A Region-Dependent Gain Condition for Asymptotic Stability

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Abstract

A sufficient condition for the stability of a system resulting from the interconnection of dynamical systems is given by the small gain theorem. Roughly speaking, to apply this theorem, it is required that the gains composition is continuous, increasing and upper bounded by the identity function. In this work, it is presented an alternative sufficient condition when such criterion fails due to either lack of continuity or the bound of the composed gain is larger than the identity function. More precisely, the local (resp. non-local) asymptotic stability of the origin (resp. global attractivity of a compact set) is ensured by a region-dependent small gain condition. Under an additional condition that implies convergence of solutions for almost all initial conditions in a suitable domain, the almost global asymptotic stability of the origin is ensured. Two examples illustrate and motivate this approach.

1 Introduction

The use of nonlinear input-output gains for stability analysis was introduced in [26] by considering a system as an input-output operator. The condition that ensures stability, called Small Gain Theorem, of interconnected systems is based on the contraction principle.

The work [22] introduces a new concept of gain relating the input to system states. This notion of stability links Zames' and Lyapunov's approaches [23]. Characterizations in terms of dissipation and Lyapunov functions are given in [24].

In [14], the contraction principle is used in the input-to-state stability notion to obtain an equivalent Small Gain Theorem. A formulation of this criterion in terms of Lyapunov functions may be found in [13].

Besides stability analysis, the Small Gain Theorem may also be used for the design of dynamic feedback laws satisfying robustness constraints. The interested reader is invited to see [9,21] and references therein. Other versions of the Small Gain theorem do exist in the literature, see [4,5,11,12] for not necessarily ISS systems.

In order to apply the Small Gain Theorem, it is required that the composition of the nonlinear gains is smaller than the argument for all of its positive values. Such a condition, called Small Gain Condition, restricts the application of the Small Gain Theorem to a composition of well chosen gains.

In this work, an alternative criterion for the stabilization of interconnected systems is provided when a single Small Gain Condition does not hold globally. It consists in showing that if a local (resp. non-local) Small Gain Condition holds in a local (resp. non-local) region of the state space, and the intersection of the local and non-local is empty. Furthermore, if outside the union of these regions, the set of initial conditions from which the associated trajectories do not converge to the local region has measure zero, then the resulting interconnected system is almost asymptotically stable (this notion is precisely defined below). In this paper, a sufficient condition guaranteeing this property to hold is presented. Moreover, for planar systems, an extension of the Bendixson's criterion to regions which are not simply connected is given. This allows to obtain global asymptotic stability of the origin.

This approach may be seen as a unification of two small gain conditions that hold in different regions: a local and a non-local. The use of a unifying approach for local and non-local properties is well known in the literature see [2] in the context of control Lyapunov functions, see [6]

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when uniting iISS and ISS properties.

This paper is organized as follows. In Section 2, the system under consideration and the problem statement are presented. Section 3 states the assumptions to solve the problem under consideration and the main results. Section 4 presents examples that illustrate the assumptions and main results. In Section 5 the proofs of the main results are presented. Section 6 collects some concluding remarks.

Notation. Let $k \in \mathbb{Z}_{>0}$. Let $\mathbf{S}$ be a subset of $\mathbb{R}^k$ containing the origin, the notation $\mathbf{S}_{\neq 0}$ stands for $\mathbf{S} \setminus \{0\}$. The *closure* of $\mathbf{S}$ is denoted by $\mathtt{cl}\{\mathbf{S}\}$. Let $x \in \mathbb{R}^k$, the notation $|x|$ stands for *Euclidean norm* of x. An *open* (resp. *closed*) *ball centered at* $x \in \mathbb{R}^k$ *with radius* $r > 0$ is denoted by $\mathbf{B}_{<r}(x)$ (resp. $\mathbf{B}_{\leq r}(x)$). A continuous function $f : \mathbb{R}^k \to \mathbb{R}$ is *positive definite* if, for every $x \in \mathbb{R}^k \setminus \{0\}$, $f(x) > 0$ and $f(0) = 0$. It is *proper* if $|f(x)| \to \infty$, as $|x| \to \infty$. By $\mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ the class of functions $\eta : \mathbb{R} \to \mathbb{R}^k$ that are *locally essentially bounded*. By $\mathcal{C}^s$ it is denoted the class of *s-times continuously differentiable functions*, by $\mathcal{P}$ it is denoted the class of *positive definite functions*, by $\mathcal{K}$ it is denoted the class of *continuous, positive definite and strictly increasing functions* $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$; it is denoted by $\mathcal{K}_\infty$ if, in addition, they are *unbounded*. Let $c \in \mathbb{R}_{>0}$, the notation $\Omega_{\diamond c}(f)$ stands for the *subset of* $\mathbb{R}^k$ defined by $\{x \in \mathbb{R}^k : f(x) \diamond c\}$, where $\diamond$ is a *comparison operator* (i.e., $=, <, \geq$ etc). The *support* of the function f is the set $\mathbf{supp} := \{x \in \mathbb{R}^k : f(x) \neq 0\}$. By $\mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$ it is denoted the class of functions $g : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$ that are locally essentially bounded. Let $x, \bar{x} \in \mathbb{R}_{\geq 0}$, the notation $x \nearrow \bar{x}$ (resp. $x \searrow \bar{x}$) stands for $x \to \bar{x}$ with $x < \bar{x}$ (resp. $x > \bar{x}$).

2 Background and problem statement

Consider the system

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where, for every $t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$, and $u \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, for some positive integers n and m. Also, $f \in \mathcal{C}^1(\mathbb{R}^{n+m}, \mathbb{R}^n)$. A solution of (1) with initial condition x, and input u at time t is denoted by $X(t, x, u)$. From now on, arguments t will be omitted, and assume that the origin is *input-to-stable stable* (ISS for short) *for* (1). For further details on this concept, the interested reader is invited to consult [23] or [25].

A locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for which there exist $\underline{\alpha}_x, \overline{\alpha}_x \in \mathcal{K}_\infty$ such that, for every $x \in \mathbb{R}^n$, $\underline{\alpha}_x(|x|) \leq V(x) \leq \overline{\alpha}_x(|x|)$ is called *storage function*.

Inspired by [7,16], here it will be used the following notion of derivative.

Definition 1. Consider the function $\xi : [a, b) \to \mathbb{R}$, the limit at $t \in [a, b)$

$$D^+\xi(t) = \limsup_{\tau \searrow 0} \tfrac{\xi(t+\tau) - \xi(t)}{\tau}$$

(if it exists) is called *Dini derivative*. Let k_1 and k_2 be positive integers, $(y_1, y_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, functions $\varphi : \mathbb{R}^{k_1+k_2} \to \mathbb{R}$, $h_1 : \mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$ and $h_2 : \mathbb{R}^{k_2} \to \mathbb{R}^{k_2}$. The limit

$$D^+_{h_1,h_2}\varphi(y_1, y_2) = \limsup_{\tau \searrow 0} \tfrac{\varphi(y_1+\tau h_1(y_1), y_2+\tau h_2(y_2)) - \varphi(y_1, y_2)}{\tau}$$

(if it exists) is called *Dini derivative of* φ *in the* h_1 *and* h_2*-directions at* (y_1, y_2). [2] $\triangleleft$

If, for a given storage function V, there exist a proper function $\lambda_x \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, and $\alpha_x \in \mathcal{K}_\infty$ called *ISS-Lyapunov gain* such that, for every $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$|x| \geq \alpha_x(|u|) \Rightarrow D^+_f V(x, u) \leq -\lambda_x(x), \tag{2}$$

then V is called *ISS-Lyapunov function* for (1). As in [7], the proof that the existence of an ISS-Lyapunov implies that (1) is ISS goes along the lines presented in [24].

Consider the system [3]

$$\dot{z} = g(v, z), \tag{3}$$

where $v \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, $z \in \mathbb{R}^m$, and $g \in \mathcal{C}^1(\mathbb{R}^{n+m}, \mathbb{R}^m)$. From now on, assume that $W : \mathbb{R}^{n+m} \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function for (3) with $\lambda_z \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^m, \mathbb{R}_{\geq 0})$, and $\alpha_z \in \mathcal{K}_\infty$ satisfying, for every $(v, z) \in \mathbb{R}^{n+m}$,

$$W(z) \geq \alpha_z(|v|) \Rightarrow D^+_g W(v, z) \leq -\lambda_z(z). \tag{4}$$

System under consideration. Interconnecting systems (1) and (3) yields the system

$$\begin{cases} \dot{x} = f(x, z), \\ \dot{z} = g(x, z). \end{cases} \tag{5}$$

Using the vectorial notation $y = (x, z)$, system (5) is denoted by $\dot{y} = h(y)$. A solution initiated from y in $\mathbb{R}^{n+m}$ and evaluated at time t is denoted $Y(t, y)$. The two ISS-Lyapunov inequalities (2) and (4) can be rephrased as follows. For every couple $(x, z) \in \mathbb{R}^{n+m}$,

$$\begin{aligned} V(x) \geq \gamma(W(z)) &\Rightarrow D^+_f V(x, z) \leq -\lambda_x(x), \\ W(z) \geq \delta(V(x)) &\Rightarrow D^+_g W(x, z) \leq -\lambda_z(z) \end{aligned} \tag{6}$$

with suitable functions $\gamma, \delta \in \mathcal{K}_\infty$.

A sufficient condition that ensures the stability of (5) is given by the small gain theorem [13]. Roughly speaking if,

$$\forall s \in \mathbb{R}_{>0}, \quad \gamma \circ \delta(s) < s, \tag{7}$$

then the origin is globally asymptotically stable for (5).

Problem statement. At this point, it is possible to explain the problem under consideration. ISS systems for

[2] When the Dini derivative is taken in only one direction, the subscript denotes only such a direction.

[3] A solution of (3) with initial condition z, and input v at time t is denoted by $Z(t, z, v)$.

which (7) does not hold in a bounded set of $\mathbb{R}_{\geq 0}$ are considered. This paper shows that by merging small gain arguments in different regions of the state space and employing some tools from measure theory, a sufficient condition ensuring almost global asymptotic stability of the origin is possible to be given. For planar interconnected systems, by using an extension of Bendixon's criterion, global asymptotic stability of the origin may be established.

3 Assumptions and main results

Assumption 1. There exist constant values $0 \leq \underline{M} < \overline{M} \leq \infty$ and $0 \leq \underline{N} < \overline{N} \leq \infty$, and class $\mathcal{K}_\infty$ functions γ and δ such that, for every $(x,z) \in \mathbf{S} \subset \mathbb{R}^n \times \mathbb{R}^m$, the implications

$$V(x) \geq \gamma(W(z)) \Rightarrow D_f^+ V(x,z) \leq -\lambda_x(x) \tag{8}$$

$$W(z) \geq \delta(V(x)) \Rightarrow D_g^+ W(x,z) \leq -\lambda_z(z) \tag{9}$$

hold, where

$$\begin{aligned}\mathbf{S} := \{&(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : \underline{M} \leq V(x) \leq \overline{M},\\ &W(z) \leq \overline{N}\} \cup \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m :\\ &V(x) \leq \overline{M}, \underline{N} \leq W(z) \leq \overline{N}\},\end{aligned} \tag{10}$$

$\triangleleft$

In other words, Assumption 1 states that the set $\Omega_{\leq \underline{M}}(V) \times \Omega_{\leq \underline{N}}(W)$ is *locally* ISS for the x and z-subsystems of (5). To see more details on locally ISS systems, the interested reader may consult [8].

Assumption 2.

$$\begin{aligned}&\text{if } \overline{M} < \infty,\ s \in [\underline{M}, \overline{M}] \setminus \{0\},\ \gamma \circ \delta(s) < s,\\ &\text{if } \overline{M} = \infty,\ s \in [\underline{M}, \overline{M}) \setminus \{0\},\ \gamma \circ \delta(s) < s.\end{aligned} \tag{11}$$

$\triangleleft$

Assumption 2 states that the small gain condition holds in the interval corresponding to the value to the value of V, when x is restricted to $\mathbf{S}$.

Proposition 1. *Under Assumptions 1 and 2, if*

$$\widetilde{M} := \max\{\gamma^{-1}(\underline{M}), \underline{N}\} < \min\{\delta(\overline{M}), \overline{N}\} =: \widehat{M}, \tag{12}$$

then there exists a proper function $U \in \mathcal{P}(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ that is locally Lipschitz on $\mathbb{R}^{n+m} \setminus \{0\}$ and such that,

$$\forall y \in \Omega_{\leq \widehat{M}}(U) \setminus \Omega_{\leq \widetilde{M}}(U), \quad \limsup_{t\to\infty} U(Y(t,y)) \leq \widetilde{M}.$$

Moreover, if $\gamma, \delta \in (\mathcal{C}^1 \cap \mathcal{K}_\infty)$, then a suitable U can be defined, for every $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$, by

$$U(x,z) = \max\left\{\tfrac{\delta(V(x)) + \gamma^{-1}(V(x))}{2}, W(z)\right\}. \tag{13}$$

Condition (12) implies that $\Omega_{\leq \widetilde{M}}(U) \subsetneq \Omega_{\leq \widehat{M}}(U)$. Proposition 1 states that solutions of (5) starting in $\Omega_{\leq \widehat{M}}(U)$ will converge to the set $\Omega_{\leq \widetilde{M}}(U)$. The proof of Proposition 1 is provided in Section 5.1.

Corollary 1. *[Local stabilization] Consider Assumptions 1 and 2 with the constant values $\underline{M} = \underline{N} = 0$, $M_\ell := \overline{M} < \infty$ or $N_\ell := \overline{N} < \infty$. The set $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ is included in the basin of attraction of the origin of (5), where U_ℓ and $\widehat{M}_\ell$ are given by Proposition 1.*

In other words, Corollary 1 states that the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ is an estimation of the set of initial conditions from which issuing solution of (5) remain close and converge to the origin.

Before stating the second corollary, some concepts regarding the asymptotic behaviour of solutions are recalled. A set $\mathbf{M} \subset \mathbb{R}^{n+m}$ is said to be *positively invariant* with respect to (5) if, for every $t \in \mathbb{R}_{\geq 0}$, $y \in \mathbf{M} \Rightarrow Y(t,y) \in \mathbf{M}$ (cf. [15, p. 127]). A compact positively invariant set $\mathbf{M} \subset \mathbb{R}^{n+m}$ is said to be *globally attractive* if, for all $y \in \mathbb{R}^{n+m}$, $\lim_{t\to\infty} |Y(t,y)|_{\mathbf{M}} = 0$.

Corollary 2. *[Global attractivity] Consider Assumptions 1 and 2 with the constant values $M_g := \underline{M} > 0$ or $N_g := \underline{N} > 0$, and $\overline{M} = \overline{N} = \infty$. The set $\Omega_{\leq \widetilde{M}_g}(U_g)$ is globally attractive for (5), where U_g and $\widetilde{M}_g$ are given by Proposition 1.*

In other words, Corollary 2 states that the set $\Omega_{\leq \widetilde{M}_g}(U_g)$ is an estimation of the global attractor of (5).

The proofs of Corollaries 1 and 2 are not provided and follow from Proposition 1. The interested reader may also consult [7,8].

Under the assumptions of Corollaries 1 and 2, if the estimation of the global attractor $\Omega_{\leq \widetilde{M}_g}(U_g)$ is contained in the estimation of the basin of attraction $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$, then global asymptotic stability of the origin for (5) follows trivially. However, when this inclusion does not hold, the set $\mathbf{R} = \Omega_{\leq \widetilde{M}_g}(U_g) \setminus \Omega_{\leq \widehat{M}_\ell}(U_\ell)$ is not empty, and solutions of (5) may converge to positively invariant sets contained in $\mathbf{R}$ instead (cf. Birkhoff's Theorem [10]). Figure 1 illustrates the region $\mathbf{R}$ obtained in this situation.

The next result provides sufficient conditions ensuring that, for almost every initial condition, issuing solutions remain close and converge to the origin. For the case in which (5) is planar, global asymptotic stability of the origin is established.

Before stating the main results, the concept of stability introduced in [3] is presented. The origin is called *almost*

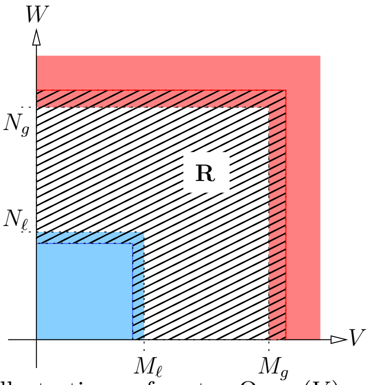

Figure 1. Illustration of sets $\Omega_{\leq M_\ell}(V) \times \Omega_{\leq N_\ell}(W)$ (blue region), $\Omega_{=\widetilde{M}_\ell}(U_\ell)$ (dark blue line), $\Omega_{\geq \underline{M}}(V) \times \Omega_{\geq \underline{N}}(W)$ (pink region), $\Omega_{=\widetilde{M}_g}(U_g)$ (red line), and $\mathbf{R} = \mathtt{cl}\{\Omega_{\leq \widehat{M}_g}(U_g) \setminus \Omega_{\leq \widetilde{M}_\ell}(U_\ell)\}$ (pattern filled).

globally asymptotically stable for (5) if it is locally stable in the Lyapunov sense and attractive for almost every initial condition. More precisely, there exists $\aleph \subset \mathbb{R}^{n+m}$, with $\mu(\aleph) = 0$ such that, for every $y \in \mathbb{R}^{n+m} \setminus \aleph$, $\lim_{t\to\infty} |Y(t,y)| = 0$, where μ is the Lebesgue measure.

Theorem 1. *Under Assumptions 1 and 2, if the constant values of Corollaries 1 and 2 are such that $M_\ell < M_g$ or $N_\ell < N_g$, there exists a function $\rho \in \mathcal{C}^1(\mathbb{R}^{n+m} \setminus \{0\}, \mathbb{R}_{\geq 0})$ with $\mathtt{supp}(\rho) \supseteq \mathbf{R}$, where $\mathbf{R} = \mathtt{cl}\{\Omega_{\leq \widetilde{M}_g}(U_g) \setminus \Omega_{\leq \widehat{M}_\ell}(U_\ell)\}$, and if for every $y \in \mathbf{R}$, $\mathtt{div}(h\rho)(y) > 0$, then the origin is almost globally asymptotically stable for (5).*

In other words, Theorem 1 states that with an extra assumption on the vector field of system (4), solutions converge to the origin for almost every initial condition and the origin is locally asymptotically stable. The proof of Theorem 1 is provided in Section 5.2.

Theorem 2. *Let $n = m = 1$. Under Assumptions 1 and 2, if the constant values of Corollaries 1 and 2 are such that $M_\ell < M_g$ or $N_\ell < N_g$, and for every $y \in \mathbf{R} = \mathtt{cl}\{\Omega_{\leq \widetilde{M}_g}(U_g) \setminus \Omega_{\leq \widehat{M}_\ell}(U_\ell)\}$, $\mathtt{div}\, h(y) \neq 0$ and $h(y) \neq 0$, then the origin is globally asymptotically stable for (5).*

Theorem 2 states that, when (5) is planar and under mild conditions on the vector field, the origin is globally asymptotically stable for (5). In other words, no ω-limit sets exists in $\mathbf{R}$. The proof of Theorem 2 is provided in Section 5.3.

4 Illustration

Here, the results given in the previous section are illustrated in two examples. The first concerns the vectorial case, while the second concerns the planar case.

4.1 A class of systems satisfying an asymptotic small-gain condition

Recall system (5), and assume that there exist locally Lipschitz and proper functions $V \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ and $W \in \mathcal{P}(\mathbb{R}^m, \mathbb{R}_{\geq 0})$ satisfying [4], for every $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\begin{cases} D_f^+ V(x,z) \leq -V(x) + \gamma(W(z)), \\ D_g^+ W(x,z) \leq -W(z) + \delta(V(x)), \end{cases} \tag{14}$$

where $\gamma, \delta \in \mathcal{K}_\infty$ are such that [5]

$$\lim_{s\to 0} \frac{\gamma \circ \delta(s)}{s} < 1 \quad \text{and} \quad \lim_{s\to\infty} \frac{\gamma \circ \delta(s)}{s} < 1. \tag{15}$$

In other words, the composition $\gamma \circ \delta$ satisfies the small-gain condition in the bi-limit: 0 and ∞. Note that to apply [5] it would be necessary to impose that, for every $b_1, b_2 \in \mathbb{R}_{\geq 0}$, and for some $\varepsilon > 0$, $\gamma(b_1)\delta(b_2) \leq (1-\varepsilon)b_1 b_2$.

From (15), there exists a positive constant M_ℓ (resp. M_g) that is sufficiently small (resp. large) and such that, for every $s \in (0, M_\ell]$ (resp. $s \in [M_g, \infty)$), $\gamma \circ \delta(s) < s$. Together with (14) and since W is continuous and proper, Assumptions 1 and 2 hold locally on the compact set $\Omega_{\leq M_\ell}(V) \times \Omega_{\leq N_\ell}(W)$ (resp. non-locally on the set $\Omega_{\geq M_g}(V) \times \Omega_{\geq N_g}(W)$).

Since condition Eq. (12) is satisfied, as formulated in Corollary 1 (resp. 2), from this result the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell) \subset \mathbb{R}^{n+m}$ (resp. $\Omega_{\leq \widetilde{M}_g}(U_g) \subset \mathbb{R}^{n+m}$) is an estimation of the basin of attraction of the origin (resp. global attractor) of (17). Also, $\Omega_{\leq \widehat{M}_\ell}(U_\ell) \subset \Omega_{\leq \widetilde{M}_g}(U_g)$.

Let [6]

$$\begin{cases} r_V(x,z) := -V(x) + \gamma(W(z)), \\ r_W(x,z) := -W(z) + \delta(V(x)). \end{cases}$$

For any (x,z) belonging to the sets $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ and $\Omega_{\geq \widetilde{M}_g}(U_g) \subset \mathbb{R}^{n+m}$ either

1. $r_V(x,z) \geq 0$ and $r_W(x,z) \leq 0$ or;
2. $r_V(x,z) \leq 0$ and $r_W(x,z) \geq 0$ or;
3. $r_V(x,z) \leq 0$ and $r_W(x,z) \leq 0$.

While in the compact set $\mathbf{R} = \Omega_{\leq \widetilde{M}_g}(U_g) \setminus \Omega_{\leq \widehat{M}_\ell}(U_\ell)$, it may happen that $r_V(x,z) \geq 0$ and $r_V(x,z) \geq 0$. In this case, the result given in [4] can not be applied, because it requires that the union of the regions described by items 1-3 forms a cover of $\mathbb{R}^{n+m}$.

Note that in contrast to [11], the existence of a Lyapunov function candidate for the system (5) whose derivative is definite negative on $\mathbb{R}^{n+m}$ is not requested.

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[4] From Remark 2.4 of [24] Eq. (14) is equivalent to (6).

[5] Note that, in contrast to [5,11,12], no information is given about the behaviour of the function $s \mapsto \gamma \circ \delta(s)$, in the interval $(0, \infty)$.

[6] Note that, in contrast to [4], here it is not assumed that $\Omega_{\geq \widetilde{M}_g}(U_g) \cup \Omega_{\leq \widehat{M}_\ell}(U_\ell) = \mathbb{R}^{n+m}$.

Let, for every $(x,y) \in \mathbb{R}^{n+m}$, $\rho(x,y) = (V(x)+W(z))^{-1}$, and assume that for every $(x,z) \in \mathbf{R}$,

$$(\mathtt{div}\, h(x,z))(V(x)+W(z)) \geq \gamma(W(z))+\delta(V(x)). \quad (16)$$

In such a compact set, note that

$$\begin{aligned}
\mathtt{div}(h\rho)(x,z) &= \rho(x,z)\,\mathtt{div}\, h(x,z) + \mathtt{grad}\,\rho(x,z)\cdot h(x,z) \\
&= \tfrac{\mathtt{div}\, h(x,z)}{V(x)+W(z)} - \tfrac{D_f^+V(x,z)+D_g^+W(x,z)}{(V(x)+W(z))^2} \\
&\geq \tfrac{\mathtt{div}\, h(x,z)+1}{V(x)+W(z)} - \tfrac{\gamma(W(z))+\delta(V(x))}{(V(x)+W(z))^2} \\
&> 0,
\end{aligned}$$

where the first inequality is due to (14): for every $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\begin{aligned}
-D_f^+V(x,z) - D_g^+W(x,z) \geq \;& V(x) \;-\; \gamma(W(z)) \\
&+W(z) \;-\; \delta(V(x)),
\end{aligned}$$

and last inequality is due to (16).

From Theorem 1, the origin is almost globally asymptotically stable for the system (5).

4.2 The planar case

Consider the system

$$\begin{cases} \dot{x} \;=\; f(x,z) \;=\; -1.5x \;+\; 2p(z), \\ \dot{z} \;=\; g(x,z) \;=\; \quad -z \quad +\; \sin(x^2/10), \end{cases} \quad (17)$$

where, for every $s \in \mathbb{R}$, $p(s) := s^3/3 - 3s^2/2 + 2s$.

Let, for every $x \in \mathbb{R}$ (resp. $z \in \mathbb{R}$), $V(x) = |x|$ (resp. $W(z) = |z|$). Taking its Lie derivative in the f-direction yields, for every $(x,z) \in \mathbb{R}^2$,

$$D_f^+V(x,z) \leq -1.5V(x) + 2|p(W(z))|. \quad (18)$$

Define, for every $s \in \mathbb{R}_{\geq 0}$, $\gamma(s) = \max\{1.3|p(r)| : 0 \leq r \leq s\}$. From (18),

$$V(x) \geq \gamma(W(z)) \Rightarrow D_f^+V(x,z) \leq -\lambda_x(x) \quad (19)$$

holds with a suitable $\lambda_x \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R},\mathbb{R}_{\geq 0})$. The Lie derivative of W in the g-direction yields, for every $(x,z) \in \mathbb{R}^2$,

$$D_g^+W(x,z) \leq -W(z) + \left|\sin\left(\tfrac{V(x)^2}{10}\right)\right|.$$

which can be rephrased as follows

$$W(z) \geq \delta(V(x)) \Rightarrow D_g^+W(x,z) \leq -\lambda_z(z)$$

with a suitable $\lambda_z \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R},\mathbb{R}_{\geq 0})$, where $\delta(s) = \max\{|\sin(r^2/10)| : 0 \leq r \leq s\}$.

The composition of the function γ and δ yields

$$\gamma\circ\delta(s) = \max\left\{1.3|p(r)| : 0 \leq r \leq \max_{0\leq a\leq s}\left\{\left|\sin\left(\tfrac{a^2}{10}\right)\right|\right\}\right\}.$$

Note that there exist values $\bar{s} > 0$ for which $\gamma\circ\delta(\bar{s}) = 1.11$. Also,

$$\lim_{s\to 0} \tfrac{\gamma\circ\delta(s)}{s} < 1, \quad \text{and} \quad \lim_{s\to\infty} \tfrac{\gamma\circ\delta(s)}{s} < 1. \quad (20)$$

From (20), and following the reasoning of the previous example, there exists $M_\ell > 0$ small (resp. $M_g > 0$ large) enough such that, for every $s \in (0, M_\ell]$ (resp. $s \in [M_g,\infty)$), $\gamma\circ\delta(s) < s$. Also, there exist [7] $\gamma_\ell, \delta_\ell \in \mathcal{K}_\infty$ (resp. [8] $\gamma_g \in \mathcal{K}$ and $\delta_g \in \mathcal{K}_\infty$) satisfying, for every $s \in [0, M_\ell]$ (resp. $s \in [M_g,\infty)$), $\gamma_\ell(s) = \gamma(s)$ and $\gamma_\ell(s) = \gamma(s)$ (resp. $\gamma_g(s) = \gamma(s)$ and $\delta_g(s) = \delta(s)$). Thus, analogously to the reasoning of the previous example, Assumptions 1 and 2 hold locally on the compact set $\Omega_{\leq M_\ell}(V) \times \Omega_{\leq N_\ell}(W)$ (resp. non-locally on the set $\Omega_{\geq M_g}(V) \times \Omega_{\geq N_g}(W)$).

Since condition Eq. (12) is satisfied, as formulated in Corollary 1 (resp. 2), from this result the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell) \subset \mathbb{R}^{n+m}$ (resp. $\Omega_{\leq \widetilde{M}_g}(U_g) \subset \mathbb{R}^{n+m}$) is estimation of the basin of attraction of the origin (resp. global attractor) of (17). Also, $\Omega_{\leq \widehat{M}_\ell}(U_\ell) \subset \Omega_{\leq \widetilde{M}_g}(U_g)$.

It now remains to check whether there exist ω-limit sets in $\mathbf{R} = \Omega_{\leq \widetilde{M}_g}(U_g) \setminus \Omega_{\leq \widehat{M}_\ell}(U_\ell)$. Since

$$\begin{aligned}
&\tfrac{\partial f}{\partial x}(x,z) + \tfrac{\partial g}{\partial z}(x,z) \equiv -2.5 \text{ and} \\
&\qquad f(x,z) = 0 = g(x,z) \Leftrightarrow (x,z) = (0,0),
\end{aligned}$$

from Theorem 2 the origin is globally asymptotically stable for (17).

5 Proofs

5.1 Proof of Proposition 1

Proof. The proof of Proposition 1 is based on the proof of [13, Theorem 3.1]. Here, it is divided into 3 parts. Firstly, the function $\sigma \in \mathcal{K}_\infty \cap \mathcal{C}^1$ is obtained. In the second part, the Dini derivative of a locally Lipschitz and proper function $U \in \mathcal{P}(\mathbb{R}^{n+m},\mathbb{R}_{\geq 0})$ is shown to be decreasing in the set $\mathbf{S}$ defined in (10). In the third part,

[7] Recall that γ and δ are continuous positive definite functions. Thus, they are strictly increasing in a neighbourhood of the origin. Note also that γ is proper.

[8] Although γ_g is of class $\mathcal{K}$, the result of Proposition 1 is still applicable. The main difference in this case would be the construction of the function $\sigma \in \mathcal{C}^1 \cap \mathcal{K}_\infty$ satisfying (21). The interested reader may consult [13] to check how this is done.

solutions of (5) starting in $\Omega_{\leq \widehat{M}}(U) \setminus \Omega_{\leq \widetilde{M}}(U)$ are shown to converge to $\Omega_{\leq \widetilde{M}}(U)$.

First Part. Under Assumptions 1 and 2, the function γ being of class $\mathcal{K}_\infty$ satisfies, for every $s \in \mathbb{R}_{>0}$, $\delta(s) < \gamma^{-1}(s)$. Together with the fact that δ is of class $\mathcal{K}_\infty$, from [13, Lemma A.1], there exists $\sigma \in \mathcal{K}_\infty \cap \mathcal{C}^1$ whose derivative is strictly positive and satisfies,

$$\forall s \in \mathbb{R}_{>0}, \quad \delta(s) < \sigma(s) < \gamma^{-1}(s). \tag{21}$$

Second Part. Define, for every $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, $U(x, z) = \max\{\sigma(V(x)), W(z)\}$. Note that $U \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ is a proper function. Pick $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, one of three cases is possible: $\sigma(V(x)) < W(z)$, $W(z) < \sigma(V(x))$ or $W(z) = \sigma(V(x))$. The proof follows by showing that the Dini derivative of U is negative definite. For each case, assume that $(x, z) \in \mathbf{S}_{\neq 0} := \mathbf{S} \setminus \{(0, 0)\}$, where $\mathbf{S}$ is defined in (10).

Case 1. Assume that $\sigma(V(x)) < W(z)$. This implies that $U(x, z) = W(z)$ and $D^+_{f,g}U(x, z) = D^+_g W(x, z)$. From (21), $\delta(V(x)) < \sigma(V(x)) < W(z)$. Since $(x, z) \in \mathbf{S}_{\neq 0}$, the inequality $D^+_g W(x, z) \leq -\lambda_z(z)$ follows from (9). Thus, $W(z) > \sigma(V(x)) \Rightarrow D^+_{f,g}U(x, z) \leq -\lambda_z(z)$.

Case 2. Assume that $W(z) < \sigma(V(x))$. This implies that $U(x, z) = \sigma(V(x))$ and $D^+_{f,g}U(x, z) = \sigma'(V(x))D^+_f V(x, z)$. Since $(x, z) \in \mathbf{S}_{\neq 0}$, and from (21),

$$W(z) < \sigma(V(x)) < \gamma^{-1}(V(x)). \tag{22}$$

From (8), the inequality $D^+_f V(x, z) \leq -\lambda_x(x)$ holds.

Case 3. Assume that $W(z) = \sigma(V(x))$. Let $U^*(x, z) := W(z) = \sigma(V(x))$. This implies

$$\begin{aligned} D^+_{f,g}U^*(x, z) &= \limsup_{t \searrow 0} \tfrac{1}{t}[\max\{\sigma(V(X(t, x, z))), \\ &\qquad W(Z(t, z, x))\} - U^*(x, z)] \\ &= \limsup_{t \searrow 0} \max\left\{\tfrac{\sigma(V(X(t,x,z))) - \sigma(V(x))}{t}, \tfrac{W(Z(t,z,x)) - W(z)}{t}\right\} \\ &= \max\{\sigma'(V(x))D^+_f V(x, z), D^+_g W(x, z)\}. \end{aligned}$$

The analysis of $D^+_{f,g}U^*$ is divided in two sub cases. In the first one, the function $D^+_g W$ is analyzed while in the last the function $D^+_f V$ is analyzed.

<u>Case 3.a</u>. *The analysis of* $D^+_g W$. From (21), and the fact that $x \neq 0$ and $z \neq 0$, the inequality $\delta(V(x)) < \sigma(V(x)) = W(z)$ holds. Moreover since $(x, z) \in \mathbf{S}_{\neq 0}$, the inequality $D^+_g W(x, z) \leq -\lambda_z(z)$ follows from (9).

<u>Case 3.b</u>. *The analysis of* $D^+_f V$. From (21), and the fact that $x \neq 0$ and $z \neq 0$, the inequality $W(z) = \sigma(V(x)) < \gamma^{-1}(V(x))$ holds. Moreover, since $(x, z) \in \mathbf{S}_{\neq 0}$, the inequality $D^+_f V(x, z) \leq -\lambda_x(x)$ follows from (8).

Summing up Case 3, $0 \neq W(z) = \sigma(V(x)) \Rightarrow D^+_{f,g}U^*(x, z) \leq -\min\{\sigma'(V(x))\lambda_x(x), \lambda_z(z)\}$.

Claim 1. *There exists $c > 0$ such that $\Omega_{\leq c}(U) \subset \Omega_{\leq \overline{M}}(V) \times \Omega_{\leq \overline{N}}(W)$. Moreover, the constants $\widetilde{M}$ and $\widehat{M}$ are such that*

$$\begin{aligned} (\Omega_{\leq \underline{M}}(V) \times \Omega_{\leq \underline{N}}(W)) \subset \Omega_{\leq \widetilde{M}}(U) \subset \Omega_{\leq \widehat{M}}(U) \\ \subset (\Omega_{\leq \overline{M}}(V) \times \Omega_{\leq \overline{N}}(W)). \end{aligned} \tag{23}$$

The proof of Claim 1 is provided in Section 5.4.

From the above case study and (23),

$$\widetilde{M} \leq U(x, z) \leq \widehat{M} \Rightarrow D^+_{f,g}U(x, z) \leq -E(x, z), \tag{24}$$

where $E \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R})$ is the proper function defined, for every $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, by $E(x, z) = \min\{\sigma'(V(x))\lambda_x(x), \lambda_z(z)\}$.

Third part. The local Lipschitz property of U on $\mathbb{R}^n \times \mathbb{R}^m \setminus \{(0, 0)\}$ is due to the fact that $\sigma(V(\cdot))$ (resp. $W(\cdot)$) is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ (resp. $\mathbb{R}^m$).

From [20, Theorem 4.3] and (24), for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, and all $t \in \mathbb{R}_{\geq 0}$, along solutions of (5), $D^+U(X(t, x, z), Z(t, z, x)) = D^+_{f,g}U(X(t, x, z), Z(t, z, x))$.

Since solutions of (5) are absolutely continuous functions and the righthand side of E is a continuous and positive definite function, from [20, Remark 4.4.b], for every (x, z) such that $\widetilde{M} \leq U(x, z) \leq \widehat{M}$, and all $t \in \mathbb{R}_{\geq 0}$, the function

$$t \mapsto U(X(t, x, z), Z(t, z, x)) \tag{25}$$

is strictly decreasing and satisfies

$$U^\infty := \lim_{t \to \infty} U(X(t, x, z), Z(t, z, x)) \leq \widetilde{M}.$$

To see this claim suppose, for purposes of contradiction, that $U^\infty > \widetilde{M}$. From the continuity of U, there exists $\varepsilon > 0$ such that $U^\infty - \varepsilon > \widetilde{M}$ and $U^\infty - \varepsilon \leq U(x, z) \leq U^\infty + \varepsilon$. Since U is proper, the constant $\xi = \min\{E(x, z) > 0 : U^\infty - \varepsilon \leq U(x, z) \leq U^\infty + \varepsilon\}$ exists. Recalling the definition of U, there exists $T > 0$ such that, for all $t \geq T$, $U(X(t, x, z), Z(t, z, x)) - U^\infty < \varepsilon$. Moreover, from the definition of the constant ξ,

$$\begin{aligned} U(X(t, x, z), Z(t, z, x)) - U(X(T, x, z), Z(T, z, x)) = \\ \int_T^t D^+U(X(s, x, z), Z(s, z, x))\, ds \leq -\xi(t - T). \end{aligned}$$

Then,

$$\begin{aligned} U^\infty &= \lim_{t \to \infty} U(X(t, x, z), Z(t, z, x)) \\ &= U(X(T, x, z), Z(T, z, x)) \\ &+ \lim_{t \to \infty} \int_T^t D^+U(X(s, x, z), Z(s, z, x))\, ds \leq -\infty \end{aligned}$$

which contradicts the fact that U is positive definite. Thus, $U^\infty \leq \widetilde{M}$. Hence, solutions of (5) starting in $\Omega_{\leq \widehat{M}}(U) \setminus \Omega_{\leq \widetilde{M}}(U)$ converge towards $\Omega_{\leq \widetilde{M}}(U)$.

To see that U can be given by (13), note that U relies on the computation of σ. Let, for every $s \in \mathbb{R}_{\geq 0}$, $\sigma(s) = (\delta(s) + \gamma^{-1}(s))/2$. Its derivative yields, for every $s > 0$, $2\sigma'(s) = \delta'(s) + 1/(\gamma' \circ \gamma(s))$ which is positive, because [9] $\delta'(s) > 0$ and $\gamma' \circ \gamma^{-1}(s) > 0$. Moreover, such a function σ satisfies (21). This concludes the proof. ■

5.2 *Proof of Theorem 1*

This proof is divided into 4 parts. The first one shows that solutions starting in $\Omega_{\geq \widetilde{M}_g}(U_g)$ converge to $\mathbf{R}$. The second part shows that almost all solutions starting in $\mathbf{R}$ converges to $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$. The third part shows that solutions starting in the latter set converge to the origin. The fourth part concludes the almost global asymptotic stability of the origin.

1ST PART. From Corollary 2, the set $\Omega_{\leq \widetilde{M}_g}(U_g)$ is globally attractive for (5), where $\widetilde{M}_g = \max\{\gamma_g^{-1}(M_g), N_g\}$, M_g and N_g are defined in Corollary 2, and γ_g is given by Assumption 1.

2ND PART. From the proof of Proposition 1, there exist proper functions $U_g, E_g \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$ (resp. $U_\ell, E_\ell \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^{n+m}, \mathbb{R}_{\geq 0})$) with U_g (resp. U_ℓ) being also locally Lipschitz and such that, for every $y \in \Omega_{\geq \widetilde{M}_g}(U_g)$, $D_h^+ U_g(y) \leq -E_g(y)$ (resp. for every $y \in \Omega_{\leq \widehat{M}_\ell}(U_\ell)$, $D_h^+ U_\ell(y) \leq -E_\ell(y)$).

To see that $\Omega_{\leq \widehat{M}_\ell}(U_\ell) \subsetneq \Omega_{\leq \widetilde{M}_g}(U_g)$. From the proof of Claim 1, $U_\ell(x,z) \leq \widehat{M}_\ell \Rightarrow \max\{V(x), W(z)\} \leq \min\{M_\ell, N_\ell\}$. Analogously, $U_g(x,z) \geq \widetilde{M}_g \Rightarrow \min\{V(x), W(z)\} \geq \max\{M_g, N_g\}$. Since $\min\{M_\ell, N_\ell\} < \max\{M_g, N_g\}$. Thus, $\Omega_{\leq \widehat{M}_\ell}(U_\ell) \subsetneq \Omega_{\leq \widetilde{M}_g}(U_g)$.

The proof proceeds by showing that, for almost every initial condition staring in $\mathbf{R} = \Omega_{\leq \widetilde{M}_g}(U_g) \setminus \Omega_{\leq \widehat{M}_\ell}(U_\ell)$, issuing solutions of (5) converge to $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$. To do so, the same lines as in [19, Theorem 1] and [3, Theorem 3] are followed. However, here a less conservative condition is required, since a set that is only positively invariant, and the divergence to be positive only in a compact set are needed.

Let $\mathbf{Z} \subset \mathbb{R}^n$ a set given by [10]

$$\mathbf{Z} = \cap_{l=1}^{\infty} \{y \in \Omega_{\leq \widetilde{M}_g}(U_g) : U_\ell(Y(t,y)) > \widehat{M}_\ell, t > l\}.$$

For every $t \in \mathbb{R}$, let $Y(t, \mathbf{Z}) = \{Y(t,z) : z \in \mathbf{Z}, t \in \mathtt{dom}(z)\}$, where $\mathtt{dom}(z)$ is the maximum time interval where $Y(t,z)$ exists. Since $\Omega_{\leq \widetilde{M}_g}(U_g)$ is positively invariant, $\mathbf{Z}$ is also positively invariant. Thus, given a fixed $\tau \in \mathbb{R}_{>0}$, for all $t \geq \tau$, $Y(t, \mathbf{Z}) \subset Y(\tau, \mathbf{Z})$. Hence, for all $t \in \mathbb{R}_{\geq 0}$,

$$\int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dy \leq 0, \tag{26}$$

where $\rho \in \mathcal{C}^1(\mathbb{R}^{n+m} \setminus \{0\}, \mathbb{R}_{\geq 0})$ and $\mathtt{supp}(\rho) \supseteq \mathbf{R}$.

From Liouville's Theorem (see [19, Lemma A.1]), for every $t \in \mathbb{R}_{\geq 0}$,

$$\int_0^t \int_{Y(s,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dyds = \int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dy.$$

Since $\mathbf{Z} \subset \mathbf{R}$, for every $t \in \mathbb{R}_{\geq 0}$, the inequality

$$\begin{aligned} t \int_{Y(t,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dy &\leq \int_0^t \int_{Y(s,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dyds \\ &\leq \int_{Y(t,\mathbf{Z})} \rho(y)\, dy - \int_{\mathbf{Z}} \rho(y)\, dy \end{aligned}$$

holds. From (26), for every $t \in \mathbb{R}_{\geq 0}$, $\int_{Y(t,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dy \leq 0$. Together with the fact that, for every $y \in \mathbf{R}$, $\mathtt{div}(h\rho)(y) > 0$, it yields $\int_{Y(t,\mathbf{Z})} \mathtt{div}(h\rho)(y)\, dy = 0$, for every $t \in \mathbb{R}_{\geq 0}$. Thus, for every $t \in \mathbb{R}_{\geq 0}$, $Y(t, \mathbf{Z})$ has Lebesgue measure zero. In particular, $\mathbf{Z}$ has also Lebesgue measure zero. Consequently, for almost every $y \in \mathbf{R}$, $\limsup_{t \to \infty} U_\ell(Y(t,y)) \leq \widehat{M}_\ell$.

It remains to check if the initial conditions belonging to $\Omega_{\geq \widetilde{M}_g}(U_g)$ from which issuing solutions converge to $\mathbf{Z}$ have also measure zero. Since $\mathbf{Z}$ is positively invariant, for all $t_1 < t_2 \leq 0$, $Y(t_2, \mathbf{Z}) \subset Y(t_1, \mathbf{Z})$. This inclusion implies that $\mathbf{Y} := \cup_{t \leq 0} \{Y(t, \mathbf{Z})\} = \cup_{l \in \mathbb{Z}_{<0}} \{Y(l, \mathbf{Z})\}$. Hence, the set $\mathbf{Y}$ is a countable union of images of $\mathbf{Z}$ by the flow. Since $\mathbf{Z}$ is measurable and, for every $t \in \mathtt{dom}(y)$, the map $\mathbf{Z} \ni y \mapsto Y(t,y)$ is a diffeomorphism [11], $\mathbf{Y}$ is also measurable.

For every $t \in \mathtt{dom}(\mathbf{Z})$, $\int_{Y(t,\mathbf{Z})} dz \leq \int_{\mathbf{Z}} |\mathtt{grad}\, Y(t,y)|\, dy = 0$, because $\mathbf{Z}$ has measure zero. This implies that, for all $t \in \mathtt{dom}(\mathbf{Z})$, the set $Y(t, \mathbf{Z})$ has measure zero. Since $\mathbf{Y}$ is the countable union of sets of measure zero, it has also measure zero. [12] Hence the set of solutions starting in $\Omega_{\geq \widetilde{M}_g}(U_g)$ that converge to $\mathbf{Z}$ have also measure zero

3RD PART. From Corollary 1, the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ is contained in the basin of attraction of the origin, where $\widehat{M}_\ell = \min\{\delta_\ell(M_\ell), N_\ell\}$, M_ℓ and N_ℓ are defined in Corollary 1, and γ_ℓ is given by Assumption 1.

4TH PART. From the above discussion, the origin is locally stable and almost globally attractive for (5). Thus,

[9] Recall that $\delta, \gamma \in (\mathcal{C}^1 \cap \mathcal{K}_\infty)$.

[10] Note that $\mathbf{Z}$ is the set of all initial conditions belonging to $\Omega_{\leq \widetilde{M}_g}(U_g)$ from which issuing solutions do not converge to $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$.

[11] Because (5) is of class $\mathcal{C}^1$ and solutions are unique.

[12] Recall that $\mathbf{Y}$ is the set of initial conditions from which issuing solutions of (5) converge $\mathbf{Z}$.

it is almost globally asymptotically stable for (5). This concludes the proof. ■

5.3 *Proof of Theorem 2*

Before proving Theorem 2, some concepts regarding the asymptotic behavior of solutions of planar systems are recalled. A point p is said to be a *positive limit point of* $Y(\cdot, y)$ if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$, with $t_n \to \infty$ as $n \to \infty$, such that $Y(t_n, y) \to p$ as $n \to \infty$ (cf. [15, p. 127]). The set $\omega(y)$ of all positive limit points of $Y(\cdot, y)$ is called *ω-limit set of y* (cf. [10, p. 517]). For planar systems, a closed curve $C \subset \mathbb{R}^2$ is called *closed orbit* if C is not an equilibrium point and there exists a time $T < \infty$ such that, for each $y \in C$, $Y(nT, y) = y$, $\forall n \in \mathbb{Z}$ (cf. [21, Definition 2.6]).

Proof. The proof of Theorem 2 follows the same line as the proof of Theorem 1. The difference here consists in the second and fourth parts.

1ST PART. Recall that from Corollary 2, the set $\Omega_{\le \widetilde{M}_g}(U_g)$ is globally attractive for (5), where $\widetilde{M}_g = \max\{\gamma_g^{-1}(M_g), N_g\}$, M_g and N_g are defined in Corollary 2, and γ_g is given by Assumption 1.

2ND PART: Bendixson's criterion for non simply connected regions. From the proof of Proposition 1, there exist proper functions $U_g, E_g \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^2, \mathbb{R}_{\ge 0})$ (resp. $U_\ell, E_\ell \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^2, \mathbb{R}_{\ge 0})$) with U_g (resp. U_ℓ) being also locally Lipschitz and such that, for every $y \in \Omega_{\ge \widetilde{M}_g}(U_g)$, $D_h^+ U_g(y) \le -E_g(y)$ (resp. for every $y \in \Omega_{\le \widehat{M}_\ell}(U_\ell)$, $D_h^+ U_\ell(y) \le -E_\ell(y)$).

Since the set $\mathbf{R} = \mathtt{cl}\{\Omega_{\le \widetilde{M}_g}(U_g) \setminus \Omega_{\le \widehat{M}_\ell}(U_\ell)\}$ is comapct, and for each $y \in \mathbf{R}$, $U_g(y) \neq 0$, from [1, Theorem 2.5]

- The set $\Omega_{= \widetilde{M}_g}(U_g)$ has finite perimeter;
- The function U_g is almost everywheredifferentiable on $\Omega_{= \widetilde{M}_g}(U_g)$;
- Let $\mathbf{N}_g \subset \Omega_{= \widetilde{M}_g}(U_g)$ be set in which U_g is not differentiable. There exists a Lipschitz parametrization $p_g : [a_g, b_g] \subset \mathbb{R} \to \Omega_{= \widetilde{M}_g}(U_g)$ that is injective and satisfies, for almost every $s \in [a_g, b_g]$, $p_g(s) \notin \mathbf{N}_g$ and $p_g'(s)$ is perpendicular to $\nabla U_g(p_g(s))$.

Recall that by assumption, for every $y \in \mathbf{R}$, $h(y) \neq 0$. Together with the fact that $h \in \mathcal{C}^1(\mathbb{R}^2)$, and almost each sublevel set of U_g has finite perimeter. From the generalized divergence theorem [17, Theorem 1.7] (see also [18])

$$\iint_{\Omega_{\le \widetilde{M}_g}(U_g)} \mathtt{div}\, h(y)\, dy = \oint_{\Omega_{= \widetilde{M}_g}(U_g)} h(y) \cdot n_g(y)\, dx dz. \quad (27)$$

Together with the above discussions and the existence of the parametrization p_g, for almost every $s \in [a_g, b_g]$, $h(p_g(s)) \cdot n_g(p_g(s)) < 0$, where for almost every $s \in [a_g, b_g]$, $n_g(p_g(s)) = \nabla U_g(p_g(s))/|\nabla U_g(p_g(s))|$,

$$\iint_{\Omega_{\le \widetilde{M}_g}(U_g)} \mathtt{div}\, h(y)\, dy = \int_{[a_g, b_g]} h(p_g(s)) \cdot n_g(p_g(s))\, ds < 0. \quad (28)$$

Analogously to the above, and by letting $p_\ell : [a_\ell, b_\ell] \to \Omega_{= \widehat{M}_\ell}(U_\ell)$ be a parametrization of $\Omega_{= \widehat{M}_\ell}(U_\ell)$ with outward unit normal n_ℓ, based on Equation (27),

$$\iint_{\Omega_{\le \widehat{M}_\ell}(U_\ell)} \mathtt{div}\, h(y)\, dy = \int_{[a_\ell, b_\ell]} h(p_\ell(s)) \cdot n_\ell(p_\ell(s))\, ds < 0. \quad (29)$$

Suppose, for purposes of contradiction, that there exists a closed orbit $C \in \mathbb{R}^2$, parametrized by $p : [a, b] \to C$ and with outward unit normal n, and contained in $\mathbf{R}$. From the generalized divergence theorem,

$$\iint_{\mathbf{D}_C} \mathtt{div}\, h(x, z)\, dx dz = \int_{[a,b]} h(p(s)) \cdot n(p(s))\, ds = 0, \quad (30)$$

where $\mathbf{D}_C$ is the simply connected region bounded by C.

Note that,

$$\begin{aligned} \iint_{\Omega_{\le \widetilde{M}_g}(U_g) \setminus \mathbf{D}_C} \mathtt{div}\, h(y)\, dy &= \iint_{\Omega_{\le \widetilde{M}_g}(U_g)} \mathtt{div}\, h(y)\, dy - \iint_{\mathbf{D}_C} \mathtt{div}\, h(y)\, dy \\ &= \iint_{\Omega_{\le \widetilde{M}_g}(U_g)} \mathtt{div}\, h(y)\, dy, \end{aligned}$$

where the last equality is due to (30). From (28),

$$\iint_{\Omega_{\le \widetilde{M}_g}(U_g) \setminus \mathbf{D}_C} \mathtt{div}\, h(y)\, dy < 0. \quad (31)$$

On the other hand,

$$\begin{aligned} \iint_{\mathbf{D}_C \setminus \Omega_{\le \widehat{M}_\ell}(U_\ell)} \mathtt{div}\, h(y)\, dy &= \iint_{\mathbf{D}_C} \mathtt{div}\, h(y)\, dy - \iint_{\Omega_{\le \widehat{M}_\ell}(U_\ell)} \mathtt{div}\, h(y)\, dy \\ &= - \iint_{\Omega_{\le \widehat{M}_\ell}(U_\ell)} \mathtt{div}\, h(y)\, dy, \end{aligned}$$

where the last equality is due to (30). From (29),

$$\iint_{\mathbf{D}_C \setminus \Omega_{\le \widehat{M}_\ell}(U_\ell)} \mathtt{div}\, h(x, z)\, dx dz > 0. \quad (32)$$

From (31), (32) and the continuity of $\mathtt{div}\, h$, the function $\mathtt{div}\, h$ changes sign in $\mathbf{R}$. Thus, there exists $\bar{y} \in \mathbf{R}$ such that $\mathtt{div}\, h(\bar{y}) = 0$ which is a contradiction with the hypothesis $\mathtt{div}(y) \neq 0$, for every $y \in \mathbf{R}$. Thus, there exist no closed orbits C contained in $\mathbf{R}$.

From the Poincaré-Bendixson Theorem [21, Theorem 2.15], the ω-limit set of a solution starting in $\mathbf{R}$ is a closed orbit or equilibrium. Since equilibria are impossible by assumption, and from above analysis there exist no ω-limit sets in $\mathbf{R}$, all solutions starting in $\mathbf{R}$ will converge to $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$.

3RD PART. Recall that from Corollary 1, the set $\Omega_{\leq \widehat{M}_\ell}(U_\ell)$ is contained in the basin of attraction of the origin, where $\widehat{M}_\ell = \min\{\delta_\ell(M_\ell), N_\ell\}$, M_ℓ and N_ℓ are defined in Corollary 1, and γ_ℓ is given by Assumption 1.

4TH PART. From the above discussion, the origin is locally stable and globally attractive for (5). Thus, it is globally asymptotically stable for (5). This concludes the proof. ■

Remark 1. Note that, if $\Omega_{\leq \widehat{M}_\ell}(U_\ell) = \{0\}$, then $\mathbf{R} = \Omega_{\leq \widetilde{M}_g}(U_g)$ is a simply connected region, and the second part of the proof of Theorem 2 can be reduced to the proof of the known Bendixson's criterion. ◦

5.4 Proof of Claim 1

Let c be a positive real number [13] such that $\Omega_{\leq c}(U) \subset \Omega_{\leq \overline{M}}(V) \times \Omega_{\leq \overline{N}}(W)$.

In the first part, it will be shown that, for all $(x,z) \in \mathbf{S}$,

$$U(x,z) \leq \widehat{M} \Rightarrow \max\{V(x), W(z)\} \leq \min\{\overline{M}, \overline{N}\}. \quad (33)$$

In the second part, it will be shown that, for all $(x,z) \in \mathbf{S}$,

$$\widetilde{M} \leq U(x,z) \Rightarrow \max\{\underline{M}, \underline{N}\} \leq \min\{V(x), W(z)\}. \quad (34)$$

Part 1. $U(x,z) \leq \widehat{M}$. This implies $U(x,z) = \max\{\sigma(V(x)), W(z)\} \leq \widehat{M} = \min\{\delta(\overline{M}), \overline{N}\}$.

Assume that $\max\{\sigma(V(x)), W(z)\} = \sigma(V(x))$ and $\min\{\delta(\overline{M}), \overline{N}\} = \delta(\overline{M})$. This implies $\sigma(V(x)) \leq \delta(\overline{M})$. From (21), $V(x) \leq \sigma^{-1} \circ \delta(\overline{M}) < \overline{M}$. Assume now that $\max\{\sigma(V(x)), W(z)\} = W(z)$ and $\min\{\delta(\overline{M}), \overline{N}\} = \delta(\overline{M})$. This implies $W(z) \leq \delta(\overline{M}) \leq \overline{N}$. The other two cases are straightforward. Thus, (33) holds. Hence, $\Omega_{\leq \widehat{M}}(U) \subset (\Omega_{\leq \overline{M}}(V) \times \Omega_{\leq \overline{N}}(W))$;

Part 2. $\widetilde{M} \leq U(x,z)$. This implies $\widetilde{M} = \max\{\gamma^{-1}(\underline{M}), \underline{N}\} \leq U(x,z) = \max\{\sigma(V(x)), W(z)\}$.

Assume that, $\max\{\gamma^{-1}(\underline{M}), \underline{N}\} = \gamma^{-1}(\underline{M})$ and $\max\{\sigma(V(x)), W(z)\} = \sigma(V(x))$. This implies $\gamma^{-1}(\underline{M}) \leq \sigma(V(x))$. From (21), $\underline{M} \leq \gamma \circ \sigma(V(x)) < V(x)$. Assume now that, $\max\{\gamma^{-1}(\underline{M}), \underline{N}\} = \gamma^{-1}(\underline{M})$ and $\max\{\sigma(V(x)), W(z)\} = W(z)$. This implies $\underline{N} \leq \gamma^{-1}(\underline{M}) \leq W(z)$. The other two cases are straightforward. Thus, (34) holds. Hence, $(\Omega_{\leq \underline{M}}(V) \times \Omega_{\leq \underline{N}}(W)) \subset \Omega_{\leq \widetilde{M}}(U)$;

Since (12) is a strict inequality, from the continuity and surjectivity of U, there exists $(x,z) \in \mathbf{S}$ such that $\widetilde{M} \leq U(x,z) \leq \widehat{M}$. From (33) and (34), $\widetilde{M} \leq U(x,z) \leq \widehat{M} \Rightarrow \max\{\underline{M}, \underline{N}\} \leq \min\{V(x), W(z)\} \leq \max\{V(x), W(z)\} \leq \min\{\overline{M}, \overline{N}\}$. Thus, the inclusion (23) holds. This concludes the proof. ■

6 Conclusion

Systems for which the small gain theorem cannot be used, a sufficient condition for the stability of the resulting interconnected system is proposed. The approach consists in verifying if the small gain condition holds in two different regions of the state space: a local and a non-local. In the gap between both regions, assuming mild properties on the vector field, a sufficient condition ensuring the convergence of solutions, for almost every initial condition, is provided. An approach is proposed for planar system for which Bendixson's criterion does not hold. Two examples illustrate the results.

The authors plan to extend the proposed approach for the case in which, in a countable number of intervals, the small gain condition holds and, between such intervals, a condition ensuring the absence of ω-limit set holds.

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References

[1] G. Alberti, S. Bianchini, and G. Crippa. Structure of level sets and Sard-type properties of Lipschitz maps. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 12(4):863–902, 2013.

[2] V. Andrieu and C. Prieur. Uniting two control Lyapunov functions for affine systems. *IEEE Trans. Autom. Control*, 55(8):1923–1927, 2010.

[3] D. Angeli. An Almost Global Notion of Input-to-State Stability. *IEEE Trans. Autom. Control*, 49(6):866–874, 2004.

[4] D. Angeli and A. Astolfi. A tight small-gain theorem for not necessarily ISS systems. *Syst. & Contr. Lett.*, 56(1):87–91, 2007.

[5] A. Astolfi and L. Praly. A weak version of the small-gain theorem. In *IEEE 51st Conference on Decision and Control (CDC), 2012*, pages 4586–4590, 2012.

[6] A. Chaillet, D. Angeli, and H. Ito. Strong iISS: combination of iISS and ISS with respect to small inputs. In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 2256–2261, 2012.

[7] S. Dashkovskiy, B. Rüffer, and F. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM J. Control Optim.*, 48(6):4089–4118, 2010.

[13] Such a positive real number always exist. Otherwise, for all $n \geq \mathbb{N}$, there exists y_n such that $y_n \in \Omega_{\leq 1/n}(U)$ and $y_n \notin \Omega_{\leq \overline{M}}(V) \times \Omega_{\leq \overline{N}}(W)$. Since U is proper, $\Omega_{\leq 1/n}(U) \subset \Omega_{\leq 1}(U)$ is compact. Hence, there exists $\{y_{n_j}\}_{j \in \mathbb{N}} \subset \{y_n\}_{n \in \mathbb{N}}$ such that $y_{n_j} \xrightarrow{j \to \infty} y^*$ and $U(y^*) = 0$. From the positive definiteness of U, $y^* = 0$. Consequently, y_{n_j} is a sequence converging to zero and outside $\Omega_{\leq \overline{M}}(V) \times \Omega_{\leq \overline{N}}(W)$. This is impossible since this set is a neighborhood of the origin.

[8] S. Dashkovskiy and B. S. Rüffer. Local ISS of large-scale interconnections and estimates for stability regions. *Syst. & Contr. Lett.*, 59:241–247, 2010.

[9] R. A. Freeman and P. V. Kokotović. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques.* Birkhäuser, 2008.

[10] A. Isidori. *Nonlinear Control Systems.* Communications and Control Engineering. Springer, 1995.

[11] H. Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Trans. Autom. Control*, 51(10):1626–1643, 2006.

[12] H. Ito and Z.-P. Jiang. Necessary and sufficient small gain conditions for integral input-to-state stable systems: A Lyapunov perspective. *IEEE Trans. Autom. Control*, 54(10):2389–2404, 2009.

[13] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8):1211–1215, 1996.

[14] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals and Systems*, 7:95–120, 1994.

[15] H. K. Khalil. *Nonlinear Systems.* Prentice Hall, 3rd edition, 2001.

[16] D. Liberzon, D. Nešić, and A. R. Teel. Lyapunov-based small-gain theorems for hybrid systems. *To appear in IEEE Trans. Autom. Control*, 2013.

[17] A. Marzocchi. Singular stresses and nonsmooth boundaries in continuum mechanics. In *XXX Ravello Summer School*, 2005.

[18] W. F. Pfeffer. *The Divergence Theorem and Sets of Finite Perimeter.* CRC Press, 2012.

[19] A. Rantzer. A dual Lyapunov's stability theorem. *Syst. & Contr. Lett.*, 42:161–168, 2001.

[20] N. Rouche, P. Habets, and M. Laloy. *Stability theory by Liapunov's direct methods*, volume 22 of *Applied mathematical sciences.* Springer, 1977.

[21] S. Sastry. *Nonlinear Systems*, volume 10 of *Interdisciplinary Applied Mathematics.* Springer, 1999.

[22] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Autom. Control*, 34(4):435–443, 1989.

[23] E. D. Sontag. The ISS philosophy as a unifying framework for stability-like behavior. In *Nonlinear control in the year 2000*, volume 259, pages 443–467. Springer London, 2001.

[24] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Syst. & Contr. Lett.*, 24(5):351–359, 1995.

[25] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Trans. Autom. Control*, 41(9):1283–1294, 1996.

[26] G. Zames. On the input-output stability of time-varying nonlinear feedback systems part I: Conditions derived using concepts of loop gain, conicity, and positivity. *IEEE Trans. Autom. Control*, 11(2):228–238, 1966.