_n|$ converges to $p(\delta^*,\theta)$ almost surely.

Hence for any $\theta\in\Theta$ and for P-almost every ω^*

$$\lim_{n\to\infty} h_n^\theta(\omega^*_{\Lambda_n}) = h^\theta$$

where

$$h^\theta = p(\delta^*,\theta) + H^\theta(P)$$

which proves assumption *i)*.

To prove Assumption *ii)*, note from [**VarPrin**] that for any $\theta\in\Theta$, any $\mu_{\delta^*,\theta}\in\mathcal{G}^{\delta^*,\theta}$ and any $\mu_{\delta^*,\theta^*}\in\mathcal{G}^{\delta^*,\theta^*}$

$$p(\delta^*,\theta) \ge -\mathcal{I}(\mu_{\delta^*,\theta^*}) - H^\theta(\mu_{\delta^*,\theta^*})$$

while $-\mathcal{I}(\mu_{\delta^*,\theta^*}) = p(\delta^*,\theta^*) + H^\theta(\mu_{\delta^*,\theta^*})$ so that

$$h^\theta \ge h^{\theta^*}.$$

From [**VarPrin**], the equality holds if and only if $\theta=\theta^*$ which proves Assumption *ii)*.

Assumption *iii)* is given by Lemma 2.

The function $\theta\mapsto H^\theta(P)$ is continuous thanks to (16) in assumption [**Regularity**]. By assumption [**VarPrin**] we have for any θ and θ' in Θ

$$p(\delta^*,\theta') \ge p(\delta^*,\theta) + H^\theta(\mu_{\delta^*,\theta}) - H^{\theta'}(\mu_{\delta^*,\theta}),$$

where $\mu_{\delta^*,\theta}\in\mathcal{G}^{\delta^*,\theta}$. By continuity of $\theta\mapsto H^\theta(P)$, it follows that $\theta\mapsto p(\delta^*,\theta)$ is lower semicontinuous. Therefore $\theta\mapsto h^\theta$ is lower semicontinuous and assumption *iv)* holds.

It remains to prove assumption *v)*. Let us start with some preliminary results.

Thanks to [**MeanEnergy**], [**Boundary**], (16) in [**Regularity**] and the ergodic Theorem, we have that, for any $\theta\in\Theta$, any $r>0$ and P-almost every ω,

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \sup_{\substack{\theta'\in\mathcal{K}\\ |\theta-\theta'|<r}} \left| H^{\theta}_{\Lambda_n}(\omega_{\Lambda_n}) - H^{\theta'}_{\Lambda_n}(\omega_{\Lambda_n}) \right| \leq g(r). \tag{38}$$

This inequality allows us to control the variation of the infimum of the specific energy in the contrast functions. The following lemma deals with the variation of the finite volume pressure.

Lemma 5 *For any $\eta > 0$ and any $\theta_0 \in \mathcal{K}$, there exists a finite subset $\mathcal{N}(\theta_0) \subset B(\theta_0, \eta) \cap \Theta$ and $0 < r(\theta_0) < \eta$ such that for $n \geq 1$ and any $\delta \in \mathcal{I}$*

$$\min_{\theta\in\mathcal{N}(\theta_0)} \frac{1}{|\Lambda_n|} \ln(Z^{\delta,\theta}_{\Lambda_n}) - \inf_{\theta\in B(\theta_0,r(\theta_0))\cap\mathcal{K}} \frac{1}{|\Lambda_n|} \ln(Z^{\delta,\theta}_{\Lambda_n}) \leq e^{\kappa+1} g(r(\theta_0)), \tag{39}$$

where g comes from assumption [**Regularity**] *and κ from assumption* [**Stability**].

Proof. For any $\delta \in \mathcal{I}$ and any θ, θ' in Θ

$$\begin{aligned}
\ln(Z^{\delta,\theta'}_{\Lambda}) - \ln(Z^{\delta,\theta}_{\Lambda}) &= \ln\left(\int \frac{1}{Z^{\delta,\theta}_{\Lambda}} e^{-H^{\theta'}_{\Lambda}(\omega_\Lambda)} \mathbb{1}_{\Omega^{\delta}_{\infty}}(\omega_\Lambda) \pi_\Lambda(d\omega_\Lambda)\right) \\
&= \ln\left(\int e^{-H^{\theta'}_{\Lambda}(\omega_\Lambda)+H^{\theta}_{\Lambda}(\omega_\Lambda)} \mathbb{1}_{\Omega^{\delta}_{\infty}}(\omega_\Lambda) \mu^{\delta,\theta}_{\Lambda}(d\omega_\Lambda)\right) \\
&= \ln E_{\mu^{\delta,\theta}_{\Lambda}}(e^{-H^{\theta'}_{\Lambda}+H^{\theta}_{\Lambda}})
\end{aligned} \tag{40}$$

where $\mu^{\delta,\theta}_{\Lambda}$ is the probability measure $(1/Z^{\delta,\theta}_{\Lambda}) e^{-H^{\theta}_{\Lambda}} \mathbb{1}_{\Omega^{\delta}_{\infty}} \pi_\Lambda$. By Jensen's inequality we have that

$$\ln(Z^{\delta,\theta'}_{\Lambda}) - \ln(Z^{\delta,\theta}_{\Lambda}) \geq E_{\mu^{\delta,\theta}_{\Lambda}}(H^{\theta}_{\Lambda} - H^{\theta'}_{\Lambda}).$$

For any $\eta > 0$ and θ_0 in $\mathcal{K}$ we choose $\mathcal{N}(\theta_0)$ and $r(\theta_0) > 0$ as in (17). Denoting $[x]_+ = \max(x, 0)$ and writing for short $\mathcal{N}$ for $\mathcal{N}(\theta_0)$ and B for $B(\theta_0, r(\theta_0)) \cap \mathcal{K}$, we obtain, for any $\delta \geq \delta_{\min}$,

$$\begin{aligned}
&\min_{\theta\in\mathcal{N}} \frac{1}{|\Lambda_n|} \ln(Z^{\delta,\theta}_{\Lambda_n}) - \inf_{\theta\in B} \frac{1}{|\Lambda_n|} \ln(Z^{\delta,\theta}_{\Lambda_n}) \\
&\quad = \min_{\theta\in\mathcal{N}} \sup_{\theta'\in B} \frac{1}{|\Lambda_n|} \left(\ln(Z^{\delta,\theta}_{\Lambda_n}) - \ln(Z^{\delta,\theta'}_{\Lambda_n})\right) \\
&\quad \leq \min_{\theta\in\mathcal{N}} \sup_{\theta'\in B} \frac{1}{|\Lambda_n|} \left(E_{\mu^{\delta,\theta}_{\Lambda_n}}(H^{\theta'}_{\Lambda_n} - H^{\theta}_{\Lambda_n})\right) \\
&\quad = \frac{1}{|\Lambda_n|} \min_{\theta\in\mathcal{N}} \sup_{\theta'\in B} E_{\mu^{\delta,\theta}_{\Lambda_n}} \left(\frac{H^{\theta'}_{\Lambda_n} - H^{\theta}_{\Lambda_n}}{N_{\Lambda_n}} N_{\Lambda_n}\right) \\
&\quad \leq \frac{1}{|\Lambda_n|} \min_{\theta\in\mathcal{N}} \sup_{\theta'\in B} \sup_{\omega_{\Lambda_n}\in\Omega^{\delta}_{\infty}} \left[\frac{H^{\theta'}_{\Lambda_n}(\omega_{\Lambda_n}) - H^{\theta}_{\Lambda_n}(\omega_{\Lambda_n})}{N_{\Lambda_n}(\omega_{\Lambda_n})}\right]_+ E_{\mu^{\delta,\theta}_{\Lambda_n}}(N_{\Lambda_n}) \\
&\quad \leq \frac{1}{|\Lambda_n|} \left(\max_{\theta\in\mathcal{N}} E_{\mu^{\delta,\theta}_{\Lambda_n}}(N_{\Lambda_n})\right) \left[\min_{\theta\in\mathcal{N}} \sup_{\theta'\in B} \sup_{\omega_{\Lambda_n}\in\Omega^{\delta}_{\infty}} \left(\frac{H^{\theta'}_{\Lambda_n}(\omega_{\Lambda_n}) - H^{\theta}_{\Lambda_n}(\omega_{\Lambda_n})}{N_{\Lambda_n}(\omega_{\Lambda_n})}\right)\right]_+ \\
&\quad = \frac{1}{|\Lambda_n|} \left(\max_{\theta\in\mathcal{N}} E_{\mu^{\delta,\theta}_{\Lambda_n}}(N_{\Lambda_n})\right) \left[-\max_{\theta\in\mathcal{N}} \inf_{\theta'\in B} \inf_{\omega_{\Lambda_n}\in\Omega^{\delta}_{\infty}} \left(\frac{H^{\theta}_{\Lambda_n}(\omega_{\Lambda_n}) - H^{\theta'}_{\Lambda_n}(\omega_{\Lambda_n})}{N_{\Lambda_n}(\omega_{\Lambda_n})}\right)\right]_+ \\
&\quad \leq \frac{1}{|\Lambda_n|} \left(\max_{\theta\in\mathcal{N}} E_{\mu^{\delta,\theta}_{\Lambda_n}}(N_{\Lambda_n})\right) g(r(\theta_0)).
\end{aligned} \tag{41}$$

Let us control $E_{\mu^{\delta,\theta}_{\Lambda_n}}(N_{\Lambda_n})$ by entropy inequalities. By definition of the entropy, the assumption **[Stability]** and the standard inequality $Z^{\delta,\theta}_\Lambda \geq \pi_\Lambda(\{\emptyset\}) = e^{-|\Lambda|}$, we find that

$$\begin{aligned}\mathcal{I}_\Lambda(\mu^{\delta,\theta}_\Lambda) &= \int \ln\left(\frac{e^{-H^\theta_\Lambda}\mathbb{1}_{\Omega^\delta_\infty}(\omega_\Lambda)}{Z^{\delta,\theta}_\Lambda}\right)\mu^{\delta,\theta}_\Lambda \\ &\leq -\ln(Z^{\delta,\theta}_\Lambda) - \int H^\theta_\Lambda\,\mu^{\delta,\theta}_\Lambda \\ &\leq |\Lambda| + \kappa E_{\mu^{\delta,\theta}_\Lambda}(N_\Lambda). \end{aligned} \tag{42}$$

The entropy is also characterised by

$$\mathcal{I}_\Lambda(\mu^{\delta,\theta}_\Lambda) = \sup_g \left(\int g\mu^{\delta,\theta}_\Lambda - \ln\left(\int e^g \pi_\Lambda\right)\right)$$

where the supremum is over all bounded measurable functions g, see for example [26]. Choosing $g = (\kappa+1)N_\Lambda \mathbb{1}_{N_\Lambda \leq C}$ for any constant $C > 0$, we get

$$\begin{aligned}\mathcal{I}_\Lambda(\mu^{\delta,\theta}_\Lambda) &\geq (\kappa+1)E_{\mu^{\delta,\theta}_\Lambda}(N_\Lambda \mathbb{1}_{N_\Lambda\leq C}) - \ln\left(\int e^{(\kappa+1)N_\Lambda \mathbb{1}_{N_\Lambda\leq C}}\pi_\Lambda\right) \\ &\geq (\kappa+1)E_{\mu^{\delta,\theta}_\Lambda}(N_\Lambda \mathbb{1}_{N_\Lambda\leq C}) - (e^{\kappa+1}-1)|\Lambda|. \end{aligned} \tag{43}$$

Let C goes to infinity and combine (42) and (43), we obtain that for any (δ,θ)

$$E_{\mu^{\delta,\theta}_\Lambda}(N_\Lambda) \leq e^{\kappa+1}|\Lambda|. \tag{44}$$

The inequalities (41) and (44) imply (39). ■

We are now in position to prove *v)*. The function g_0 in *v)* depends on g in (38) and Lemma 5 as follows. For any $x > 0$, we choose $g_0(x) > 0$ such that

$$g(2\,g_0(x)) + e^{\kappa+1}g(g_0(x)) < x.$$

This choice is always possible since $g(u)$ tends to 0 when u goes to 0.

Let $\varepsilon > 0$ and $\theta_0 \in \mathcal{K}$ and consider the finite subset $\mathcal{N}(\theta_0) \subset B(\theta_0,\eta)\cap\Theta$ and the positive number $r(\theta_0)$, with $r(\theta_0) < \eta$, given by Lemma 5 where $\eta = g_0(\varepsilon)$. In the following, we write for short $\mathcal{N}$ for $\mathcal{N}(\theta_0)$ and B for $B(\theta_0, r(\theta_0))\cap\mathcal{K}$. Using (38), Lemma 5 and the definition of g_0, we have that for P-almost every ω,

$$\begin{aligned}&\limsup_{n\to\infty}\left(\min_{\theta\in\mathcal{N}} h^\theta_n(\omega_{\Lambda_n}) - \inf_{\theta\in B} h^\theta_n(\omega_{\Lambda_n})\right) \\ &= \limsup_{n\to\infty}\frac{1}{|\Lambda_n|}\left(\min_{\theta\in\mathcal{N}}\left(H^\theta_{\Lambda_n}(\omega_{\Lambda_n}) + \ln(Z^{\hat\delta_n,\theta}_{\Lambda_n})\right) - \inf_{\theta\in B}\left(H^\theta_{\Lambda_n}(\omega_{\Lambda_n}) + \ln(Z^{\hat\delta_n,\theta}_{\Lambda_n})\right)\right) \\ &\leq \limsup_{n\to\infty}\frac{1}{|\Lambda_n|}\left(\max_{\theta\in\mathcal{N}} H^\theta_{\Lambda_n}(\omega_{\Lambda_n}) - \inf_{\theta\in B} H^\theta_{\Lambda_n}(\omega_{\Lambda_n}) + \min_{\theta\in\mathcal{N}}\ln(Z^{\hat\delta_n,\theta}_{\Lambda_n}) - \inf_{\theta\in B}\ln(Z^{\hat\delta_n,\theta}_{\Lambda_n})\right) \\ &\leq \max_{\theta\in\mathcal{N}}\limsup_{n\to\infty}\frac{1}{|\Lambda_n|}\sup_{\theta'\in B}\left|H^\theta_{\Lambda_n}(\omega_{\Lambda_n}) - H^{\theta'}_{\Lambda_n}(\omega_{\Lambda_n})\right| + \limsup_{n\to\infty}\frac{1}{|\Lambda_n|}\left(\min_{\theta\in\mathcal{N}}\ln(Z^{\hat\delta_n,\theta}_{\Lambda_n}) - \inf_{\theta\in B}\ln(Z^{\hat\delta_n,\theta}_{\Lambda_n})\right) \\ &\leq g(2\,g_0(\varepsilon)) + e^{\kappa+1}g(g_0(\varepsilon)) < \varepsilon,\end{aligned}$$

which proves *v)*.

5.2 Proof of Corollary 1

To apply Theorem 1, we have to prove that **[Argmax]** and **[Regularity]** hold true in the setting of exponential models. Assumptions **[Argmax]** and the inequality (16) in **[Regularity]** are obviously satisfied since the energy H^θ_Λ and the mean energy H^θ_0 are linear in θ.

It remains to check (17) in [**Regularity**]. Let $\mathcal{K}$ be a compact subset of Θ and $\epsilon > 0$ such that $\mathcal{K} \oplus B(0,\epsilon) \subset \Theta$. For any $\eta > 0$ and $\theta_0 \in \mathcal{K}$, we choose r sufficiently small such that $\theta := (1 + r/\epsilon)\theta_0 \in B(\theta_0, \eta) \cap \Theta$ and we set $\mathcal{N}(\theta_0) = \{\theta\}$. Note that for any $\theta' \in \mathcal{B}(\theta_0, r)$, $\frac{\epsilon}{r}(\theta - \theta') \in B(\theta_0, \epsilon) \subset \mathcal{K} \oplus B(0,\epsilon)$. Then from (20) and the assumption [**Stability**], for any $\theta' \in B(\theta_0, r)$, for any $\omega_\Lambda \in \Omega_\infty^{\delta_{\min}}$,

$$\frac{\epsilon}{r}(H_\Lambda^\theta(\omega_\Lambda) - H_\Lambda^{\theta'}(\omega_\Lambda)) = H_\Lambda^{\epsilon(\theta_0-\theta')/r}(\omega_\Lambda) \geq -\kappa N_\Lambda(\omega_\Lambda)$$

which implies $(H_\Lambda^\theta(\omega_\Lambda) - H_\Lambda^{\theta'}(\omega_\Lambda)) \geq -r\frac{\kappa}{\epsilon}N_\Lambda(\omega_\Lambda)$ and thus (17).

5.3 Proof of Theorem 2

We check the assumptions of the more general Theorem 1 with the choice $\Theta = \mathcal{Z} \times \Upsilon$ where $\mathcal{Z}$ and Υ are bounded open subsets of $(0,\infty)$ and $\mathcal{B} \times \mathcal{R}$ respectively satisfying $\mathcal{Z} \times \Upsilon \supset \mathcal{K}$. We denote in the following $\theta = (z, \beta, R)$, where $z \in \mathcal{Z}$ and $(\beta, R) \in \Upsilon$.

For any $\delta \in \mathcal{I}$ and $(\beta, R) \in \Upsilon$, the pairwise interaction $\infty \mathbb{I}_{[0,\delta)} + \phi^{\beta,R}$ is superstable, finite range and bounded from above. The existence of Gibbs measures in this setting, i.e. assumption [**Existence**], is proved for example in [24]. Moreover, we can deduce the following useful result from Corollary 2.9 in [24]: for any $P \in \mathcal{G}^{\delta,\theta}$, for any bounded set Δ and any $c > 0$,

$$E_P\left(e^{cN_\Delta}\right) < \infty. \tag{45}$$

Superstability implies [**Stability**], where the lower bound can be chosen uniformly over Θ since this set is bounded. The decomposition in [**MeanEnergy**] can be done by

$$H_{\Lambda_n}^\theta(\omega) = \sum_{k \in \mathcal{I}_n} H_0^\theta \circ \tau_{-k}(\omega) + \partial H_{\Lambda_n}^\theta(\omega) \tag{46}$$

where

$$H_0^\theta(\omega) = z\, N_{\Delta_0}(\omega) + \sum_{\{x,y\} \in \omega_{\Delta_0}} \phi^{\beta,R}(x-y) + \frac{1}{2} \sum_{x \in \omega_{\Delta_0}, y \in \omega_{\Delta_0^c}} \phi^{\beta,R}(x-y) \tag{47}$$

and

$$\partial H_{\Lambda_n}^\theta(\omega) = \frac{1}{2} \sum_{x \in \omega_{\Lambda_n}, y \in \omega_{\Lambda_n^c}} \phi^{\beta,R}(x-y).$$

The finiteness of the mean energy given by (15) is easily deduced from the finite range and boundedness of $\phi^{\beta,R}$ along with (45) that implies finite second order moments for P. On the other hand from (2) we have

$$H_{\Lambda_n}^\theta(\omega) - H_{\Lambda_n}^\theta(\omega_{\Lambda_n}) = \sum_{x \in \omega_{\Lambda_n}, y \in \omega_{\Lambda_n^c}} \phi^{\beta,R}(x-y).$$

Therefore [**MeanEnergy**] and [**Boundary**] reduce to prove that for P-almost every $\omega \in \Omega_T$

$$\sup_{(\beta,R) \in \Upsilon} \frac{1}{|\Lambda_n|} \sum_{x \in \omega_{\Lambda_n}, y \in \omega_{\Lambda_n^c}} \phi^{\beta,R}(x-y) \to 0. \tag{48}$$

Recall the definition of $I_n = \{-n, \ldots, n-1\}^d$. Since $\phi^{\beta,R}$ has finite range and is continuous, there exist two constant c and $\bar{r}$ large enough such that

$$\begin{aligned}\sup_{(\beta,R) \in \Upsilon} \frac{1}{|\Lambda_n|} \sum_{x \in \omega_{\Lambda_n}, y \in \omega_{\Lambda_n^c}} |\phi^{\beta,R}(x-y)| &\leq \frac{c}{|\Lambda_n|} \sum_{i \in I_n \setminus I_{n-1}} N_{B(i,\bar{r}) \cap \Lambda_n}(\omega) N_{B(i,\bar{r}) \cap \Lambda_n^c}(\omega) \\ &\leq \frac{c}{|\Lambda_n|} \sum_{i \in I_n \setminus I_{n-1}} N_{B(i,\bar{r})}^2(\omega).\end{aligned}$$

From (45), for any $P \in \mathcal{G}^{\delta,\theta}$, $E_P(N^2_{B(i,\bar{r})})$ is finite and by stationarity $E_P(N^2_{B(i,\bar{r})}) = E_P(N^2_{B(0,\bar{r})})$. The ergodic theorem thus implies that P almost surely

$$\frac{1}{|\Lambda_n|} \sum_{i \in I_n} N^2_{B(i,\bar{r})}(\omega) \to E_P(N^2_{B(0,\bar{r})})$$

which yields (48).

Assumption [**Argmax**] holds because $(\beta, R) \mapsto \phi^{\beta,R}(x)$ is lower semicontinuous by construction and the number of discontinuities are finite, so it is easy to change the sense of the brackets in (22) to obtain an upper semicontinuous version.

It remains to check (a) [**Regularity**] and (b) [**VarPrin**].

(a) We start to prove (16) in [**Regularity**]. Note from (47) that

$$\begin{aligned} H_0^{\theta}(\omega) - H_0^{\theta'}(\omega) &= (z - z')\, N_{\Delta_0}(\omega) \\ &\quad + \sum_{\{x,y\} \in \omega_{\Delta_0}} (\phi^{\beta,R} - \phi^{\beta',R'})(x-y) + \frac{1}{2} \sum_{x \in \omega_{\Delta_0}, y \in \omega_{\Delta_0^c}} (\phi^{\beta,R} - \phi^{\beta',R'})(x-y). \end{aligned} \tag{49}$$

If $|\theta - \theta'| < r$, obviously there exists a positive function g with $\lim_{s \to 0} g(s) = 0$ such that

$$|z - z'|\, N_{\Delta_0}(\omega) \le g(r) N_{\Delta_0}(\omega). \tag{50}$$

The two sums in the right hand side of (49) can be handled similarly. We give the details for the first one only, which reduces from (22) to consider the following generic term for P-almost every ω

$$\begin{aligned} &\sum_{\{x,y\} \in \omega_{\Delta_0}} |\mathbb{1}_{(R_{k-1},R_k)}(|x-y|)\, \varphi_k^{\beta,R}(x-y) - \mathbb{1}_{(R'_{k-1},R'_k)}(|x-y|)\, \varphi_k^{\beta',R'}(x-y)| \\ &\le \sum_{\{x,y\} \in \omega_{\Delta_0}} |\varphi_k^{\beta,R} - \varphi_k^{\beta',R'}|(x-y) + c \sum_{\{x,y\} \in \omega_{\Delta_0}} |\mathbb{1}_{(R_{k-1},R_k)} - \mathbb{1}_{(R'_{k-1},R'_k)}|(|x-y|) \end{aligned}$$

where c is a positive constant that comes from the boundedness of $\varphi_k^{\beta,R}$. Since the map $(\beta, R, x) \mapsto \varphi_k^{\beta,R}(|x|)$ is continuous it is uniformly continuous on any compact set and there exists a positive function g, independent of x, y and θ, such that $\lim_{s \to 0} g(s) = 0$, and if $|\theta - \theta'| < r$ then

$$\sum_{\{x,y\} \in \omega_{\Delta_0}} |\varphi_k^{\beta,R} - \varphi_k^{\beta',R'}|(x-y) \le \sum_{\{x,y\} \in \omega_{\Delta_0}} g(r) \le g(r) N^2_{\Delta_0}(\omega). \tag{51}$$

On the other hand, denoting ΔR_k the symmetric difference $(R_{k-1}, R_k) \Delta (R'_{k-1}, R'_k)$, we have for any $R \in \mathcal{R}$

$$\begin{aligned} E_P\left(\sup_{\substack{R' \in \mathcal{R} \\ |R-R'| \le r}} \sum_{\{x,y\} \in \omega_{\Delta_0}} \mathbb{1}_{\Delta R_k}(|x-y|) \right) &= E_P\left(\sum_{x \in \omega_{\Delta_0}} \sup_{\substack{R' \in \mathcal{R} \\ |R-R'| \le r}} \sum_{y \in \omega_{\Delta_0}, y \ne x} \mathbb{1}_{\Delta R_k}(|x-y|) \right) \\ &= E_P\left(\int_{\Delta_0} e^{-h^{\theta}(x|\omega)} \sup_{\substack{R' \in \mathcal{R} \\ |R-R'| \le r}} \sum_{y \in \omega_{\Delta_0}} \mathbb{1}_{\Delta R_k}(|x-y|)\, dx \right) \end{aligned}$$

where the last equality comes from the Georgii-Nguyen-Zessin (GNZ) formula [6, 19] and $h^{\theta}(x|\omega)$ denotes the local energy needed to insert the point x into the configuration ω, defined for any

$\Lambda \ni x$ by $h^\theta(x|\omega) = H^\theta_\Lambda(\omega \cup x) - H^\theta_\Lambda(\omega)$. Applying again the GNZ formula we obtain

$$\begin{aligned}
&E_P\left(\sup_{\substack{R'\in\mathcal{R}\\|R-R'|\le r}} \sum_{\{x,y\}\in\omega_{\Delta_0}} \mathbb{1}_{\Delta R_k}(|x-y|)\right)\\
&= E_P\left(\int_{\Delta_0^2} e^{-h^\theta(y|\omega)} e^{-h^\theta(x|\omega\cup\{y\})} \sup_{\substack{R'\in\mathcal{R}\\|R-R'|\le r}} \mathbb{1}_{\Delta R_k}(|x-y|)dxdy\right)\\
&\le E_P\left(e^{cN_{\Delta_0\oplus B(0,\bar r)}}\right)\int_{\Delta_0^2} \sup_{\substack{R'\in\mathcal{R}\\|R-R'|\le r}} \mathbb{1}_{\Delta R_k}(|x-y|)dxdy\\
&\le E_P\left(e^{cN_{\Delta_0\oplus B(0,\bar r)}}\right)|\Delta_0||\mathcal{A}_k(r)| \qquad (52)
\end{aligned}$$

where c and $\bar r$ are two constants large enough and $\mathcal{A}_k(r)$ is the union of the two rings $\{y \in \mathbb{R}^d,\ R_{k-1} - r \le |y| \le R_{k-1} + r\}$ and $\{y,\ R_k - r \le |y| \le R_k + r\}$. Thanks to (45) and since $|\mathcal{A}_k(r)| \to 0$ as $r \to 0$, the upper bound in (52) tends to 0 as $r \to 0$. The same conclusion holds true in (50) and (51) because $E_P(N_{\Delta_0}) < \infty$ and $E_P(N^2_{\Delta_0}) < \infty$. The combination of these results shows (16).

Let us now prove (17) in [**Regularity**]. Let $\eta > 0$ and $\theta_0 = (z_0, \beta_0, R_0) \in \mathcal{K}$ where $R_0 = (R_{0,1}, \dots, R_{0,q})$. We choose r sufficiently small to ensure that

(i) $r < \eta$,

(ii) $B(\theta_0, r) \subset \Theta$,

(iii) $2r < \inf_{k=1,\dots,q} |R_{0,k} - R_{0,k-1}|$,

(iv) for all $k = 1, \dots, q$, if $\varphi_k^{\beta_0,R_0}(R_{0,k}) \ne \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$, then for all $\theta', \theta'' \in B(\theta_0, r)$ and for all x such that $||x| - R_{0,k}| < r$, we have

$$\left(\varphi_k^{\beta',R'}(|x|) - \varphi_{k+1}^{\beta'',R''}(|x|)\right)\left(\varphi_k^{\beta_0,R_0}(R_{0,k}) - \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})\right) > 0,$$

which is possible by continuity of $(\beta, R, x) \mapsto \varphi_k^{\beta,R}(|x|)$.

Then we fix $\mathcal{N}(\theta_0) = \{(z_0, \beta_0, R)\}$ where $R = (R_1, \dots, R_q)$ is defined as follows : if $\varphi_k^{\beta_0,R_0}(R_{0,k}) > \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$ then $R_k = R_{0,k} + r$, if $\varphi_k^{\beta_0,R_0}(R_{0,k}) < \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$ then $R_k = R_{0,k} - r$, and if $\varphi_k^{\beta_0,R_0}(R_{0,k}) = \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$ then $R_k = R_{0,k}$.

For $\theta = (z_0, \beta_0, R)$ and $\theta' \in B(\theta_0, r)$, the proof of (17) amounts to control

$$\frac{H^\theta_\Lambda(\omega_\Lambda) - H^{\theta'}_\Lambda(\omega_\Lambda)}{N_\Lambda(\omega_\Lambda)} = (z_0 - z') + \frac{1}{N_\Lambda(\omega_\Lambda)} \sum_{\{x,y\}\in\omega,\{x,y\}\cap\omega_\Lambda\ne\emptyset} \left(\phi^{\beta,R}(x-y) - \phi^{\beta',R'}(x-y)\right). \quad (53)$$

The first term $(z_0 - z')$ is greater than $-g(r)$ for a positive function g with $\lim_{s\to 0} g(s) = 0$. For the second term, let us introduce the middle points $\bar R_{0,k} = \frac{1}{2}(R_{0,k-1} + R_{0,k})$ for $k = 1, \dots, q-1$, $\bar R_{0,0} = 0$ and $\bar R_{0,q} = R_{0,q} + 2r$. Note that from our choice of r, for any k both R_k and R'_k belong to $[\bar R_{0,k}, \bar R_{0,k+1}]$. From (22), we can write

$$\begin{aligned}
\phi^{\beta,R}(x) - \phi^{\beta',R'}(x) &= \sum_{k=0}^{q} \mathbb{1}_{[\bar R_{0,k},\bar R_{0,k+1}]}(|x|)(\phi^{\beta,R}(x) - \phi^{\beta',R'}(x))\\
&= \sum_{k=0}^{q} \Big(\mathbb{1}_{[\bar R_{0,k},R_k)}(|x|)\varphi_k^{\beta,R}(|x|) + \mathbb{1}_{[R_k,\bar R_{0,k+1}]}(|x|)\varphi_{k+1}^{\beta,R}(|x|)\\
&\quad - \mathbb{1}_{[\bar R_{0,k},R'_k)}(|x|)\varphi_k^{\beta',R'}(|x|) - \mathbb{1}_{[R'_k,\bar R_{0,k+1}]}(|x|)\varphi_{k+1}^{\beta',R'}(|x|)\Big). \qquad (54)
\end{aligned}$$

If $\varphi_k^{\beta_0,R_0}(R_{0,k}) > \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$, then $R_k > R'_k$ and the k-th term in the sum above writes

$$\mathbb{1}_{[\bar{R}_{0,k},R'_k)}(|x|)\left(\varphi_k^{\beta,R}(|x|)-\varphi_k^{\beta',R'}(|x|)\right)+\mathbb{1}_{[R'_k,R_k]}(|x|)\left(\varphi_k^{\beta,R}(|x|)-\varphi_{k+1}^{\beta',R'}(|x|)\right)$$
$$+\mathbb{1}_{[R_k,\bar{R}_{0,k+1}]}(|x|)\left(\varphi_{k+1}^{\beta,R}(|x|)-\varphi_{k+1}^{\beta',R'}(|x|)\right).$$

From our choice of r, see (iv), the second term is always positive, and by uniform continuity of $(\beta,R,x)\mapsto\varphi_k^{\beta,R}(|x|)$, there exists a positive function g, independent of x, with $\lim_{s\to 0}g(s)=0$, such that the two remaining terms are greater than $-g(r)\mathbb{1}_{[0,\bar{R}_{0,q}]}(|x|)$. We thus obtain that the k-th term in (54) is greater than $-g(r)\mathbb{1}_{[0,\bar{R}_{0,q}]}(|x|)$.

If $\varphi_k^{\beta_0,R_0}(R_{0,k}) < \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$, we obtain the same lower bound by using the fact that $R_k < R'_k$. If $\varphi_k^{\beta_0,R_0}(R_{0,k}) = \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$, then by uniform continuity $\left|\varphi_k^{\beta,R}(|x|)-\varphi_{k'}^{\beta',R'}(|y|)\right| < g(r)$ for any $k'=k,k+1$, any $|x|,|y|$ in $[\bar{R}_{0,k},\bar{R}_{0,k+1}]$ and any θ,θ' in $B(\theta_0,r)$, and the same lower bound holds for the the k-th term in (54).

Coming back to (53), we deduce that

$$\frac{H_\Lambda^\theta(\omega_\Lambda)-H_\Lambda^{\theta'}(\omega_\Lambda)}{N_\Lambda(\omega_\Lambda)} > -g(r)-(q+1)g(r)\frac{1}{N_\Lambda(\omega_\Lambda)}\sum_{\{x,y\}\in\omega,\{x,y\}\cap\omega_\Lambda\neq\emptyset}\mathbb{1}_{[0,\bar{R}_{0,q}]}(|x-y|).$$

For any $\omega\in\Omega_\infty^{\delta_{\min}}$, there exists $c>0$ such that for any $x\in\omega$, $\sum_{y\in\omega,y\neq x}\mathbb{1}_{[0,\bar{R}_{0,q}]}(|x-y|)\le c$ because of the hardcore distance $\delta_{\min}>0$. Consequently the lower bound above is greater than $-g(r)$, up to a positive constant, and this completes the proof.

(b) The variational principle in this setting is proved in [8], which implies (19) with equality if and only if $\mu^{\delta',\theta'}=\mu^{\delta,\theta}$. When $(\delta',\theta')=(\delta^*,\theta^*)$, $\mu^{\delta^*,\theta^*}=\mu^{\delta,\theta}$ is equivalent to $(\delta,\theta)=(\delta^*,\theta^*)$ from our identifiability assumption (23). Hence, in order to verify **[VarPrin]**, it remains to prove that the pressure (18) is right continuous in δ.

For any $\delta>0$ and $\theta\in\Theta$, from the rescaling $\omega\to\delta\omega$,

$$\begin{aligned}Z_{\Lambda_n}^{\delta,\theta}&=\int e^{-H_{\Lambda_n}^\theta(\omega_{\Lambda_n})}\mathbb{1}_{\Omega_\infty^\delta}(\omega_{\Lambda_n})\pi_{\Lambda_n}(d\omega)\\&=e^{-|\Lambda_n|(1-1/\delta^d)}\int e^{-H_{\Lambda_n}^\theta((\delta\omega)_{\Lambda_n})}\mathbb{1}_{\Omega_\infty^1}(\omega_{\Lambda_n})\delta^{N_{\frac{1}{\delta}\Lambda_n}(\omega)}\pi_{\frac{1}{\delta}\Lambda_n}(d\omega)\\&=e^{-|\Lambda_n|(1-1/\delta^d)}\tilde{Z}_{\frac{1}{\delta}\Lambda_n}^{\delta,\theta},\end{aligned}$$

where $\tilde{Z}_\Lambda^{\delta,\theta}=\int e^{-\tilde{H}_\Lambda^{\delta,\theta}(\omega_\Lambda)}\mathbb{1}_{\Omega_\infty^1}(\omega_\Lambda)\pi_\Lambda(d\omega_\Lambda)$ is the partition function associated to the Gibbs measure defined on Ω_∞^1 with energy function $\tilde{H}_\Lambda^{\delta,\theta}(\omega_\Lambda)=H_{\delta\Lambda}^\theta(\delta(\omega_\Lambda))-N_\Lambda(\omega)\ln(\delta)$. Denoting $\tilde{p}(\delta,\theta)$ the pressure associated to the latter measure, we deduce

$$p(\delta,\theta)=\frac{1}{\delta^d}\,\tilde{p}(\delta,\theta)-1+1/\delta^d. \tag{55}$$

The energy function $\tilde{H}_{\Lambda_n}^{\delta,\theta}$ follows a decomposition like (14) where

$$\tilde{H}_0^{\delta,\theta}(\omega)=(z-\ln(\delta))\,N_{\Delta_0}(\omega)+\sum_{\{x,y\}\in\omega_{\delta\Delta_0}}\phi^{\beta,R}(\delta x-\delta y)+\frac{1}{2}\sum_{x\in\omega_{\delta\Delta_0},y\in\omega_{\delta\Delta_0^c}}\phi^{\beta,R}(\delta x-\delta y).$$

By similar arguments as used earlier in (a) to prove (16), we deduce that $\delta\mapsto E_P(\tilde{H}_0^{\delta,\theta})$ is continuous. Moreover, $\tilde{H}_{\Lambda_n}^{\delta,\theta}$ inherits the superstable, lower regular and regular properties of $\phi^{\beta,R}$, which shows from [8] that a variational principle holds, i.e. for any (δ,θ) and (δ',θ')

$$\tilde{p}(\delta',\theta')\ge\tilde{p}(\delta,\theta)+\tilde{H}^{\delta,\theta}(\tilde{\mu}_{\delta,\theta})-\tilde{H}^{\delta',\theta'}(\tilde{\mu}_{\delta,\theta}),$$

where $\tilde{\mu}_{\delta,\theta}$ denotes a Gibbs measure associated to $\tilde{H}^{\delta,\theta}_{\Lambda_n}$. Note that contrary to (19), this inequality is valid for any δ, δ' because $\tilde{\mu}_{\delta,\theta}$ is supported on Ω^1_∞ for any (δ, θ). We therefore deduce that $\delta \mapsto \tilde{p}(\delta, \theta)$ is lower semicontinuous, and so is $\delta \mapsto p(\delta, \theta)$ from (55). Since $\delta \mapsto Z^{\delta,\theta}_{\Lambda_n}$ is decreasing, $\delta \mapsto p(\delta, \theta)$ turns out to be both decreasing and lower semicontinuous, which shows that it is right continuous.

5.4 Proof of Theorem 3

As in the previous section, it is sufficient to check the assumption of the main Theorem 1. The assumption [**Stability**] is included in the assumptions of Theorem 3 and has not to be showed. The proofs of assumptions [**Existence**], [**MeanEnergy**], [**Boundary**], [**Argmax**] and the first part of [**Regularity**], i.e. (16), are the same as in the proof of Theorem 2. The variational principle is shown in [4] for any finite range stable pair potential. Since $\mathcal{I} = \{0\}$, the right continuity of the pressure is not required here, so Assumption [**VarPrin**] follows. It remains to prove the second part of [**Regularity**], i.e. (17).

Let $\eta > 0$ and $\theta_0 = (z_0, \beta_0, R_0) \in \mathcal{K}$ where $R_0 = (R_{0,1}, \dots, R_{0,q})$. We denote by $\bar{R}$ the maximal value of R_q when $\theta = (z, \beta, R) \in \mathcal{K}$. We choose $r > 0$ small enough such that

(i) $r < \eta$,

(ii) $B(\theta_0, r) \subset \Theta$,

(iii) $2r < \inf_{k=1,\dots,q} |R_{0,k} - R_{0,k-1}|$,

(iv) there exists $\beta_1 \in \mathcal{B}$ such that for any $\theta' \in B(\theta_0, r)$, any $\theta = (z, \beta, R) \in B((z_0, \beta_1, R_0), \sqrt{qr})$, any $k = 1, \dots, q$ and any $x \in B(0, \bar{R})$

$$\varphi_k^{(\beta,R)}(x) \geq \varphi_k^{(\beta',R')}(x),$$

which is possible by continuity of $(\beta, R, x) \mapsto \varphi_k^{\beta,R}(|x|)$ and thanks to the local positivity assumption (24).

(v) for all $k = 1, \dots, q$, if $\varphi_k^{\beta_0,R_0}(R_{0,k}) \neq \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$, then for all $\theta', \theta'' \in B(\theta_0, r) \cup B((z_0, \beta_1, R_0), \sqrt{qr})$ and for all x such that $||x| - R_{0,k}| < r$, we have

$$\left(\varphi_k^{\beta',R'}(|x|) - \varphi_{k+1}^{\beta'',R''}(|x|)\right)\left(\varphi_k^{\beta_0,R_0}(R_{0,k}) - \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})\right) > 0,$$

which is possible by continuity of $(\beta, R, x) \mapsto \varphi_k^{\beta,R}(|x|)$ and since β_1 can be chosen as close as we want to β_0.

(vi) for all $k = 1, \dots, q$, if $\varphi_k^{\beta_0,R_0}(R_{0,k}) = \varphi_{k+1}^{\beta_0,R_0}(R_{0,k})$, then for all $\theta' \in B(\theta_0, r)$, $\theta \in B((z_0, \beta_1, R_0), \sqrt{qr})$ and for all x such that $||x| - R_{0,k}| < r$,

$$\varphi_k^{(\beta,R)}(x) \geq \varphi_{k+1}^{(\beta',R')}(x) \quad \text{and} \quad \varphi_{k+1}^{(\beta,R)}(x) \geq \varphi_k^{(\beta',R')}(x),$$

which is possible for the same reasons as in (iv).

Now we choose $\mathcal{N}(\theta_0) = \{(z_0, \beta_1, R)\}$ where $R = (R_1, R_2, \dots, R_q)$ is as in the proof of Theorem 2, see (53) and before. Note that (z_0, β_1, R) is in $B((z_0, \beta_1, R_0), \sqrt{qr})$ and that β_1 can be chosen sufficiently close to β_0 to ensure $\mathcal{N}(\theta_0) \subset B(\theta_0, \eta)$. Following the same calculus as in (54), we obtain that for any $\theta' = (z', \beta', R') \in B(\theta_0, r)$

$$\phi^{\beta_1,R}(x) \geq \phi^{\beta',R'}(x).$$

We deduce that

$$\frac{H_\Lambda^{\theta_0}(\omega_\Lambda) - H_\Lambda^{\theta'}(\omega_\Lambda)}{N_\Lambda(\omega_\Lambda)} > z_0 - z' > -r$$

and the second part of [**Regularity**] is proved.

5.5 Proof of Theorem 4

We check the assumptions of Theorem 1. For any $\beta \in \mathcal{B}$ the pair potential ϕ^β is regular and non-integrably divergent at the origin. As it is well known (see [23] for instance), ϕ^β is superstable and the existence of an associated Gibbs measures follows, i.e. [**Existence**] holds true. Superstability implies stability which shows assumption [**Stability**]. Since the map $\beta \mapsto \phi^\beta$ is differentiable and so continuous, assumption [**Argmax**] is obvious. The decomposition in [**MeanEnergy**] is done as in (46). In particular the mean energy (15) is

$$H^\theta(P) = E_P\left(z\, N_{\Delta_0} + \sum_{\{x,y\}\in\omega_{\Delta_0}} \phi^\beta(x-y) + \frac{1}{2}\sum_{x\in\omega_{\Delta_0}, y\in\omega_{\Delta_0^c}} \phi^\beta(x-y)\right) \tag{56}$$

where $\theta = (z, \beta)$. To check assumptions [**MeanEnergy**] and [**Boundary**], we thus have to prove that (56) is finite and that for any compact set $\mathcal{K} \subset \mathcal{B}$,

$$\sup_{\beta\in\mathcal{K}} \frac{1}{|\Lambda_n|} \sum_{x\in\omega_{\Lambda_n}, y\in\omega_{\Lambda_n^c}} \phi^\beta(x-y) \to 0. \tag{57}$$

For this purpose, we need the following Ruelle estimates.

Proposition 3 ([24]) *We define, for any $i \in \mathbb{Z}^d$, $\psi_i = \psi(max(|i|-1, 0)$, where the function ψ comes from assumption* (25)*. Then $\sum_{i\in\mathbb{Z}^d} \psi_i < +\infty$ and for any $c > 0$*

$$E_P\left(e^{c\sum_{i\in\mathbb{Z}^d}\psi_i N_{\tau_i(\Delta_0)}}\right) < +\infty.$$

Let us show (57). Note that

$$\sup_{\beta\in\mathcal{K}} \frac{1}{|\Lambda_n|} \sum_{x\in\omega_{\Lambda_n}, y\in\omega_{\Lambda_n^c}} \phi^\beta(x-y) \le A_1 + A_2$$

with

$$A_1 = \frac{1}{|\Lambda_n|} \sum_{\substack{x\in\omega_{\Lambda_n\setminus\Lambda_{n-n_0}} \\ y\in\omega,\, y\neq x \\ |x-y|\le r_0}} \sup_{\beta\in\mathcal{K}} |\phi^\beta|(x-y) \qquad \text{and} \qquad A_2 = \frac{1}{|\Lambda_n|} \sum_{\substack{x\in\omega_{\Lambda_n} \\ y\in\omega_{\Lambda_n^c} \\ |x-y|>r_0}} \sup_{\beta\in\mathcal{K}} |\phi^\beta|(x-y),$$

where r_0 comes from assumption (25) and n_0 is an integer greater than r_0.

By the spatial ergodic theorem, P-almost surely

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{\substack{x\in\omega_{\Lambda_n} \\ y\in\omega,\, y\neq x \\ |x-y|\le r_0}} \sup_{\beta\in\mathcal{K}} |\phi^\beta|(x-y) = E\Bigg(\sum_{\substack{x\in\omega_{\Delta_0} \\ y\in\omega,\, y\neq x \\ |x-y|\le r_0}} \sup_{\beta\in\mathcal{K}} |\phi^\beta|(x-y)\Bigg).$$

Therefore, A_1 goes P-almost surely to 0 if the expectation above is finite. By the GNZ equation, stationarity of P and assumption (25)

$$
\begin{aligned}
E\Big(\sum_{\substack{x\in\omega_{\Delta_0}\\ y\in\omega,\, y\neq x\\ |x-y|\leq r_0}} \sup_{\beta\in\mathcal{K}} |\phi^\beta|(x-y)\Big) &= z^* E\Big(e^{-\sum_{z\in\omega}\phi^{\beta^*}(z)} \sum_{\substack{y\in\omega\\ |y|\leq r_0}} \sup_{\beta\in\mathcal{K}} \phi^\beta(y)\Big)\\
&\leq z^* E\Big(e^{\sum_{i\in\mathbb{Z}^d}\psi_i N_{\tau_i(\Delta_0)}} \sum_{\substack{y\in\omega\\ |y|\leq r_0}} e^{-\sum_{z\in\omega,|z|\leq r_0}\phi^{\beta^*}(z)} \sup_{\beta\in\mathcal{K}} \phi^\beta(y)\Big)\\
&\leq z^* E\Big(e^{\sum_{i\in\mathbb{Z}^d}\psi_i N_{\tau_i(\Delta_0)}} \sum_{\substack{y\in\omega\\ |y|\leq r_0}} e^{-\phi^{\beta^*}(y)} \sup_{\beta\in\mathcal{K}} \phi^\beta(y)\Big)\\
&\leq z^* E\Big(e^{\sum_{i\in\mathbb{Z}^d}\psi_i N_{\tau_i(\Delta_0)}} N_{B(0,r_0)}\Big) \sup_{\beta\in\mathcal{K}} \sup_{y\in\mathcal{B}(0,r_0)} \sup_{\beta'\in\mathcal{K}} e^{-\phi^{\beta'}(y)}\phi^\beta(y)
\end{aligned}
\tag{58}
$$

which is finite by assumption (28) and Proposition 3.

Regarding the term A_2, for any integer K we have

$$
\begin{aligned}
A_2 &\leq \frac{1}{|\Lambda_n|} \sum_{i\in\Lambda_n, j\in\Lambda_n^c} \psi_{|i-j|} N_{\tau_i(\Delta_0)}(\omega) N_{\tau_j(\Delta_0)}(\omega)\\
&\leq \frac{1}{|\Lambda_n|} \sum_{i\in\Lambda_n\setminus\Lambda_{n-K}, j\in\mathbb{Z}^d} \psi_{|i-j|} N_{\tau_i(\Delta_0)}(\omega) N_{\tau_j(\Delta_0)}(\omega) + \frac{1}{|\Lambda_n|} \sum_{\substack{i\in\Lambda_{n-K}, j\in\mathbb{Z}^d\\ |j-i|\geq K}} \psi_{|i-j|} N_{\tau_i(\Delta_0)}(\omega) N_{\tau_j(\Delta_0)}(\omega).
\end{aligned}
\tag{59}
$$

By Proposition 3, $N_{\Delta_0}\sum_{j\in\mathbb{Z}^d}\psi_j N_{\tau_j(\Delta_0)}$ is integrable which implies thanks to the ergodic theorem that, when n goes to infinity, the first term in (59) goes to zero and the second term goes to

$$
E\Big(N_{\Delta_0} \sum_{j\in\mathbb{Z}^d, |j|\geq K} \psi_j N_{\tau_j(\Delta_0)}\Big).
$$

Choosing K large enough this last expectation can be made smaller than any positive value. Therefore the term A_2 tends P-almost surely to zero as well and (57) holds true.

By similar computations as above, using the Ruelle estimates in Proposition 3, we obtain that the mean energy (56) is finite. Assumptions **[MeanEnergy]** and **[Boundary]** are verified.

The existence and finiteness of the pressure in assumption **[VarPrin]** is proved in Theorem 0.2 of [24]. The variational principle for such Gibbs measures is proved in [7, 8] and thanks to the identifiability assumption(30) we get **[VarPrin]**.

It remains to prove **[Regularity]**. Let us show (16). For any compact set $\mathcal{Z}\times\mathcal{K}\subset\Theta$ with $\mathcal{Z}\subset\mathbb{R}_+$ and $\mathcal{K}\subset\mathcal{B}$, for any $r>0$,

$$
\begin{aligned}
E_P\Big(\sup_{\substack{\theta'\in\mathcal{Z}\times\mathcal{K}\\ |\theta-\theta'|\leq r}} \Big|H_0^\theta - H_0^{\theta'}\Big|\Big) &\leq rE_P(N_{\Delta_0}) + E_P\Big(\sum_{x\in\omega_{\Delta_0}} \sum_{y\in\omega\setminus\{x\}} \sup_{\substack{\beta'\in\mathcal{K}\\ |\beta-\beta'|\leq r}} |\phi^\beta - \phi^{\beta'}|(x-y)\Big)\\
&\leq rE_P(N_{\Delta_0}) + rE_P\Big(\sum_{x\in\omega_{\Delta_0}} \sum_{y\in\omega\setminus\{x\}} \sup_{\beta\in\mathcal{K}} |\nabla\phi^\beta|(x-y)\Big).
\end{aligned}
$$

The proof of (16) is completed if we show that the two expectations above are finite. This is clear for the first one. We split the second one according to $|y|\leq r_0$ or $|y|>r_0$ where r_0 comes from

assumption (25). Thanks to similar computations as in (58) we get

$$E_P\Big(\sum_{x\in\omega_{\Delta_0}}\sum_{\substack{y\in\omega\setminus\{x\}\\|y|\le r_0}}\sup_{\beta\in\mathcal{K}}|\nabla\phi^\beta|(x-y)\Big)$$
$$\le z^*E\Big(e^{\sum_{i\in\mathbb{Z}^d}\psi_i N_{\tau_i(\Delta_0)}}N_{B(0,r_0)}\Big)\sup_{\beta\in\mathcal{K}}\sup_{y\in\mathcal{B}(0,r_0)}\sup_{\beta'\in\mathcal{K}}e^{-\phi^{\beta'}(y)}|\nabla\phi^\beta|(y)$$

which is finite by (28) and Proposition 3, while from (27)

$$E_P\Big(\sum_{x\in\omega_{\Delta_0}}\sum_{\substack{y\in\omega\setminus\{x\}\\|y|>r_0}}\sup_{\beta\in\mathcal{K}}|\nabla\phi^\beta|(x-y)\Big)\le E\Big(N_{\Delta_0}\sum_{j\in\mathbb{Z}^d}\psi_j N_{\tau_j(\Delta_0)}\Big)$$

which is also finite. This proves (16) in [**Regularity**].

To prove (17), let $\eta>0$ and $\theta_0=(z_0,\beta_0)\in\mathcal{Z}\times\mathcal{K}$. There exists $0<\xi<\eta$ such that $B(\beta_0,\xi)\subset\mathcal{B}$. Consider the open set U associated to $\mathcal{K}\oplus B(0,\xi)$ through the assumption (29). Let us fix $u_0\in U$ and $\epsilon>0$ such that $B(u_0,\epsilon)$ is a ball included in U. We choose r sufficiently small to ensure that $r<\xi$ and $\beta:=\beta_0+\frac{r}{\epsilon}u_0$ belongs to $B(\beta_0,\eta)$. By assumption, for any $\beta'\in B(\beta_0,r)$ and $x\in\mathbb{R}^d$, there exists $t\in[0,1]$ such that

$$\phi^\beta(x)-\phi^{\beta'}(x)=\nabla\phi^{\beta'+t(\beta-\beta')}(x).(\beta-\beta').$$

Note that the above choices ensure that $\beta'+t(\beta-\beta')$ belongs to $B(\beta_0,\xi)\subset\mathcal{K}\oplus B(0,\xi)$ and that $\frac{\epsilon}{r}(\beta-\beta')$ belongs to $B(u_0,\epsilon)\subset U$. Hence by assumption (29), for any $x\in\mathbb{R}^d$,

$$\frac{\epsilon}{r}(\phi^\beta(x)-\phi^{\beta'}(x))\ge\tilde\phi(x)$$

where $\tilde\phi$ is a stable pair potential. We finally choose $\mathcal{N}(\theta_0)=\{\theta\}:=\{(z_0,\beta)\}$. Therefore for any bounded set Λ, any $\theta'=(z',\beta')\in[z_0-r,z_0+r]\times B(\beta_0,r)$ and any configuration ω, denoting by $A>0$ the stability constant of $\tilde\phi$, we have

$$H^\theta_\Lambda(\omega_\Lambda)-H^{\theta'}_\Lambda(\omega_\Lambda)\ge(z_0-z')N_\Lambda(\omega_\Lambda)+\sum_{\{x,y\}\in\omega_\Lambda}\frac{r}{\epsilon}\,\tilde\phi(x-y)\ge -r\,(1+A/\epsilon)N_\Lambda(\omega_\Lambda)$$

which proves (17) and concludes the proof.

5.6 Proof of Proposition 1

We apply Theorem 1. The local energy $h^\theta(x|\omega)$ associated to the area-interaction process is, for any Λ containing x,

$$h^\theta(x|\omega)=H^\theta_\Lambda(\omega\cup x)-H^\theta_\Lambda(\omega)=z+\beta\lambda^d\left(B(x,R)\setminus\bigcup_{y\in\omega}B(y,R)\right).\tag{60}$$

Since $\mathcal{K}$ is compact, $\bar z=\sup_{\theta\in\mathcal{K}}z$, $\bar R=\sup_{\theta\in\mathcal{K}}R$ and $\bar\beta=\sup_{\theta\in\mathcal{K}}|\beta|$ are all finite and h^θ is uniformly bounded:

$$\sup_{x\in\mathbb{R}^d}\sup_{\theta\in\mathcal{K}}\sup_{\omega\in\Omega}|h^\theta(x|\omega)|\le\bar z+\bar\beta(2\bar R)^d.$$

This local stability property implies [**Stability**]. Since h^θ has also finite range $2\bar R$, an associated Gibbs measure exists and the variational principle holds with the existence of the pressure, see

[22] and [4]. This gives **[Existence]** and **[VarPrin]**. Since the application $\theta \mapsto H^\theta_\Lambda(\omega)$ is continuous, the assumption **[Argmax]** is obviously satisfied. As for **[MeanEnergy]** we suggest this decomposition (other one could have been considered)

$$H^\theta_{\Lambda_n} = \sum_{k\in\mathcal{I}_n} H^\theta_0 \circ \tau_{-k} + \partial H^\theta_{\Lambda_n}$$

with

$$H^\theta_0(\omega) = zN_{\Delta_0}(\omega) + \lambda^d\left(\Delta_0 \cap \bigcup_{x\in\omega} B(x,R)\right). \tag{61}$$

Clearly $E_P(H^\theta_0)$ is finite. Moreover for any $n \geq 1$

$$|\partial H^\theta_{\Lambda_n}| \leq \bar\beta\lambda^d\left((\Lambda_n \oplus B(0,\bar R)) \setminus (\Lambda_n \ominus B(0,\bar R))\right) \leq 4d\bar\beta\bar R(2n+2\bar R)^{d-1}$$

which implies **[MeanEnergy]**. Similar computations show that assumption **[Boundary]** holds as well. Finally, Assumption **[Regularity]** is a consequence of the following uniform continuity of H^θ_0 coming from a simple geometric analysis: for any $\theta = (z,R,\beta)$, $\theta' = (z',R',\beta')$ in Θ (we assume $R \leq R'$)

$$\begin{aligned}|H^\theta_0 - H^{\theta'}_0| &\leq |H^{(z,R,\beta)}_0 - H^{(z',R,\beta)}_0| + |H^{(z',R,\beta)}_0 - H^{(z',R',\beta)}_0| + |H^{(z',R',\beta)}_0 - H^{(z',R',\beta')}_0| \\ &\leq |z-z'|N_{\Delta_0} + 2d|\beta|(2R')^d|R-R'|N_{\Delta_0\oplus B(0,R')} + |\beta-\beta'|.\end{aligned}$$

5.7 Proof of Proposition 2

Since the Quermass-interaction model has an exponential form, it is sufficient to check the assumptions of Corollary 1. The existence of the model for any $z > 0$ and any $\beta \in \mathbb{R}^3$ is proved in [3] while stability of the interaction follows from [14]. The assumptions **[MeanEnergy]** and **[Boundary]** can be proved as in Section 5.6, see (61), thanks to the additivity of the Minkowski functionals. The assumption **[VarPrin]** follows from [4] since the interaction is stable and finite range.

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