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A MULTIDIMENSIONAL BORG-LEVINSON THEOREM FOR MAGNETIC SCHRÖDINGER OPERATORS WITH PARTIAL SPECTRAL DATA

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Abstract. We consider the multidimensional Borg-Levinson theorem of determining both the magnetic field dA and the electric potential V, appearing in the Dirichlet realization of the magnetic Schrödinger operator $H = (-\mathrm{i}\nabla + A)^2 + V$ on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, from partial knowledge of the boundary spectral data of H. The full boundary spectral data are given by the set $\{(\lambda_k, \partial_\nu \varphi_{k|\partial\Omega}) : \ k \geq 1\}$, where $\{\lambda_k : \ k \in \mathbb{N}^*\}$ is the non-decreasing sequence of eigenvalues of H, $\{\varphi_k : \ k \in \mathbb{N}^*\}$ an associated Hilbertian basis of eigenfunctions and ν is the unit outward normal vector to $\partial\Omega$. We prove that some asymptotic knowledge of $(\lambda_k, \partial_\nu \varphi_{k|\partial\Omega})$ with respect to $k \geq 1$ determines uniquely the magnetic field dA and the electric potential V.

Keywords: Inverse spectral problem, Borg-Levinson theorem, magnetic Schrödinger operators.

2010 AMS Subject Classification: Primary: 35R30, 35J10; secondary: 35P99.

1. Introduction

1.1. **Statement of the problem.** We consider $\Omega \subset \mathbb{R}^n$, $n \geq 2$, a $\mathcal{C}^{1,1}$ bounded domain and we set $\Gamma = \partial\Omega$. Let $A \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, $V \in L^\infty(\Omega, \mathbb{R})$ and consider the magnetic Schrödinger operator $H = (-\mathrm{i}\nabla + A)^2 + V$ acting on $L^2(\Omega)$ with domain $D(H) = \{v \in H_0^1(\Omega) : \ (-\mathrm{i}\nabla + A)^2 v \in L^2(\Omega)\}$.

Let $A_j \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, $V_j \in L^\infty(\Omega, \mathbb{R})$, $j = 1, 2$, and consider the magnetic Schrödinger operators $H_j = H$ for $A = A_j$ and $V = V_j$, $j = 1, 2$. We say that H_1 and H_2 are gauge equivalent if there exists $p \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$ such that $H_2 = e^{-\mathrm{i}p} H_1 e^{\mathrm{i}p}$.

It is well known that H is a selfadjoint operator. By the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the spectrum of H is purely discrete. We note $\{\lambda_k : \ k \in \mathbb{N}^*\}$ the non-decreasing sequence of eigenvalues of H and $\{\varphi_k : \ k \in \mathbb{N}^*\}$ an associated Hilbertian basis of eigenfunctions. In the present paper we consider the Borg-Levinson inverse spectral problem of determining uniquely H, modulo gauge equivalence, from partial knowledge of the boundary spectral data $\{(\lambda_k, \partial_\nu \varphi_{k|\Gamma}) : \ k \in \mathbb{N}^*\}$ with ν the outward unit normal vector to Γ. Namely, we prove that some asymptotic knowledge of $(\lambda_k, \partial_\nu \varphi_{k|\Gamma})$ with respect to $k \in \mathbb{N}^*$ determines uniquely the operator H modulo gauge transformation.

1.2. **Borg-Levinson inverse spectral problems.** It is Ambartasumyan who first investigated in 1929 the inverse spectral problem of determining the real potential V appearing in the Schrödinger operator $H = -\Delta + V$, acting in $L^2(\Omega)$, from partial spectral data of H. For $\Omega = (0, 1)$, he proved in [1] that $V = 0$ if the spectrum of the Neumann realization of H equals $\{k^2 : \ k \in \mathbb{N}\}$. For the same operator, but endowed with homogeneous Dirichlet boundary conditions, Borg [6] and Levinson [19] established that the Dirichlet

spectrum $\{\lambda_k : k \in \mathbb{N}^*\}$ does not uniquely determine V. They showed that additional spectral data, namely $\{\|\varphi_k\|_{L^2(0,1)} : k \in \mathbb{N}^*\}$, where $\{\varphi_k : k \in \mathbb{N}^*\}$ is an $L^2(0,1)$-orthogonal basis of eigenfunctions of H obeying the condition $\varphi_k'(0) = 1$, is needed. Gel'fand and Levitan proved in [13] that uniqueness is still valid upon substituting the terminal velocity $\varphi_k'(1)$ for $\|\varphi_k\|_{L^2(0,1)}$ in the one-dimensional Borg and Levinson theorem.

In 1988, Nachman, Sylvester, Uhlmann [21] and Novikov [23] proposed a multidimensional formulation of the result of Borg and Levinson. Namely, they proved that the boundary spectral data $\{(\lambda_k, \partial_\nu \varphi_{k|\partial\Omega}) : k \in \mathbb{N}^*\}$, where ν denotes the outward unit normal vector to $\partial\Omega$ and (λ_k, φ_k) is the kth eigenpair of $-\Delta + V$, determines uniquely the Dirichlet realization of the operator $-\Delta + V$. The initial formulation of the multidimensional Borg-Levinson theorem by [21] and [23] has been improved in several ways by various authors. Isozaki [14] (see also [9]) extended the result of [21] when finitely many eigenpairs remain unknown, and, recently, Choulli and Stefanov [10] claimed uniqueness in the determination of V from the asymptotic behavior of $(\lambda_k, \partial_\nu \varphi_{k|\Gamma})$ with respect to k. Moreover, Canuto and Kavian [7, 8] considered the determination of the conductivity c, the electric potential V and the weight ρ from the boundary spectral data of the operator $\rho^{-1}(-div(c\nabla\cdot)+V)$ acting on the weighted space $L^2_\rho(\Omega)$ endowed with either Dirichlet or Neumann boundary conditions. Namely, [7, 8] proved that the boundary spectral data of $\rho^{-1}(-div(c\nabla\cdot)+V)$ determines uniquely two of the three coefficients c, V and ρ. The case of magnetic Schrödinger operator has been treated by [25] who determined both the magnetic field dA and the electric potential V of the operator $H = (-\mathrm{i}\nabla + A)^2 + V$. Here the 2-form dA of a vector valued function $A = (a_1, \dots, a_n)$ is defined by

$$dA = \sum_{i,j=1}^{n} (\partial_{x_j} a_i - \partial_{x_i} a_j) dx_j \wedge dx_i.$$

All the above mentioned results were obtained with Ω bounded and operators of purely discrete spectral type. In some recent work [16] examined a Borg-Levinson inverse problem stated in an infinite cylindrical waveguide for Schrödinger operators with purely absolutely continuous spectral type. More precisely, [16] proved that a real potential V which is 2π-periodic along the axis of the waveguide is uniquely determined by some asymptotic knowledge of the boundary Floquet spectral data of the Schrödinger operator $-\Delta + V$ with Dirichlet boundary conditions.

Finally, let us mention for the sake of completeness that the stability issue in the context of Borg-Levinson inverse problems was examined in [4, 5, 9, 10, 16] and that related results on Riemannian manifolds were examined by [2, 5, 15].

1.3. **Main result.** Let $A_j \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, $V_j \in L^\infty(\Omega, \mathbb{R})$ and consider the magnetic Schrödinger operators $H_j = H$ for $A = A_j$ and $V = V_j$, $j = 1, 2$. Further we note $(\lambda_{j,k}, \varphi_{j,k})$, $k \geq 1$, the kth eigenpair of H_j, for $j = 1, 2$. Our main result can be stated as follows.

Theorem 1.1. *For $j = 1, 2$, let $V_j \in L^\infty(\Omega, \mathbb{R})$ and let $A_j \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ fulfill*

$$A_1(x) = A_2(x), \quad x \in \Gamma. \tag{1.1}$$

Assume that the conditions

$$\lim_{k \to +\infty} |\lambda_{1,k} - \lambda_{2,k}| = 0, \quad \sum_{k=1}^{+\infty} \|\partial_\nu \varphi_{1,k} - \partial_\nu \varphi_{2,k}\|^2_{L^2(\Gamma)} < \infty \tag{1.2}$$

hold simultaneously. Then we have $dA_1 = dA_2$ and $V_1 = V_2$.

Let us observe that, as mentioned by [10, 16], Theorem 1.1 can be considered as a uniqueness theorem under the assumption that the spectral data are asymptotically "very close". Conditions (1.2) are similar to the one considered by [16] and they are weaker than the requirement that

$$|\lambda_{1,k} - \lambda_{2,k}| \leq Ck^{-\alpha}, \quad \|\partial_\nu \varphi_{1,k} - \partial_\nu \varphi_{2,k}\|_{L^2(\Gamma)} \leq Ck^{-\beta}$$

for some $\alpha > 1$ and $\beta > 1 - \frac{1}{2n}$, considered in [10, Theorem 2.1]. Note also that conditions (1.2) are weaker than the knowledge of the boundary spectral data with a finite number of data missing considered by [14].

Let us remark that there is an obstruction to uniqueness given by the gauge invariance of boundary spectral data for magnetic Shrödinger operators. More precisely, let $p \in \mathcal{C}_0^\infty(\Omega) \setminus \{0\}$ and assume that $A_1 = \nabla p + A_2 \neq A_2$, $V_1 = V_2$. Then, we have $H_1 = e^{-\mathrm{i}p} H_2 e^{\mathrm{i}p}$ and one can check that we can choose the spectral data of H_1 and H_2 in such a way that the conditions

$$\partial_\nu \varphi_{1,k|\Gamma} = \partial_\nu \varphi_{2,k|\Gamma}, \quad \lambda_{1,k} = \lambda_{2,k}, \quad k \in \mathbb{N}^*$$

is fulfilled. Therefore, conditions (1.1)-(1.2) are fulfilled but $H_1 \neq H_2$. Nevertheless, assuming (1.1) fulfilled, the conditions $dA_1 = dA_2$ and $V_1 = V_2$ imply that H_1 and H_2 are gauge equivalent. Therefore, Theorem 1.1 is equivalent to the unique determination of magnetic Schrödinger operators modulo gauge transformation from the asymptotic knowledge of the boundary spectral data given by conditions (1.2).

We stress out that the problem under examination in this text is a Borg-Levinson inverse problem for the magnetic Schrödinger operator $H = (-\mathrm{i}\nabla + A)^2 + V$. To our best knowledge, there are only two multi-dimensional Borg-Levinson uniqueness result for magnetic Schrödinger operators available in the mathematical literature, [15, Theorem B] and [25, Theorem 3.2]. In [15], the authors considered general magnetic Schrödinger operators with smooth coefficients on a smooth connected Riemanian manifold and they proved unique determination of this operator modulo gauge invariance from the knowledge of the boundary spectral data with a missing finite number of data. In [25], Serov treated this problem on a bounded domain of $\mathbb{R}^n$, and he proved that, for $A \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ and $V \in L^\infty(\Omega, \mathbb{R})$, the full boundary spectral data $\{(\lambda_k, \partial_\nu \varphi_{k|\Gamma}) : \ k \in \mathbb{N}^*\}$ determines uniquely dA and V. In contrast to [15, 25], in the present paper we prove that the asymptotic knowledge of the boundary spectral data, given by the conditions (1.2), is sufficient for the unique determination of dA and V. To our best knowledge, conditions (1.2) are the weakest conditions on boundary spectral data that guaranty uniqueness of magnetic Schrödinger operators modulo gauge transformation. Moreover, our uniqueness result is stated with conditions similar to [16, Theorem 1.4], which seems to be the most precise Borg-Levinson uniqueness result so far for Schrödinger operator without magnetic potential ($A = 0$).

The main ingredient in our analysis is a suitable representation that allows to express the magnetic potential A and the electric potential V in terms of Dirichlet-Neumann map associated to the equations $(-\mathrm{i}\nabla + A)^2 u + Vu - \lambda u = 0$ for some $\lambda \in \mathbb{C}$. In [10, 14, 16] the authors applied a similar approach to the Schrödinger operator $-\Delta + V$ with Dirichlet boundary condition[1]. Inspired by the construction of complex geometric optics solutions of [3, 11, 17, 18, 22, 24, 27], we prove that the approach of [10, 14, 16] can be extended to magnetic Schrödinger operators.

We believe that the approach developed in the present paper can be used for results of stability in the determination of the magnetic field dA and the electric potential V similar to [16, Theorem 1.3]. Indeed, following the strategy set in this paper we expect a stability estimate associated to the the determination of the magnetic field dA. The main issue comes from the stability in the determination of the electric potential V. Nevertheless, we expect that this problem can be solved by adapting suitably the argument developed in [28] related to the inversion of the d operator on differential forms restricted to the right subspaces.

1.4. **Outline.** This paper is organized as follows. In Section 2 we consider some useful preliminary results concerning solutions of equations of the form $(-\mathrm{i}\nabla + A)^2 u + Vu - \lambda u = 0$ for some $\lambda \in \mathbb{C} \setminus \sigma(H)$. In Section 3 we introduce a representation formula making the connection between Dirichlet-Neumann map associated to the previous equations and the couple (A, V) of magnetic and electric potential. Finally, in section 4 we combine all these results and we prove Theorem 1.1.

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[1]This argument was initially introduced by [14] and was inspired by the Born approximation method of the scattering theory.

2. Notations and preliminary results

We denote by $\langle f,\psi\rangle$ the duality between $\psi\in H^{1/2}(\Gamma)$ and f belonging to the dual $H^{-1/2}(\Gamma)$ of $H^{1/2}(\Gamma)$. However, when in $\langle f,\psi\rangle$ both f and ψ belong to $L^2(\Gamma)$, to make things simpler $\langle\cdot,\cdot\rangle$ can be interpreted as the scalar product of $L^2(\Gamma)$, namely

$$\langle f,\psi\rangle=\int_\Gamma f(\sigma)\,\overline{\psi(\sigma)}\,d\sigma.$$

We introduce the operator H defined as

$$Hu:=(-\mathrm{i}\nabla+A)^2u+Vu,\quad u\in D(H):=\left\{\psi\in H^1_0(\Omega)\ ;\ (-\mathrm{i}\nabla+A)^2\psi\in L^2(\Omega)\right\}. \tag{2.3}$$

Recall that H is associated to the quadratic form b given by

$$b(u,v)=\int_\Omega(-\mathrm{i}\nabla+A)u(x)\cdot\overline{(-\mathrm{i}\nabla+A)v(x)}\,dx+\int_\Omega V(x)\,u(x)\overline{v(x)}\,dx,\quad u,v\in H^1_0(\Omega).$$

Moreover, the spectrum of H is discrete and composed of the non decreasing sequence of eigenvalues denoted by $\sigma(H)=\{\lambda_k\ ;\ k\geq 1\}$. If we write $V=V^+-V^-$, with $V^\pm:=\max(0,\pm V)$, we have that the spectrum $\sigma(H)$ of H is contained into $[-\|V^-\|_{L^\infty(\Omega)},+\infty)$. According to [12, Theorem 2.2.2.3], we can show that $D(H)$ embedded continuously into $H^2(\Omega)$. Therefore the eigenfunctions $(\varphi_k)_{k\geq 1}$ of H, that form an Hilbertian basis, are lying in $H^2(\Omega)$ and we have $\partial_\nu\varphi_{k|\Gamma}\in H^{1/2}(\Gamma)$.

Lemma 2.1. *For any $f\in H^{1/2}(\Gamma)$ and $\lambda\in\mathbb{C}\setminus\sigma(H)$, there exists a unique solution $u\in H^1(\Omega)$ to the equation*

$$\left\{\begin{array}{rcll}(-\mathrm{i}\nabla+A)^2u+Vu-\lambda u&=&0,&\text{in }\Omega,\\ u(x)&=&f(x),&x\in\Gamma,\end{array}\right. \tag{2.4}$$

which can be written as

$$u_\lambda:=u=\sum_{k\geq 1}\frac{\alpha_k}{\lambda-\lambda_k}\,\varphi_k, \tag{2.5}$$

where for convenience we set

$$h_k:=\partial_\nu\varphi_{k|\Gamma},\qquad\text{and}\qquad\alpha_k:=\langle f,h_k\rangle. \tag{2.6}$$

Moreover

$$\|u_\lambda\|^2_{L^2(\Omega)}=\sum_{k\geq 1}\frac{|\alpha_k|^2}{|\lambda-\lambda_k|^2}\to 0\qquad\text{as }\lambda\to-\infty.$$

Finally, there exists $\mu_*<0$ such that $(-\infty,\mu_*)\subset\mathbb{C}\setminus\sigma(H)$ and

$$\sup_{\lambda\leq\mu_*}\|u_\lambda\|_{H^1(\Omega)}<\infty.$$

Proof. Since $\lambda\notin\sigma(H)$, one can easily check that (2.4) admits a unique solution $u_\lambda\in H^1(\Omega)$. Moreover, u_λ can be written in terms of the eigenvalues and eigenfunctions λ_k,φ_k. Indeed, $u_\lambda\in L^2(\Omega)$ can be expressed in the Hilbert basis $(\varphi_k)_{k\geq 1}$ as

$$u_\lambda=\sum_{k\geq 1}(u_\lambda|\varphi_k)\,\varphi_k$$

with $(\cdot,\cdot)$ the scalar product with respect to $L^2(\Omega)$. Since $u_\lambda\in H^1(\Omega)$ and $\Delta u_\lambda=-2iA\cdot\nabla u_\lambda+(-\mathrm{i}\,div(A)+|A|^2+V)u_\lambda\in L^2(\Omega)$, we have $\nabla u_\lambda\in H_{div}(\Omega)=\{v\in L^2(\Omega;\mathbb{R}^n):\ div(v)\in L^2(\Omega)\}$. Thus, taking the scalar product of the first equation in (2.4) with φ_k and applying the Green formula we obtain

$$\langle f,h_k\rangle=(\lambda-\lambda_k)\,(u|\varphi_k),$$

which yields the expression given by (2.5). The fact that $\|u_\lambda\| \to 0$ as $\lambda \to -\infty$ is a consequence of the fact that we may fix $c_0 > \|V\|_{L^\infty(\Omega)}$ large enough so that if λ is real and such that $\lambda \leq -c_0$, we have $|\lambda - \lambda_k|^2 \geq |c_0 - \lambda_k|^2$ for all $k \geq 1$, and thus

$$\frac{|\alpha_k|^2}{|\lambda - \lambda_k|^2} \leq \frac{|\alpha_k|^2}{|c_0 - \lambda_k|^2},$$

so that we may apply Lebesgue's dominated convergence as $\lambda \to -\infty$.

Now let us consider the last part of the lemma. Since $\nabla u_\lambda \in H_{div}(\Omega)$, multiplying (2.4) by $\overline{u_\lambda}$ and applying the Green formula we obtain

$$\int_\Omega |(-\mathrm{i}\nabla + A)u_\lambda|^2 \, dx + \int_\Omega (V - \lambda) |u_\lambda|^2 \, dx = - \langle \partial_\nu u_\lambda + \mathrm{i}(A \cdot \nu) f, f \rangle . \tag{2.7}$$

Then, since $A \in W^{1,\infty}(\Omega, \mathbb{R}^n) \subset \mathcal{C}(\overline{\Omega}, \mathbb{R}^n)$ and $f \in H^{\frac{1}{2}}(\Gamma)$, we get

$$\langle A \cdot \nu f, f \rangle = \int_\Gamma (A(x) \cdot \nu(x)) |f(x)|^2 \, d\sigma(x) \in \mathbb{R}.$$

Thus, for $\lambda \in \mathbb{R} \setminus \sigma(H)$, taking the real part of (2.7), we find

$$\int_\Omega |(-\mathrm{i}\nabla + A)u_\lambda|^2 \, dx + \int_\Omega (V - \lambda) |u_\lambda|^2 \, dx = -\Re \langle \partial_\nu u_\lambda, f \rangle \leq \|\partial_\nu u_\lambda\|_{H^{-\frac{1}{2}}(\Gamma)} \|f\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \|u_\lambda\|_{H^1(\Omega)} \|f\|_{H^{\frac{1}{2}}(\Gamma)} \tag{2.8}$$

with C a constant depending only on Ω. On the other hand, we have

$$\int_\Omega |(-\mathrm{i}\nabla + A)u_\lambda|^2 \, dx + \int_\Omega (V - \lambda) |u_\lambda|^2 \, dx \geq \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 \, dx - \Big[(32 \|A\|_{L^\infty(\Omega, \mathbb{R}^n)} + \|V\|_{L^\infty(\Omega)}) + \lambda \Big] \int_\Omega |u_\lambda|^2 \, dx.$$

Therefore, choosing $\mu_* = -(32 \|A\|_{L^\infty(\Omega, \mathbb{R}^n)} + \|V\|_{L^\infty(\Omega)}) - \frac{1}{2}$, we have $(-\infty, \mu_*) \subset \mathbb{R} \setminus \sigma(H)$ and for $\lambda < \mu_*$ we obtain

$$\int_\Omega |(-\mathrm{i}\nabla + A)u_\lambda|^2 \, dx + \int_\Omega (V - \lambda) |u_\lambda|^2 \, dx \geq \frac{\|u_\lambda\|^2_{H^1(\Omega)}}{2}.$$

Combining this estimate with (2.8) we get that

$$\|u_\lambda\|_{H^1(\Omega)} \leq 2C \|f\|_{H^{\frac{1}{2}}(\Gamma)}, \quad \lambda < \mu_*.$$

This completes the proof the lemma since the right hand side of this inequality is independent of λ.

□

It is clear that the series (2.5) giving u_λ in terms of α_k, λ_k and φ_k, converges only in $L^2(\Omega)$ and thus we cannot deduce an expression of the normal derivative $\partial_\nu u_\lambda$ in terms of α_k, λ_k and h_k. To avoid this difficulty, in a similar way to [16], we have the following lemma:

Lemma 2.2. *Let $f \in H^{1/2}(\Gamma)$ be fixed and for $\lambda, \mu \in \mathbb{C} \setminus \sigma(H)$ let u_λ and u_μ be the solutions given by Lemma 2.1. If we set $v := v_{\lambda,\mu} := u_\lambda - u_\mu$, then*

$$\partial_\nu v = \sum_k \frac{(\mu - \lambda)\alpha_k}{(\lambda - \lambda_k)(\mu - \lambda_k)} h_k , \tag{2.9}$$

the convergence taking place in $H^{1/2}(\Gamma)$.

Proof. Let $v_{\lambda,\mu} := u_\lambda - u_\mu$; One verifies that $v_{\lambda,\mu}$ solves

$$\begin{cases} (-\mathrm{i}\nabla + A)^2 v_{\lambda,\mu} + V v_{\lambda,\mu} - \lambda v_{\lambda,\mu} &= (\lambda - \mu) u_\mu, & \text{in } \Omega, \\ v_{\lambda,\mu}(\sigma) &= 0, & \sigma \in \Gamma. \end{cases} \tag{2.10}$$

Since $(u_\mu|\varphi_k) = \alpha_k/(\mu - \lambda_k)$, it follows that

$$v_{\lambda,\mu} = \sum_k \frac{(\lambda-\mu)\alpha_k}{(\lambda_k-\lambda)(\mu-\lambda_k)}\,\varphi_k,$$

the convergence taking place in $D(H)$. Since the operator $v \mapsto \partial_\nu v$ is continuous from $D(H)$ into $H^{1/2}(\Gamma)$, the result of the lemma follows. □

The next lemma states essentially that if for $j = 1$ or $j = 2$ we have two magnetic potentials A_j, two electric potentials V_j and $u_j := u_{j,\mu}$ solutions of

$$\left\{\begin{array}{rcll} (-\mathrm{i}\nabla + A_j)^2 u_j + V_j u_j - \mu u_j & = & 0, & \text{in } \Omega, \\ u_j(x) & = & f(x), & x \in \Gamma, \end{array}\right. \tag{2.11}$$

then $u_{1,\mu}$ and $u_{2,\mu}$ are *close* as $\mu \to -\infty$: in some sense the influence of the potentials A_j and V_j are dimmed when $\mu \to -\infty$. More precisely we have:

Lemma 2.3. *Let $V_j \in L^\infty(\Omega,\mathbb{R})$ and $A_j \in W^{1,\infty}(\Omega,\mathbb{R}^n)$ be given for $j = 1$ or $j = 2$, and denote by H_j the corresponding operator defined by* (2.3)*. For $f \in H^{1/2}(\Gamma)$ and $\mu \in (-\infty,\mu_*) \subset \mathbb{C}\setminus\sigma(H)$, let $u_{j,\mu} := u_j$ be the solution of* (2.11)*. Then, for any $0 < \varepsilon < 1/4$, $z_\mu := u_{1,\mu} - u_{2,\mu} \in H^2(\Omega)$ converge to 0 in $H^{2(1-\varepsilon)}(\Omega)$ as $\mu \to -\infty$. In particular $\partial_\nu z_\mu \to 0$ in $L^2(\Gamma)$ as $\mu \to -\infty$.*

Proof. Since the trace map $v \mapsto \partial_\nu v$ is continuous from $H^{2(1-\varepsilon)}(\Omega)$ to $L^2(\Gamma)$ (e.g. Theorem 9.4 in Chapter 4 of [20]), it is enough to show that $z_\mu \in H^2(\Omega)$ and $\|z_\mu\|_{H^{2(1-\varepsilon)}(\Omega)} \to 0$ when $\mu \to -\infty$. We fix $\mu < \mu_*$ with $\mu_* < 0$ the minimal value of the constant given by Lemma 2.1 for $A = A_j$, $V = V_j$, $j = 1, 2$. Without lost of generality we assume that $H_j - \mu_*$ is strictly positive, $j = 1, 2$. One verifies that z_μ solves the equation

$$\left\{\begin{array}{rcll} (-\mathrm{i}\nabla + A_1)^2 z_\mu + V_1 z_\mu - \mu z_\mu & = & -2\mathrm{i}(A_2 - A_1)\cdot\nabla u_{2,\mu} + (p_2 - p_1)u_{2,\mu}, & \text{in } \Omega, \\ z_\mu(\sigma) & = & 0, & \sigma \in \Gamma, \end{array}\right. \tag{2.12}$$

with $p_j = -\mathrm{i}div(A_j) + |A_j|^2 + V_j$, $j = 1, 2$. That is, denoting by $R_{1,\mu} = (H_1 - \mu I)^{-1}$ the resolvent of the operator $H_1 := (-\mathrm{i}\nabla + A_1)^2 + V_1$, we have

$$z_\mu = R_{1,\mu}(-2\mathrm{i}(A_2 - A_1)\cdot\nabla u_{2,\mu} + (p_2 - p_1)u_{2,\mu}) = \sum_{k=1}^{+\infty} \frac{(w_\mu, \varphi_{1,k})}{(\lambda_{1,k} - \mu)}\varphi_{1,k}$$

with $w_\mu = -2\mathrm{i}(A_2 - A_1)\cdot\nabla u_{2,\mu} + (p_2 - p_1)u_{2,\mu}$ and $(\lambda_{1,k})_{k\geq 1}$, $(\varphi_{1,k})_{k\geq 1}$ respectively the eigenvalues of H_1 and an Hilbertian basis of eigenfunctions associated to these eigenvalues. Since $w_\mu \in L^2(\Omega)$, z_μ is lying in $D(H_1)$ and by the same way in $H^2(\Omega)$. It remains to show that $\|z_\mu\|_{H^{2(1-\varepsilon)}(\Omega)} \to 0$ when $\mu \to -\infty$. Since $D(H_1)$ embedded continuously into $H^2(\Omega)$ there exists a constant C_2 depending on A_1, V_1 and Ω such that

$$\|v\|^2_{H^2(\Omega)} \leq C_2 \sum_{k=1}^{\infty} |\lambda_{1,k} - \mu_*|^2 \, |(v, \varphi_{1,k})|^2, \quad v \in D(H_1).$$

In a same way, one can find a constant C_1 depending on A_1, V_1 and Ω such that

$$\|v\|^2_{H^1(\Omega)} \leq C_1 \sum_{k=1}^{\infty} |\lambda_{1,k} - \mu_*| \, |(v, \varphi_{1,k})|^2, \quad v \in D(H_1).$$

Therefore, by interpolation (e.g. Theorem 5.1 in Chapter 1 of [20]), there exists a constant $C_{2(1-\varepsilon)}$ depending on ε, A_1, V_1 and Ω such that

$$\|v\|^2_{H^{2(1-\varepsilon)}(\Omega)} \leq C_{2(1-\varepsilon)} \sum_{k=1}^{\infty} |\lambda_{1,k} - \mu_*|^{2(1-\varepsilon)} \, |(v, \varphi_{1,k})|^2, \quad v \in D(H_1).$$

It follows

$$\|z_\mu\|^2_{H^{2(1-\varepsilon)}(\Omega)} \leq C_{2(1-\varepsilon)} \sum_{k=1}^{\infty} \frac{|\lambda_{1,k}-\mu_*|^{2(1-\varepsilon)} |(w_\mu, \varphi_{1,k})|^2}{|\lambda_{1,k}-\mu|^2} \leq C_{2(1-\varepsilon)} \frac{\|w_\mu\|^2_{L^2(\Omega)}}{|\mu-\mu_*|^{2\varepsilon}}. \tag{2.13}$$

On the other hand, we have

$$\|w_\mu\|^2_{L^2(\Omega)} \leq C \|u_{2,\mu}\|_{H^1(\Omega)}$$

with C independent of μ. Then, according to Lemma 2.1, we obtain

$$\sup_{\mu\leq\mu_*} \|w_\mu\|^2_{L^2(\Omega)} \leq C \sup_{\mu\leq\mu_*} \|u_{2,\mu}\|_{H^1(\Omega)} < \infty.$$

Combining this with estimate (2.13) we find

$$\|z_\mu\|^2_{H^{2(1-\varepsilon)}(\Omega)} \leq C_{2(1-\varepsilon)} \frac{\sup_{\lambda\leq\mu_*} \|w_\lambda\|^2_{L^2(\Omega)}}{|\mu-\mu_*|^{2\varepsilon}}.$$

We complete the proof by remarking that the right hand side of this inequality converge to 0 as $\mu \to -\infty$. □

3. A representation formula

From now on, for all $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, we denote by $x \cdot y$ the quantity

$$x \cdot y = \sum_{k=1}^{n} x_k y_k$$

and for all $x \in \mathbb{R}^n$ we denote by $x^\perp$ the subspace of $\mathbb{R}^n$ defined by $\{y \in \mathbb{R}^n : \ y \cdot x = 0\}$.

For $j = 1, 2$, we associate to the problem

$$\begin{cases} (-\mathrm{i}\nabla + A_j)^2 u_j + V_j u_j - \lambda u_j &= 0, \quad \text{in } \Omega, \\ u_j(x) &= f(x), \quad x \in \Gamma \end{cases} \tag{3.14}$$

the Dirichlet-Neumann map

$$\Lambda_{j,\lambda} : H^{\frac{1}{2}}(\partial\Omega) \ni f \mapsto (\partial_\nu + \mathrm{i}A_j \cdot \nu) u_{j,\lambda|\Gamma},$$

where $u_{j,\lambda}$ solves (3.14).

The goal of this section is to apply the Dirichlet-Neumann maps $\Lambda_{j,\lambda}$ to some suitable ansatz in order to get a representation formula involving the magnetic potentials A_j and the electric potentials V_j, $j = 1, 2$. A similar approach was considered by [10, 14, 16]. All these authors considered this representation formula for Schrödinger operators $-\Delta + V$ with an electric potential V, in other words for Schrödinger operators with a variable coefficient of order zero. In our case we need to extend this strategy to Schrödinger operators with both magnetic and electric potentials, which means an extension to Schrödinger operators with variable coefficients of order zero and one. Therefore, in accordance with results related to the determination of magnetic Schrödinger operators from boundary measurements (e.g. [3, 11, 17, 18, 22, 24, 27]), we consider some ansatzs of the form

$$\Phi_{j,\lambda}(x) = e^{\zeta_j \cdot x} g_j(x), \quad \zeta_j \in \mathbb{C}^n, \ x \in \Omega, \ j = 1, 2 \tag{3.15}$$

with ζ_j chosen in such way that $(-\Delta - \lambda)e^{\zeta_j \cdot x} = 0$ and with g_1 and g_2 respectively a solution of

$$\zeta_1 \cdot \nabla g_1 + (\mathrm{i}\zeta_1 \cdot A_{1,\sharp}) g_1 = 0, \quad \zeta_2 \cdot \nabla g_2 - (\mathrm{i}\zeta_2 \cdot A_{2,\sharp}) g_2 = 0$$

with $A_{j,\sharp}$ some smooth functions close to the magnetic potential A_j, $j = 1, 2$. More precisely, we fix $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $\eta_1, \eta_2 \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n, \ |y| = 1\}$ and $A_{j,\sharp} \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^n)$, $j = 1, 2$, and we define the ansatzs

$$\Phi_{1,\lambda}(x) := e^{\mathrm{i}\sqrt{\lambda}\eta_1 \cdot x} e^{\mathrm{i}\psi_1(x)}, \quad \Phi_{2,\lambda}(x) := e^{-\mathrm{i}\sqrt{\lambda}\eta_2 \cdot x} b_2(x) e^{-\mathrm{i}\psi_2(x)}, \quad x \in \Omega, \tag{3.16}$$

where ψ_j is a solution lying in $W^{2,\infty}(\Omega)$ of

$$\eta_j \cdot \nabla\psi_j(x) = -\eta_j \cdot A_{j,\sharp}, \quad j = 1, 2$$

and $b_2 \in W^{2,\infty}(\mathbb{R}^n)$ satisfies $\eta_2 \cdot \nabla b_2 = 0$. In the construction of our ansatzs we consider some smooth approximation of the magnetic potentials instead of the magnetic potentials to obtain sufficiently smooth functions $\Phi_{j,\lambda}$, $j = 1, 2$. Further, for $j = 1, 2$, we put

$$S_j(\lambda, \eta_1, \eta_2) = \left\langle \Lambda_{j,\lambda}\Phi_{1,\lambda}, \overline{\Phi_{2,\lambda}} \right\rangle = \int_\Gamma (\Lambda_{j,\lambda}\Phi_{1,\lambda})\Phi_{2,\lambda}(x)d\sigma(x). \tag{3.17}$$

In other words we apply $\Lambda_{j,\lambda}$, $j = 1, 2$, to ansatzs of the form (3.15) with $\zeta_1 = \mathrm{i}\sqrt{\lambda}\eta_1$, $\zeta_2 = -\mathrm{i}\sqrt{\lambda}\eta_2$, $g_1 = e^{\mathrm{i}\psi_1}$ and $g_2 = b_2 e^{-\mathrm{i}\psi_2}$. From now on, for the sake of simplicity we will systematically omit the subscripts λ in $\Phi_{j,\lambda}$, $j = 1, 2$, in the remaining of this text. In view of determining the behavior of S_j, $j = 1, 2$, as $\Im\lambda \to +\infty$ we first establish the following lemma.

Lemma 3.1. *For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\eta_j \in \mathbb{S}^{n-1}$, $j = 1, 2$, the scalar products $S_j(\lambda, \eta_1, \eta_2)$ have the following expression*

$$\begin{aligned}
&S_1(\lambda, \eta_1, \eta_2)\\
&= 2\sqrt{\lambda}\int_\Omega \eta_2 \cdot (A_1 - A_{2,\sharp})e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} b_2 e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}dx\\
&+ \int_\Omega (V_1 - q_{12})e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} b_2 e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}dx\\
&-\mathrm{i}\int_\Gamma e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}(b_2\sqrt{\lambda}\eta_2 + b_2\nabla\psi_2 + \mathrm{i}\nabla b_2 + b_2 A_1)\cdot \nu d\sigma(x)\\
&- \int_\Omega \left[(H_1-\lambda)^{-1}\left(2\sqrt{\lambda}\eta_1\cdot(A_1 - A_{1,\sharp}) + q_{11}\right)\Phi_1\right]\left(2\sqrt{\lambda}\eta_2\cdot(A_1 - A_{2,\sharp})b_2 + V_1 b_2 - q_{12}\right)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{-\mathrm{i}\psi_2}dx,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&S_2(\lambda, \eta_1, \eta_2)\\
&= \int_\Omega \left[2\sqrt{\lambda}\eta_2\cdot(A_2 - A_{2,\sharp}) + V_2 - q_{22}\right]e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} b_2 e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}dx\\
&-\mathrm{i}\int_{\partial\Omega} e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}(b_2\sqrt{\lambda}\eta_2 + b_2\nabla\psi_2 + \mathrm{i}\nabla b_2 + b_2 A_2)\cdot \nu d\sigma(x)\\
&- \int_\Omega \left[(H_2-\lambda)^{-1}\left(2\sqrt{\lambda}\eta_1\cdot(A_2 - A_{1,\sharp}) + q_{21}\right)\Phi_1\right](2\sqrt{\lambda}\eta_2\cdot(A_2 - A_{2,\sharp})b_2 + V_2 b_2 - q_{22})e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{-\mathrm{i}\psi_2}dx.
\end{aligned} \tag{3.19}$$

Here we denote by q_{11}, q_{12}, q_{21}, q_{22} the expressions

$$q_{11} = -\mathrm{i}divA_1 + |A_1|^2 + V_1(x) + 2A_1\cdot\nabla\psi_1 - \mathrm{i}\Delta\psi_1 + |\nabla\psi_1|^2,$$

$$q_{12} = \Delta b_2 - 2\mathrm{i}\nabla\psi_2\cdot\nabla b_2 - 2\mathrm{i}\nabla b_2\cdot A_1 + \left(-\mathrm{i}\Delta\psi_2 - |\nabla\psi_2|^2 - 2\nabla\psi_2\cdot A_1 - \mathrm{i}divA_1 - |A_1|^2\right)b_2,$$

$$q_{21} = -\mathrm{i}divA_2 + |A_2|^2 + V_2(x) + 2A_2\cdot\nabla\psi_1 - \mathrm{i}\Delta\psi_1 + |\nabla\psi_1|^2,$$

$$q_{22} = \Delta b_2 - 2\mathrm{i}\nabla\psi_2\cdot\nabla b_2 - 2\mathrm{i}\nabla b_2\cdot A_2 + \left(-\mathrm{i}\Delta\psi_2 - |\nabla\psi_2|^2 - 2\nabla\psi_2\cdot A_2 - \mathrm{i}divA_2 - |A_2|^2\right)b_2.$$

Moreover, H_j, $j = 1, 2$, denotes the selfadjoint operator $(-\mathrm{i}\nabla + A_j) + V_j$ acting on $L^2(\Omega)$ with domain

$$D(H_j) = \{v \in H_0^1(\Omega) : \ (-\mathrm{i}\nabla + A_j)v \in L^2(\Omega)\}.$$

Proof. We start with the expression of $S_1(\lambda, \eta_1, \eta_2)$. Recall that

$$(-\mathrm{i}\nabla + A_1)^2 u + V_1 u - \lambda u = -\Delta u - 2\mathrm{i}A_1\cdot\nabla u + qu - \lambda u$$

with $q(x) = -\mathrm{i}div A_1(x) + |A_1(x)|^2 + V_1(x)$. Therefore, in light of (3.16) we have

$$\begin{aligned}&(-\mathrm{i}\nabla + A_1)^2\Phi_1 + V_1\Phi_1 - \lambda\Phi_1\\ &= (\lambda + 2\sqrt{\lambda}\eta_1\cdot\nabla\psi_1 - \mathrm{i}\Delta\psi_1 + |\nabla\psi_1|^2)\Phi_1 + (2\sqrt{\lambda}\eta_1\cdot A_1 + 2A_1\cdot\nabla\psi_1)\Phi_1 + q\Phi_1 - \lambda\Phi_1\\ &= 2\sqrt{\lambda}(\eta_1\cdot\nabla\psi_1 + \eta_1\cdot A_1)\Phi_1 + q_{11}\Phi_1\end{aligned}$$

with $q_{11} = q + 2A_1\cdot\nabla\psi_1 - \mathrm{i}\Delta\psi_1 + |\nabla\psi_1|^2$. On the other hand, since ψ_1 satisfies $\eta_1\cdot\nabla\psi_1 + \eta_1\cdot A_{1,\sharp} = 0$, we deduce that

$$(-\mathrm{i}\nabla + A_1)^2\Phi_1 + V_1\Phi_1 - \lambda\Phi_1 = \left[2\sqrt{\lambda}\eta_1\cdot(A_1 - A_{1,\sharp}) + q_{11}\right]\Phi_1. \tag{3.20}$$

Now consider u_1 the solution of

$$\begin{cases} (-\mathrm{i}\nabla + A_1)^2 u_1 + V_1 u_1 - \lambda u_1 &= 0, & \text{in } \Omega,\\ u_1(x) &= \Phi_1(x), & x\in\partial\Omega.\end{cases}$$

Note that, with our assumptions one can check that $D(H_1) = H_0^1(\Omega)\cap H^2(\Omega)$. In view of (3.20), we can split u_1 into two terms $u_1 = \Phi_1 + v_1$ with v_1 the solution of

$$\begin{cases} (-\mathrm{i}\nabla + A_1)^2 v_1 + V_1 v_1 - \lambda v_1 &= -\left[2\sqrt{\lambda}\eta_1\cdot(A_1 - A_{1,\sharp}) + q_{11}\right]\Phi_1, & \text{in } \Omega,\\ v_1(x) &= 0, & x\in\partial\Omega.\end{cases}$$

Then, we have

$$u_1 = \Phi_1 - (H_1 - \lambda)^{-1}\left[2\sqrt{\lambda}\eta_1\cdot(A_1 - A_{1,\sharp}) + q_{11}\right]\Phi_1. \tag{3.21}$$

Further, as

$$S_1 = \int_{\partial\Omega}(\partial_\nu + \mathrm{i}A_1\cdot\nu)u_1(x)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2(x)}d\sigma(x), \tag{3.22}$$

from (3.17), and $(\nabla + \mathrm{i}A_1)u_1 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{\mathrm{i}\psi_2}\in H(div,\Omega)$, we get

$$\begin{aligned}S_1 &= \int_\Omega div\left((\nabla + \mathrm{i}A_1(x))u_1(x)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2(x)}\right)dx\\ &= \int_\Omega(\nabla + \mathrm{i}A_1)^2 u_1 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2}dx + \int_\Omega(\nabla + \mathrm{i}A_1)u_1\cdot(\nabla - \mathrm{i}A_1)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2}dx\end{aligned} \tag{3.23}$$

by applying Stokes formula. Doing the same with $(\nabla - \mathrm{i}A_1)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{\mathrm{i}\psi_2}\in H(div;\Omega)$ we find out that

$$\begin{aligned}&\int_\Omega(\nabla + \mathrm{i}A_1)u_1\cdot(\nabla - \mathrm{i}A_1)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2}dx\\ =\ & -\mathrm{i}\int_\Gamma u_1(x)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{-\mathrm{i}\psi_2}(\sqrt{\lambda}b_2\eta_2 + b_2\nabla\psi_2 + \mathrm{i}\nabla b_2 + b_2 A_1)\cdot\nu d\sigma(x)\\ & -\int_\Omega u_1(x)(\nabla - \mathrm{i}A_1)^2 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2}dx.\end{aligned}$$

In light of (3.16) and the identity $u_{1|\Gamma} = \varphi_1$, this entails

$$\begin{aligned}&\textstyle\int_\Omega(\nabla + \mathrm{i}A_1)u_1\cdot(\nabla - \mathrm{i}A_1)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2}dx\\ &= -\mathrm{i}\textstyle\int_\Gamma e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x}e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}(\sqrt{\lambda}b_2\eta_2 + b_2\nabla\psi_2 + \mathrm{i}\nabla b_2 + b_2 A_1)\cdot\nu d\sigma(x)\\ &\quad -\textstyle\int_\Omega u_1(x)(\nabla - \mathrm{i}A_1)^2 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2}dx.\end{aligned}$$

Moreover, one can check that

$$(\nabla - \mathrm{i}A_1)^2 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2} = \left(-\lambda b_2 - 2\sqrt{\lambda}(\eta_2\cdot\nabla\psi_2 + A_1\cdot\eta_2)b_2 - 2\mathrm{i}\sqrt{\lambda}\eta_2\cdot\nabla b_2 + q_{12}\right)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{-\mathrm{i}\psi_2}$$

with $q_{12} = \Delta b_2 - 2\mathrm{i}\nabla\psi_2\cdot\nabla b_2 - 2\mathrm{i}\nabla b_2\cdot A_1 + \left(-\mathrm{i}\Delta\psi_2 - |\nabla\psi_2|^2 - 2\nabla\psi_2\cdot A_1 - \mathrm{i}div A_1 - |A_1|^2\right)b_2$. Combining this with the fact that ψ_2 satisfies $\eta_2\cdot\nabla\psi_2 + \eta_2\cdot A_{2,\sharp} = 0$ and b_2 solves $\eta_2\cdot\nabla b_2 = 0$, we deduce that

$$(\nabla - \mathrm{i}A_1)^2 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2 e^{-\mathrm{i}\psi_2} = \left([-\lambda - 2\sqrt{\lambda}\eta_2\cdot(A_1 - A_{2,\sharp})]b_2 + q_{12}\right)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{-\mathrm{i}\psi_2}.$$

Therefore, we find

$$\begin{aligned}&\int_\Omega(\nabla+\mathrm{i}A_1)u_1\cdot(\nabla-\mathrm{i}A_1)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx\\&=-\mathrm{i}\int_\Gamma e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x}e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}(\sqrt{\lambda}b_2\eta_2+b_2\nabla\psi_2+\mathrm{i}\nabla b_2+b_2A_1)\cdot\nu d\sigma(x)\\&\quad-\int_\Omega u_1(x)\left(-\lambda b_2-2\sqrt{\lambda}\eta_2\cdot(A_1-A_{2,\sharp})b_2+q_{12}\right)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{-\mathrm{i}\psi_2}dx.\end{aligned}$$

Combining this with (3.21) we find

$$\begin{aligned}&\int_\Omega(\nabla+\mathrm{i}A_1)u_1\cdot(\nabla-\mathrm{i}A_1)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx\\&=-\mathrm{i}\int_\Gamma e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x}e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}(\sqrt{\lambda}b_2\eta_2+b_2\nabla\psi_2+\mathrm{i}\nabla b_2+b_2A_1)\cdot\nu d\sigma(x)\\&\quad+\lambda\int_\Omega u_1e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx+2\sqrt{\lambda}\int_\Omega\eta_2\cdot(A_1-A_{2,\sharp})e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x}b_2e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}dx\\&\quad-\int_\Omega q_{12}e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x}b_2e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}dx\\&\quad-\int_\Omega[(H_1-\lambda)^{-1}\left(2\sqrt{\lambda}\eta_1\cdot(A_1-A_{1,\sharp})+q_{11}\right)\Phi_1]\left(2\sqrt{\lambda}\eta_2\cdot(A_1-A_{2,\sharp})b_2-q_{12}\right)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{-\mathrm{i}\psi_2}dx.\end{aligned}\tag{3.24}$$

Next, taking into account the fact that $(\nabla+\mathrm{i}A_1)^2u_1=(V_1-\lambda)u_1$ in Ω, we obtain

$$\begin{aligned}\int_\Omega(\nabla+\mathrm{i}A_1)^2u_1e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx&=\int_\Omega(V_1-\lambda)u_1e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx\\&=-\lambda\int_\Omega u_1e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx+\int_\Omega V_1e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x}b_2e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}dx\\&\quad-\int_\Omega V_1\left[(H_1-\lambda)^{-1}\left(2\sqrt{\lambda}\eta_1\cdot(A_1-A_{1,\sharp})+q_{11}\right)\Phi_1\right]e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx.\end{aligned}$$

Finally, we deduce (3.18) from (3.23)-(3.24). In the same way we have

$$(-\mathrm{i}\nabla+A_2)^2\Phi_1+V_2\Phi_1-\lambda\Phi_1=2\sqrt{\lambda}(\eta_1\cdot\nabla\psi_1+A_2\cdot\eta_1)\Phi_1+q_{21}\Phi_1$$

with $q_{21}=-\mathrm{i}divA_2+|A_2|^2+V_2(x)+2A_2\cdot\nabla\psi_1-\mathrm{i}\Delta\psi_1+|\nabla\psi_1|^2$. Then, since ψ_1 is a solution of $\eta_1\cdot\nabla\psi_1+\eta_1\cdot A_{1,\sharp}=0$, we deduce that

$$(-\mathrm{i}\nabla+A_2)^2\Phi_1+V_2\Phi_1-\lambda\Phi_1=\left(2\sqrt{\lambda}\eta_1\cdot(A_2-A_{1,\sharp})+q_{21}\right)\Phi_1.$$

Moreover, the solution u_2 of

$$\begin{cases}(-\mathrm{i}\nabla+A_2)^2u_2+V_2u_2-\lambda u_2 &= 0, & \text{in }\Omega,\\ u_2(x) &= \Phi_1(x), & x\in\partial\Omega\end{cases}$$

is given by

$$u_2=\Phi_1-(H_2-\lambda)^{-1}\left(2\sqrt{\lambda}\eta_1\cdot(A_2-A_{1,\sharp})+q_{21}\right)\Phi_1.\tag{3.25}$$

Repeating our previous arguments, we deduce

$$S_2=\int_\Omega(\nabla+\mathrm{i}A_2)^2u_2e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx+\int_\Omega(\nabla+\mathrm{i}A_2)u_2\cdot(\nabla-\mathrm{i}A_2)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx.\tag{3.26}$$

On the other hand, using the fact that ψ_2 is a solution of the equation $\eta_2\cdot\nabla\psi_2+\eta_2\cdot A_{2,\sharp}=0$, we get

$$\begin{aligned}&\int_\Omega(\nabla+\mathrm{i}A_2)u_2\cdot(\nabla-\mathrm{i}A_2)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}dx\\&=-\mathrm{i}\int_\Gamma e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x}e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}(\sqrt{\lambda}b_2\eta_2+b_2\nabla\psi_2+\mathrm{i}\nabla b_2+b_2A_2)\cdot\nu d\sigma(x)\\&\quad-\int_\Omega u_2(x)\left(-\lambda b_2-2\sqrt{\lambda}\eta_2\cdot(A_2-A_{2,\sharp})b_2+q_{22}\right)e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}e^{-\mathrm{i}\psi_2}dx\end{aligned}$$

with $q_{22}=\Delta b_2-2\mathrm{i}\nabla\psi_2\cdot\nabla b_2-2\mathrm{i}\nabla b_2\cdot A_2+\left(-\mathrm{i}\Delta\psi_2-|\nabla\psi_2|^2-2\nabla\psi_2\cdot A_2-\mathrm{i}divA_2-|A_2|^2\right)b_2$. Combining this with (3.25)-(3.26) and repeating our previous arguments we obtain (3.19). □

In order to get a suitable expression of the functions ψ_j, $A_{j,\sharp}$, we first need to extend identically the magnetic potentials A_j, $j=1,2$.

Lemma 3.2. *Assume that condition* (1.1) *is fulfilled and let* $\tilde{\Omega}$ *be an open bounded set of* $\mathbb{R}^n$ *such that* $\overline{\Omega}\subset\tilde{\Omega}$. *Then, there exist* $\tilde{A}_j\in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$ *with* $\mathrm{supp}\tilde{A}_j\subset\tilde{\Omega}$, $j=1,2$, *such that*

$$\tilde{A}_1(x)=\begin{cases} A_1(x), \text{ for } x\in\Omega,\\ \tilde{A}(x), \text{ for } x\in\tilde{\Omega}\setminus\Omega,\end{cases},\quad \tilde{A}_2(x)=\begin{cases} A_2(x), \text{ for } x\in\Omega,\\ \tilde{A}(x), \text{ for } x\in\tilde{\Omega}\setminus\Omega,\end{cases} \tag{3.27}$$

and

$$\left\|\tilde{A}_j\right\|_{W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)}\leq C\max_{j=1,2}\left\|A_j\right\|_{W^{1,\infty}(\Omega,\mathbb{R}^n)}, \tag{3.28}$$

where C *depends only on* Ω *and* $\tilde{\Omega}$.

Proof. In light of Theorem 5 page 181 of [26], we first define $\tilde{A}_1\in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, with $\mathrm{supp}\tilde{A}_1\subset\tilde{\Omega}$, satisfying (3.27) and (3.28). Then, we define $\tilde{A}_2$ by

$$\tilde{A}_2(x)=\begin{cases} A_2(x), \text{ for } x\in\Omega,\\ \tilde{A}_1(x), \text{ for } x\in\tilde{\Omega}\setminus\Omega.\end{cases}$$

In view of (1.1), it is clear that $\tilde{A}_2\in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$ and that (3.28) is fulfilled for $j=2$. □

From now on we set $\tilde{\Omega}$ a bounded open set of $\mathbb{R}^n$ such that $\overline{\Omega}\subset\tilde{\Omega}$ and $\tilde{A}_j\in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $j=1,2$, supported on $\tilde{\Omega}$ and satisfying (3.27)-(3.28). Without lost of generality, we assume that $\mathrm{Diam}(\tilde{\Omega})=2\mathrm{Diam}(\Omega)$. We now introduce the following quantities: we consider an arbitrary $\xi\in\mathbb{R}^n\setminus\{0\}$ and pick $\eta\in\mathbb{S}^{n-1}$ such that $\eta\cdot\xi=0$. Then for $\tau>|\xi|$ we put

$$B_\tau=\sqrt{1-\frac{|\xi|^2}{4\tau^2}},\ \eta_1(\tau)=B_\tau\eta-\frac{\xi}{2\tau},\ \eta_2(\tau)=B_\tau\eta+\frac{\xi}{2\tau}\text{ and }\sqrt{\lambda(\tau)}=\tau+i, \tag{3.29}$$

in such a way that

$$\begin{cases} \eta_1,\eta_2\in\mathbb{S}^{n-1},\\ \sqrt{\lambda}(\eta_1-\eta_2)\to-\xi,\quad\text{as }\tau\to+\infty,\\ \Im\lambda\to+\infty,\quad\text{as }\tau\to+\infty,\\ \Im\sqrt{\lambda}\eta_1,\Im\sqrt{\lambda}\eta_2\ \text{ are bounded wrt }\tau>|\xi|.\end{cases} \tag{3.30}$$

We define the functions $A_{j,\sharp}\in\mathcal{C}_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$, $j=1,2$, by

$$A_{j,\sharp}(x):=\chi_\delta*\tilde{A}_j(x)=\int_{\mathbb{R}^n}\chi_\delta(x-y)\tilde{A}_j(y)dy,$$

where $\chi_\delta(x)=\delta^{-n}\chi(\delta^{-1}x)$ is the usual mollifer with $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$, $\mathrm{supp}\chi\subset\{x\in\mathbb{R}^n:\ |x|\leq 1\}$, $0\leq\chi\leq 1$ and $\int_{\mathbb{R}^n}\chi dx=1$. From now on we set $\delta=\tau^{-\frac{1}{3}}$ and we fix

$$\psi_j(x)=-\int_{-\infty}^{x\cdot\eta_j}A_{j,\sharp}(x+(s-x\cdot\eta_j)\eta_j)ds.$$

We set also

$$b_2=e^{\mathrm{i}\omega\cdot x}\partial_y\left[\exp\left(-\mathrm{i}\int_{\mathbb{R}}\eta_2\cdot A_\sharp(x+s\eta_2)ds\right)e^{-\mathrm{i}\omega\cdot x}\right], \tag{3.31}$$

where $A_\sharp=A_{2,\sharp}-A_{1,\sharp}$, $\omega=B_\tau\xi-\frac{|\xi|^2\eta}{2\tau}\in\eta_2^\perp$, $B_\tau=\sqrt{1-\frac{|\xi|^2}{4\tau^2}}$, and

$$b=e^{\mathrm{i}x\cdot\xi}\partial_y\left[\exp\left(-\mathrm{i}\int_{\mathbb{R}}\eta\cdot A(x+s\eta)ds\right)e^{-\mathrm{i}x\cdot\xi}\right],\quad\psi=\int_{-\infty}^{x\cdot\eta}\eta\cdot A(x+(s-x\cdot\eta)\eta)ds.$$

Here $y \in \mathbb{S}^{n-1}$ denotes a vector lying in $\eta^{\perp}$, $\partial_y = y \cdot \nabla$ and A is the function defined by $A_2 - A_1$ on Ω and extended by 0 outside of Ω. By the Sobolev embedding theorem (e.g. Theorem 1.4.4.1 in [12]), for any $\varepsilon \in (0,1)$, we have $\tilde{A}_j \in \mathcal{C}^{0,1-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$. Therefore, we find

$$\|A_{j,\sharp} - A_j\|_{L^\infty(\Omega)} \leq \left\|A_{j,\sharp} - \tilde{A}_j\right\|_{L^\infty(\mathbb{R}^n)} \leq C_\varepsilon \delta^{1-\varepsilon} = C_\varepsilon \tau^{-\frac{1-\varepsilon}{3}} \tag{3.32}$$

with C_ε depending on ε, Ω and any $M \geq \|A_j\|_{W^{1,\infty}(\Omega)}$. On the other hand, one can check that

$$\|\partial_x^\alpha A_{j,\sharp}\|_{L^\infty(\mathbb{R}^m)} \leq C\delta^{|\alpha|-1} = C\tau^{\frac{|\alpha|-1}{3}}, \quad \alpha \in \mathbb{N}^n \setminus \{0\}, \tag{3.33}$$

where C depends on Ω and any $M \geq \max\limits_{j=1,2} \|A_j\|_{W^{1,\infty}(\Omega)}$. Applying (3.32) and (3.33), we obtain the following.

Lemma 3.3. *Let the condition introduced above be fulfilled. Then, we have*

$$\sup_{\tau > |\xi|+1} \|b_2\|_{L^\infty(\mathbb{R}^n)} < \infty \tag{3.34}$$

and there exists an increasing and unbounded sequence $(\tau_m)_{m\geq 1}$ of $(1,+\infty)$ such that, for $\tau = \tau_m$, we find

$$\lim_{m\to+\infty} b_2(x) = b(x), \quad \lim_{m\to+\infty} \psi_1(x) - \psi_2(x) = \psi(x) = \int_{-\infty}^{x\cdot\eta} A(x + (s - x\cdot\eta)\eta)ds, \ ae\ x \in \mathbb{R}^n. \tag{3.35}$$

Proof. Note first that

$$b_2(x) = \left(-\mathrm{i}\omega \cdot y + -\mathrm{i}\int_{\mathbb{R}} \eta_2 \cdot \partial_y A_\sharp(x + s\eta_2)ds\right) \exp\left(-\mathrm{i}\int_{\mathbb{R}} \eta_2 \cdot A_\sharp(x + s\eta_2)ds\right). \tag{3.36}$$

On the other hand, we have $|\omega| \leq 1 + |\xi|$ and, since $\tilde{A}_2 - \tilde{A}_1$ is compactly supported and $\tilde{A}_2 - \tilde{A}_1 \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, we find $\partial_y A_\sharp = \chi_\delta * \partial_y(\tilde{A}_2 - \tilde{A}_1)$. Therefore, we obtain

$$\|b_2\|_{L^\infty(\mathbb{R}^n)} \leq 1 + |\xi| + C\|\chi_\delta\|_{L^1(\mathbb{R}^n)} \left\|\partial_y(\tilde{A}_2 - \tilde{A}_1)\right\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \leq 1 + |\xi| + CM$$

with C a generic constant depending only on Ω and $M \geq \max\limits_{j=1,2} \|A_j\|_{W^{1,\infty}(\Omega)}$. From this last estimate we deduce (3.34). Now let us prove (3.35). According to (3.27), we have $\tilde{A}_2 - \tilde{A}_1 = A$. Therefore, we deduce that $A_\sharp = \chi_\delta * A$ and

$$|\partial_y A_\sharp(x + s\eta_2) - \partial_y A(x + s\eta)| \leq |\partial_y A_\sharp(x + s\eta_2) - \partial_y A_\sharp(x + s\eta)| + |\partial_y A_\sharp(x + s\eta) - \partial_y A(x + s\eta)|. \tag{3.37}$$

The second term on the right hand side of this estimate can be rewritten as

$$\partial_y A_\sharp(x + s\eta) - \partial_y A(x + s\eta) = \chi_\delta * \partial_y A(x + s\eta) - \partial_y A(x + s\eta)$$

and we have

$$\lim_{\tau\to+\infty} \|\partial_y A_\sharp - \partial_y A\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)} = 0.$$

Therefore, we can find an increasing and unbounded sequence $(\tau_m)_{m\geq 1}$ of $(1,+\infty)$ such that, for $\tau = \tau_m$,

$$\lim_{m\to+\infty} \partial_y A_\sharp(x + s\eta) - \partial_y A(x + s\eta) = 0, \quad \text{a.e. } x \in \mathbb{R}^n. \tag{3.38}$$

For the first term on the right hand side of (3.37), using the fact that for τ sufficiently large we have

$$\eta_2 = \eta + \frac{\xi}{2\tau} + \underset{\tau\to+\infty}{o}\left(\frac{1}{\tau}\right)$$

and applying (3.33), we get

$$|\partial_y A_\sharp(x + s\eta_2) - \partial_y A_\sharp(x + s\eta)| \leq \|A_\sharp\|_{W^{2,\infty}(\mathbb{R}^n)} |s(\eta - \eta_1)| \leq C|s|\tau^{-\frac{2}{3}}$$

with C depending on ξ, Ω, A_1 and A_2. In view of this estimate we have

$$\lim_{\tau\to+\infty} \partial_y A_\sharp(x + s\eta_2) - \partial_y A_\sharp(x + s\eta) = 0, \quad x \in \mathbb{R}^n.$$

Combining this last result with (3.37)-(3.38), we deduce that $\tau = \tau_m$ we have

$$\lim_{m\to+\infty} \partial_y A_\sharp(x+s\eta) - \partial_y A(x+s\eta) = 0, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Then, using the fact that $\mathrm{supp} A_\sharp \subset \Omega + \{x \in \mathbb{R}^n : \ |x| \le \delta\}$ and (3.33), by the dominate convergence theorem we get that for $\tau = \tau_m$,

$$\lim_{m\to+\infty} \int_{\mathbb{R}} \partial_y A_\sharp(x+s\eta_2)ds = \int_{\mathbb{R}} \partial_y A(x+s\eta)ds, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Putting this together with (3.36) and the fact that $\omega \to \xi$, $\eta_2 \to \eta$ as $\tau \to +\infty$, for $\tau = \tau_m$, we obtain

$$\lim_{m\to+\infty} b_2(x) = \left(-\mathrm{i}\xi\cdot y + -\mathrm{i}\int_{\mathbb{R}} \eta\cdot\partial_y A(x+s\eta)ds\right)\exp\left(-\mathrm{i}\int_{\mathbb{R}} \eta\cdot A(x+s\eta)ds\right) = b(x), \text{ a.e. } x \in \mathbb{R}^n.$$

Using similar arguments we deduce that

$$\lim_{\tau\to+\infty} \psi_1(x) - \psi_2(x) = \psi(x) = \int_{-\infty}^{x\cdot\eta} A(x+(s-x\cdot\eta)\eta)ds, \quad x \in \mathbb{R}^n.$$

This completes the proof of the lemma. □

Applying (3.18)-(3.19) and (3.32)-(3.35), we obtain the following representation

Lemma 3.4. *Fix $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{S}^{n-1}$ such that $\eta\cdot\xi = 0$. Fix also $(\tau_m)_{m\ge1}$ the sequence introduced in Lemma 3.3. Let λ, η_1 and η_2 be defined by* (3.29) *with $\tau = \tau_m$ and let b_2 be defined by* (3.31)*. Then, we have*

$$\lim_{m\to+\infty} \frac{S_1 - S_2}{\sqrt{\lambda}} = 2\int_\Omega \eta\cdot(A_1 - A_2)e^{-\mathrm{i}\xi\cdot x} b e^{\mathrm{i}\psi(x)}dx. \tag{3.39}$$

Proof. With reference to (3.16) and (3.29) we have $|\Phi_1(x)| = e^{-\eta_1\cdot x}$ and $\left|e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}\right| = e^{\eta_2\cdot x}$ for all $x \in \Omega$, hence $\|\Phi_1\|^2_{L^2(\Omega)} = \int_\Omega e^{-2\eta_1\cdot x}dx \le |\Omega|e^{2|\Omega|}$ and $\left\|e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}\right\|^2_{L^2(\Omega)} \le |\Omega|e^{2|\Omega|}$ since $|\eta_1| = |\eta_2| = 1$. Moreover, in view of (3.29), we have the estimate

$$\left\|(H_j - \lambda)^{-1}\right\|_{\mathcal{B}(L^2(\Omega))} = \frac{1}{(\mathrm{dist}(\lambda, \sigma(H_j)))} \le \frac{1}{|\Im\lambda|} = \frac{1}{2\tau}, \quad j = 1, 2.$$

In addition, in light of (3.33), we get

$$\|\psi_j\|_{W^{2,\infty}(\Omega)} \le C\delta = C\tau^{\frac{1}{3}}, \quad \|b_j\|_{W^{2,\infty}(\Omega)} \le C\delta^2 = C\tau^{\frac{2}{3}}$$

with C a generic constant depending on ξ, A_j, $j = 1, 2$, and Ω. Putting these estimates together with (1.1), (3.18)-(3.19) and (3.32) , we deduce that

$$\frac{S_1 - S_2}{\sqrt{\lambda}} = 2\int_\Omega \eta_2\cdot(A_1 - A_2)e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} b_2 e^{\mathrm{i}(\psi_1(x)-\psi_2(x))}dx + \underset{\tau\to+\infty}{\mathcal{O}}\left(\tau^{-\frac{1-\varepsilon}{3}}\right).$$

Combining this with (3.34)-(3.35) and applying the dominate convergence theorem we deduce (3.39). □

Using similar arguments we obtain the following.

Lemma 3.5. *Assume that $A_1 = A_2$. Fix $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{S}^{n-1}$ such that $\eta\cdot\xi = 0$. Let λ, η_1 and η_2 be defined by* (3.29) *and $b_2 = 1$. Then, we have*

$$\lim_{\tau\to+\infty} S_1 - S_2 = \int_\Omega (V_1 - V_2)e^{-\mathrm{i}\xi\cdot x}dx. \tag{3.40}$$

Proof. Note that for $A_1 = A_2$ we have $q_{11} - V_1 = q_{21} - V_2$, $q_{12} = q_{22}$, $A_{1,\sharp} = A_{2,\sharp}$. Therefore, we deduce that (3.18)-(3.19) imply

$$\begin{aligned} S_1 - S_2 = & \int_\Omega (V_1 - V_2) e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} e^{\mathrm{i}(\psi_1(x)-\psi_2(x))} dx - \int_\Omega \left[\lambda \left((H_1-\lambda)^{-1} - (H_2-\lambda)^{-1}\right) Q_1\right] Q_2 dx \\ & - \int_\Omega \left[\sqrt{\lambda}(H_1-\lambda)^{-1} Q_1\right] V_1 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x} e^{-\mathrm{i}\psi_2} dx - \int_\Omega \left[\sqrt{\lambda}(H_1-\lambda)^{-1} V_1 \Phi_1\right] Q_2 dx \\ & - \int_\Omega \left[(H_1-\lambda)^{-1} V_1 \Phi_1\right] V_1 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x} e^{-\mathrm{i}\psi_2} dx + \int_\Omega \left[\sqrt{\lambda}(H_2-\lambda)^{-1} Q_1\right] V_2 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x} e^{-\mathrm{i}\psi_2} dx \\ & + \int_\Omega \left[\sqrt{\lambda}(H_2-\lambda)^{-1} V_2 \Phi_1\right] Q_2 dx + \int_\Omega \left[(H_2-\lambda)^{-1} V_2 \Phi_1\right] V_2 e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x} e^{-\mathrm{i}\psi_2} dx, \end{aligned} \tag{3.41}$$

where

$$Q_1 = 2\eta_1 \cdot (A_1 - A_{1,\sharp})\Phi_1 + \frac{(q_{11} - V_1)\Phi_1}{\sqrt{\lambda}}, \quad Q_2 = \left(2\eta_2 \cdot (A_1 - A_{1,\sharp}) - \frac{q_{12}}{\sqrt{\lambda}}\right) e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x} e^{-\mathrm{i}\psi_2}.$$

On the other hand, since $H_2 - \lambda = H_1 - \lambda - (V_1 - V_2)$, for τ sufficiently large we have

$$\begin{aligned} (H_1-\lambda)^{-1} - (H_2-\lambda)^{-1} &= (H_1-\lambda)^{-1} \left(\mathrm{Id} - \left(\mathrm{Id} - (V_1-V_2)(H_1-\lambda)^{-1}\right)^{-1}\right) \\ &= -(H_1-\lambda)^{-1} \sum_{k=1}^{\infty} \left((V_1-V_2)(H_1-\lambda)^{-1}\right)^k. \end{aligned}$$

Combining this with the fact that $\Im\lambda = 2\tau$, $|\lambda| \leq |\tau^2 - 1| + 2\tau$, and the fact that

$$\left\|(H_1-\lambda)^{-1}\right\|_{\mathcal{B}(L^2(\Omega))} + \left\|(V_1-V_2)(H_1-\lambda)^{-1}\right\|_{\mathcal{B}(L^2(\Omega))} \leq \frac{C}{|\Im\lambda|} = \frac{C}{2\tau}$$

with C depending only on V_1, V_2 and Ω, we deduce that

$$\sup_{\tau > |\xi|+1} \left\|\lambda \left((H_1-\lambda)^{-1} - (H_2-\lambda)^{-1}\right)\right\|_{\mathcal{B}(L^2(\Omega))} < \infty. \tag{3.42}$$

In addition, (3.32)-(3.33) imply

$$\lim_{\tau\to+\infty} \|Q_1\|_{L^\infty(\Omega)} = \lim_{\tau\to+\infty} \|Q_2\|_{L^\infty(\Omega)} = 0.$$

Putting this result together with (3.30), (3.41)-(3.42), we obtain

$$\limsup_{\tau\to+\infty} \left|(S_1 - S_2) - \int_\Omega (V_1 - V_2) e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} e^{\mathrm{i}(\psi_1(x)-\psi_2(x))} dx\right| = 0.$$

On the other hand, repeating the arguments of Lemma 3.3, we find

$$\lim_{\tau\to+\infty} \psi_1(x) - \psi_2(x) = \psi(x) = \int_{-\infty}^{x\cdot\eta} A(x + (s - x\cdot\eta)\eta) ds = 0$$

since $A_1 = A_2$. Thus, applying the dominate convergence theorem we obtain

$$\lim_{\tau\to+\infty} \int_\Omega (V_1 - V_2) e^{\mathrm{i}\sqrt{\lambda}(\eta_1-\eta_2)\cdot x} e^{\mathrm{i}(\psi_1(x)-\psi_2(x))} dx = \int_\Omega (V_1 - V_2) e^{\mathrm{i}x\cdot\xi} dx$$

and we deduce (3.40). □

4. Proof of the main result

This section is devoted to the proof of our main result. In all this section, for $j = 1$ and $j = 2$, we consider two magnetic potentials A_j and electric potentials V_j satisfying the assumptions of Theorem 1.1, and denote by H_j the associated operators defined by (2.3) for $A = A_j$ and $V = V_j$. Let $(\lambda_{j,k}, \varphi_{j,k})_{k\geq 1}$ be a sequence of eigenvalues and eigenfunctions of H_j. We start with two intermediate results.

Lemma 4.1. *Let $\eta_1(\tau)$, $\eta_2(\tau)$ and $\lambda(\tau)$ be fixed by* (3.29) *and b_2 be defined by* (3.31)*. Assume that*

$$\lim_{\tau\to+\infty} \frac{S_1(\lambda(\tau), \eta_1(\tau), \eta_2(\tau)) - S_2(\lambda(\tau), \eta_1(\tau), \eta_2(\tau))}{\sqrt{\lambda(\tau)}} = 0. \tag{4.43}$$

Then we have $dA_1 = dA_2$.

Proof. Combining (4.43) with (3.39) we deduce that for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{S}^{n-1}$, satisfying $\eta \cdot \xi = 0$, we get

$$\int_\Omega \eta \cdot (A_2 - A_1) e^{-\mathrm{i}\xi\cdot x} b(x) e^{\mathrm{i}\psi(x)} dx = 0.$$

Here b takes the form

$$b(x) = e^{\mathrm{i}x\cdot\xi} \partial_y \left[\exp\left(-\mathrm{i} \int_{\mathbb{R}} \eta \cdot A(x + s\eta) ds \right) e^{-\mathrm{i}x\cdot\xi} \right]$$

with $y \in \mathbb{S}^{n-1} \cap \eta^\perp$. Then, we obtain

$$0 = \int_{\mathbb{R}^n} \eta \cdot A(x) e^{-\mathrm{i}\xi\cdot x} b(x) e^{\mathrm{i}\psi(x)} dx = \int_{\eta^\perp} \int_{\mathbb{R}} \eta \cdot A(x' + t\eta) e^{\mathrm{i}\psi(x'+t\eta)} b(x') e^{-\mathrm{i}\xi\cdot x'} dt dx'.$$

Here we use the fact that $b(x) = b(x - (x\cdot\eta)\eta)$ and $\xi \cdot \eta = 0$. On the other hand, for all $x' \in \eta^\perp$ and $t \in \mathbb{R}$, we have

$$\eta \cdot A(x' + t\eta) e^{\mathrm{i}\psi(x'+t\eta)} = \eta \cdot A(x' + t\eta) \exp\left(\mathrm{i} \int_{-\infty}^t \eta \cdot A(x' + s\eta) ds \right) = -\mathrm{i}\partial_t \exp\left(\mathrm{i} \int_{-\infty}^t \eta \cdot A(x' + s\eta) ds \right).$$

Therefore, we find

$$\begin{aligned} \int_{\mathbb{R}^n} \eta \cdot A(x) e^{-\mathrm{i}\xi\cdot x} b(x) e^{\mathrm{i}\psi(x)} dx &= -\mathrm{i} \int_{\eta^\perp} \left[\int_{\mathbb{R}} \partial_t \exp\left(\mathrm{i} \int_{-\infty}^t \eta \cdot A(x' + s\eta) ds \right) dt \right] b(x') e^{-\mathrm{i}\xi\cdot x'} dx' \\ &= -\mathrm{i} \int_{\eta^\perp} \left[\exp\left(\mathrm{i} \int_{\mathbb{R}} \eta \cdot A(x' + s\eta) ds \right) - 1 \right] b(x') e^{-\mathrm{i}\xi\cdot x'} dx'. \end{aligned}$$

It follows

$$\int_{\eta^\perp} \left[\exp\left(\mathrm{i} \int_{\mathbb{R}} \eta \cdot A(x' + s\eta) ds \right) - 1 \right] b(x') e^{-\mathrm{i}\xi\cdot x'} dx' = 0. \tag{4.44}$$

Now assume that $\xi \in \{\xi = (\xi_1, \dots, \xi_n) : \ \xi_l > 0, \ l = 1, \dots, n\}$. Fix $i, j \in \{1, \dots, n\}$ such that $i \neq j$. We can choose $\eta = \frac{\xi_j e_i - \xi_i e_j}{\sqrt{\xi_i^2 + \xi_j^2}}$ and $y = \frac{\xi_i e_i + \xi_j e_j}{\sqrt{\xi_i^2 + \xi_j^2}} \in \eta^\perp$. Here $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$ defined by $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$. Then, (4.44) implies

$$\int_{\eta^\perp} \left[\exp\left(\mathrm{i} \int_{\mathbb{R}} \eta \cdot A(x' + s\eta) ds \right) - 1 \right] \partial_y \left[\exp\left(-\mathrm{i} \int_{\mathbb{R}} \eta \cdot A(x' + s\eta) ds \right) e^{-ix'\cdot\xi} \right] dx' = 0.$$

Integrating by parts we get

$$\frac{-\mathrm{i}}{\sqrt{\xi_i^2 + \xi_j^2}} \cdot \int_{\mathbb{R}^n} (\xi_j \partial_y a_i(x) - \xi_i \partial_y a_j(x)) e^{-\mathrm{i}x\cdot\xi} dx = -\mathrm{i} \int_{\eta^\perp} \left(\int_{\mathbb{R}} \eta \cdot \partial_y A(x' + s\eta) ds \right) e^{-\mathrm{i}x'\cdot\xi} dx' = 0$$

with $A=(a_1,\dots,a_n)$. Integrating again by parts, we find

$$\begin{aligned}\int_{\mathbb{R}^n}(\xi_j a_i-\xi_i a_j)e^{-\mathrm{i}x\cdot\xi}dx &= \frac{y\cdot\xi}{\sqrt{\xi_i^2+\xi_j^2}}\int_{\mathbb{R}^n}(\xi_j a_i-\xi_i a_j)e^{-\mathrm{i}x\cdot\xi}dx\\ &= \frac{-\mathrm{i}}{\sqrt{\xi_i^2+\xi_j^2}}\cdot\int_{\mathbb{R}^n}(\xi_j\partial_y a_i(x)-\xi_i\partial_y a_j(x))e^{-\mathrm{i}x\cdot\xi}dx=0\end{aligned}$$

and it follows that for all $\xi\in\{\xi=(\xi_1,\dots,\xi_n):\ \xi_l>0,\ l=1,\dots,n\}$ we have $\mathcal{F}[\partial_{x_j}a_i-\partial_{x_i}a_j](\xi)=0$. On the other hand, since $\partial_{x_j}a_i-\partial_{x_i}a_j$ is compactly supported, $\mathcal{F}(\partial_{x_j}a_i-\partial_{x_i}a_j)(\xi)$ is analytic in $\xi\in\mathbb{R}^n$ and it follows $\mathcal{F}(\partial_{x_j}a_i-\partial_{x_i}a_j)=0$ on $\mathbb{R}^n$. Here we use the fact that $\{\xi=(\xi_1,\dots,\xi_n):\ \xi_l>0,\ l=1,\dots,n\}$ is an open set of $\mathbb{R}^n$. From this last result, we deduce that $\partial_{x_j}a_i-\partial_{x_i}a_j=0$ which implies that $dA_1=dA_2$. □

Lemma 4.2. *Let $\eta_1(\tau)$, $\eta_2(\tau)$ and $\lambda(\tau)$ be fixed by* (3.29) *and $b_2=1$. Assume that $A_1=A_2$ and*

$$\lim_{\tau\to+\infty}S_1(\lambda(\tau),\eta_1(\tau),\eta_2(\tau))-S_2(\lambda(\tau),\eta_1(\tau),\eta_2(\tau))=0. \tag{4.45}$$

Then we have $V_1=V_2$.

Proof. Fix $\xi\in\mathbb{R}^n\setminus\{0\}$ and choose $\eta\in\mathbb{S}^{n-1}\cap\xi^\perp$. Fix also $b=1$. Thus, combining (3.40) and (4.45), we find

$$\int_{\mathbb{R}^n}V(x)e^{-\mathrm{i}x\cdot\xi}dx=0$$

with $V=V_1-V_2$ extended by 0 outside of Ω. It follows that $V_1=V_2$. □

Now let $\lambda\in\mathbb{C}$ and $\mu\in\mathbb{R}$, such that $\lambda,\mu\notin\sigma(H_1)\cup\sigma(H_2)$, and $f\in H^{1/2}(\Gamma)$ consider $u_{j,\lambda}$ solution to the equation (2.4) where $V:=V_j$ and $A=A_j$, and also denote

$$h_{j,k}:=\partial_\nu\varphi_{j,k|\Gamma},\qquad \alpha_{j,k}:=\langle f,h_{j,k}\rangle.$$

According to Lemma 4.1, 4.2, the proof of Theorem 1.1 will be completed if we show that conditions (1.2) imply conditions (4.43)-(4.45). For this purpose, we adapt the approach of [16] to magnetic Schrödinger operators.

Let $f\in H^{\frac12}(\Gamma)$ being fixed, with the notations of Lemmas 2.1 and 2.2, we denote by $v_{j,\lambda,\mu}:=u_{j,\lambda}-u_{j,\mu}$ the solution of (2.10) where V is replaced by V_j and A by A_j. Recalling that in Lemma 2.3 we have set $z_\mu=u_{1,\mu}-u_{2,\mu}$, in a similar way to [16], writing the above identity for $j=1$ and $j=2$, applying (1.1) and then subtracting the resulting equations, we end up with a new relation, namely

$$(\partial_\nu+\mathrm{i}A_1\cdot\nu)u_{1,\lambda|\Gamma}-(\partial_\nu+\mathrm{i}A_2\cdot\nu)u_{2,\lambda|\Gamma}=\mathrm{i}(A_1-A_2)\cdot\nu f+\partial_\nu u_{1,\lambda}-\partial_\nu u_{2,\lambda}=\partial_\nu z_\mu+\partial_\nu v_{1,\lambda,\mu}-\partial_\nu v_{2,\lambda,\mu}. \tag{4.46}$$

Now let us set

$$F_j(\lambda,\mu,f):=\partial_\nu v_{j,\lambda,\mu|\Gamma},\quad j=1,2.$$

According to (2.9), we have

$$F(\lambda,\mu,f):=F_1(\lambda,\mu,f)-F_2(\lambda,\mu,f)=\sum_{k=1}^{+\infty}\left[\frac{(\mu-\lambda)\alpha_{1,k}}{(\lambda-\lambda_{1,k})(\mu-\lambda_{1,k})}\,h_{1,k}-\frac{(\mu-\lambda)\alpha_{2,k}}{(\lambda-\lambda_{2,k})(\mu-\lambda_{2,k})}\,h_{2,k}\right] \tag{4.47}$$

and consider the following intermediate results.

Lemma 4.3. *Let η_1,η_2,λ be given by* (3.29)*. Consider Φ_j, $j=1,2$, with Φ_1 introduced in the previous section and $\Phi_2=e^{-\mathrm{i}\sqrt{\lambda}\eta_2\cdot x}b_2e^{-\mathrm{i}\psi_2}$. Then, we have*

$$\sup_{\tau>1}\sum_{k=1}^{\infty}\left|\frac{\langle\Phi_1,h_{j,k}\rangle}{\lambda_{j,k}-\lambda}\right|^2<\infty,\quad \sup_{\tau>1}\sum_{k=1}^{\infty}\left|\frac{\langle\overline{\Phi_2},h_{2,k}\rangle}{\lambda_{2,k}-\lambda}\right|^2<\infty,\ j=1,2. \tag{4.48}$$

Proof. We start with the first estimate of (4.48) for $j=1$. According to Lemma 2.1 the solution $u_{1,\lambda}$ of (2.4) for $f=\Phi_1$, $A=A_1$ and $V=V_1$, is given by

$$u_{1,\lambda}=\sum_{k=1}^{\infty}\frac{\langle\Phi_1,h_{1,k}\rangle}{\lambda-\lambda_{1,k}}\varphi_{1,k}.$$

Therefore, we have

$$\|u_{1,\lambda}\|^2_{L^2(\Omega)}=\sum_{k=1}^{\infty}\left|\frac{\langle\Phi_1,h_{1,k}\rangle}{\lambda_{j,k}-\lambda}\right|^2. \tag{4.49}$$

On the other hand, we can split $u_{1,\lambda}$ into two terms $u_{1,\lambda}=\Phi_1+v_{1,\lambda}$ where $v_{1,\lambda}$ solves

$$\begin{cases}(-\mathrm{i}\nabla+A_1)^2v_1+V_1v_1-\lambda v_1 &= -G_1, \quad \text{in } \Omega,\\ v(x) &= 0, \quad x\in\Gamma,\end{cases}$$

where according to (3.20)

$$G_1=(-\mathrm{i}\nabla+A_1)^2\Phi_1+V_1\Phi_1-\lambda\Phi_1=\sqrt{\lambda}\left[2\eta_1\cdot(A_1-A_{1,\sharp})+K_1\right]\Phi_1$$

with

$$K_1=\frac{-\mathrm{i}divA_1+|A_1|^2+V_1(x)+2A_1\cdot\nabla\psi_1-\mathrm{i}\Delta\psi_1+|\nabla\psi_1|^2}{\sqrt{\lambda}}.$$

Thus, we have $v_{1,\lambda}=-(H_1-\lambda)^{-1}G_1$ and we deduce that

$$\|u_{1,\lambda}\|_{L^2(\Omega)}\le\|\Phi_1\|_{L^2(\Omega)}+\left\|\sqrt{\lambda}(H_1-\lambda)^{-1}\left[2\eta_1\cdot(A_1-A_{1,\sharp})+K_1\right]\right\|_{L^2(\Omega)}.$$

Combining this with the fact that

$$\left\|\sqrt{\lambda}(H_1-\lambda)^{-1}\right\|_{\mathcal{B}(L^2(\Omega)}\le\frac{|\tau+\mathrm{i}|}{|\Im\lambda|}=\frac{|\tau+\mathrm{i}|}{2\tau}\le 1$$

and the fact that, according to (3.32)-(3.33), we have

$$\lim_{\tau\to+\infty}\|\eta_1\cdot(A_1-A_{1,\sharp})\|_{L^\infty(\Omega)}=\lim_{\tau\to+\infty}\|K_1\|_{L^\infty(\Omega)}=0$$

and we deduce the first estimate of (4.48) for $j=1$. In a same way, for $j=2$ using the fact that according to (3.33) we have

$$(-\mathrm{i}\nabla+A_2)^2\Phi_1+V_2\Phi_1-\lambda\Phi_1=\underset{\tau\to+\infty}{\mathcal{O}}(\tau)$$

and repeating our previous arguments we deduce the first estimate (4.48) for $j=2$. For the second estimate of (4.48), repeating the previous arguments we find

$$(-\mathrm{i}\nabla+A_2)^2\overline{\Phi_2}+V_2\overline{\Phi_2}-\overline{\lambda}\,\overline{\Phi_2}=\overline{(\mathrm{i}\nabla+A_2)^2\Phi_2+V_2\Phi_2-\lambda\Phi_2}=\underset{\tau\to+\infty}{\mathcal{O}}(\tau).$$

Combining this estimate with the fact that

$$\left|\frac{\langle\overline{\Phi_2},h_{2,k}\rangle}{\lambda_{2,k}-\lambda}\right|=\left|\frac{\langle\overline{\Phi_2},h_{2,k}\rangle}{\lambda_{2,k}-\overline{\lambda}}\right|$$

since $\lambda_{2,k}\in\mathbb{R}$, we deduce the second estimate of (4.48) by repeating the above arguments. □

From now on we set

$$\begin{aligned}G(\lambda,\mu,\Phi_1,\Phi_2)&:=\langle F(\lambda,\mu,\Phi_1),\overline{\Phi_2}\rangle\\&=\sum_{k=1}^{+\infty}(\mu-\lambda)\left[\frac{\langle\Phi_1,h_{1,k}\rangle\left\langle h_{1,k},\overline{\Phi_2}\right\rangle}{(\lambda-\lambda_{1,k})(\mu-\lambda_{1,k})}-\frac{\langle\Phi_1,h_{2,k}\rangle\left\langle h_{2,k},\overline{\Phi_2}\right\rangle}{(\lambda-\lambda_{2,k})(\mu-\lambda_{2,k})}\right].\end{aligned}$$

Combining estimates (4.48) with Lemma 4.3, 4.4, 4.5 of [16], we obtain the following.

Lemma 4.4. *Let the conditions of Theorem 1.1 be fulfilled and let η_1, η_2, λ be given by* (3.29)*. Then, $G(\lambda, \mu, \Phi_1, \Phi_2)$ converge to $G_*(\lambda, \Phi_1, \Phi_2)$ as $\mu \to -\infty$ and $G_*(\lambda, \Phi_1, \Phi_2)$ converge to* 0 *as $\tau \to +\infty$. Here we consider both the case b_2 given by* (3.31) *and the case $b_2 = 1$.*

Armed with Lemma 4.4, we are now in position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Note first that according (4.46), for $M = \|V_1\|_{L^\infty(\Omega)} + \|V_2\|_{L^\infty(\Omega)}$, we have

$$S_1(\lambda, \eta_1, \eta_2) - S_2(\lambda, \eta_1, \eta_2) = \left\langle \partial_\nu z_\mu, e^{\mathrm{i}\sqrt{\lambda}\eta_2 \cdot x} \overline{b_2} e^{\mathrm{i}\psi_2} \right\rangle + G(\lambda, \mu, \Phi_1, \Phi_2), \quad \mu \in (-\infty, -M),$$

where λ, η_1, η_2 are fixed by(3.29), b_2 is given by (3.31) or $b_2 = 1$ and $z_\mu = u_{1,\mu} - u_{2,\mu}$ with $u_{j,\mu}$, $j = 1, 2$, the solution of (2.10) where λ is replaced by μ, V by V_j, A by A_j and f by Φ_1. In view of Lemma 2.3 and Lemma 4.4, sending $\mu \to -\infty$ we get

$$S_1(\lambda, \eta_1, \eta_2) - S_2(\lambda, \eta_1, \eta_2) = G_*(\lambda, \Phi_1, \Phi_2).$$

Then, in view of Lemma 4.4, conditions (4.43) and (4.45) are fulfilled. Therefore, for $A = A_2 - A_1$ extended by 0 outside of Ω we have $dA = 0$ on $\mathbb{R}^n$. Thus there exists $p \in W^{2,\infty}(\mathbb{R}^n)$ given by

$$p(x) = \int_0^1 x \cdot A(tx) dt$$

such that $A = \nabla p$ on $\mathbb{R}^n$. Applying the fact that $A = 0$ on $\mathbb{R}^n \setminus \Omega$, upon eventually subtracting a constant we may assume that $p_{|\mathbb{R}^n \setminus \Omega} = 0$ which implies that $p_{|\Gamma} = 0$. Now let us consider the operator $H_3 = (-\mathrm{i}\nabla + A_1) + V_2$ acting on $L^2(\Omega)$ with Dirichlet boundary condition and let $(\lambda_{3,k}, \varphi_{3,k})_{k \geq 1}$ be a sequence of eigenvalues and eigenfunctions of H_3. Since $A_1 = A_2 - \nabla p$ one can check that $H_3 = e^{\mathrm{i}p} H_2 e^{-\mathrm{i}p}$. From this identity we deduce that

$$\lambda_{3,k} = \lambda_{2,k}, \quad k \geq 1.$$

Moreover, for all $k \geq 1$ we can choose $\varphi_{3,k} = e^{ip}\varphi_{2,k}$ and deduce that the condition

$$\partial_\nu \varphi_{3,k} = \partial_\nu \varphi_{2,k}, \quad k \geq 1$$

is also fulfilled. Thus conditions (1.2) imply that

$$\lim_{k \to +\infty} |\lambda_{1,k} - \lambda_{3,k}| = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \|\partial_\nu \varphi_{1,k} - \partial_\nu \varphi_{3,k}\|^2_{L^2(\Gamma)} < \infty.$$

Then repeating the arguments of Lemma 4.4 we obtain

$$\lim_{\tau \to +\infty} \tilde{S}_1(\lambda(\tau), \eta_1(\tau), \eta_2(\tau)) - \tilde{S}_3(\lambda(\tau), \eta_1(\tau), \eta_2(\tau)) = 0$$

where

$$\tilde{S}_j(\lambda, \eta_1, \eta_2) = \left\langle \Lambda_{j,\lambda} \Phi_1, e^{\mathrm{i}\sqrt{\lambda}\eta_2 \cdot x} e^{\mathrm{i}\tilde{\psi}_2} \right\rangle, \quad j = 1, 3$$

with

$$\tilde{\psi}_2(x) = \int_{-\infty}^{x \cdot \eta_2} \eta_2 \cdot A_{1,\sharp}(x + (s - x \cdot \eta_2)\eta_2) ds, \quad b_2 = 1$$

and $\Lambda_{3,\lambda}$ the Dirichlet Neumann map associated to problem (2.4) for $A = A_1$ and $V = V_2$. Then in view of Lemma 4.2 we have $V_1 = V_2$. This completes the proof of Theorem 1.1. □

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