Quantum ergodicity for sub-Riemannian Laplacians. I: the contact 3D case

Yves Colin de Verdière, Luc Hillairet, Emmanuel Trélat

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To the memory of Louis Boutet de Monvel (1941–2014)

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Abstract

Let us consider a sub-Riemannian (sR) metric on a closed three-dimensional manifold with an oriented contact distribution. There exists a privileged choice of the contact form, with an associated Reeb vector field and a canonical volume form that coincides with the Popp measure. We establish a Quantum Ergodicity (QE) theorem for the eigenfunctions of any associated sR Laplacian under the assumption that the Reeb flow is ergodic. The limit measure is given by the normalized Popp measure.

This is the first time that such a result is established for a hypoelliptic operator,whereas the usual Shnirelman theorem yields QE for the Laplace-Beltrami operator on a closed Riemannian manifold with ergodic geodesic flow.

To prove our main theorem, we first establish a microlocal Weyl law, which allows us to identify the limit measure and to prove the microlocal concentration of the eigenfunctions on the characteristic manifold of the sR Laplacian. Then, we derive a Birkhoff normal form, thus showing that, in some sense, all contact 3D sR structures are microlocally equivalent. The quantum version of this normal form provides a useful microlocal factorization of the sR Laplacian. Using the normal form, the factorization and the ergodicity assumption, we finally establish a variance estimate, from which QE follows.

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Contents

1 Introduction and main result

Quantum ergodicity (QE) theorems started with the seminal note [32] by A. Shnirelman (see also [33]). His arguments were made precise in [8, 38], and then extended to the case of manifolds with boundary in [16, 40], to the case of discontinuous metrics in [21] and to the semiclassical regime in [17]. A weak version of QE is the following: let M be a compact metric space, endowed with a measure μ, and let T be a self-adjoint operator on $L^2(M,\mu)$, bounded below and having a compact resolvent (and hence a discrete spectrum). Let $(\phi_n)_{n\in\mathbb{N}^*}$ be a (complex-valued) Hilbert basis of $L^2(M,\mu)$, consisting of eigenfunctions of T, associated with the ordered sequence of eigenvalues $\lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$. We say that QE holds for T is there exist a probability measure ν on M and a density-one sequence $(n_j)_{j\in\mathbb{N}^*}$ of positive integers such that the sequence of probability measures $|\phi_{n_j}|^2 d\mu$ converges weakly to ν. The measure ν may be different from some scalar multiple of μ, and may even be singular with respect to μ (see [9]). Microlocal versions of QE hold true in general and are stated in terms of pseudo-differential operators.

Such a property provides some insight on the way eigenfunctions of the operator T *concentrate* in the limit of large eigenvalues: roughly speaking, it says that almost every eigenfunction equidistributes with respect to ν. When M is a compact Riemannian manifold and T is the usual Laplace-Beltrami operator, QE is established under the assumption that the geodesic flow is ergodic, and the limit measure ν is then the projection on M of the normalized Liouville measure on the unit cotangent bundle of M.

To our knowledge, similar results are known in different contexts, but always for elliptic operators. In the present paper, we establish a QE theorem for *sub-Riemannian Laplacians* on a closed three-dimensional contact sub-Riemannian manifold without boundary. Let us describe our main result.

Let M be a smooth connected compact three-dimensional manifold, equipped with an arbitrary smooth volume form $d\mu$ (the corresponding measure being denoted by μ). Let $D \subset TM$ be a smooth oriented subbundle of codimension one. Let g be a smooth Riemannian metric on D. We assume that D is a contact structure, so that there exists a unique contact form α_g defining D such that $(d\alpha_g)_{|D}$ coincides with the oriented volume form induced by g on D. Let Z be the associated Reeb vector field, defined by $\iota_Z \alpha_g = 1$ and $\iota_Z\, d\alpha_g = 0$. The flow generated by Z on M is called the *Reeb flow*.

Let $\triangle_{sR}$ be the sub-Riemannian (sR) Laplacian associated with the contact sub-Riemannian structure (M, D, g) and with the volume form $d\mu$ (see Section 2 for some reminders and for the precise definition). The operator $\triangle_{sR}$ is self-adjoint on $L^2(M,\mu)$, is hypoelliptic, has a compact resolvent and thus has a discrete spectrum $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$, with $\lambda_n \to +\infty$ as $n \to +\infty$.

Using the contact form α_g, we define the Popp density $dP = |\alpha_g \wedge d\alpha_g|$ on M, and we define the probability measure $d\nu = \frac{1}{P(M)} dP$. Note that the corresponding *probability Popp measure* ν, defined on M, is invariant under Z and differs from μ in general.

Here, and throughout the paper, the notation $\langle\ ,\ \rangle$ stands for the (Hermitian) inner product in $L^2(M,\mu)$.

Theorem 1. *We assume that the Reeb flow is* ergodic[1].

Then, for any real-valued *orthonormal Hilbert basis* $(\phi_n)_{n\in\mathbb{N}^*}$ *of* $L^2(M,\mu)$ *consisting of eigenfunctions of* $\triangle_{sR}$ *associated with the eigenvalues* $(\lambda_n)_{n\in\mathbb{N}^*}$*, there exists a density-one sequence* $(n_j)_{j\in\mathbb{N}^*}$ *of positive integers such that*

$$\lim_{j\to+\infty}\left\langle A\phi_{n_j},\phi_{n_j}\right\rangle=\frac{1}{2}\int_M\big(a(q,\alpha_g(q))+a(q,-\alpha_g(q))\big)\,d\nu,$$

for every classical pseudo-differential operator A *of order* 0 *with principal symbol* a*.*

In particular, we have

$$\lim_{j\to+\infty}\int_M f|\phi_{n_j}|^2\,d\mu=\int_M f\,d\nu,$$

for every continuous function f *on* M*.*

Remark 1. In Appendix A.1, we show how to drop the assumption that the distribution D is oriented. We also treat the case of a complex-valued eigenbasis.

Note that our result is valid for any choice of a smooth volume form $d\mu$ on M. This is due to the fact that, up to an explicit unitary gauge transform, changing the volume form modifies the sub-Riemannian Laplacian with an additional bounded operator (see Remark 5 for a precise statement).

The classical Shnirelman theorem is established in the Riemannian setting under the assumption that the Riemannian geodesic flow is ergodic on $(S^\star M,\lambda_L)$, where the limit measure is the Liouville measure λ_L on the unit cotangent bundle $S^\star M$ of M. In contrast, here the Liouville measure on the unit bundle $g^\star=1$ has infinite total mass (where $g^\star$ is the co-metric on $T^\star M$ associated with g), and hence the QE property cannot be formulated in terms of the geodesic flow.

Another interesting difference is that, in the Riemannian setting, QE says that most eigenfunctions equidistribute in the phase space, whereas here, in the contact 3D case, they concentrate on $\Sigma=D^\perp$, the contact cone that is the characteristic manifold of $\triangle_{sR}$ (see the microlocal Weyl formula established in Theorem 2).

Remark 2. To our knowledge, Theorem 1 is the first QE result established for a hypoelliptic operator. Our proof is specific to the 3D contact case, but the result can probably be extended to other sub-Riemannian Laplacians. It is likely that the microlocal Weyl formula (Theorem 2) can be generalized to equiregular sub-Riemannian structures. The simplest nonregular case, the Martinet case, is already more sophisticated (see [9, 28]), and it is difficult to identify the adequate dynamics, the appropriate invariant measure being given by the microlocal Weyl formula. The relationship with abnormal geodesics is, in particular, an interesting issue.

Remark 3. It is interesting to note that, as a byproduct of our analysis, the Popp volume $P(M)$ of M is a spectral invariant of the sub-Riemannian structure (and this, for any choice of $d\mu$). When $M=\mathbb{S}^3$, then $1/P(M)$ is the asymptotic Hopf invariant of the Reeb vector field Z (with respect to the probability Popp measure ν) introduced in [2]. It follows that this quantity is also a spectral invariant.

[1] The Reeb flow $(\mathcal{R}_t)_{t\in\mathbb{R}}$ generated on M by the vector field Z lets the measure ν invariant. We say that this flow is ergodic on (M,ν) if any measurable invariant subset of M is of measure 0 or 1. This implies, by the von Neumann ergodic theorem, that, for every continuous function f on M, we have

$$\frac{1}{T}\int_0^T f\circ\mathcal{R}_t(x)\,dt\to\int_M f\,d\nu,$$

in $L^2(M,\nu)$, as $T\to+\infty$.

A general path towards QE. Let us indicate some ideas behind the proof of Theorem 1. We will follow the general path towards establishing the QE property, as clarified, e.g., in [39]. We set $N(\lambda) = \#\{n \mid \lambda_n \leqslant \lambda\}$.

The first step consists in establishing a *microlocal Weyl law*:

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} \langle A\phi_n, \phi_n\rangle = \bar{a} = \int_{ST^\star M} a \, dW_\triangle, \tag{1}$$

for every pseudo-differential operator A of order 0 with a positively homogeneous principal symbol a, where $ST^\star M$ is the unit sphere bundle of $T^\star M$, and $dW_\triangle$ is a probability measure that we call the *microlocal Weyl measure*. This Cesáro convergence property can usually be established under weak assumptions, without any ergodicity property.

The second step consists in proving the *variance estimate*

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} |\langle (A - \bar{a}\,\mathrm{id})\phi_n, \phi_n\rangle|^2 = 0. \tag{2}$$

The variance estimate usually follows by combining the microlocal Weyl law (1) with ergodicity properties of some associated classical dynamics and with an Egorov theorem.

Then QE follows from the two properties above. Indeed, for a fixed pseudo-differential operator A of order 0, it follows from (2) and from a well known lemma[2] due to Koopman and Von Neumann (see, e.g., [30, Chapter 2.6, Lemma 6.2]) that there exists a density-one sequence $(n_j)_{j\in\mathbb{N}^*}$ of positive integers such that

$$\lim_{j\to+\infty} \left\langle A\phi_{n_j}, \phi_{n_j}\right\rangle = \bar{a}.$$

Using the fact that the space of symbols of order 0 admits a countable dense subset, QE is then established with a diagonal argument.

Organization of the paper. In Section 2, we provide the precise and complete framework that is relevant both to the sub-Riemannian geometric setting and to the spectral analysis of the associated sub-Riemannian Laplacian.

In Section 3, we recall the spectral theory of compact quotients of the Heisenberg group with an invariant metric. The study made in [7] serves as a guiding model for our general result. We also give some examples of three-dimensional contact manifolds on which the Reeb flow is ergodic.

In Section 4, we establish the microlocal Weyl law. No ergodicity assumption is required here. The local Weyl law has been established by several authors. We recall it and we present in Appendix A.3 a proof using the sR heat kernel, which is adapted from [3]. The limit measure that appears in the local Weyl law is the Popp probability measure. Therefore, this measure is the good candidate for a QE result, and the fact that the Popp measure is invariant under the Reeb flow makes it natural to expect that the Reeb vector field is relevant for that purpose. For the same reason, it also indicates that the sub-Riemannian geodesic flow *is not* a good candidate, because it does not preserve the Popp measure in general. The microlocal Weyl law is derived thanks to a general argument proving the average concentration of the eigenfunctions on Σ.

In Section 5, we establish a classical (Birkhoff) normal form and then a quantum normal form microlocally near Σ. Such normal forms have proved to be relevant in the semiclassical literature to obtain fine spectral results. Our normal form is essentially due to [25] and is closely related to the one of [31] for large magnetic fields in dimension two (see also [5]). This normal form implies

[2] This lemma states that, given a bounded sequence $(u_n)_{n\in\mathbb{N}}$ of nonnegative real numbers, the Cesáro mean $\frac{1}{n}\sum_{k=0}^{n-1} u_k$ converges to 0 if and only if there exists a subset $S \subset \mathbb{N}$ of density one such that $(u_k)_{k\in S}$ converges to 0. We recall that S is of density one if $\frac{1}{n}\#\{k \in S \mid k \leqslant n-1\}$ converges to 1 as n tends to $+\infty$.

that, microlocally near a point of Σ, all contact three-dimensional sub-Riemannian structures are equivalent, and in particular can (almost) be conjugated to the local model of the Heisenberg flat case. In the quantum version, this conjugation is performed with a Fourier Integral Operator, and we infer from that result an almost factorization of $\triangle_{sR}$ microlocally near Σ. Here, "almost" means that the factorization is exact only along Σ, with remainder terms that are flat along Σ.

In Section 6, we use this quantum Birkhoff normal form to prove the variance estimate, and, using pseudo-differential calculus techniques (in particular, averaging and brackets) and the ergodicity assumption on the Reeb flow, we finally infer the QE property for $\triangle_{sR}$.

Several appendices gather some useful technical statements. Note that one of the appendices is devoted to provide, as a byproduct, an interesting consequence of our Birkhoff normal form to the classical geodesic dynamics of the 3D contact case.

2 Geometric and spectral preliminaries

Let us start with several notations.

We denote by ω the canonical symplectic form on the cotangent bundle $T^\star M$ of M. In local coordinates (q,p) of $T^\star M$, we have $\omega = dq \wedge dp = -d\lambda$ with $\lambda = p\,dq$. We denote by $\{\ ,\ \}_\omega$ the Poisson bracket associated with ω.

For every vector field X on M, we denote by $h_X = \lambda(X)$ the Hamiltonian function (momentum map) on $T^\star M$ associated with X. Let $\vec{h}_X$ be the Hamiltonian vector field on $T^\star M$ associated with the function h_X, defined by $\iota_{\vec{h}_X}\omega = dh_X$, and let $\exp(t\vec{h}_X)$ be the flow at time t generated by $\vec{h}_X$ on $T^\star M$.

Throughout the paper, the notation orth_ω stands for the symplectic ω-orthogonal.

2.1 Sub-Riemannian Laplacians

Let (M, D, g) be a sub-Riemannian (sR) structure, where M is a smooth connected compact three-dimensional manifold, D is a smooth subbundle of TM of rank two (called horizontal distribution), and g is a fibered Riemannian metric on D. We assume that D is a contact distribution, that is, we can write $D = \ker\alpha$ locally around any point, for some one-form α such that $\alpha \wedge d\alpha \neq 0$ (locally). At this step, we do not need to normalize the contact form.

In order to define a sub-Riemannian Laplacian $\triangle_{sR}$, let us choose a smooth volume form $d\mu$ on M, the associated measure being denoted by μ. The choice of $d\mu$ is independent of that of g.[3] Let $L^2(M,\mu)$ be the set of complex-valued functions u such that $|u|^2$ is μ-integrable over M. Then $-\triangle_{sR}$ is the nonnegative self-adjoint operator on $L^2(M,\mu)$ defined as the Friedrichs extension of the Dirichlet integral

$$Q(\phi) = \int_M \|d\phi\|^2_{g^*}\, d\mu,$$

where the norm of $d\phi$ is calculated with respect to the (degenerate) dual metric $g^\star$ (also called "co-metric") on $T^\star M$ associated with g. The sub-Riemannian Laplacian $\triangle_{sR}$ depends on the choice of g and of $d\mu$.

We consider the divergence operator div_μ associated with the volume form $d\mu$, defined by $\mathcal{L}_X d\mu = \mathrm{div}_\mu(X)\, d\mu$ for any vector field X on M. Besides, the horizontal gradient $\nabla_{sR}\phi$ of a smooth function ϕ is the unique section of D such that $g_q(\nabla_{sR}\phi(q), v) = d\phi(q).v$, for every $v \in D_q$. Then, we have

$$\triangle_{sR}\phi = \mathrm{div}_\mu(\nabla_{sR}\phi),$$

[3] As we will see, the choice of μ plays no role in what follows. Beyond this paper, we expect that this fact is important in the non-equiregular cases where there is no canonical choice of μ.

for every smooth function ϕ on M.

Since $\|d\phi\|_{g^\star}^2 = \|\nabla_{sR}\phi\|_g^2$, if (X,Y) is a local g-orthonormal frame of D, then $\nabla_{sR}\phi = (X\phi)X + (Y\phi)Y$, and $Q(\phi) = \int_M \left((X\phi)^2 + (Y\phi)^2\right) d\mu$. It follows that

$$\triangle_{sR} = -X^\star X - Y^\star Y = X^2 + Y^2 + \mathrm{div}_\mu(X)X + \mathrm{div}_\mu(Y)Y,$$

where the adjoints are taken in $L^2(M,\mu)$.

Remark 4. The co-metric $g^\star$ induces on $T^\star M$ an Hamiltonian vector field $\vec{G}$. The projections onto M of the integral curves of $\vec{G}$ are the (normal) geodesics of the sub-Riemannian metric g (see [29]).

Remark 5. Let us express the difference between two sub-Riemannian Laplacians $\triangle_{\mu_1}$ and $\triangle_{\mu_2}$ associated with two different volume forms (but with the same metric g). Assume that $\mu_2 = h^2\mu_1$ with h a positive smooth function on M. It is easy to see that $\mathrm{div}_{\mu_2}(X) = \mathrm{div}_{\mu_1}(X) + \frac{X(h^2)}{h^2}$, for every vector field X. By simple computations, we then establish that $a\triangle_{\mu_2}(\phi) = \triangle_{\mu_1}(a\phi) - \phi\triangle_{\mu_1}(h) = \triangle_{\mu_1}(h\phi) + h^2\phi\triangle_{\mu_2}(h^{-1})$. To settle this identity in a more abstract way, we define the isometric bijection $J : L^2(M,\mu_2) \to L^2(M,\mu_1)$ by $J\phi = h\phi$. Then, we have

$$J\triangle_{\mu_2}J^{-1} = \triangle_{\mu_1} - \frac{1}{h}\triangle_{\mu_1}(h)\,\mathrm{id} = \triangle_{\mu_1} + h\triangle_{\mu_2}(h^{-1})\,\mathrm{id}.$$

It follows that $\triangle_{\mu_1}$ is unitarily equivalent to $\triangle_{\mu_2} + W$, where W is a bounded operator.

This remark is important because, for a given metric, it allows us to work with the sub-Riemannian Laplacian associated with any measure. Usually, this kind of fact is abstracted by using half-densities (see [12]) which give a canonical Hilbert space on M. A way of rephrasing the previous remark is then to say that $\triangle_{sR}$ is any self-adjoint second-order differential operator whose principal symbol is $g^\star$, whose sub-principal symbol vanishes, and such that $\lambda_1 = 0$ (first eigenvalue).

2.2 Microlocal aspects

We are going to use some microlocal analysis in order to study the operator $\triangle_{sR}$, and thus we are going to use the symbolic calculus of pseudo-differential operators.

For every $m \in \mathbb{R}$, we consider the class $\mathcal{S}^m$ of classical symbols of order m. Any classical symbol $a : T^\star\mathbb{R}^3 \setminus \{0\} \to \mathbb{C}$ of order m admits an asymptotic expansion $a \sim \sum_{j=0}^{+\infty} a_{m-j}$ where a_{m-j} is smooth and homogeneous of degree $m-j$, meaning that, for any N, $a_N = a - \sum_{j=0}^{N-1} a_{m-j}$ satisfies

$$|\partial_q^\alpha \partial_p^\beta a_N(q,p)| \leqslant C_{\alpha,\beta}(1+|p|)^{m-N-|\beta|},$$

for all elements α and β of $\mathbb{N}^d$, for some constants $C_{\alpha\beta} > 0$. This class of symbol is independent of the choice of local canonical coordinates.

We denote by Ψ^m the set of pseudo-differential operators on M whose symbol in local coordinates belongs to $\mathcal{S}^m$.

We recall that the notion of principal symbol of a pseudo-differential operator is defined in an intrinsic way, as well as the notion of sub-principal symbol since the measure μ is fixed, the latter being viewed as acting on functions rather than on half-densities (see, e.g., [12]).

The principal symbol of $-\triangle_{sR}$ is

$$\sigma_P(-\triangle_{sR}) = h_X^2 + h_Y^2 = g^\star,$$

(it coincides with the co-metric $g^\star$), and its sub-principal symbol is zero. It also follows that, for any other volume form $d\mu'$ on M, the corresponding sub-Riemannian Laplacian is unitarily equivalent to $\triangle_{sR} + V\,\mathrm{id}_M$, where V is a smooth function (see also Remark 5).

Since we are in the contact case, the vector fields $(X, Y, [X, Y])$ generate TM, and it follows from [18] that the operator $-\triangle_{sR}$ is hypoelliptic and has a compact resolvent (see also [34]). It thus has a discrete real spectrum $(\lambda_n)_{n\in\mathbb{N}^*}$, with $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$, with $\lambda_n \to +\infty$ as $n \to +\infty$.

Throughout the paper, we consider an orthonormal Hilbert basis $(\phi_n)_{n\in\mathbb{N}}$ of $L^2(M, \mu)$, consisting of eigenfunctions of $\triangle_{sR}$, associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$. Using that $\triangle_{sR}$ commutes with complex conjugation, we may assume that the eigenfunctions are real-valued.

2.3 Popp measure and Reeb vector field

In order to define the Popp measure and the Reeb vector field, we need to normalize the contact form defining D, and we need this contact form to be defined globally on M. Since D is assumed to be orientable, and since we are in dimension 3, D is co-orientable as well and hence the contact form is defined globally.

Then, in the sequel, we assume that $D = \ker \alpha_g$, where α_g is a (global) real-valued one-form such that $\alpha_g \wedge d\alpha_g$ does not vanish, or equivalently, $(d\alpha_g)_{|D} \neq 0$. The contact form α_g is chosen such that $d\alpha_g(X, Y) = 1$ for any positive g-orthonormal local frame (X, Y) of D, and this property determines α_g in a unique way (equivalently, $(d\alpha_g)_{|D}$ coincides with the volume form induced by g on D).

We define the density $dP = |\alpha_g \wedge d\alpha_g|$ on M. In general, dP differs from $d\mu$. The corresponding measure P is called the *Popp measure* in the existing literature (see [29], where it is defined in the general equiregular case). Of course, here, this measure is the canonical contact measure associated with the normalized contact form α_g. In what follows, we consider the *probability Popp measure*

$$\nu = \frac{P}{P(M)}.$$

The *Reeb vector field* Z of the contact form α_g is defined as the unique vector field such that $\iota_Z \alpha_g = 1$ and $\iota_Z\, d\alpha_g = 0$. Equivalently, Z is the unique vector field such that

$$[X, Z] \in D, \quad [Y, Z] \in D, \quad [X, Y] = -Z \mod D, \tag{3}$$

for any positive orthonormal local frame (X, Y) of D.

Using the Cartan formula, we have $\mathcal{L}_Z \nu = d\iota_Z \nu + \iota_Z\, d\nu = d\iota_Z \nu$, and $\iota_Z \nu = d\alpha_g$ by definition of Z, hence $\mathcal{L}_Z \nu = 0$. This computation shows that the Popp measure ν is invariant under the vector field Z, and equivalently, under the Reeb flow generated by Z. As already said, it is crucial to identify such an invariance property in view of deriving a QE result.

Remark 6. The Reeb vector field Z has the following dynamical interpretation. If $(q_0, v_0) \in D$, then there exists a one-parameter family of geodesics associated with these Cauchy data: they are the projections of the integral curves of the Hamiltonian vector field $\vec{G}$ (defined in Remark 4) whose Cauchy data are (q_0, p_0) with $(p_0)_{|D_{q_0}} = g(v_0, \cdot)$. For every $u \in \mathbb{R}$, the projections on M of the integral curves of $\vec{G}$ with Cauchy data $(q_0, p_0 + u\alpha_g)$ in the cotangent space have the same Cauchy data (q_0, v_0) in the tangent space. As $u \to \pm\infty$, they spiral around the integral curves of $\mp Z$ (see also Appendix A.7 where we give more precise results). From the point of view of semiclassical analysis, this part of the dynamics is expected to be the dominant one.

2.4 The characteristic cone and the Hamiltonian interpretation of the Reeb flow

Let $\Sigma \subset T^\star M$ be the characteristic manifold of $-\triangle_{sR}$. We have $\Sigma = (g^\star)^{-1}(0) = D^\perp$ (annihilator of D), and Σ coincides with the cone of contact forms defining D. In particular, we have

$$\Sigma = \{(q, s\alpha_g(q)) \in T^\star M \mid q \in M, s \in \mathbb{R}\}. \tag{4}$$

Note that Σ can be seen as a foliation parametrized by s, in the sense that $\Sigma = \cup_{s\in\mathbb{R}}\Sigma_s$, with $\Sigma_s = \{(q, s\alpha_g(q)) \in T^\star M \mid q \in M\}$.

An important feature of the contact situation is that the characteristic cone is symplectic: the restriction $\omega_{|\Sigma}$ is symplectic. The Reeb dynamics play a crucial role in what follows, due to the following result.

Proposition 1. *The Hamiltonian vector field $\hat{Z}$ on Σ that is associated to the Hamiltonian $\rho : (q, s\alpha_g) \mapsto s$ leaves Σ_s invariant and projects onto Z.*

Proof. Denoting by $j : M \times \mathbb{R} \to \Sigma$ the parametrization of Σ given by $j(q, s) = (q, s\alpha_g(q))$, we have $\omega_\Sigma = j^\star(\omega) = -(ds \wedge \alpha_g + s\alpha_g)$, hence $\iota_Z\omega_\Sigma = d\rho$ by the definition of Z. The result follows. □

We denote by ρ_t the Hamiltonian flow of ρ on Σ_s, which projects onto the Reeb flow. We define $\Sigma^\pm = j(M \times \{\pm s > 0\})$. Note that Σ^+ and Σ^- are permuted when one changes the orientation of D, while the definition of the sR Laplacian remains unchanged. We also denote by $\hat{\nu}_s$ the lift of the measure ν to Σ_s. Note that Σ_1 and Σ_{-1} are connected manifolds and that the Reeb flow acts on each of these copies of M. In the Riemannian setting, the unit tangent bundle is connected for $d \geqslant 2$. This is why our main theorem applies only to a real eigenbasis.

3 Examples

3.1 The Heisenberg flat case

The simplest example is given by an invariant metric on a compact quotient of the Heisenberg group. The spectral decomposition of the Heisenberg Laplacian is then explicit (see [7]). Moreover, as we will show in Section 5, this example serves as a microlocal normal form model for all sub-Riemannian Laplacians of contact type in dimension three.

Let G be the three-dimensional Heisenberg group defined as $G = \mathbb{R}^3$ with the product rule

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - xy').$$

The contact form $\alpha_H = dz + x\,dy$ and the vector fields $X_H = \partial_x$ and $Y_H = \partial_y - x\partial_z$ are left-invariant on G. Let Γ be the discrete co-compact subgroup of G defined by

$$\Gamma = \{(x, y, z) \in G \ \mid \ x, y \in \sqrt{2\pi}\mathbb{Z},\ z \in 2\pi\mathbb{Z}\}.$$

We define the three-dimensional compact manifold manifold $M_H = \Gamma\backslash G$, and we consider the horizontal distribution $D_H = \ker \alpha_H$, endowed with the metric g_H such that (X_H, Y_H) is a g_H-orthonormal frame of D_H. With this choice, we have $\alpha_{H\,g} = \alpha_H$.

The Reeb vector field is given by $Z_H = -[X_H, Y_H] = \partial_z$. The Lebesgue volume $d\mu = dx\,dy\,dz$ coincides with the Popp volume dP, and we consider the corresponding sub-Riemannian Laplacian $\triangle_H = X_H^2 + Y_H^2$ (here, the vector fields have divergence zero).

We refer to this sub-Riemannian case as the *Heisenberg flat case*.

Spectrum of $\triangle_H$. The sub-Riemannian Laplacian $\triangle_H$ commutes with the vector field $-iZ_H$ whose spectrum is the set of integers $m \in \mathbb{Z}$. After some computations, whose detail is in [7], we get that the spectrum of $-\triangle_H$ is given by

$$\{\lambda_{\ell,m} = (2\ell+1)|m| \mid m \in \mathbb{Z} \setminus \{0\},\ \ell \in \mathbb{N}\} \cup \{\mu_{j,k} = 2\pi(j^2+k^2) \mid j,k \in \mathbb{Z}\},$$

where $\lambda_{\ell,m}$ is of multiplicity $|m|$. The spectral counting function is given by $N(\lambda) = \sum_{\ell\in\mathbb{N}} N_\ell(\lambda)$, with $N_0(\lambda) \sim C\lambda$ and

$$N_\ell(\lambda) = \sum_{\substack{m\neq 0 \\ |m|\leqslant \lambda/(2\ell+1)}} |m|.$$

From this, we get the asymptotics

$$N(\lambda) \sim \sum_{l=0}^{\infty} \frac{\lambda^2}{(2l+1)^2} = \frac{\pi^2}{8}\lambda^2 = \frac{P(M_H)}{32}\lambda^2,$$

as $\lambda \to +\infty$. It is interesting to compare that result with the corresponding result for a Riemannian Laplacian on a three-dimensional closed Riemannian manifold, for which $N(\lambda) \sim C\lambda^{3/2}$.

Normal form. Let us prove that the sub-Riemannian Laplacian can be written as a product $-\triangle_H = R_H\Omega_H$ with two commuting operators. The pseudo-differential operator $R_H = \sqrt{Z_H^\star Z_H}$ acts by multiplication by $|m|$ on the functions $e^{imz}f(x,y)$. We define the two pseudo-differential operators U_H and V_H of order $1/2$ in the cone $C_c = \{p_z^2 > c(p_x^2+p_y^2)\}$ for some $c > 0$ (where we denote by $p = (p_x, p_y, p_z)$ the local coordinates in the cotangent space), by $U_H = \frac{1}{i}R_H^{-\frac{1}{2}}X_H$ and $V_H = \frac{1}{i}R_H^{-\frac{1}{2}}Y_H$, and we set $\Omega_H = U_H^2 + V_H^2$. Then we have $-\triangle_H = R_H\Omega_H = \Omega_H R_H$ and $[U_H, V_H] = \pm\mathrm{id}$ (according to the sign of h_{Z_H}), and moreover $\exp(2i\pi\Omega_H) = \mathrm{id}$ (harmonic oscillator).

In terms of symbols, since X_H and Y_H have divergence zero, we have $\sigma(-\triangle_H) = h_{X_H}^2 + h_{Y_H}^2$ (full symbol), which can be factorized as

$$|h_{Z_H}|\left(\left(\frac{h_{X_H}}{\sqrt{|h_{Z_H}|}}\right)^2 + \left(\frac{h_{Y_H}}{\sqrt{|h_{Z_H}|}}\right)^2\right).$$

Then, microlocally in a neighborhood of the cone (in which R_H is a positive pseudo-differential operator) we have $U_H = \mathrm{Op}^{\mathcal{W}}\left(\frac{h_{X_H}}{\sqrt{|h_{Z_H}|}}\right)$ and $V_H = \mathrm{Op}^{\mathcal{W}}\left(\frac{h_{Y_H}}{\sqrt{|h_{Z_H}|}}\right)$, where the notation $\mathrm{Op}^{\mathcal{W}}$ stands for the Weyl quantization.

Heat kernel. As established in [14], the heat kernel on the Heisenberg group (presented in a different way in that reference) is given by

$$H_t((x,y,z),(0,0,0)) = \frac{1}{8\pi^2t^2}\int_{\mathbb{R}} \frac{s}{\sinh s}\exp\left(-\frac{s(x^2+y^2)}{4t\tanh s}\right)\cos\left(\frac{zs}{t}\right)\,ds.$$

We will however not use this formula. The interesting fact is that the asymptotics is in $1/t^2$ as $t \to 0^+$.

3.2 2D magnetic fields

Following [28], we start with a two-dimensional closed Riemannian manifold X endowed with a magnetic two-form whose integral is an integer multiple of 2π. We associate to these data an Hermitian line bundle L (the choice is not unique if the manifold is not simply connected) with a connection whose curvature is B, and a magnetic Schrödinger operator H_B acting on the sections of L. Let M be the associated principal S^1-bundle. We can write $L^2(M,\mu)$ as an orthogonal sum of $L^2(X, L^{\otimes n})$ with the action of $\oplus_n H_{nB} = \triangle$. Then $\triangle$ is a sub-Riemannian Laplacian on M associated with the horizontal distribution defined by the connection. This example is a generalization of the case of the Heisenberg quotient defined in Section 3.1. The associated Reeb vector field is the generator of the $\mathbb{S}^1$-action on the principal bundle.

3.3 Examples of ergodic Reeb flows in dimension 3

In this section, we give examples of contact three-dimensional manifolds for which the Reeb flow is ergodic. We use the well known fact that a Reeb flow can be realized, in some appropriate context, as a geodesic flow or as an Hamiltonian flow. We provide two constructions.

Geodesic flows. Let X be a two-dimensional compact Riemannian surface, endowed with a Riemannian metric h, and let $M = S^\star X$ be the unit cotangent bundle of X. The closed three-dimensional manifold M is then naturally endowed with the contact form α defined as the restriction to M of the Liouville 1-form $\lambda = p\,dq$. Let Z be the associated Reeb vector field. Identifying the tangent and cotangent bundles of X with the metric h, the set M is viewed as the unit tangent bundle of X. Then Z is identified with the vector field on the unit tangent bundle of X generating the geodesic flow on $S^\star X$. Therefore, with this identification, the Reeb flow is the geodesic flow on M.

Now, the geodesic flow is ergodic (and hence, the Reeb flow is ergodic with respect to the Popp measure) in the following cases:

- If the curvature of X is negative, then the geodesic flow is ergodic. In that case, the Reeb flow is Anosov and the contact distribution is generated by the stable and unstable lines of the geodesic flow.
- It is known that any compact orientable surface can be endowed with a Riemannian metric having an ergodic geodesic flow (see [10]). Applied to X, this implies that there exists a metric h for which the geodesic flow is ergodic.

There are explicit examples with $M = \mathbb{P}^3(\mathbb{R})$ seen as the unit tangent bundle of $(\mathbb{S}^2, g)$ where the geodesic flow of g is ergodic (see [10]).

We refer to [13] for other recent examples of Anosov contact flows (which can be realized as Reeb flows) on three-dimensional manifolds, obtained by surgery from unit cotangent bundles of hyperbolic surfaces.

Hamiltonian flows. We use the relation with symplectic geometry (see, e.g., [15]).

Let (W,ω) be a symplectic manifold of dimension 4, and let M be a submanifold of W of dimension 3, such that there exists a vector field v on a neighborhood of M in W, satisfying $\mathcal{L}_v\omega = \omega$ (Liouville vector field), and transverse to M. Then the one-form $\alpha = \iota_v\omega$ is a global contact form on M, and we have $d\alpha = \omega$. Note that, if $\omega = d\lambda$ is exact, then the vector field v defined by $\iota_v\omega = \lambda$ is Liouville (in local symplectic coordinates (q,p) on W, we have $v = p\,\partial_p$). If the manifold M is moreover a level set of an Hamiltonian function h on W, then the Reeb flow on M (associated with α) is a reparametrization of the Hamiltonian flow restricted to M.

If $D = \ker \alpha$ is moreover endowed with a Riemannian metric g, then the contact form α_g constructed in Section 2.3 is collinear to α, that is, $\alpha_g = h\alpha$ for some smooth function h (never vanishing). Let us then choose the metric g such that $h = 1$ (it suffices to take g such that $(d\alpha)_{|D}$ coincides with the Riemannian volume induced by g on D). The Popp measure is defined as in Section 2.3.

Then, the Reeb flow is ergodic on (M, ν) if and only if the Hamiltonian flow is ergodic on (W, ω^2). This gives many possible examples of an ergodic Reeb flow.

4 The local and microlocal Weyl laws

4.1 Definitions

We recall that $(\phi_n)_{n\geqslant 0}$ is a fixed orthonormal basis of (real-valued) eigenfunctions of $\triangle_{sR}$. The definitions and results in this subsection are general.

Definition 1. *For every bounded linear operator A on $L^2(M, \mu)$, we define the Cesàro mean $E(A) \in \mathbb{C}$ by*

$$E(A) = \lim_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} \langle A\phi_n, \phi_n\rangle \,,$$

whenever this limit exists. We define the variance $V(A) \in \mathbb{R}^+$ by

$$V(A) = \limsup_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} |\langle A\phi_n, \phi_n\rangle|^2 \,.$$

Note that every pseudo-differential operator of order 0 on M is a bounded operator on $L^2(M, \mu)$, and that every pseudo-differential operator of negative order is a compact operator on $L^2(M, \mu)$.

The following definitions then follow by restricting either to operators A that are given by the multiplication by a function f or to operators that are obtained by quantizing a symbol in $S^\star M$ into an element of $\Psi^0(M)$.

Definition 2. *The* local Weyl measure *$w_\triangle$ is the probability measure on M defined by*

$$\int_M f\, dw_\triangle = \lim_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} \int_M f|\phi_n|^2\, d\mu,$$

for every continuous function $f : M \to \mathbb{R}$, whenever the limit exists. In other words, $w_\triangle$ is the weak limit (in the sense of measures) of $\frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} |\phi_n|^2 \mu$ as $\lambda \to +\infty$.

The microlocal Weyl measure *$W_\triangle$ is the probability measure on $S^\star M = S(T^\star M)$, the co-sphere bundle, defined as follows: we identify positively homogeneous functions of degree 0 on $T^\star M$ with functions on the bundle $S^\star M$. Then, for every symbol $a : S^\star M \to \mathbb{R}$ of order zero, we have*

$$\int_{S^\star M} a\, dW_\triangle = \lim_{\lambda\to+\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} \langle \mathrm{Op}_+(a)\phi_n, \phi_n\rangle,$$

where Op_+ is a positive quantization, whenever the limit exists. In other words $W_\triangle$ is the weak limit of the probability measures on $S^\star M$ defined by $a \mapsto \frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} \langle \mathrm{Op}_+(a)\phi_n, \phi_n\rangle$.

It can be seen that, if it exists, the microlocal Weyl measure is independent of the choice of the quantization and of the choice of the orthonormal eigenbasis (because it is a trace). Choosing a positive quantization allows to deal with probability measures for any λ and not only in the limit

$\lambda \to +\infty$. Using the fact that we are dealing with probability measures also allows us to extend the definition to continuous functions.

We finish this section by proving several general lemmas that will be useful in the sequel.

Lemma 1. *1. The set of $A \in \Psi^0$ such that $V(A) = 0$ is a linear subspace.*

2. Denoting by $A^\star$ the adjoint of $A \in \Psi^0$ in $L^2(M, \mu)$, when the right-hand side is defined, we have

$$V(A) \leqslant E(A^\star A). \tag{5}$$

3. Let A and B be bounded linear operators on $L^2(M, \mu)$. If $V(B) = 0$ then $V(A + B) = V(A)$.

Proof. Using the Cauchy-Schwarz inequality and the Young inequality we have $V(A + B) \leqslant 2\,(V(A) + V(B))$, for all bounded linear operators A and B on $L^2(M, \mu)$. The first claim follows. The second also follows from the Cauchy-Schwarz inequality, since $|\,\langle A\phi_n, \phi_n\rangle\,|^2 \leqslant \langle A^\star A\phi_n, \phi_n\rangle$, for every integer n. Let us prove the third point. Using the generalized Young inequality, we have

$$\langle (A + B)\phi_n, \phi_n\rangle^2 \leqslant (1 + \varepsilon)\langle A\phi_n, \phi_n\rangle^2 + (1 + \frac{1}{\varepsilon})\langle B\phi_n, \phi_n\rangle^2,$$

for every $\varepsilon > 0$, from which we infer that $V(A + B) - V(A) \leqslant \varepsilon V(A) + (1 + \frac{1}{\varepsilon})V(B)$. Using that $V(B) = 0$ and letting ε go to 0, we get that $V(A+B) \leqslant V(A)$. The converse inequality is obtained by writing $A = A + B - B$ and by following the same argument. □

Lemma 2. *If A is a compact operator on $L^2(M, \mu)$, then $E(A) = 0$ and $V(A) = 0$.*

This lemma can in particular be applied to pseudo-differential operators of negative order.

Proof. If A is compact, then $\|A\phi_n\| \to 0$ as $n \to +\infty$. The lemma follows, using the Cauchy-Schwarz inequality. □

Lemma 3. *The microlocal Weyl measure $W_\triangle$ is even with respect to the canonical involution of $S^\star M$.*

Proof. Since the eigenfunctions ϕ_n are real-valued, we have

$$\langle \mathrm{Op}^{\mathcal{W}}(a)\phi_n, \phi_n\rangle = \langle \mathrm{Op}^{\mathcal{W}}(\tilde{a})\phi_n, \phi_n\rangle, \tag{6}$$

for every integer n, with $\tilde{a}(q, p) = a(q, -p)$, and hence $\langle \mathrm{Op}_+(a)\phi_n, \phi_n\rangle = \langle \mathrm{Op}_+(\tilde{a})\phi_n, \phi_n\rangle + \mathrm{o}(1)$. The lemma follows. □

The above definitions and lemmas make sense in a very general setting. Our next objective is to identify the microlocal Weyl measure in the 3D contact case.

4.2 The microlocal Weyl law

Theorem 2. *Let $A \in \Psi^0$ be a pseudo-differential operator of order 0 with principal symbol $a \in \mathcal{S}^0$. In the 3D contact case, we have*

$$\sum_{\lambda_n \leqslant \lambda} \langle A\phi_n, \phi_n\rangle = \frac{P(M)}{64}\lambda^2(1 + \mathrm{o}(1)) \int_M \big(a(q, \alpha_g(q)) + a(q, -\alpha_g(q))\big)\, d\nu, \tag{7}$$

as $\lambda \to +\infty$.

In particular, it follows that $N(\lambda) \sim \frac{P(M)}{32}\lambda^2$, and that

$$E(A) = \frac{1}{2}\int_M \big(a(q, \alpha_g(q)) + a(q, -\alpha_g(q))\big)\, d\nu. \tag{8}$$

Remark 7. Theorem 2 (proved in Section 4.3) says that the eigenfunctions "concentrate" on Σ. In our proof, we are able to infer the microlocal Weyl measure from the local one, because the distribution D is of codimension one. It should be noticed that the local Weyl measure has been computed in the 3D contact case by many authors: for instance, its explicit expression is at least contained in [26] (see also [27]). For the convenience of the reader and for the sake of completeness, we present in Appendix A.3 an elementary proof of the local Weyl law, inspired by [3].

Remark 8. The right-hand side of (7) can be rewritten as

$$\frac{P(M)}{64}\lambda^2(1+\mathrm{o}(1))\left(\int_{\Sigma_1} a\, d\hat{\nu}_1 + \int_{\Sigma_{-1}} a\, d\hat{\nu}_{-1}\right).$$

Therefore (8) gives

$$E(A) = \int_{S\Sigma} a\, dW_\triangle,$$

where $S\Sigma = (\Sigma \setminus \{0\})/(0,+\infty)$ is the co-sphere bundle of Σ.

Note that, due to (8), $E(A)$ only depends on $a_{|\Sigma}$. For $V(A)$, we have the following.

Corollary 1. *1. For every $A \in \Psi^0$ whose principal symbol r vanishes on Σ, we have $V(A) = 0$.*

2. Let $A, B \in \Psi^0$, with principal symbols $a, b \in \mathcal{S}^0$ such that $a_{|\Sigma} = b_{|\Sigma}$. Then $V(A) = V(B)$.

Proof of Corollary 1. The principal symbol of $A^\star A$ is $|a|^2$. If $a_{|\Sigma} = 0$, then it follows from the microlocal Weyl formula (8) that $E(A^\star A) = 0$, and hence $V(A) = 0$. For the second point, writing $A = B + A - B$, we have $V(A - B) = 0$ by the first point, and the conclusion follows by using Lemma 1. □

4.3 Proof of the microlocal Weyl law

In this section, we provide a general argument showing how one can derive the microlocal Weyl measure from the local one (which is proved in Appendix A.3 – see also Remark 7). Since this argument actually works in a more general setting than the 3D contact case, we provide hereafter results in a general sub-Riemannian context.

Let (M, D, g) be a sub-Riemannian structure, where M is a compact manifold of dimension d. Let $\triangle_{sR}$ be a sub-Riemannian Laplacian on M (see Section 2.1). We assume that the horizontal distribution D is *equiregular*, which means that, if we define the sequence of subbundles $(D^i)_{i\in\mathbb{N}^*}$ by $D^1 = D$, $D^{i+1} = D^i + [D^i, D]$, then the dimension of D^i_q dot not depend on $q \in M$, for every integer i. We also assume that $\mathrm{Lie}(D) = TM$ (this is Hörmander's assumption, which implies hypoellipticity). Let r be the smallest positive integer such that $D^r = TM$.

In this general context, the characteristic manifold of $\triangle_{sR}$ is still defined by $\Sigma = D^\perp$. We have the following result.

Proposition 2. *If the microlocal Weyl measure $W_\triangle$ exists, then* $\mathrm{Supp}(W_\triangle) \subset S\Sigma$.

Proof. It is proved in [27] that, in the equiregular case,

$$N(\lambda) \sim C\lambda^{Q/2},$$

where $Q = \sum_{k=1}^r k(\dim D^k - \dim D^{k-1})$ (Hausdorff dimension of the sR structure). We have the following lemma.

Lemma 4. *Let $A \in \Psi^0$ be a pseudo-differential operator of order 0 whose principal symbol a vanishes in a neighborhood of $\Sigma = D^\perp$. Then*

$$\sum_{\lambda_n \leqslant \lambda} |\langle A\phi_n, \phi_n\rangle| = \mathrm{O}\big(\lambda^{d/2}\big),$$

as $\lambda \to +\infty$.

Remark 9. The exponent $\frac{d}{2}$ is the one that we would obtain in the classical elliptic case. Outside of Σ, the operator $\triangle_{sR}$ is an elliptic operator.

Proof of Lemma 4. We can assume that A is positive. The proof follows closely the arguments of [19] (see, in particular, the end of Section 3) and [11, Section 2]. We consider $Z_A(t) = \mathrm{Tr}\left(\exp(-it\sqrt{-\triangle_{sR}})A\right)$ as a distribution on a small interval $(-\varepsilon, \varepsilon)$. Let ρ be a positive function in the Schwartz space $\mathscr{S}(\mathbb{R})$ whose Fourier transform $\hat{\rho}(\omega) = \int_{\mathbb{R}} e^{-it\omega}\rho(t)\,dt$ is compactly supported in $(-\varepsilon, \varepsilon)$. It follows from [11] that the estimate of the lemma is related to the singularity at $t = 0$ of Z_A and that this singularity can be understood by studying the half-wave propagator $U(t) = \exp(-it\sqrt{-\triangle_{sR}})$. Note that, although $U(t)$ is not a Fourier Integral Operator (FIO) as in the elliptic case, it is however well defined as an operator smoothly depending on t that acts on smooth functions (as follows for instance from the functional calculus).

We now choose $\varepsilon > 0$ such that $\phi_t(WF'(A)) \cap \Sigma = \{0\}$ for $|t| < \varepsilon$, where ϕ_t is the sub-Riemannian geodesic flow (homogeneous of degree 0). This condition implies that, when we consider the operators $U(t)A$, we do not leave the elliptic region. Thus, using FIO's as in the elliptic case, we get an approximate solution $K_A(t)$ that is given by a FIO. More precisely, for every $t \in (-\varepsilon, \varepsilon)$, we have $(\partial_t + i\sqrt{-\triangle_{sR}})K_A = r$ with r smoothing and $K_A(0) - A = s$ with s smoothing as in [19, Section 3]. Using the Duhamel formula, hypoellipticity, and the calculus of FIO's to take the trace, we obtain that $\mathrm{Tr}(K_A(t)) - Z_A(t)$ is smooth.

It follows that the singularity of $Z_A(t)$ at $t = 0$ is obtained exactly as in the elliptic case. We obtain the asymptotics

$$\sum_{n=1}^{\infty} a_n \rho(\omega - \omega_n) \sim C(A)\omega^{d-1},$$

as $\omega \to +\infty$, where $\omega_n = \sqrt{\lambda_n}$ and $a_n = \langle A\phi_n, \phi_n\rangle$. The result then follows in a standard way (see [11]). □

Since $Q > d$, it follows from Lemma 4 that, if A is microlocally supported in $T^\star M \setminus \Sigma$, then

$$\frac{1}{N(\lambda)} \sum_{\lambda_n \leqslant \lambda} |\langle A\phi_n, \phi_n\rangle| \to 0,$$

as $\lambda \to +\infty$. The conclusion follows. □

Remark 10. In the 3D contact case, we have $N(\lambda) \sim C\lambda^2$ (see Appendix A.3).

Corollary 2. *If the horizontal distribution D is of codimension 1 in TM, and if the local Weyl measure $w_\triangle$ exists, then the microlocal Weyl measure $W_\triangle$ exists and is equal to half of the pullback of $w_\triangle$ by the double covering $S\Sigma \to M$ which is the restriction of the canonical projection of $T^\star M$ onto M.*

Proof. Using Proposition 2 and a density argument, we obtain that, if a is continuous and vanishes on Σ, then $W_\triangle(a)$ is well defined and vanishes. Now, let a be a general continuous function. Using the fact that D is of codimension one, we write $a = a - \bar{a} + \bar{a}$ where $\bar{a}$ is the microlocal lift to Σ of a function on the base. By the above reasoning, $W_\triangle(a - \bar{a})$ is well defined and vanishes. Now, by construction, $W_\triangle(\bar{a})$ is well defined and is expressed with the local Weyl measure $w_\triangle$. □

Let us now conclude the proof of Theorem 2, in the 3D contact case. It follows from Proposition 5 in Appendix A.3 that the local Weyl measure exists and coincides with the probability Popp measure. Then Theorem 2 follows from Corollary 2.

5 Birkhoff normal form

In this section, we are going to establish a normal form for the principal symbol of a contact 3D sub-Riemannian Laplacian, in the spirit of a result by Melrose in [25, Section 2]. This normal form implies in particular that, microlocally near the characteristic cone, all contact 3D sub-Riemannian Laplacians (associated with different metrics and/or measures) are equivalent.

Recall that the characteristic cone Σ of $-\triangle_{sR}$ is defined by (4), and is an embedded conic submanifold of $T^\star M$ which is parametrized by $(q,s) \in M \times \mathbb{R}$. Note that Σ depends on the metric g but not on the volume form $d\mu$ (as well as the principal symbol $\sigma_P(-\triangle_{sR})$, which is equal to the co-metric). We define

$$\Sigma^+ = \{(q, s\alpha_g(q)) \in T^\star M \mid s > 0\},$$
$$\Sigma^- = \{(q, s\alpha_g(q)) \in T^\star M \mid s < 0\}\}.$$

Accordingly, Σ^+ and Σ^- are positive conic submanifolds of $T^\star M$. For every $q \in M$, we denote by $\Sigma^\pm_q$ the fiber in $\Sigma^\pm$ above q, that is, the half line generated in $\Sigma^\pm$ by $\alpha_g(q)$.

Recall that $(M_H, \triangle_H)$ is the Heisenberg flat case considered in Section 3.1. The cotangent space $T^\star M_H$ is endowed with its canonical symplectic form. The characteristic cone Σ_H is defined accordingly, as well as the positive cones $\Sigma^\pm_H$.

Given $k \in \mathbb{N} \cap \{+\infty\}$ and given a smooth function f on $T^\star M$, the notation $f = \mathrm{O}_\Sigma(k)$ means that f vanishes along Σ at order k (at least). The word flat is used when $k = +\infty$.

In what follows we will work only with Σ^+, the results for Σ^- being similar.

5.1 Classical normal form

Theorem 3. *Let $q \in M$ be arbitrary. There exist a conic neighborhood C_q of Σ^+_q in $(T^\star M, \omega)$ and a homogeneous symplectomorphism χ from C_q to $(T^\star M_H, \omega_H)$, satisfying $\chi(q) = 0$ and $\chi(\Sigma^+ \cap C_q) \subset \Sigma^+_H$, such that*

$$\sigma_P(-\triangle_H) \circ \chi = \sigma_P(-\triangle_{sR}) + \mathrm{O}_\Sigma(\infty).$$

The proof of Theorem 3 is quite long and is done in Sections 5.1.1, 5.1.2, 5.1.3 and 5.1.4. The main steps are the following.

First of all, in Section 5.1.1, we endow $\mathbb{R}^6$ with a symplectic form $\tilde{\omega}$, with an appropriate conic structure, and with an Hamiltonian function H_2, such that, for any given contact structure and any $q \in M$, there exists a homogeneous diffeomorphism χ_1 from a conic neighborhood C_q of Σ^+_q to $\mathbb{R}^6$ such that

$$\chi_1^*\tilde{\omega} = \omega + \mathrm{O}_\Sigma(1),$$
$$H_2 \circ \chi_1 = \sigma_P(-\triangle_{sR}).$$

In a second step, using a conic version of the Darboux-Weinstein lemma, we modify χ_1 into a homogeneous diffeomorphism χ_2 such that

$$\chi_2^*\tilde{\omega} = \omega,$$
$$H_2 \circ \chi_2 = \sigma_P(-\triangle_{sR}) + \mathrm{O}_\Sigma(3).$$

In other words, we kill the remainder term in the pullback and thus we obtain a symplectomorphism, but at the price of adding a remainder term of order 3 along Σ in the equality of symbols (see Section 5.1.2).

In a third step, we improve the latter remainder to a flat remainder $\mathrm{O}_\Sigma(\infty)$, by solving cohomological equations in the model $(\mathbb{R}^6, \tilde{\omega})$ (see Section 5.1.3). This is the most technical and lengthy part of the proof.

Using $\mathbb{R}^6$ as a pivot space, we finally obtain the theorem (see Section 5.1.4).

5.1.1 Construction of the model

Preliminaries on the symplectic conic manifold Σ. We define the conic manifold $W^+ = \{(q,s) \mid q \in M,\ s > 0\}$ (of dimension 4). The conic structure is defined by $\lambda \cdot (q,s) = (q, \lambda s)$ for $\lambda > 0$. We define the mapping $i_{W^+} : W^+ \to \Sigma^+$ by $i_{W^+}(q,s) = (q, s\alpha_g(q))$, and $\pi_{W^+} : \{(q,p) \in T^\star M \mid h_Z(q,p) > 0\} \to W^+$ by $\pi_{W^+}(q,p) = (q, h_Z(q,p))$. Then, clearly, we have $i_{W^+} \circ \pi_{W^+|\Sigma^+} = \mathrm{id}_{\Sigma^+}$ and $\pi_{W^+|\Sigma^+} \circ i_{W^+} = \mathrm{id}_{W^+}$, and hence Σ^+ and W^+ can be identified with the charts defined by the diffeomorphisms i_{W^+} and $\pi_{W^+|\Sigma^+}$, in some conic neighborhood of Σ^+.

Since we are in the contact case, the characteristic manifold Σ is symplectic, which means that the restriction $\omega_{|\Sigma}$ is symplectic. Recall that, writing $T_\psi(T^\star M) = T_\psi\Sigma \oplus \mathrm{orth}_{\omega_\psi}(T_\psi\Sigma)$ at any point $\psi \in \Sigma$, the bilinear form $(\omega_{|\Sigma})_\psi$ is defined as the restriction of ω_ψ to $T_\psi\Sigma \times T_\psi\Sigma$.

Endowing W with the symplectic form $\omega_W = \alpha_g \wedge ds - s\, d\alpha_g$, we claim that

$$i_W^\star\, \omega_{|\Sigma} = \omega_W.$$

Indeed, taking local coordinates, we have $\omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z$, and $\alpha_g = a_1\, dx + a_2\, dy + a_3\, dz$. Then $d\alpha_g = da_1 \wedge dx + da_2 \wedge dy + da_3 \wedge dz$. Along $\Sigma \sim W$, we have $p_i = sa_i$ and hence $dp_i = a_i\, ds + s\, da_i$, and the claim follows.

Finally, note that we have

$$\mathrm{orth}_\omega(T\Sigma) = \mathrm{Span}\{\vec{h}_X, \vec{h}_Y\},$$

at any point $q \in M$, where (X,Y) is an arbitrary local g-orthonormal frame of D. Indeed, this follows from the fact that

$$\begin{aligned} T\Sigma &= \ker dh_X \cap \ker dh_Y \\ &= \{\xi \in T(T^\star M) \mid dh_X.\xi = dh_Y.\xi = 0\} \\ &= \{\xi \in T(T^\star M) \mid \omega(\xi, \vec{h}_X) = \omega(\xi, \vec{h}_Y) = 0\}. \end{aligned}$$

The canonical symplectic form ω along Σ. We are going to compute the canonical symplectic form ω at any point of Σ.

Let $\psi = (q,p)$ be any point of Σ, identified with $(q,s) \in W$ with the chart described above ($s = h_Z(q,p)$). We have $T_\psi(T^\star M) = T_\psi\Sigma \oplus \mathrm{orth}_{\omega_\psi}(T_\psi\Sigma)$, and we already know that the restriction $(\omega_\psi)_{|T_\psi\Sigma}$ of the bilinear form ω_ψ to $T_\psi\Sigma \times T_\psi\Sigma$ is $(\omega_\psi)_{|T_\psi\Sigma} = \alpha_g(\psi) \wedge ds - s\, d\alpha_g(\psi)$. Let us now compute the restriction $(\omega_\psi)_{|\mathrm{orth}_{\omega_\psi}(T_\psi\Sigma)}$ of the bilinear form ω_ψ to $\mathrm{orth}_{\omega_\psi}(T_\psi\Sigma) \times \mathrm{orth}_{\omega_\psi}(T_\psi\Sigma)$.

Let (X,Y) be an arbitrary local g-orthonormal frame of D in a neighborhood of q.

For every $\xi \in \mathrm{orth}_{\omega_\psi}(T_\psi\Sigma)$, we set $\xi = U\vec{h}_X(\psi) + V\vec{h}_Y(\psi)$. Since $\omega(\vec{h}_X, \vec{h}_Y) = \{h_X, h_Y\}_\omega$, the matrix of the bilinear form $(\omega_\psi)_{|\mathrm{orth}_{\omega_\psi}(T_\psi\Sigma)}$ in the basis $(\vec{h}_X(\psi), \vec{h}_Y(\psi))$ is

$$\begin{pmatrix} 0 & \{h_X, h_Y\}_\omega(\psi) \\ -\{h_X, h_Y\}_\omega(\psi) & 0 \end{pmatrix}.$$

According to (3), we have

$$\{h_X, h_Y\}_\omega = -h_{[X,Y] \bmod D} = h_Z + O_\Sigma(1).$$

Hence, since $\psi \in \Sigma$, we have $\{h_X, h_Y\}_\omega(\psi) = h_Z(\psi)$, and therefore $(\omega_\psi)_{|\mathrm{orth}_\omega(T\Sigma)} = h_Z(\psi)\, dU \wedge dV$. Now, using the facts that $dh_X(\psi).\xi = \omega_\psi(\vec{h}_X(\psi), \xi) = V\omega_\psi(\vec{h}_X(\psi), \vec{h}_Y(\psi)) = Vh_Z(\psi)$ and that, similarly, $dh_Y(\psi).\xi = -Uh_Z(\psi)$, it follows that $h_Z(\psi)\, dV = dh_X(\psi)$ and $h_Z(\psi)\, dU = -dh_Y(\psi)$.

At this step, we define the functions u and v on $\{(q,p) \in T^\star M \mid h_Z(q,p) \neq 0\}$ by

$$u(q,p) = \frac{h_X(q,p)}{\sqrt{|h_Z(q,p)|}}, \qquad v(q,p) = \frac{h_Y(q,p)}{\sqrt{|h_Z(q,p)|}}. \tag{9}$$

Note that, since (X, Y, Z) is a local frame of TM, we have $h_Z(q,p) \neq 0$ for every $(q,p) \in \Sigma$ with $p \neq 0$. Note also that u and v are positively homogeneous of order $1/2$ with respect to p. With these notations, we get immediately that $(\omega_\psi)_{|\mathrm{orth}_\omega(T\Sigma)} = \mathrm{sign}(h_Z(q,p))\, du(q,p) \wedge dv(q,p)$.

We conclude that, for any $\psi = (q,p) \in \Sigma^+$ with $p \neq 0$, we have

$$\omega_\psi = \alpha_g(q) \wedge dh_Z(q,p) - h_Z(q,p)\, d\alpha_g(q) + du(q,p) \wedge dv(q,p). \tag{10}$$

Construction of the diffeomorphism χ_1. We consider the positive symplectic cone $P^+ = \mathbb{R}^2$, with local coordinates denoted by (u,v), endowed with the symplectic form $du \wedge dv$. Its conic structure is defined by $\lambda \cdot (u,v) = (\sqrt{\lambda}u, \sqrt{\lambda}v)$ for $\lambda > 0$.

The conic manifold $W^+ \times P^+$, with local coordinates (q,s,u,v), is endowed with the symplectic form $\tilde{\omega} = \omega_W + du \wedge dv$.

We define the mapping $\chi_1 : \{(q,p) \in T^\star M \mid h_Z(q,p) \neq 0\} \to W \times P$ by

$$\chi_1(q,p) = (q, h_Z(q,p), u(q,p), v(q,p)), \tag{11}$$

with the functions u and v defined by (9). It is clear that χ_1 is a local homogeneous diffeomorphism in some conic neighborhood of any point $(q,p) \in \Sigma$, and that χ_1 maps Σ^+ to $W^+ \times \{0\}$.

It follows from (10) and from the definition of χ_1 that the symplectic forms $\chi_1^\star\tilde{\omega}$ and ω agree along Σ^+, that is, $(\chi_1^\star\tilde{\omega})_\psi = \omega_\psi$ for every $\psi \in \Sigma^+$. In other words, we have $\chi_1^\star\tilde{\omega} = \omega + O_\Sigma(1)$.

Note that, by definition of the functions u and v, the principal symbol of $-\triangle_{sR}$ can be written in $\{(q,p) \in T^\star M \mid h_Z(q,p) \neq 0\}$ as

$$\sigma_P(-\triangle_{sR})(q,p) = h_X(q,p)^2 + h_Y(q,p)^2 = |h_Z(q,p)| \left(u(q,p)^2 + v(q,p)^2\right). \tag{12}$$

In the Heisenberg flat case, we recover the factorization of $\sigma_P(-\triangle_{sR})$ done in Section 3.1 (and in that case we have exactly $\chi_1^\star\tilde{\omega} = \omega$, without any remainder term).

Defining the Hamiltonian function H_2 on $W \times P$ by $H_2(q,s,u,v) = |s|(u^2+v^2)$, we have $\sigma_P(-\triangle_{sR}) = H_2 \circ \chi_1$. Note that $\chi_1(q, \lambda p) = \lambda \cdot \chi_1(q,p)$ for every $\lambda > 0$. Hence, at this step, we have obtained the following intermediate result.

Lemma 5. *Let $q \in M$ be arbitrary. We consider the symplectic positive cone $W^+ = \{(q,s) \mid q \in M,\ s > 0\}$, with the conic structure defined by $\lambda \cdot (q,s) = (q, \lambda s)$ for $\lambda > 0$, endowed with the symplectic form $\omega_W = \alpha_g \wedge ds - s\, d\alpha_g$. We also consider the symplectic two-dimensional space $P^+ = \mathbb{R}^2$, with the conic structure defined by $\lambda \cdot (u,v) = (\sqrt{\lambda}u, \sqrt{\lambda}v)$, endowed with the symplectic form $u \wedge dv$. The mapping χ_1 defined by (11) is a homogeneous diffeomorphism in a conic neighborhood C_q of Σ_q^+ in $(T^\star M, \omega)$ (for the canonical conic structure of $T^\star M$), with values in the conic manifold $W^+ \times P^+$. Moreover, defining the symplectic form $\tilde{\omega}_+ = \omega_W + du \wedge dv$ and the Hamiltonian function H_2 on $W^+ \times P^+$ by $H_2(q,s,u,v) = |s|(u^2+v^2)$, we have*

$$\chi_1^\star\tilde{\omega} = \omega + \mathrm{O}_\Sigma(1), \tag{13}$$

and

$$\sigma_P(-\triangle_{sR}) = H_2 \circ \chi_1. \tag{14}$$

The diffeomorphism χ_1 is almost symplectic, but there is a remainder term $\mathrm{O}_\Sigma(1)$ in (13). In the Heisenberg flat case, there is no remainder term and hence χ_1 is actually a symplectomorphism. In the general case, we are going to modify slightly χ_1 in order to obtain a homogeneous symplectomorphism χ_2, but this will be at the price of introducing a remainder term in the equality (14) for the principal symbol. The method used to normalize the symplectic form is standard and known as Darboux-Weinstein lemma.

5.1.2 Using the Darboux-Weinstein lemma

In order to remove the remainder term $O_\Sigma(1)$ in (13), we use a conic version of the Darboux-Weinstein lemma. This version is stated in a general version, in Lemma 16 (see Appendix A.5).

It follows from that lemma (applied with $N = T^\star M$, $P = \Sigma$ and $k = 1$) that there exists a homogeneous diffeomorphism f defined in a conic neighborhood of Σ^+, such that $f^\star(\chi_1^\star\tilde{\omega}) = \omega$, and such that f is tangent to the identity along Σ^+, that is, such that $f = \mathrm{id} + O_\Sigma(2)$.

We define $\chi_2 = \chi_1 \circ f$. Then χ_2 is a local homogeneous diffeomorphism in some conic neighborhood of any point $(q, p) \in \Sigma$, mapping Σ^+ to $W^+ \times \{0\}$, and satisfying $\chi_2^\star\tilde{\omega} = \omega$ (and thus χ_2 is a symplectomorphism).

Since $f = \mathrm{id} + O_\Sigma(2)$, we have

$$\chi_2(q, p) = \chi_1((q, p) + O_\Sigma(2)) = (q, h_Z(q, p), u(q, p), v(q, p)) + O_\Sigma(2). \tag{15}$$

Using (12), (15) and the fact that $u\, O_\Sigma(2) = O_\Sigma(3)$ and $v\, O_\Sigma(2) = O_\Sigma(3)$, we get that

$$H_2 \circ \chi_2 = \sigma_P(-\triangle_{sR}) + O_\Sigma(3),$$

in any conic neighborhood of W^+.

Taking local Darboux coordinates for the contact structure on M, in which $q = (x, y, z)$ and $\alpha_g = x\, dy + dz$, we get the following result.

Lemma 6. *We consider the symplectic positive conic space $\mathbb{R}^6$, with local coordinates (x, y, z, s, u, v), endowed with the conic structure given by $\lambda \cdot (x, y, z, s, u, v) = (x, y, z, \lambda s, \sqrt{\lambda}u, \sqrt{\lambda}v)$, and with the symplectic form $\tilde{\omega}_+ = (xdy + dz) \wedge ds - s\, dx \wedge dy + du \wedge dv$. We define the Hamiltonian function H_2 on $\mathbb{R}^6$ by $H_2(x, y, z, s, u, v) = |s|(u^2 + v^2)$. Let $\tilde{\Sigma}^+ \subset \mathbb{R}^6$ be the characteristic cone defined by $\tilde{\Sigma}^+ = \{u = v = 0,\ s > 0\}$. Then, for any $q \in M$, there exist a conic neighborhood C_q of Σ_q^+ in $(T^\star M, \omega)$ and a homogeneous symplectomorphism χ_2 from C_q to $\mathbb{R}^6$, satisfying $\chi_2(q) = 0$ and $\chi_2(\Sigma^+ \cap C_q) \subset \tilde{\Sigma}^+$, such that*

$$\chi_2^\star\tilde{\omega} = \omega,$$
$$H_2 \circ \chi_2 = \sigma_P(-\triangle_{sR}) + \mathrm{O}_\Sigma(3).$$

At this step, we have thus obtained (up to a homogeneous canonical transform) the normal form with a remainder term $O_\Sigma(3)$. In order to improve this normal form at the infinite order, we are next going to solve a series of cohomological equations in $(\mathbb{R}^6, \tilde{\omega})$.

5.1.3 Cohomological equations

We consider the function $H = \sigma_P(-\triangle_{sR}) \circ \chi_2^{-1}$ defined on $\mathbb{R}^6$. According to Lemma 6, we have

$$H = H_2 + \mathrm{O}_\Sigma(3) = H_2 + \mathrm{O}((u^2 + v^2)^{\frac{3}{2}}), \tag{16}$$

and H is homogeneous of order 2 with respect to the cone structure of $\mathbb{R}^6$ under consideration in that lemma.

Our objective is to construct, near 0, a local symplectomorphism φ from $(\mathbb{R}^6, \tilde{\omega})$ into itself such that

$$H \circ \varphi = H_2 + \mathrm{O}_\Sigma(\infty) = H_2 + \mathrm{O}((u^2+v^2)^\infty).$$

The usual procedure, due to Birkhoff, consists in constructing φ iteratively, by composing flows at time 1 associated with an appropriate Hamiltonian function (also called Lie transforms), chosen by identifying the Taylor expansions at increasing orders, and by solving a series of (so-called) cohomological equations in $(\mathbb{R}^6, \tilde{\omega})$. Then the canonical transform is constructed by using the Borel theorem.

In the present setting, we have to adapt this general method and to define appropriate spaces of homogeneous functions and polynomials, sharing nice properties in terms of Poisson brackets. The procedure goes as follows.

Recall that we consider local coordinates (q, s, u, v) in $\mathbb{R}^6$ with $q = (x, y, z)$. Let C be a conic neighborhood of $(0, 0, 0, 1, 0, 0)$ (that will be taken sufficiently small in the sequel). For every integer j, we denote by $\mathcal{F}_j$ the set of functions g that are smooth in C and homogeneous of order j for the conic structure of $\mathbb{R}^6$, meaning that $g(q, \lambda s, \sqrt{\lambda}u, \sqrt{\lambda}v) = \lambda^j \cdot g(q, s, u, v)$, for all $\lambda > 0$ and $(q, s, u, v) \in C$. For every integer k, let $\mathcal{F}_{j,k}$ be the subspace of functions g of $\mathcal{F}_j$ that can be factorized by an homogeneous (in the classical sense) polynomial of degree k in (u, v), i.e., such that there exists a homogeneous polynomial Q of degree k and a function a (smooth in C) such that $g(q, s, u, v) = a(q, s)Q(u, v)$. Note that, by definition, we must have $a \in \Gamma_{j-k/2}$, where Γ_γ is the set of functions of (q, s), smooth in C, satisfying $a(q, \lambda s) = \lambda^\gamma a(q, s)$ for every $\lambda > 0$.

For every integer k, we set

$$\mathcal{F}^0_{j,k} = \left\{ g \in \mathcal{F}_{j,k} \ \mid \ \int_0^{2\pi} g(q, s, R\cos(\theta), R\sin(\theta))\, d\theta = 0, \quad \forall (q, s) \in \mathbb{R}^4, \quad \forall R > 0 \right\},$$
$$\mathcal{F}^{\mathrm{inv}}_{j,k} = \left\{ (q, s, u, v) \mapsto a(q, s)(u^2+v^2)^{\frac{k}{2}} \ \mid \ a \in \Gamma_{j-k/2} \right\}.$$

The set $\mathcal{F}^0_{j,k}$ is the subset of functions of $\mathcal{F}_{j,k}$ with average 0 on circles. Using polar coordinates, it is easy to prove that

$$\mathcal{F}_{j,k} = \mathcal{F}^0_{j,k} \oplus \mathcal{F}^{\mathrm{inv}}_{j,k}, \qquad \text{and} \qquad \mathcal{F}^{\mathrm{inv}}_{j,k} = \{0\} \text{ if } k \text{ is odd.} \tag{17}$$

In the sequel, we denote by $\mathcal{F}_{j,\geqslant k}$ (and accordingly, $\mathcal{F}^0_{j,\geqslant k}$ and $\mathcal{F}^{\mathrm{inv}}_{j,\geqslant k}$) the set of functions of $\mathcal{F}_j$ that can be factorized by a homogeneous polynomial of degree greater than or equal to k.

Note that $H_2 \in \mathcal{F}^{\mathrm{inv}}_{2,2}$ and that $H \in \mathcal{F}_{2,\geqslant 2}$.

In what follows, we organize the procedure in two steps, by solving cohomological equations, first in $\mathcal{F}^0$ modulo $\mathcal{F}^{\mathrm{inv}}$, and then in $\mathcal{F}^{\mathrm{inv}}$. This choice is due to the specific Poisson bracket properties in those spaces, stated in the following lemma.

Lemma 7. *For all integers j and k, we have*

$$\{H_2, \mathcal{F}^{\mathrm{inv}}_{j,k}\}_{\tilde{\omega}} = \mathcal{F}^{\mathrm{inv}}_{j+1,k+2}, \tag{18}$$
$$\{H_2, \mathcal{F}^0_{j,k}\}_{\tilde{\omega}} = \mathcal{F}^0_{j+1,k} \mod \mathcal{F}^0_{j+1,k+2}, \tag{19}$$
$$\{\mathcal{F}_{j,k}, \mathcal{F}_{j',k'}\}_{\tilde{\omega}} \subset \mathcal{F}_{j+j'-1,k+k'-2} \mod \mathcal{F}_{j+j'-1,k+k'}, \tag{20}$$
$$\{\mathcal{F}^{\mathrm{inv}}_{j,k}, \mathcal{F}^{\mathrm{inv}}_{j',k'}\}_{\tilde{\omega}} \subset \mathcal{F}^{\mathrm{inv}}_{j+j'-1,k+k'}. \tag{21}$$

Proof. Throughout the proof, we assume that we are in Σ^+, so that $H_2 = s(u^2+v^2)$. Recall that $\tilde{\omega} = \omega_W + du \wedge dv$, that the coordinates u and v are symplectically conjugate, and that the coordinates (q, s) and (u, v) are symplectically orthogonal. It follows that

$$\{a(q, s)P(u, v), b(q, s)Q(u, v)\}_{\tilde{\omega}} = \{a, b\}_{\omega_W} PQ + ab(\partial_u P \partial_v Q - \partial_v P \partial_u Q), \tag{22}$$

for all smooth functions a, b, Q.

Taking $a(q,s)=s$, $P(u,v)=u^2+v^2$, and $Q(u,v)=(u^2+v^2)^{\frac{k}{2}}$, (22) gives

$$\{H_2, b(q,s)(u^2+v^2)^{\frac{k}{2}}\}_{\tilde{\omega}} = \{s,b\}_{\omega_W}(u^2+v^2)^{\frac{k+2}{2}}.$$

Assuming that $bQ \in \mathcal{F}^{\mathrm{inv}}_{j,k}$, we have $b \in \Gamma_{j-k/2}$. By definition of ω_W, we have $\{s,b\}_{\omega_W} = \partial_{\tilde{s}} b$, where $\tilde{s}$ is some coordinate symplectically conjugate to s, and then $\{s,b\}_{\omega_W} \in \Gamma_{j-k/2}$. Hence $\{s,b\}_{\omega_W}(u^2+v^2)^{\frac{k+2}{2}} \in \mathcal{F}^{\mathrm{inv}}_{j+1,k+2}$. We have thus proved that $\{H_2, S^{\mathrm{inv}}_{j,k}\} \subset S^{\mathrm{inv}}_{j+1,k+2}$. To establish the equality, it suffices to consider $c(u^2+v^2)^{\frac{k+2}{2}} \in \mathcal{F}^{\mathrm{inv}}_{j+1,k+2}$, with $c \in \Gamma_{j-k/2}$, and to note that one can find $b \in \Gamma_{j-k/2}$ such that $\{s,b\}_{\omega_W} = c$. We have thus proved (18).

Taking $a(q,s)=s$, $P(u,v)=u^2+v^2$, and Q homogeneous polynomial of degree k in (u,v) and of average 0 on circles, (22) gives

$$\{H_2, b(q,s)Q(u,v)\}_{\tilde{\omega}} = \{s,b\}_{\omega_W}(u^2+v^2)Q + 2sb(u\partial_v Q - v\partial_u Q).$$

Assuming that $bQ \in \mathcal{F}^0_{j,k}$, we have $b \in \Gamma_{j-k/2}$. Then, clearly, we have $\{s,b\}_{\omega_W}(u^2+v^2)Q \in \mathcal{F}^0_{j+1,k+2}$ and $2sb(u\partial_v Q - v\partial_u Q) \in \mathcal{F}^0_{j+1,k}$ (indeed, the polynomial $u\partial_v Q - v\partial_u Q$ is, like Q, homogeneous of degree k and of average 0 on circles by an obvious computation in polar coordinates). We have proved that $\{H_2, \mathcal{F}^0_{j,k}\} \subset \mathcal{F}^0_{j+1,k} \mod \mathcal{F}^0_{j+1,k+2}$. Let us prove the equality. Take $cR \in \mathcal{F}^0_{j+1,k}$, with $c \in \Gamma_{j+1-k/2}$ and R polynomial of degree k in (u,v), of average 0 on circles. We set $b = c/2s \in \Gamma_{j-k/2}$, and since R is of average 0 on circles, there exists Q polynomial of degree k in (u,v) such that $R = u\partial_v Q - v\partial_v Q$.[4] The equality follows. We have thus proved (19).

The inclusion (20) follows obviously from (22), noticing that, if $a \in \Gamma_{j-k/2}$ and $b \in \Gamma_{j'-k'/2}$, then $\{a,b\}_{\omega_W} \in \Gamma_{j+j'-k/2-k'/2-1}$ and $ab \in \Gamma_{j+j'-k/2-k'/2}$, and if P (resp., Q) is homogeneous of degree k (resp., k') in (u,v), then $\{a,b\}_{\omega_W}PQ \in \mathcal{F}_{j+j'-1,k+k'}$ and $ab(\partial_u P\partial_v Q - \partial_v P\partial_u Q) \in \mathcal{F}_{j+j'-1,k+k'-2}$.

To prove (21), it suffices to see that, in the latter argument, if moreover $P(u,v) = (u^2+v^2)^{\frac{k}{2}}$ and $Q(u,v) = (u^2+v^2)^{\frac{k'}{2}}$, then $\partial_u P\partial_v Q - \partial_v P\partial_u Q = 0$ and $\{a,b\}_{\omega_W}PQ \in \mathcal{F}^{\mathrm{inv}}_{j+j'-1,k+k'}$. □

Using (16), and since $\mathcal{F}^{\mathrm{inv}}_{2,3} = \{0\}$ (see (17)), we start with the fact that

$$\begin{aligned} H &= H_2 + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}_{2,\geqslant 3} \\ &= H_2 + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}^0_{2,\geqslant 3} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right). \end{aligned} \tag{23}$$

As said previously, we proceed in two steps, by first removing all terms in $\mathcal{F}^0_2$, thus obtaining the normal form $H \circ \varphi = H_2 + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}$, by using Lie transforms generated by appropriate elements of $\mathcal{F}^0_1$. In a second step, we remove the (invariant) terms in $(u^2+v^2)^k$ by using Lie transforms generated by appropriate elements of $\mathcal{F}^{\mathrm{inv}}_1$.

First step. In the first step, our objective is to construct a symplectomorphism allowing us to remove from (23) all terms in $\mathcal{F}^0_2$.

Lemma 8. *There exist a conic neighborhood C of $(0,0,0,1,0,0)$ in $(\mathbb{R}^6, \tilde{\omega})$ and a homogeneous symplectomorphism φ from C to $\mathbb{R}^6$ such that, in C,*

$$\varphi = \mathrm{id} + \mathrm{O}_\Sigma(2), \tag{24}$$

[4] Indeed, by linear algebra considerations, it is easily seen that the operator $A = u\partial_v - v\partial_u$ is an endomorphism of the set of homogeneous polynomials of degree k, with the following properties. If k is odd then A is invertible, and any homogeneous polynomials of degree k has average 0 on circles. If k is even then the range of A is of codimension one, and coincides with the set of homogeneous polynomials of degree k, of average 0 on circles.

and

$$H \circ \varphi = H_2 + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}. \tag{25}$$

Proof. We proceed by (strong) recurrence on k, starting at $k = 3$, by constructing, at each step, a local homogeneous symplectomorphism φ_k satisfying (24) and such that

$$H \circ \varphi_3 \circ \cdots \circ \varphi_k = H_2 + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}^0_{2,\geqslant k+1} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right),$$

and we search each φ_k in the form $\varphi_k = \varphi_k(1) = \exp(\vec{F}^0_k)$, where $\varphi_k(t) = \exp(t\vec{F}^0_k)$ is the flow at time t generated by an adequate Hamiltonian function $F^0_k \in \mathcal{F}^0_{1,k}$ (note that φ_k is then symplectic, as desired).

Before going to the recurrence, let us note that, for every $k \geqslant 3$, there exists a (small enough) conic neighborhood C_k of $(0,0,0,1,0,0)$ such that the flow φ_k is well defined on $[0,1]\times C_k$; moreover, since we are going to compose these symplectomorphisms, we choose C_k such that φ_{k+1} maps C_{k+1} to C_k, for every $k \geqslant 3$. Indeed, since $F^0_k = \mathrm{O}_\Sigma(k)$, we have $\varphi_k(t) = \mathrm{id} + \int_0^t \vec{F}^0_k \circ \varphi_k(s)\, ds = \mathrm{id} + \mathrm{O}_\Sigma(k-1)$ uniformly with respect to t on compact intervals, and the claim follows (as well as the expansion (24)).

Let us now make the construction by recurrence.

For $k = 3$, we want to prove that there exists $F^0_3 \in \mathcal{F}^0_{1,3}$ such that, setting $\varphi_3 = \exp(\vec{F}^0_3)$, we have $H \circ \varphi_3 = H_2 \mod \left(\mathcal{F}^0_{2,\geqslant 4} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right)$. Using that $\frac{d}{dt}(H \circ \varphi_3(t)) = \{F^0_3, H\}_{\tilde{\omega}} \circ \varphi_3(t)$, and since the flow is well defined on $[0,1] \times C_3$, we have

$$H \circ \varphi_3(t) = H + \{F^0_3, H\}_{\tilde{\omega}} + \mathrm{O}\left(\frac{t^2}{2}\{F^0_3, \{F^0_3, H\}_{\tilde{\omega}}\}_{\tilde{\omega}} \circ \varphi_3(t)\right), \tag{26}$$

on $[0,1] \times C_3$. Using (23), we have $H = H_2 + H^0_3 \mod \left(\mathcal{F}^0_{2,\geqslant 4} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right)$, with $H^0_3 \in \mathcal{F}^0_{2,4}$. Hence, taking $t = 1$ in (26), using (19) and (20), we infer that

$$H \circ \varphi_3 = H_2 + H^0_3 + \{F^0_3, H_2\}_{\tilde{\omega}} + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}^0_{2,\geqslant 4} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right).$$

Therefore, we have to solve the cohomological equation

$$\{H_2, F^0_3\}_{\tilde{\omega}} = H^0_3 + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}^0_{2,\geqslant 4} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right),$$

which has a solution $F^0_3 \in \mathcal{F}^0_{2,3}$ by using (19).

Let us assume that we have constructed $\varphi_3, \ldots, \varphi_{k-1}$ such that $H \circ \varphi_3 \circ \cdots \circ \varphi_{k-1} = H_2 + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}^0_{2,\geqslant k} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right)$. We want to prove that there exists $F^0_k \in \mathcal{F}^0_{2,k}$ such that, setting $\varphi_k = \exp(\vec{F}^0_k)$, we have

$$H \circ \varphi_3 \circ \cdots \circ \varphi_k = H_2 + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}^0_{2,\geqslant k+1} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right).$$

Using that $\frac{d}{dt}(H \circ \varphi_3 \circ \cdots \circ \varphi_k(t)) = \{F^0_k, H \circ \varphi_3 \circ \cdots \circ \varphi_{k-1}\}_{\tilde{\omega}} \circ \varphi_k(t)$, and since the flow φ_k is well defined on $[0,1] \times C_k$, we have

$$\begin{aligned} H \circ \varphi_3 \circ \cdots \circ \varphi_k(t) = H \circ \varphi_3 \circ \cdots \circ \varphi_{k-1} &+ \{F^0_k, H \circ \varphi_3 \circ \cdots \circ \varphi_{k-1}\}_{\tilde{\omega}} \\ &+ \mathrm{O}\left(\frac{t^2}{2}\{F^0_k, \{F^0_k, H \circ \varphi_3 \circ \cdots \circ \varphi_{k-1}\}_{\tilde{\omega}}\}_{\tilde{\omega}}\right), \end{aligned} \tag{27}$$

on $[0,1] \times C_k$. Using the recurrence assumption, we have

$$H \circ \varphi_3 \circ \cdots \circ \varphi_{k-1} = H_2 + \tilde{H}^0_k + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}^0_{2,\geqslant k+1} \oplus \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 4}\right),$$

for some $\tilde{H}_k^0 \in \mathcal{F}_{2,k}^0$. Hence, taking $t = 1$ in (27), using (19) and (20), we infer that

$$H \circ \varphi_3 \circ \cdots \circ \varphi_k = H_2 + \tilde{H}_k^0 + \{F_k^0, H_2\}_{\tilde{\omega}} + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}_{2,\geqslant k+1}^0 \oplus \mathcal{F}_{2,\geqslant 4}^{\mathrm{inv}}\right).$$

Therefore, we have to solve the cohomological equation

$$\{H_2, F_k^0\}_{\tilde{\omega}} = \tilde{H}_k^0 + \mathrm{O}_\Sigma(\infty) \mod \left(\mathcal{F}_{2,\geqslant k+1}^0 \oplus \mathcal{F}_{2,\geqslant 4}^{\mathrm{inv}}\right),$$

which has a solution $F_k^0 \in \mathcal{F}_{2,k}^0$ by using (19).

The recurrence is established.

By definition, φ_k is the flow at time 1 generated by the Hamiltonian function $F_k^0 \in \mathcal{F}_1$. By definition of $\mathcal{F}_1$, we have $F_k^0(\lambda \cdot (q, s, u, v)) = \lambda F_k^0(q, s, u, v)$, that is, F_k^0 is homogeneous for the conic structure of $\mathbb{R}^6$. Then φ_k is indeed homogeneous, as a consequence of the following general lemma, that we recall for completeness.

Lemma 9. *Let (N, ω) be a conic symplectic manifold, with a conic structure $x \mapsto \lambda \cdot x$, for $\lambda > 0$ and $x \in N$. Let H be a smooth Hamiltonian function on N, which is homogeneous, meaning that $H(\lambda \cdot x) = \lambda H(x)$ for every $\lambda > 0$ and every $x \in N$. Then the associated Hamiltonian vector field $\vec{H}$ is homogeneous, in the sense that $\vec{H}(\lambda \cdot x) = \lambda \cdot \vec{H}(x)$, and as a consequence, the generated flow $\exp(t\vec{H})$ is homogeneous as well.*

Proof of Lemma 9. By definition, the symplectic form ω is conic, in the sense that $\omega_{\lambda \cdot x}(\lambda \cdot v_1, \lambda \cdot v_2) = \lambda \omega_x(v_1, v_2)$, for all $\lambda > 0$, $x \in N$ and $v_1, v_2 \in T_x N$. The Hamiltonian vector field $\vec{H}$ is defined at any point $x \in N$ by $\omega_x(\vec{H}(x), v) = dH(x).v$, for every $v \in T_x N$. Since H is homogeneous, by differentiation we get that $dH(\lambda \cdot x).(\lambda \cdot v) = \lambda dH(x).v$, for every $v \in T_x N$, and therefore,

$$\omega_{\lambda \cdot x}(\vec{H}(\lambda \cdot x), \lambda \cdot v) = dH(\lambda \cdot x).(\lambda \cdot v) = \lambda dH(x).v = \lambda \omega_x(\vec{H}(x), v) = \omega_{\lambda \cdot x}(\lambda \cdot \vec{H}(x), \lambda \cdot v),$$

from which it follows that $\vec{H}(\lambda \cdot x) = \lambda \cdot \vec{H}(x)$. □

Let us finish the proof of Lemma 8.

We consider the formal infinite composition $\varphi_3 \circ \cdots \circ \varphi_k \cdots$ in the Fréchet space of smooth homogeneous mappings from $\mathbb{R}^6$ to $\mathbb{R}^6$. By the Borel theorem[5], there exists a smooth homogeneous mapping φ that is the Borel summation of that formal composition, i.e., such that $\varphi = \varphi_3 \circ \cdots \circ \varphi_k \cdots + \mathrm{O}_\Sigma(\infty)$. Clearly, φ is a local homogeneous diffeomorphism (because it is tangent to the identity), and we have (24) and (25). Note that φ may not be a symplectomorphism, however, by construction we have $\varphi^\star \tilde{\omega} = \tilde{\omega} + \mathrm{O}_\Sigma(\infty)$. It is however possible to modify the homogeneous diffeomorphism φ, by composing it with a homogeneous diffeomorphism tangent to identity at infinite order, so as to obtain exactly $\varphi^\star \tilde{\omega} = \tilde{\omega}$ (and thus, φ is a homogeneous symplectomorphism). This is done thanks to Lemma 16 in Appendix A.5, applied with $k = +\infty$. □

At the end of this first step, we have therefore constructed, near $(0, 0, 0, 1, 0, 0)$, a local homogeneous symplectomorphism φ such that

$$H \circ \varphi = H_2 + \sum_{k=2}^{+\infty} w_{2k}(q, s)(u^2 + v^2)^k + \mathrm{O}((u^2 + v^2)^\infty), \tag{28}$$

with $w_{2k}(q, s)(u^2 + v^2)^k \in \mathcal{F}_{2,2k}^{\mathrm{inv}}$ for every $k \geqslant 2$. We set $\tilde{H} = H \circ \varphi$.

[5] Let n be a nonzero integer, let E be a Fréchet space, and let $(a_\alpha)_{\alpha \in \mathbb{N}^n}$ be a (multiindex) sequence in E. There exists a smooth function $f : \mathbb{R}^n \to E$ whose infinite Taylor expansion at 0 is $\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$. If E has a conic structure, then f can be chosen to be homogeneous for that structure.

Second step. In the second step, our objective is to construct a symplectomorphism allowing us to remove from (28) all terms in $\mathcal{F}_2^{\mathrm{inv}}$.

Lemma 10. *There exist a conic neighborhood C' of $(0,0,0,1,0,0)$ in $(\mathbb{R}^6,\tilde{\omega})$ and a homogeneous symplectomorphism ψ from C' to $\mathbb{R}^6$ such that $\psi(C')\subset C$, and such that, in C',*

$$\psi = \mathrm{id} + \mathrm{O}_\Sigma(2), \tag{29}$$

and

$$H\circ\varphi\circ\psi = \tilde{H}\circ\psi = H_2 + \mathrm{O}_\Sigma(\infty).$$

Although the proof of Lemma 10 is similar to the one of Lemma 8, there are several differences and subtleties which make it preferable to write the whole proof in details.

Proof. As in the first step, we proceed by (strong) recurrence on k, starting at $k=1$, by constructing, at each step, a local homogeneous symplectomorphism ψ_{2k} satisfying (29) and such that

$$H\circ\psi_2\circ\cdots\psi_{2k} = H_2 + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 2k+2},$$

and we search each ψ_{2k} in the form $\psi_{2k}=\psi_{2k}(1)=\exp(\vec{F}^{\mathrm{inv}}_{2k})$, where $\psi_{2k}(t)=\exp(t\vec{F}^{\mathrm{inv}}_{2k})$ is the flow at time t generated by an adequate Hamiltonian function $F^{\mathrm{inv}}_{2k}\in\mathcal{F}^{\mathrm{inv}}_{1,2k}$ (note that ψ_{2k} will indeed be homogeneous by Lemma 9).

Before going to the recurrence, let us note that, as in the first step, for every $k\geqslant 1$, there exists a (small enough) conic neighborhood C'_{2k} of $(0,0,0,1,0,0)$ such that the flow ψ_{2k} is well defined on $[0,1]\times C'_{2k}$; moreover, since we are going to compose these symplectomorphisms, we choose C'_{2k} such that ψ_{2k+2} maps C'_{2k+2} to C'_{2k}, for every $k\geqslant 1$. Indeed, since $F^{\mathrm{inv}}_{2k}=\mathrm{O}_\Sigma(2k)$, we have $\psi_{2k}(t)=\mathrm{id}+\int_0^t \vec{F}^{\mathrm{inv}}_{2k}\circ\psi_{2k}(s)\,ds = \mathrm{id}+\mathrm{O}_\Sigma(2k-1)=\mathrm{id}+\mathrm{O}_\Sigma(1)$ uniformly with respect to t on compact intervals, and the claim follows. This argument is however not sufficient in order to establish (29), because we start with $k=1$. To prove (29), we use temporarily the notation $x=(q,s,u,v)=(x_1,\dots,x_6)$, and we consider the six functions $\pi_i(x)=x_i$ in $\mathbb{R}^6$, $i=1,\dots,6$. Writing $F^{\mathrm{inv}}_{2k}(q,s,u,v)=a_{2k}(q,s)(u^2+v^2)^k$ with $a\in\Gamma_{1-k}$, and using (22), we infer that

$$\frac{d}{dt}\pi_i\circ\psi_{2k}(t) = \{F^{\mathrm{inv}}_{2k},\pi_i\}_{\tilde{\omega}}\circ\psi_{2k}(t) = \begin{cases} \{a_{2k},\pi_i\}_{\omega_W}(\pi_5^2+\pi_6^2)^k\circ\psi_{2k}(t) = \mathrm{O}_\Sigma(2) & \text{if } i\leqslant 4,\\ 0 & \text{if } i=5,6,\end{cases}$$

and therefore $\pi_i\circ\psi_{2k}=\pi_i+\mathrm{O}_\Sigma(2)$, for $i=1,\dots,6$. We conclude that $\psi_{2k}=\mathrm{id}+\mathrm{O}_\Sigma(2)$.

Let us now make the construction by recurrence.

For $k=1$, we want to prove that there exists $F^{\mathrm{inv}}_2\in\mathcal{F}^{\mathrm{inv}}_{1,2}$ such that, setting $\psi_2=\exp(\vec{F}^{\mathrm{inv}}_2)$, we have $\tilde{H}\circ\psi_2 = H_2 \mod \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 6}$. Using that $\frac{d}{dt}(\tilde{H}\circ\psi_2(t)) = \{F^{\mathrm{inv}}_2,\tilde{H}\}_{\tilde{\omega}}\circ\psi_2(t)$, and since the flow is well defined on $[0,1]\times C'_2$, we have

$$\tilde{H}\circ\psi_2(t) = \tilde{H} + \{F^{\mathrm{inv}}_2,\tilde{H}\}_{\tilde{\omega}} + \mathrm{O}\left(\frac{t^2}{2}\{F^{\mathrm{inv}}_2,\{F^{\mathrm{inv}}_2,\tilde{H}\}_{\tilde{\omega}}\}_{\tilde{\omega}}\circ\psi_2(t)\right), \tag{30}$$

on $[0,1]\times C'_2$. Using (28), we have $\tilde{H}=H_2+\tilde{H}^{\mathrm{inv}}_4+\mathrm{O}_\Sigma(\infty) \mod \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 6}$, with $\tilde{H}^{\mathrm{inv}}_4\in\mathcal{F}^{\mathrm{inv}}_{2,4}$. Hence, taking $t=1$ in (30), using (18) and (21), we infer that

$$\tilde{H}\circ\psi_2 = H_2 + \tilde{H}^{\mathrm{inv}}_4 + \{F^{\mathrm{inv}}_2,H_2\}_{\tilde{\omega}} + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 6}.$$

Therefore, we have to solve the cohomological equation

$$\{H_2,F^{\mathrm{inv}}_2\}_{\tilde{\omega}} = \tilde{H}^{\mathrm{inv}}_4 + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}^{\mathrm{inv}}_{2,\geqslant 6},$$

which has a solution $F_4^{\rm inv} \in \mathcal{F}_{2,4}^{\rm inv}$ by using (18).

It is important to note that, thanks to (21), the whole procedure done in this second step takes place in $\mathcal{F}_{2,\geqslant 2}^{\rm inv}$ (if terms in $\mathcal{F}_{2,\geqslant 2}^{0}$ were to appear again then our two-steps procedure would fail). This kind of triangular stability is crucial.

Let us assume that we have constructed $\psi_2, \dots, \psi_{2k-2}$ such that $\tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k-2} = H_2 \mod \mathcal{F}_{2,\geqslant 2k}^{\rm inv}$. We want to prove that there exists $F_{2k}^{\rm inv} \in \mathcal{F}_{2,2k}^{\rm inv}$ such that, setting $\varphi_{2k} = \exp(\vec{F}_{2k}^{\rm inv})$, we have

$$\tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k} = H_2 + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}_{2,\geqslant 2k+2}^{\rm inv}.$$

Using that $\frac{d}{dt}(\tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k}(t)) = \{F_{2k}^{\rm inv}, \tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k-2}\}_{\tilde{\omega}} \circ \psi_{2k}(t)$, and since the flow ψ_{2k} is well defined on $[0,1] \times C'_{2k}$, we have

$$\begin{aligned}\tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k}(t) = \tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k-2} + \{F_{2k}^{\rm inv}, \tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k-2}\}_{\tilde{\omega}} \\ + \mathrm{O}\left(\frac{t^2}{2}\{F_{2k}^{\rm inv}, \{F_{2k}^{\rm inv}, \tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k-2}\}_{\tilde{\omega}}\}_{\tilde{\omega}}\right), \quad (31)\end{aligned}$$

on $[0,1] \times C'_{2k}$. Using the recurrence assumption, we have

$$\tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k-2} = H_2 + \tilde{\tilde{H}}_{2k}^{\rm inv} + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}_{2,\geqslant 2k+2}^{\rm inv},$$

for some $\tilde{\tilde{H}}_{2k}^{\rm inv} \in \mathcal{F}_{2,2k}^{\rm inv}$. Hence, taking $t = 1$ in (31), using (18) and (21), we infer that

$$\tilde{H} \circ \psi_2 \circ \cdots \circ \psi_{2k} = H_2 + \tilde{\tilde{H}}_{2k}^{\rm inv} + \{F_{2k}^{\rm inv}, H_2\}_{\tilde{\omega}} + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}_{2,\geqslant 2k+2}^{\rm inv}.$$

Therefore, we have to solve the cohomological equation

$$\{H_2, F_{2k}^{\rm inv}\}_{\tilde{\omega}} = \tilde{\tilde{H}}_{2k}^{\rm inv} + \mathrm{O}_\Sigma(\infty) \mod \mathcal{F}_{2,\geqslant 2k+2}^{\rm inv},$$

which has a solution $F_{2k}^{\rm inv} \in \mathcal{F}_{2,2k}^{\rm inv}$ by using (18).

The recurrence is established.

Considering, as in the first step, the formal infinite composition $\psi_2 \circ \cdots \circ \psi_{2k} \cdots$, by the Borel theorem, there exists a smooth homogeneous mapping ψ such that $\psi = \psi_2 \circ \cdots \circ \psi_{2k} \cdots + \mathrm{O}_\Sigma(\infty)$. By construction we have $\psi^\star \tilde{\omega} = \tilde{\omega} + \mathrm{O}_\Sigma(\infty)$, and using again Lemma 16 (Appendix A.5), we modify slightly ψ, by composing it with a homogeneous diffeomorphism tangent to identity at infinite order, so that $\psi^\star \tilde{\omega} = \tilde{\omega}$. Lemma 10 is proved. □

At the end of these two steps, setting $\chi_3 = \psi^{-1} \circ \varphi^{-1} \circ \chi_2$, using (15) and Lemmas 8 and 10, we have obtained the following result, improving Lemma 6.

Lemma 11. *For every $q \in M$, there exist a conic neighborhood C_q of Σ_q^+ in $(T^\star M, \omega)$ and a homogeneous symplectomorphism χ_3 from C_q to $(\mathbb{R}^6, \tilde{\omega})$, satisfying $\chi_3(q) = 0$ and $\chi_3(\Sigma^+ \cap C_q) \subset \tilde{\Sigma}^+$, such that*

$$\chi_3(q,p) = (q, h_Z(q,p), u(q,p), v(q,p)) + O_\Sigma(2), \quad (32)$$

where the functions u and v are defined by (9), and

$$\sigma_P(-\triangle_{sR}) = H_2 \circ \chi_3 + \mathrm{O}_\Sigma(\infty), \quad (33)$$

where $H_2(q,s,u,v) = |s|(u^2 + v^2)$.

Remark 11. Note that, in the Heisenberg flat case (Section 3.1), there are no remainder terms in (32) and (33) in Lemma 11 above.

5.1.4 End of the proof of the normal form

To prove Theorem 3, it suffices to perform all the previous construction (in particular, we apply Lemma 11) two times: we apply it to the manifold M on the one hand, and to the manifold M_H (Heisenberg flat case) on the other hand. The conclusion follows easily.

5.2 Quantum normal form

By quantizing the Birkhoff normal form obtained in Theorem 3, we obtain the following quantum normal form for the sub-Riemannian Laplacian. We use the notion of a pseudo-differential operator that is flat along Σ.

Theorem 4. *For every $q \in M$, there exists a (conic) microlocal neighborhood $\tilde{U}$ of Σ_q in $T^\star M$ such that, considering all the following pseudo-differential operators operators as acting on functions microlocally supported in $\tilde{U}$, we have*

$$-\triangle_{sR} = R\Omega + V_0 + \mathrm{O}_\Sigma(\infty), \tag{34}$$

where

- *$V_0 \in \Psi^0$ is a self-adjoint pseudo-differential operator of order 0,*
- *$R \in \Psi^1$ is a self-adjoint pseudo-differential operator of order 1, with principal symbol*

$$\sigma_P(R) = |h_Z| + \mathrm{O}_\Sigma(2), \tag{35}$$

- *$\Omega \in \Psi^1$ is a self-adjoint pseudo-differential operator of order 1, with principal symbol*

$$\sigma_P(\Omega) = u^2 + v^2 + \mathrm{O}_\Sigma(3), \tag{36}$$

  where the functions u and v are defined by (9),
- $[R, \Omega] = 0 \mod \Psi^{-\infty}$,
- $\exp(2i\pi\Omega) = \mathrm{id} \mod \Psi^{-\infty}$.

Remark 12. In the flat Heisenberg case, there are no remainder terms in (34), (35) and (36), and we recover the operators R_H and Ω_H defined in Section 3.1. The pseudo-differential operators R and Ω can be seen as appropriate perturbations of R_H and Ω_H, designed such that the last two items of Theorem 4 are satisfied.

Remark 13. We stress that the last two items are valid only if we consider both sides as acting on functions that are microlocally supported in $\tilde{U}$. If one wants to drop this assumption then all the above operators have to be extended (almost arbitrarily) outside $\tilde{U}$, and then the equalities hold only modulo remainder terms in $\mathrm{O}_\Sigma(\infty)$.

Note that the operators R and Ω depend on the microlocal neighborhood $\tilde{U}$ under consideration. This neighborhood can then be understood as a chart in the manifold $T^\star M$, in which the quantum normal form is valid.

In the sequel we will call *normal* any (conic) microlocal neighborhood $\tilde{U}$ in which the conclusions of Theorem 4 hold true. We also speak of a *normal chart* in $T^\star M$.

Proof of Theorem 4. Let $q \in M$ be arbitrary. We have established in Theorem 3 the existence of a local symplectomorphism

$$\begin{array}{rcl} \chi : C_q \subset (T^\star M, \omega) & \longrightarrow & C_q^H \subset (T^\star M_H, \omega_H) \\ (q,p) & \longmapsto & (q_H, p_H), \end{array}$$

defined on a conic neighborhood C_q of Σ_q, such that $\sigma_P(-\triangle_H) \circ \chi = \sigma_P(-\triangle_{sR}) + \mathrm{O}_\Sigma(\infty)$. Let U_χ be the unitary Fourier Integral Operator associated with the canonical transformation χ (see [12, 20]). Setting $-\tilde{\triangle}_{sR} = -U_\chi^\star \triangle_H U_\chi$, we have (generalized Egorov theorem)

$$\sigma_P(-\tilde{\triangle}_{sR}) = \sigma_P(-\triangle_H) \circ \chi = \sigma_P(-\triangle_{sR}) + \mathrm{O}_\Sigma(\infty).$$

Actually, since the sub-principal symbol of $-\triangle_H$ vanishes, it follows from an argument due to Weinstein that the sub-principal symbol of $-\tilde{\triangle}_{sR}$ vanishes as well. This argument is explained in Proposition 6 in Appendix A.4. Using the Weyl quantization, it follows that, defining $V_0 = -\triangle_{sR} + \tilde{\triangle}_{sR}$, we have $V_0 \in \Psi^0$ and V_0 is clearly self-adjoint. Setting $R = U_\chi^\star R_H U_\chi$ and $\Omega = U_\chi^\star \Omega_H U_\chi$, we have $\sigma_P(R) = \sigma_P(R_H) \circ \chi$ and $\sigma_P(\Omega) = \sigma_P(\Omega_H) \circ \chi$. Denoting by χ_3 (resp., by χ_3') the local symplectomorphism from $T^\star M$ to $\mathbb{R}^6$ (resp., from $T^\star M_H$ to $\mathbb{R}^6$), constructed in Lemma 11, we have $\chi_3(q,p) = (q, h_Z(q,p), u(q,p), v(q,p)) + O_\Sigma(2)$ by (32), and, using Remark 12, $\chi_3'(q_H, p_H) = (q_H, h_{Z_H}(q_H, p_H), u_H(q_H, p_H), v_H(q_H, p_H))$ (where the functions u_H and v_H are defined as in (9), in the Heisenberg flat case). Since $\chi = (\chi_3')^{-1} \circ \chi_3$, (35) and (36) are easily established. The rest follows from the corresponding relations in the Heisenberg flat case. □

Remark 14. It follows from the above proof and from the definition of R and Ω that

$$\sigma_P(R) \circ \chi_3^{-1}(q,s,u,v) = \sigma_P(R_H) \circ (\chi_3')^{-1}(q,s,u,v) = |s|,$$

and

$$\sigma_P(\Omega) \circ \chi_3^{-1}(q,s,u,v) = \sigma_P(\Omega_H) \circ (\chi_3')^{-1}(q,s,u,v) = u^2 + v^2.$$

Note also that, using (33), we have

$$\sigma_P(-\triangle_{sR}) = \sigma_P(\Omega)\sigma_P(R) + \mathrm{O}_\Sigma(\infty). \tag{37}$$

6 The variance estimate

In this section, we are going to establish the following result (from which Theorem 1 follows).

Proposition 3. *For every pseudo-differential operator $A \in \Psi^0$ whose principal symbol vanishes on Σ^-, we have $V(A - \hat{A}_+) = 0$, where*

$$\hat{A}_+ = \hat{a}_+ \Pi_+, \qquad \hat{a}_+ = \int_M a(q, \alpha_g(q))\, d\nu, \tag{38}$$

and Π_+ is a pseudo-differential operator whose principal symbol is equal to 1 in a conical neighborhood of Σ^+ and equal to 0 in a conical neighborhood of Σ^-.

Similarly, for every $A \in \Psi^0$ whose principal symbol vanishes on Σ^+, we have $V(A - \hat{A}_-) = 0$, with similar notations. This can be inferred directly from Proposition 3 by using (6) (already used in the proof of Lemma 3: this is because ϕ_n is real-valued).

Admitting temporarily Proposition 3, let us prove Theorem 1. As explained in the introduction, in order to establish the QE property, it suffices to prove that, for every pseudo-differential operator $A \in \Psi^0$, we have

$$V(A - \bar{a}\,\mathrm{id}) = 0,$$

where we have set

$$\bar{a} = \frac{1}{2}\int_M \big(a(q,\alpha_g(q)) + a(q,-\alpha_g(q))\big)\, d\nu.$$

In order to prove that fact, we write $A = A_+ + A_-$, with the principal symbol of A_+ (resp., of A_-) vanishing on Σ_- (resp., on Σ_+). Using the above results, we have $V(A_+ - \hat{a}_+\Pi_+) = 0$ and $V(A_- - \hat{a}_-\Pi_-) = 0$. Since $V(A - \hat{A}_+ - \hat{A}_-) \leqslant 2\Big(V(A_+ - \hat{A}_+) + V(A_- - \hat{A}_-)\Big)$, we infer that $V(A - \hat{a}_+\Pi_+ - \hat{a}_-\Pi_-) = 0$. Besides, noting that $\Pi_+ + \Pi_- = \mathrm{id} + O_\Sigma(1)$ and that $\bar{a} = \frac{1}{2}(\hat{a}_+ + \hat{a}_-)$, we have

$$\hat{a}_+\Pi_+ + \hat{a}_-\Pi_- = \bar{a}\,\mathrm{id} + O_\Sigma(1) + \frac{\hat{a}_+ - \hat{a}_-}{2}(\Pi_+ - \Pi_-),$$

and using again (6), we get that $V(\Pi_+ - \Pi_-) = 0$. Indeed, the principal symbol of $\Pi_+ - \Pi_-$ is odd with respect to Σ, which implies that $\langle(\Pi_+ - \Pi_-)\phi_n, \phi_n\rangle = \mathrm{o}(1)$. Using Lemma 1, we conclude that $V(A - \bar{a}\,\mathrm{id}) = 0$.

The proof of Proposition 3 is done in Section 6.3. We first establish in Sections 6.1 and 6.2 two useful preliminary lemmas.

6.1 Averaging in a normal chart

Given $A \in \Psi^0$, according to Corollary 1, $V(A)$ depends only on the restriction $a_{|\Sigma}$ (where $a = \sigma_p(A) \in \mathcal{S}^0$), in the sense that, if the principal symbols of two pseudo-differential operators A_1 and A_2 of order 0 agree on Σ, then $V(A_1 - A_2) = 0$. This property gives us the possibility to modify A without changing $a_{|\Sigma}$, and we can use this latitude to impose the extra condition $[A, \Omega] = 0 \mod \Psi^{-\infty}$.

Lemma 12. *Let $A \in \Psi^0$ be supported in a normal chart U and let Ω be given by Theorem 4 (in the microlocal neighborhood U). Assuming U small enough, the pseudo-differential operator defined by*

$$B = \frac{1}{2\pi}\int_0^{2\pi} \exp(is\Omega)A\exp(-is\Omega)\, ds,$$

is in Ψ^0, is microlocally supported in U, and satisfies

$$\sigma_P(B) = \sigma_p(A) + O_\Sigma(1),$$
$$[B, \Omega] = 0 \mod \Psi^{-\infty}.$$

Proof. The proof follows an argument introduced by Weinstein in [37] (see also [6]). For every $s \in [0, 2\pi]$, we set $B_s = \exp(is\Omega)A\exp(-is\Omega)$. By the Egorov theorem, we have $B_s \in \Psi^0$ and $\sigma_P(B_s) = a \circ \exp(s\vec{w})$, where $w = \sigma_P(\Omega)$ and $\exp(s\vec{w})$ is the flow generated by the Hamiltonian vector field $\vec{w}$ associated with the Hamiltonian function w. Here, the microlocal neighborhood U in which this construction is performed must be chosen small enough, so that it is invariant under the flow $\exp(s\vec{w})$, for $s \in [0, 2\pi]$. This is possible because, using (36), we have $w = \mathrm{O}_\Sigma(2)$, and therefore $\exp(s\vec{w})_{|\Sigma} = \mathrm{id}$. Moreover, we infer that $\sigma_P(B_s)_{|\Sigma} = a_{|\Sigma}$.

Setting $B = \frac{1}{2\pi}\int_0^{2\pi} B_s\, ds \in \Psi^0$, we have $\sigma_P(B)_{|\Sigma} = a_{|\Sigma}$. By Theorem 4, we have $\exp(2i\pi\Omega) = \mathrm{id} \mod \Psi^{-\infty}$, and thus $B_{2\pi} = B_0 \mod \Psi^{-\infty}$. Now, since $\frac{d}{ds}B_s = i[\Omega, B_s]$, integrating over $[0, 2\pi]$ yields $[B, \Omega] = 0 \mod \Psi^{-\infty}$. □

6.2 The main lemma

The following lemma may be seen as a substitute for the invariance properties with respect to the geodesic flow, that are used in the proof of the classical Shnirelman theorem.

Lemma 13. *Let $B \in \Psi^0$ whose principal symbol is supported in a normal chart U. We consider the pseudo-differential operators Ω and R given by Theorem 4 (in the microlocal neighborhood U). If $[B, \Omega] = 0 \mod \Psi^{-\infty}$ then $V([B, R]) = 0$.*

Proof. In order to prove that $V([B, R]) = 0$, it suffices to prove that $\langle [B, R]\phi_n, \phi_n \rangle \to 0$ as $n \to +\infty$. For every $n \in \mathbb{N}^*$, we have

$$\begin{aligned}\langle [B, R]\phi_n, \phi_n \rangle &= \langle BR\phi_n, \phi_n \rangle - \langle RB\phi_n, \phi_n \rangle \\ &= \frac{1}{\lambda_n} \langle BR\phi_n, -\triangle_{sR}\phi_n \rangle - \frac{1}{\lambda_n} \langle RB(-\triangle_{sR})\phi_n, \phi_n \rangle ,\end{aligned}$$

because $-\triangle_{sR}\phi_n = \lambda_n\phi_n$. Now, using (34), we have $-\triangle_{sR} = R\Omega + V_0 + C$, with $V_0 \in \Psi^0$ and $C = \mathrm{O}_\Sigma(\infty)$. Note that $C \in \Psi^2$ and that C is self-adjoint (but we cannot say that $C \in \Psi^1$ because of the remainder term in (37)). It follows that

$$\langle [B, R]\phi_n, \phi_n \rangle = I_n + J_n + K_n,$$

with

$$\begin{aligned}I_n &= \frac{1}{\lambda_n} \langle BR\phi_n, R\Omega\phi_n \rangle - \frac{1}{\lambda_n} \langle RBR\Omega\phi_n, \phi_n \rangle , \\ J_n &= \frac{1}{\lambda_n} \langle BR\phi_n, V_0\phi_n \rangle - \frac{1}{\lambda_n} \langle RBV_0\phi_n, \phi_n \rangle , \\ K_n &= \frac{1}{\lambda_n} \langle BR\phi_n, C\phi_n \rangle - \frac{1}{\lambda_n} \langle RBC\phi_n, \phi_n \rangle .\end{aligned}$$

Since R and Ω are self-adjoint, we have

$$I_n = \frac{1}{\lambda_n} \langle \Omega RBR\phi_n, \phi_n \rangle - \frac{1}{\lambda_n} \langle RBR\Omega\phi_n, \phi_n \rangle = \frac{1}{\lambda_n} \langle [\Omega, RBR]\phi_n, \phi_n \rangle .$$

Since $[R, \Omega] = 0 \mod \Psi^{-\infty}$ and $[B, \Omega] = 0 \mod \Psi^{-\infty}$, we infer that $[\Omega, RBR] = 0 \mod \Psi^{-\infty}$, and hence $I_n = \mathrm{o}(1)$ as $n \to +\infty$.

Let us now focus on the second term. Since V_0 is self-adjoint, this term can be written as

$$J_n = \frac{1}{\lambda_n} \langle V_0[B, R]\phi_n, \phi_n \rangle + \frac{1}{\lambda_n} \langle [V_0, RB]\phi_n, \phi_n \rangle .$$

The two pseudo-differential operators $V_0[B, R]$ and $[V_0, RB]$ are of order 0 and therefore are bounded. Since $\lambda_n \to +\infty$ as $n \to +\infty$, it follows that $J_n = \mathrm{o}(1)$ as $n \to +\infty$.

Finally, the third term can be written as

$$K_n = \frac{1}{\lambda_n} \langle (C[B, R] + [C, RB])\phi_n, \phi_n \rangle = \frac{1}{\lambda_n} \langle D\phi_n, \phi_n \rangle ,$$

with $D = C[B, R] + [C, RB]$. Clearly, we have $D \in \Psi^2$ and $D = \mathrm{O}_\Sigma(\infty)$. Therefore, by Lemma 17 (see Appendix A.6), there exists $D_1 \in \Psi^0$, with $D_1 = \mathrm{O}_\Sigma(\infty)$, such that $D = -D_1\triangle_{sR}$. It follows from this factorization that $K_n = \langle D_1\phi_n, \phi_n \rangle$, with $\sigma_P(D_1)_{|\Sigma} = 0$.

We conclude from the study of these three terms that

$$\langle [B, R]\phi_n, \phi_n \rangle = \langle D_1\phi_n, \phi_n \rangle + \mathrm{o}(1),$$

as $n \to +\infty$, and hence $V([B, R]) = V(D_1)$. Since $D_1 \in \Psi^0$ has a principal symbol vanishing along Σ, it follows from Corollary 1 that $V(D_1) = 0$. The conclusion follows. □

6.3 Proof of Proposition 3

Let $A \in \Psi^0$ whose principal symbol a vanishes in Σ^-. The objective is to prove that $V(A - \hat{A}_+) = 0$.

We consider the pseudo-differential operator $|Z| = \mathrm{Op}^{\mathcal{W}}(|h_Z|) \in \Psi^1$. It is globally well defined. For every $t \in \mathbb{R}$, we define $A_t \in \Psi^0$ by

$$A_t = \exp(it|Z|)A\exp(-it|Z|). \tag{39}$$

Its principal symbol $a_t \in \mathcal{S}^0$ satisfies $a_t = a \circ \rho_t$ along Σ (where ρ_t is defined in Section 2.4), and vanishes in Σ^-. For every $T > 0$, we set $A_T = \frac{1}{T}\int_0^T A_t\,dt$. The principal symbol $\bar{a}_T \in \mathcal{S}^0$ of $\bar{A}_T$ is $\bar{a}_T = \frac{1}{T}\int_0^T a_t\,dt$.

Remark 15. Alternatively, we could take any $A_t \in \Psi^0$ whose principal symbol $a_t \in \mathcal{S}^0$ satisfies $a_t = a \circ \rho_t$ along Σ. We also note that, since $a_0 = a$ vanishes in Σ^-, the operator A_t defined by (39) is equal to $A_t = \exp(itZ)A\exp(-itZ)$, for which we also have $A_t f = A\,(f \circ \mathcal{R}_{-t}) \circ \mathcal{R}_t$, for every $f \in L^2(M, \mu)$, where $\mathcal{R}_t$ is the Reeb flow.

Our objective is to prove that $V(A - \hat{A}_+) = 0$. The proof goes in two steps:

1. Prove that $V(A - A_t) = 0$ for every t, and hence that $V\left(A - \bar{A}_T\right) = 0$ (this step does not require any ergodicity assumption);
2. Using the ergodicity of the Reeb flow and the Von Neumann mean ergodic theorem, prove that $\lim_{T\to+\infty} V\left(\bar{A}_T - \hat{A}_+\right) = 0$.

First step: $V(A - \bar{A}_T) = 0$**.**

Lemma 14. *For every $t \in \mathbb{R}$, we have $V\left(\frac{d}{dt}A_t\right) = 0$.*

Proof. Let us fix a normal chart U. There exist $V \subset U$ and $\varepsilon > 0$ such that, for any $A \in \Psi^0$ that is microlocally supported in V, the pseudo-differential operator A_t defined by (39) is microlocally supported in U for every $t \in [-\varepsilon, \varepsilon]$. Following Section 6.1, we define

$$B_t = \frac{1}{2\pi}\int_0^{2\pi} \exp(is\Omega)A_t\exp(-is\Omega)\,ds,$$

for every $t \in [-\varepsilon, \varepsilon]$. Here, we consider the pseudo-differential operators Ω and R given by Theorem 4 (in the microlocal neighborhood U). According to Lemma 12, $B_t \in \Psi^0$ is microlocally supported in U, and we have $\sigma_P(B_t)_{|\Sigma} = \sigma_P(A_t)_{|\Sigma}$ and $[B_t, \Omega] = 0 \mod \Psi^{-\infty}$, for every $t \in [0, \varepsilon]$. Lemma 13 implies that $V([B_t, R]) = 0$.

Now, since $\sigma_P\left(\frac{d}{dt}A_t\right)_{|\Sigma} = \sigma_P\left(\frac{d}{dt}B_t\right)_{|\Sigma}$, we have $V\left(\frac{d}{dt}A_t - \frac{d}{dt}B_t\right) = 0$, and therefore, using Lemma 1, we infer that $V\left(\frac{d}{dt}A_t\right) = V\left(\frac{d}{dt}B_t\right)$.

By definition of A_t, we have $\frac{d}{dt}A_t = i[|Z|, A_t]$, and hence

$$\begin{aligned}\frac{d}{dt}B_t &= \frac{i}{2\pi}\int_0^{2\pi} \exp(is\Omega)[|Z|, A_t]\exp(-is\Omega)\,ds \\ &= \frac{i}{2\pi}\int_0^{2\pi} \exp(is\Omega)[R, A_t]\exp(-is\Omega)\,ds + \frac{i}{2\pi}\int_0^{2\pi} \exp(is\Omega)[|Z| - R, A_t]\exp(-is\Omega)\,ds.\end{aligned}$$

On the one part, since $[R, \Omega] = 0 \mod \Psi^{-\infty}$, we have

$$\int_0^{2\pi} \exp(is\Omega)[R, A_t]\exp(-is\Omega)\,ds = \left[R, \int_0^{2\pi} \exp(is\Omega)A_t\exp(-is\Omega)\,ds\right] \mod \Psi^{-\infty}.$$

On the other part, using (35), the principal symbol of the second integral term is $\mathrm{O}_\Sigma(1)$. We conclude that

$$\frac{d}{dt}B_t = i[R, B_t] + \mathrm{O}_\Sigma(1) \mod \Psi^{-\infty}.$$

Using Corollary 1, we infer that $V\left(\frac{d}{dt}B_t\right) = V([R, B_t]) = 0$.

At this step, we have proved that $V\left(\frac{d}{dt}A_t\right) = 0$, for every $t \in [-\varepsilon, \varepsilon]$. By compactness and using a partition of unity, we get the property for every $t \in \mathbb{R}$. □

Using this lemma we find as a corollary the following proposition.

Proposition 4. *Let $A_0 \in \Psi^0$ whose principal symbol vanishes on Σ^- and let A_t and $\bar{A}_T$ be defined as above. Then, for any time t, we have $V(A - A_t) = 0$, and as a consequence, for every $T > 0$, $V(A - \bar{A}_T) = 0$.*

Proof. We start from

$$\langle (A - A_t)\phi_n, \phi_n \rangle = -\int_0^t \left\langle \frac{dA_s}{ds}\phi_n, \phi_n \right\rangle ds,$$

for every time t, and hence, by the Cauchy-Schwarz inequality,

$$\langle (A - A_t)\phi_n, \phi_n \rangle^2 \leqslant t \int_0^t \left| \left\langle \frac{dA_s}{ds}\phi_n, \phi_n \right\rangle \right|^2 ds.$$

Summing with respect to n, and since all terms are non-negative, we get that $V(A - A_t) \leqslant t\int_0^t V\left(\frac{d}{ds}A_s\right) ds$. By Lemma 14, we infer that $V(A - A_t) = 0$, for every $t \in \mathbb{R}$.

Let us now prove that $V(A - \bar{A}_T) = 0$, with $\bar{A}_T = \frac{1}{T}\int_0^T A_t\, dt$. We have, by the Fubini theorem,

$$\left\langle (A - \bar{A}_T)\phi_n, \phi_n \right\rangle = \frac{1}{T}\int_0^T \langle (A - A_t)\phi_n, \phi_n \rangle\, dt,$$

for every integer n. Using again the Jensen inequality, and summing with respect to n, we get $V(A - \bar{A}_T) \leqslant \frac{1}{T}\int_0^T V(A - A_t)\, dt$. It follows that $V(A - \bar{A}_T) = 0$. □

Second step: $V(\bar{A}_T - \hat{A}_+) \to 0$ as $T \to +\infty$. Here, we are going to use the ergodicity assumption on the Reeb flow.

Using (5), we have $V(\bar{A}_T - \hat{A}_+) \leqslant E((\bar{A}_T - \hat{A}_+)^\star(\bar{A}_T - \hat{A}_+))$, and it follows from Theorem 2 (microlocal Weyl law) that

$$E((\bar{A}_T - \hat{A}_+)^\star(\bar{A}_T - \hat{A}_+)) = \frac{1}{2}\int_{\Sigma_1} |\bar{a}_T - \hat{a}_+|^2\, d\hat{\nu}_1,$$

with $\bar{a}_T = \frac{1}{T}\int_0^T a \circ \rho_t\, dt$ and $\hat{a}_+$ defined by (38). Since the flow ρ_t of $\hat{Z}$ (which is the lift to Σ_1 of the flow of Z) is ergodic on $(\Sigma_1, \hat{\nu}_1)$, it follows from the Von Neumann mean ergodic theorem (see, e.g., [30]) that $\bar{a}_T$ converges to $\hat{a}_+$ in $L^2(\Sigma_1, \hat{\nu}_1)$ as $T \to +\infty$. Therefore $V(\bar{A}_T - \hat{A}_+)$ converges to 0 as $T \to +\infty$.

Using the inequality $V(A - \hat{A}_+) \leqslant 2\left(V(A - \bar{A}_T) + V(\bar{A}_T - \hat{A}_+)\right)$, and the results of the two steps above, we conclude the proof of Proposition 3.

A Appendix

A.1 Complex-valued eigenbasis, and the non-orientable case

Complex-valued eigenbasis, with D oriented. We can extend Theorem 1 to the case of a complex-valued eigenbasis, to the price of requiring that the principal symbol a of A satisfies the evenness condition

$$a(q, \alpha_g(q)) = a(q, -\alpha_g(q)), \tag{40}$$

for every $q \in M$. The proof is the same.

D not orientable. Let us assume that the subbundle D is not orientable. Then there exists a double covering $\tilde{M}$ of M with an involution J, so that we can lift all data to $\tilde{M}$, and then the subbundle $\tilde{D}$ of $T\tilde{M}$ is orientable. The Reeb vector field $\tilde{Z}$ on $\tilde{M}$ is odd with respect to the involution.

Definition 3. *The Reeb dynamics is ergodic if every measurable subset of $\tilde{M}$ which is invariant under $\tilde{Z}$ and invariant under J is of measure 0 or 1.*

Theorem 5. *We assume that the Reeb dynamics is ergodic. Then we have QE for any eigenbasis of $\triangle_{sR}$.*

The proof is an adaptation of the orientable case. Note that $\Sigma \setminus \{0\}$ is connected. We can remove the assumption that the eigenfunctions are real-valued: indeed, any eigenfunction on M, real-valued or complex-valued, is lifted to $\tilde{M}$ to an even function. Moreover, denoting by V_M (resp., by $V_{\tilde{M}}$) the variance on M (resp., on $\tilde{M}$), we have $V_M(A) = V_{\tilde{M}}(\tilde{A})$, with $\tilde{A}$ even with respect to the involution J. In particular, the principal symbol of $\tilde{A}$ satisfies (40) along Σ.

A.2 Pseudo-differential operators flat on Σ

Let us define the notion of flatness that we needed, within the following lemma.

Lemma 15. *Let $(q_0, p_0) \in T^\star M$ with $p_0 \neq 0$, let $d \geqslant 0$, and let $A \in \Psi^d$ be arbitrary. The following properties are equivalent:*

- *For any oscillatory function $u(q) = b(q)e^{i\tau S(q)}$ with $dS(q_0) = p_0$, with b a smooth function on M, we have $Au(q_0) = \mathrm{O}(\tau^{-\infty})$.*
- *For any local canonical coordinate system around q_0, the full left (or right, or Weyl) symbol of A is flat at (q_0, p_0).*

Moreover, if such a property is satisfied, and if U_χ is an elliptic Fourier Integral Operator associated with a canonical transformation χ, then the same properties are satisfied as well for $U_\chi^{-1} A U_\chi$ at $\chi(q_0, p_0)$.

Proof. Using the classical formulas permitting to pass from one quantization to another one (see, e.g., [41]), it is clear that the flatness property does not depend on the quantization. Now, the first item implies the second one, by taking $u(q) = e^{i\tau q \cdot p_0}$. The second item implies the first one by a stationary phase argument. The invariance under Fourier Integral Operators (see [20]) follows as well from a stationary phase argument. □

Definition 4. *For $k \in \mathbb{N} \cap \{\infty\}$, we use the notation $\mathrm{O}_\Sigma(k)$ for a smooth function on $T^\star M$ that vanishes on Σ up to order k. The word flat is used when $k = +\infty$.*

If the full (left, or right, or Weyl) symbol of $A \in \Psi^d$ is flat on Σ (i.e., at any point of Σ), then we say that the pseudo-differential operator A is flat on Σ, and we write $A \in \mathrm{O}_\Sigma(\infty)$. The set of pseudo-differential operators that are flat on Σ is a bilateral ideal in the algebra of classical pseudo-differential operators on M.

A.3 A proof of the local Weyl law

We assume that we are in the conditions of Section 1 (3D contact case). We have the following result, called the local Weyl law.

Proposition 5. *Let f be a continuous function on M. Then*

$$\sum_{\lambda_n \leqslant \lambda} \int_M f|\phi_n|^2 \, d\mu = \frac{P(M)}{32} \lambda^2 (1 + \mathrm{o}(1)) \int_M f \, d\nu, \tag{41}$$

as $\lambda \to +\infty$.

As already said in Remark 7, this result is already known in several contexts. The sketch of proof that we give below follows arguments of [3].

Proof. We denote by $\triangle_{g,\mu}$ is the sR Laplacian associated with the measure μ. Since $-\triangle_{g,\mu}$ is hypoelliptic and nonnegative, and using the maximum principle for hypoelliptic operators (see [4]), the sub-Riemannian heat semi-group $e^{t\triangle_{g,\mu}}$ has, for $t > 0$, a positive symmetric Schwartz kernel of class C^∞ (see [34]), and thus is of trace-class. We use the heat equation method, which consists in studying the asymptotic behavior, as $t \to 0^+$, of the integrals $\int_M f(q) e_{\triangle_{g,\mu}}(t, q, q) \, d\mu(q)$, where $e_{\triangle_{g,\mu}}(t, q_1, q_2)$ is the sR heat kernel, that is the Schwartz kernel of $e^{t\triangle_{g,\mu}}$

Thanks to the gauge transform introduced in Remark 5 (see Section 2.1), we can actually assume that μ is the Popp measure and the operator is now $\triangle_{g,\mathrm{Popp}} + W\mathrm{id}$, where W is a smooth potential. We have then

$$e_{\triangle_{g,\mu}}(t, q, q) \, d\mu = e_{\triangle_{g,\mathrm{Popp}} + W\mathrm{id}}(t, q, q) \, dP. \tag{42}$$

Given a local g-orthonormal frame (X, Y) of D, the triple (X, Y, Z) (where Z is the Reeb vector field) is a local frame of TM. Let (ν_X, ν_Y, ν_Z) be the dual basis of (X, Y, Z) (this means that ν_X is the 1-form defined by $\nu_X(X) = 1$, $\nu_X(Y) = \nu_X(Z) = 0$, and similarly for the other 1-forms ν_Y and ν_Z). Then the Popp measure is given by $dP = |d\nu_X \wedge d\nu_Y \wedge d\nu_Z|$ (see [1]). In the Heisenberg flat case (studied in Section 3.1), we have then $dP = |dx \, dy \, dz|$, the Lebesgue volume.

It is a standard fact (see, e.g., [29]) that, around any point $q \in M$, there exist local coordinates (called privileged coordinates) in which $q = 0$ and in which the frame (X, Y) is a perturbation of the so-called nilpotent approximation $(\hat{X}, \hat{Y})$, in the following precise sense. For every $s > 0$, we define the dilation mapping δ_s by $\delta_s(x, y, z) = (sx, sy, s^2 z)$, in the privileged coordinates. Moreover, the privileged coordinates can be chosen such that $\hat{X} = \partial_x$ and $\hat{Y} = \partial_y - x\partial_z$ (Heisenberg flat case, studied in Section 3.1).

We define $\triangle_\varepsilon = \varepsilon^2 \delta_\varepsilon^\star (\triangle_{g,\mathrm{Popp}} + W)$, and $\triangle_H = -\hat{X}\hat{X}^\star - \hat{Y}\hat{Y}^\star$. It is easy to see that the heat kernel $e^\varepsilon(t, q_1, q_2)$ of $\triangle_\varepsilon$ satisfies the scaling relation

$$e^\varepsilon(t, q_1, q_2) = \varepsilon^4 e_{\triangle_{g,\mathrm{Popp}} + W\mathrm{id}}(\varepsilon^2 t, \delta_\varepsilon q_1, \delta_\varepsilon q_2),$$

in a local chart. Moreover, we have $\triangle_\varepsilon = \triangle_H + O(\varepsilon^2)$. From the Duhamel formula, we infer that, for $t > 0$ fixed, $e^{t\triangle_\varepsilon} = e^{t\triangle_H} + O(\varepsilon^2)$, and then,

$$e^\varepsilon(t, q_1, q_2) = e_H(t, q_1, q_2) + O(\varepsilon^2),$$

where $e_H(t, q_1, q_2)$ is the heat kernel associated with $\triangle_H$. In particular, in the local coordinates around $q = 0$, taking $\varepsilon = \sqrt{\tau}$ and $t = 1$, we get

$$e_{\triangle_{g,\mathrm{Popp}}+W\mathrm{id}}(\tau, 0, 0) = \frac{1}{\tau^2} e^{\sqrt{\tau}}(1, 0, 0) = \frac{1}{\tau^2} \left(e_H(1, 0, 0) + \mathrm{O}(\tau)\right).$$

Note that the constant $e_H(1, 0, 0)$ is positive.

Since the point q was arbitrary, we have proved, at this step, that

$$e_{\triangle_{g,\mathrm{Popp}}+W\mathrm{id}}(t, q, q) = \frac{e_H(1, 0, 0)}{t^2}(1 + \mathrm{O}(t)), \tag{43}$$

as $t \to 0^+$, for every $q \in M$. Note that the remainder term in (43) is uniform with respect to $q \in M$. It is also important to note that the constant $e_H(1, 0, 0)$ does not depend on the point q under consideration, and this, because we have considered the Popp measure, given in local coordinates by the Lebesgue measure, which itself does not depend on q.

Let f be a continuous function on M, and let A be the operator on $L^2(M, dP)$ of multiplication by f. Without loss of generality, we assume that f is nonnegative. The operator $Ae^{t\triangle_{g,\mu}}$ is of trace-class, and we have, using (42),

$$\begin{aligned}\mathrm{Tr}(Ae^{t\triangle_{g,\mu}}) = \sum_{n=1}^{+\infty} e^{-\lambda_n t} \int_M f|\phi_n|^2 \, d\mu &= \int_M f(q) e_{\triangle_{g,\mu}}(t, q, q) \, d\mu(q) \\ &= \int_M f(q) e_{\triangle_{g,\mathrm{Popp}}+W\mathrm{id}}(t, q, q) \, dP(q).\end{aligned}$$

Note that the latter integral is done with respect to the Popp measure $dP = P(M) \, d\nu$, which coincides with the Lebesgue measure at the origin of any local chart with privileged coordinates. Using (43) and the dominated convergence theorem, it follows that

$$\sum_{n=1}^{+\infty} e^{-\lambda_n t} \int_M f|\phi_n|^2 \, d\mu = \frac{P(M) e_H(1, 0, 0)}{t^2}(1 + \mathrm{O}(t)) \int_M f \, d\nu,$$

as $t \to 0^+$. We infer from the Karamata tauberian theorem (see [22]) that

$$\sum_{\lambda_n \leqslant \lambda} \int_M f|\phi_n|^2 \, d\mu = \frac{P(M) e_H(1, 0, 0)}{2} \lambda^2 (1 + \mathrm{o}(1)) \int_M f \, d\nu.$$

The constant $\hat{e}(1, 0, 0)$ can be fitted to $\hat{e}(1, 0, 0) = 1/16$ by using the Weyl formula for the Heisenberg quotients given in Section 3.1. □

A.4 The Weinstein argument

We provide here an argument of Weinstein given in [36], leading to the following result.

Proposition 6. *Let X be a smooth manifold. Let Δ be a pseudo-differential operator defined in some cone $C \subset T^\star X$. We denote by p the principal symbol of Δ, and we assume that the sub-principal symbol of Δ vanishes. Let $\chi : C \to C' \subset T^\star Y$ be a canonical transformation, where Y is another smooth manifold. Then, there exists a microlocally unitary Fourier Integral Operator U_χ, associated with χ, such that $U\Delta U^\star = B$, where B is a pseudo-differential operator in C' whose principal symbol is $p \circ \chi^{-1}$ (general Egorov theorem) and whose sub-principal symbol vanishes.*

Proof. The proof uses in a strong way the symbolic calculus of Fourier Integral Operators, for which we refer to the book [12] or to the paper [36]. Let us sketch the argument. We choose the Fourier Integral Operator U_χ associated with the canonical transformation χ such that its principal symbol is constant of modulus 1 and U is microlocally unitary, i.e., $U^\star U = \mathrm{id}$ in the cone C. This is possible as follows: we choose a first U_0 with only the prescription of the principal symbol, then $U_0^\star U_0 = \mathrm{id} + A$ where A is a self-adjoint pseudo-differential operator in $\Psi^{-1}(C)$. If $D = (\mathrm{id} + A)^{-1/2}$ in C, then we take $U = U_0 D$.

Denoting by $K(x, y)$ the Schwartz kernel of U_χ, if $B = U\Delta U^\star$, the relation $BU - U\Delta \sim 0$ is written as

$$(\Delta_x \otimes \mathrm{id}_Y - \mathrm{id}_X \otimes B_y)\, K \sim 0.$$

The distribution K is a Lagrangian distribution associated with a submanifold of $C \times C'$ which is the graph of χ. If we assume that the principal symbol of U_χ is a constant of modulus 1, then the sub-principal symbol of the right-hand side, which is 0, is the sum of Lie derivatives of the principal symbol of K, which vanish due to the choice of U_χ and of the product of the sub-principal symbol of $\mathrm{id}_X \otimes B_y$ by the non-vanishing symbol of U_χ. This implies that the latter vanishes. This is the argument of Weinstein. We have only to take care of the fact that tensor products are pseudo-differential operators only in some cones of the product of the cotangent spaces where ξ and η are of comparable sizes. □

A.5 Darboux-Weinstein lemma

We have the following easy generalization of the well-known Darboux-Weinstein lemma (see [35]).

Lemma 16. *Let N be a manifold endowed with two symplectic forms ω_1 and ω_2, and let P be a compact submanifold of N along which $\omega_1 = \omega_2 + \mathrm{O}_P(k)$, for some $k \in \mathbb{N}^* \cup \{+\infty\}$. Then there exist open neighborhoods U and V of P in N and a diffeomorphism $f : U \to V$ such that $f = \mathrm{id}_N + \mathrm{O}_P(k+1)$ and $f^\star\omega_2 = \omega_1$. Moreover, if N has a conic structure, then the diffeomorphism f can be chosen to be homogeneous with respect to that conic structure.*

The most usual statement of that lemma is when $k = 1$, and then the usual conclusion is that $f = \mathrm{id}_N + O_P(1)$; actually, already in that case we have the better conclusion that $f = \mathrm{id}_N + O_P(2)$ (as is well known, and as it is proved for instance in [23, Lemma 43.11 p. 462]), i.e., $df(q) = \mathrm{id}$ for every $q \in N$, or in other words, f is tangent to the identity.

Proof. We follow the standard argument (see, e.g., [24]). We define the closed two-form $\omega(t) = \omega_1 + t(\omega_2 - \omega_1)$, for every $t \in [0, 1]$. Let U be a neighborhood of P in which $\omega(t)$ is nondegenerate for every t. By the relative Poincaré lemma, since ω_1 and ω_2 agree along P, shrinking U if necessary, there exists a one-form η on U such that $\omega_1 - \omega_2 = d\eta$, with $\eta_x = 0$ for every $x \in P$. Since $\omega_1 = \omega_2 + \mathrm{O}_P(k)$, we have actually $\eta = \mathrm{O}_P(k + 1)$. Note that, as it is well known in the relative Poincaré lemma, we can choose $\eta = Q(\omega_1 - \omega_2)$, where Q is defined by $Q\omega = \int_0^1 \rho(t)^\star \iota_{Y(t)}\omega\, dt$, where $Y(t)$, at the point $y = \rho(t, x)$, is the vector tangent to the curve $\rho(s, x)$ at $s = t$, and $(\rho(t))_{0\leqslant t\leqslant 1}$ is a smooth homotopy from the local projection onto P (in a tubular neighborhood of P) to the identity, fixing P.

The diffeomorphism f is then constructed by the Moser trick. The time-dependent vector field $X(\cdot)$ defined for every t by $\iota_{X(t)}\omega(t) = \eta$ generates the time-dependent flow $f(\cdot)$ (satisfying $\dot f(t) = X(t) \circ f(t)$, $f(0) = \mathrm{id}_N$), and we have

$$\frac{d}{dt} f(t)^\star\omega(t) = f(t)^\star\mathcal{L}_{X(t)}\omega(t) + f(t)^\star\dot\omega(t) = f(t)^\star d(\iota_{X(t)}\omega(t) - \eta) = 0,$$

whence $\omega_1 = f(1)^\star\omega_2$. We set $f = f(1)$.

Since $\iota_{X(t)}\omega(t) = \eta = \mathrm{O}_P(k+1)$, it follows that $X(t) = \mathrm{O}_P(k+1)$ and hence $f(t) = \mathrm{id}_N + \int_0^t X(s) \circ f(s)\, ds = \mathrm{id}_N + \mathrm{O}_P(k+1)$ for every $t \in [0,1]$. The lemma is proved.

Let us now prove that, if N is conic, with a conic structure $x \mapsto \lambda \cdot x$, for $\lambda > 0$ and $x \in N$, then f is homogeneous. The two-form $\omega = \omega_1 - \omega_2$ is then conic, meaning that $\omega_{\lambda \cdot x}(\lambda \cdot v_1, \lambda \cdot v_2) = \lambda \omega_x(v_1, v_2)$, for all $\lambda > 0$, $x \in N$ and $v_1, v_2 \in T_x N$. It is easy to see that the homotopy operator Q considered above can be chosen to be homogeneous. Then $\eta = Q\omega$ is homogeneous as well. It easily follows that the time-dependent vector field $X(\cdot)$ of the Moser trick is homogeneous (meaning that $X(t, \lambda \cdot x) = \lambda \cdot X(t,x)$) and hence that its flow is homogeneous. The conclusion follows. □

A.6 A factorization lemma

The following lemma is required in Section 6.2.

Lemma 17. *If $C \in \Psi^m$ is flat along Σ, then there exists $C_1 \in \Psi^{m-2}$, which is flat along Σ, such that $C = \triangle_{sR} C_1 \mod \Psi^{-\infty}$.*

Proof. Let c' be the quotient of the principal symbol of C by $g^\star = \sigma_P(-\triangle_{sR})$. Then c' is flat along Σ and the operator $C' = \mathrm{Op}(c')$ is flat along Σ, and $C = \triangle_{sR} C' + R_1$ with $R_1 \in \Psi^{m-1}$ flat along Σ (because $\triangle_{sR} C'$ is flat along Σ). Then we iterate the construction on R_1. □

A.7 Globalization of the normal and the sub-Riemannian geodesic flow dynamics

In this section, we describe a semi-global version of the Birkhoff normal form derived in Section 5. We will then show that this normal form has strong consequences on the spiraling behavior of some geodesics around the Reeb trajectories.

Let us first note that we have a semi-global normal form which is stated as follows. We denote by $\pi_M : T^\star M \to M$ the canonical projection.

Theorem 6. *We assume that the horizontal bundle D is trivial in some open subset Ω of M. Then there exist a conical neighborhood U of $\Sigma_\Omega = \Sigma \cap \pi_M^{-1}(\Omega)$ and a homogeneous canonical transformation $\chi : U \to \Sigma \times \mathbb{R}^2_{u,v}$, such that*

$$g^\star \circ \chi = \sum_{j=1}^{+\infty} r_j (u^2 + v^2)^j + O_\Sigma(\infty),$$

with $r_j : \Sigma_\Omega \to \mathbb{R}$ homogeneous of degree $2 - j$.

Remark 16. The fiber bundle D is trivial on M in any of the following situations:

- The Reeb flow is Anosov. Indeed by using the stable and unstable directions, we get a continuous trivialization of D which can by smoothened.
- $M = S^\star X$, with (X, h) an orientable Riemannian surface with the contact distribution induced by the Liouville form $p\, dq$. A possible trivialization is (e_1, e_2), where $e_1(q,p)$ is the horizontal lift of the direct normal to v, which is the Legendre transform of p, and e_2 is the generator of the S^1-action on the fibers of $S^\star X \to X$.
- M is the principal bundle associated with a non-vanishing magnetic field on a surface X where the horizontal distribution is the horizontal distribution of the magnetic connection. Then the bundle D is trivial on any open set $\Omega = \pi_M^{-1}(U)$ with $U \subset X$ and the Euler characteristic of U vanishes.

- The flat Heisenberg manifold.

Moreover, the fiber bundle D is trivial on Ω if Ω is a small tubular neighborhood of a curve (open or closed). Let us note moreover that the triviality of D is structurally stable and remains valid under continuous deformation.

The proof of Theorem 6 follows the proof given in Section 5, Step 1, but one cannot perform Step 2 (i.e., remove the invariant part), because the corresponding cohomological equation, which is of the form $Zu = f$, where f is given, cannot be solved in u without having strong properties for the right-hand side f.

In the next statement, we denote by (σ, u, v) the local coordinates in $\Sigma \times \mathbb{R}^2_{u,v}$, and we use the notation $I = u^2 + v^2$.

Theorem 7. *We consider an integral trajectory $\gamma : [0, +\infty) \to M$ of the Reeb flow, which lives in a compact subset K of Ω. Let $\sigma_0 = \tilde{\gamma}(0)$, where $\tilde{\gamma}$ is the lift of γ to Σ_1. We denote by $\rho(t)$ the image of the geodesic flow by χ. Then, for every integer N, there exists ε_0 such that, for every $\varepsilon \in (0, \varepsilon_0]$, if $I_0 \leqslant \varepsilon$ then $\rho(t)(\sigma_0, u_0, v_0) \in \chi(U)$ and $|I(t) - I_0| \leqslant \varepsilon^N$, for every $t \in (0, \varepsilon^{-N}]$.*

Moreover, denoting by $\rho_0(t)$ the flow of the normal form, for every $t \in [0, C|\log \varepsilon|/\varepsilon]$ we have $\|\rho(t)(\sigma_0, u_0, v_0) - \rho_0(t)(\sigma_0, u_0, v_0)\| \leqslant \varepsilon^N$.

The proof of that result follows the arguments given in [31, Section 3]. We do not provide any details.

The first statement says that the geodesic flow has an approximate first integral over a long horizon of time if the initial data are close enough to Σ, and this integral enforces the geodesics to remain close to the Reeb flow. The second statement describes the way in which the orbits spiral around the Reeb trajectories over a logarithmic time horizon. This logarithmic time cannot be improved in general due to the possible instability of the Reeb dynamics.

Remark 17. If $\gamma(t)$ stays in Ω only for $t \in (0, T)$, then we get a similar result over an horizon of time of the order of T/ε.

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