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CAUCHY PROBLEM AND EXPONENTIAL STABILITY FOR THE INHOMOGENEOUS LANDAU EQUATION

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Abstract. This work deals with the inhomogeneous Landau equation on the torus in the cases of hard, Maxwellian and moderately soft potentials. We first investigate the linearized equation and we prove exponential decay estimates for the associated semigroup. We then turn to the nonlinear equation and we use the linearized semigroup decay in order to construct solutions in a close-to-equilibrium setting. Finally, we prove an exponential stability for such a solution, with a rate as close as we want to the optimal rate given by the semigroup decay.

Contents

1. Introduction

1.1. **The model.** In this paper, we investigate the Cauchy theory associated to the *spatially inhomogeneous Landau equation*. This equation is a kinetic model in plasma physics that describes the evolution of the density function $F = F(t, x, v)$ in the phase space of position and velocities of the particles. In the torus, the equation is given by, for $F = F(t, x, v) \geq 0$ with

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$t \in \mathbb{R}^+$, $x \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ (that we assume without loss of generality to have volume one $|\mathbb{T}^3| = 1$) and $v \in \mathbb{R}^3$,

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F,F) \\ F_{|t=0} = F_0 \end{cases} \tag{1.1}$$

where the Landau operator Q is a bilinear operator that takes the form

$$Q(G,F)(v) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v-v_*) \left[G_* \partial_j F - F \partial_j G_* \right] dv_*, \tag{1.2}$$

and we use the convention of summation of repeated indices, and the derivatives are in the velocity variable, i.e. $\partial_i = \partial_{v_i}$. Hereafter we use the shorthand notations $G_* = G(v_*)$, $F = F(v)$, $\partial_j G_* = \partial_{v_{*j}} G(v_*)$, $\partial_j F = \partial_{v_j} F(v)$, etc.

The matrix a_{ij} is symmetric semi-positive, depends on the interaction between particles and is given by

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right). \tag{1.3}$$

We define (see [21]) in 3-dimension the following quantities

$$\begin{aligned} &b_i(v) = \partial_j a_{ij}(v) = -2\,|v|^\gamma\, v_i, \\ &c(v) = \partial_{ij} a_{ij}(v) = -2(\gamma+3)\,|v|^\gamma \quad \text{or} \quad c = 8\pi\delta_0 \ \text{ if } \ \gamma = -3. \end{aligned} \tag{1.4}$$

We can rewrite the Landau operator (1.2) in the following way

$$Q(G,F) = (a_{ij} *_v G)\partial_{ij} F - (c *_v G)F = \nabla_v \cdot \{ (a *_v g)\nabla_v f - (b *_v g) f \}. \tag{1.5}$$

We have the following classification: we call hard potentials if $\gamma \in (0,1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2,0)$, very soft potentials if $\gamma \in (-3,-2)$ and Coulombian potential if $\gamma = -3$. Hereafter we shall consider the cases of hard potentials, Maxwellian molecules and moderately soft potentials, i.e. $\gamma \in [-2,1]$.

The Landau equation conserves mass, momentum and energy. Indeed, at least formally, for any test function φ, we have

$$\int_{\mathbb{R}^3} Q(F,F)\varphi\, dv = -\frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v-v_*) F F_* \left(\frac{\partial_i F}{F} - \frac{\partial_i F_*}{F_*} \right) (\partial_j \varphi - \partial_j \varphi_*)\, dv\, dv_*,$$

from which we deduce that

$$\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F\varphi(v)\, dx\, dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left[Q(F,F) - v \cdot \nabla_x F \right] \varphi(v)\, dx\, dv = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2. \tag{1.6}$$

Moreover, the Landau version of the Boltzmann H-theorem asserts that the entropy

$$H(F) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} F \log F\, dx\, dv$$

is non increasing. Indeed, at least formally, since a_{ij} is nonnegative, we have the following inequality for the entropy dissipation $D(F)$:

$$\begin{aligned} D(F) &:= -\frac{d}{dt} H(F) \\ &= \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v-v_*) F F_* \left(\frac{\partial_i F}{F} - \frac{\partial_i F_*}{F_*} \right) \left(\frac{\partial_j F}{F} - \frac{\partial_j F_*}{F_*} \right) dv\, dv_*\, dx \geq 0. \end{aligned}$$

It is known that the global equilibria of (1.1) are global Maxwellian distributions that are independent of time t and position x. We shall always consider initial data F_0 verifying

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} F_0\, dx\, dv = 1, \quad \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_0\, v\, dx\, dv = 0, \quad \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_0\, |v|^2\, dx\, dv = 3,$$

therefore we consider the Maxwellian equilibrium

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

with same mass, momentum and energy of the initial data.

We linearize the Landau equation around μ with the perturbation

$$F = \mu + f.$$

The Landau equation (1.1) for $f = f(t, x, v)$ takes the form

$$\begin{cases} \partial_t f = \Lambda f + Q(f, f) := \mathcal{L} f - v \cdot \nabla_x f + Q(f, f) \\ f_{|t=0} = f_0 = F_0 - \mu, \end{cases} \tag{1.7}$$

where $\Lambda = \mathcal{L} - v \cdot \nabla_x$ is the inhomogeneous linearized Landau operator and the homogeneous linearized Landau operator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} f &:= Q(\mu, f) + Q(f, \mu) \\ &= (a_{ij} * \mu)\partial_{ij} f - (c * \mu) f + (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu. \end{aligned} \tag{1.8}$$

Through the paper we introduce the following notation

$$\bar{a}_{ij}(v) = a_{ij} * \mu, \quad \bar{b}_i(v) = b_i * \mu, \quad \bar{c}(v) = c * \mu. \tag{1.9}$$

The conservation laws (1.6) can then be rewritten as, for all $t \geq 0$,

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, x, v) \varphi(v)\, dx\, dv = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2. \tag{1.10}$$

1.2. **Notations.** Through all the paper we shall consider function of two variables $f = f(x, v)$ with $x \in \mathbb{T}^3$ and $v \in \mathbb{R}^3$. Let $m = m(v)$ be a positive Borel weight function and $1 \leq p, q \leq \infty$. We define the space $L^q_x L^p_v(m)$ as the Lebesgue space associated to the norm, for $f = f(x, v)$,

$$\begin{aligned} \|f\|_{L^q_x L^p_v(m)} &:= \big\| \|f\|_{L^p_v(m)} \big\|_{L^q_x} := \big\| \|m\, f\|_{L^p_v} \big\|_{L^q_x} \\ &= \left(\int_{\mathbb{T}^3_x} \|f(x, \cdot)\|^q_{L^p_v(m)}\, dx \right)^{1/q} \\ &= \left(\int_{\mathbb{T}^3_x} \left(\int_{\mathbb{R}^3_v} |f(x, v)|^p\, m(v)^p\, dv \right)^{q/p} dx \right)^{1/q}. \end{aligned}$$

We also define the high-order Sobolev spaces $W^{n,q}_x W^{\ell,p}_v(m)$, for $n, \ell \in \mathbb{N}$:

$$\|f\|_{W^{n,q}_x W^{\ell,p}_v(m)} = \sum_{0 \leq |\alpha| \leq \ell,\, 0 \leq |\beta| \leq n,\, |\alpha| + |\beta| \leq \max(\ell, n)} \|\partial^\alpha_v \partial^\beta_x f\|_{L^q_x L^p_v(m)}.$$

This definition reduces to the usual weighted Sobolev space $W^{\ell,p}_{x,v}(m)$ when $p = q$ and $\ell = n$, and we recall the shorthand notation $H^\ell = W^{\ell,2}$. We shall denote $W^{\ell,p}(m) = W^{\ell,p}_{x,v}(m)$ when considering spaces in the two variables (x, v).

Let X, Y be Banach spaces and consider a linear operator $\Lambda : X \to X$. We shall denote by $\mathcal{S}_\Lambda(t) = e^{t\Lambda}$ the semigroup generated by Λ. Moreover we denote by $\mathscr{B}(X, Y)$ the space of bounded linear operators from X to Y and by $\| \cdot \|_{\mathscr{B}(X,Y)}$ its norm operator, with the usual simplification $\mathscr{B}(X) = \mathscr{B}(X, X)$.

For simplicity of notations, hereafter, we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$; $a \sim b$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 b \leq a \leq c_2 b$; we abbreviate " $\leq C$ " to " $\lesssim$ ", where C is a positive constant depending only on fixed number.

1.3. **Main results.**

1.3.1. *Cauchy theory and convergence to equilibrium.* We develop a Cauchy theory of perturbative solutions in "large" spaces for $\gamma \in [-2, 1]$. We also deal with the problem of convergence to equilibrium of the constructed solutions, we prove an exponential convergence to equilibrium. Let us now state our assumptions for the main result.

(H0) Assumptions for Theorem 1.1:

- **Hard potentials $\gamma \in (0, 1]$ and Maxwellian molecules $\gamma = 0$:**
  - (i) *Polynomial weight*: $m = \langle v\rangle^k$ with $k > \gamma + 7 + 3/2$.
  - (ii) *Stretched exponential weight*: $m = e^{r\langle v\rangle^s}$ with $r > 0$ and $s \in (0, 2)$.
  - (iii) *Exponential weight*: $m = e^{r\langle v\rangle^2}$ with $r \in (0, 1/2)$.
- **Moderately soft potentials $\gamma \in [-2, 0)$:**
  - (i) *Stretched exponential weight*: $m = e^{r\langle v\rangle^s}$ with $r > 0$, $s \in (-\gamma, 2)$.
  - (ii) *Exponential weight*: $m = e^{r\langle v\rangle^2}$ with $r \in (0, 1/2)$.

Through the paper, we shall use the notation $\sigma = 0$ when $m = \langle v\rangle^k$ and $\sigma = s$ when $m = e^{r\langle v\rangle^s}$.

We define the space $\mathcal{H}^3_x L^2_v(m)$ (for m a polynomial or exponential weight) associated to the norm

$$\begin{aligned}\|h\|^2_{\mathcal{H}^3_x L^2_v(m)} &= \|h\|^2_{L^2_x L^2_v(m)} + \|\nabla_x h\|^2_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad + \|\nabla^2_x h\|^2_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} + \|\nabla^3_x h\|^2_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})}.\end{aligned} \tag{1.11}$$

We also introduce the velocity space $H^1_{v,*}(m)$ through the norm

$$\|h\|^2_{H^1_{v,*}(m)} = \|h\|^2_{L^2_v(m\langle v\rangle^{(\gamma+\sigma)/2})} + \|P_v\nabla_v h\|^2_{L^2_v(m\langle v\rangle^{\gamma/2})} + \|(I - P_v)\nabla_v h\|^2_{L^2_v(m\langle v\rangle^{(\gamma+2)/2})}, \tag{1.12}$$

with P_v the projection onto v, namely $P_v\xi = \left(\xi \cdot \frac{v}{|v|}\right)\frac{v}{|v|}$, as well as the space $\mathcal{H}^3_x(H^1_{v,*}(m))$ associated to

$$\begin{aligned}\|h\|^2_{\mathcal{H}^3_x(H^1_{v,*}(m))} &= \|h\|^2_{L^2_x(H^1_{v,*}(m))} + \|\nabla_x h\|^2_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)}))} \\ &\quad + \|\nabla^2_x h\|^2_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)}))} + \|\nabla^3_x h\|^2_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)}))} \\ &= \int_{\mathbb{T}^3_x} \|h\|^2_{H^1_{v,*}(m)} + \int_{\mathbb{T}^3_x} \|\nabla_x h\|^2_{H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad + \int_{\mathbb{T}^3_x} \|\nabla^2_x h\|^2_{H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} + \int_{\mathbb{T}^3_x} \|\nabla^3_x h\|^2_{H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}.\end{aligned} \tag{1.13}$$

Here are the main results on the fully nonlinear problem (1.7) that we prove in what follows. For simplicity denote $X := \mathcal{H}^3_x L^2_v(m)$ and $Y := \mathcal{H}^3_x(H^1_{v,*}(m))$ (see (1.11) and (1.13)).

Theorem 1.1. *Consider assumption* **(H0)** *with some weight function m. We assume that f_0 satisfies* (1.10) *and also that $F_0 = \mu + f_0 \geq 0$. There is a constant $\epsilon_0 = \epsilon_0(m) > 0$ such that if $\|f_0\|_X \leq \epsilon_0$, then there exists a unique global weak solution f to the Landau equation* (1.7)*, which satisfies, for some constant $C > 0$,*

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \leq C\epsilon_0.$$

Moreover, this solution verifies an exponential decay: for any $0 < \lambda_2 < \lambda_1$ there exists $C > 0$ such that

$$\forall\, t \geq 0, \quad \|f(t)\|_X \leq C\, e^{-\lambda_2 t}\, \|f_0\|_X,$$

where $\lambda_1 > 0$ is the optimal rate given by the semigroup decay of the associated linearized operator in Theorem 2.1.

Let us comment our result and give an overview on the previous works on the Cauchy theory for the inhomogeneous Landau equation. For general large data, we refer to the papers of DiPerna-Lions [7] for global existence of the so-called renormalized solutions in the case of the Boltzmann equation. This notion of solution have been extend to the Landau equation by Alexandre-Villani [1] where they construct global renormalized solutions with a defect measure. We also mention the work of Desvillettes-Villani [6] that proves the convergence to equilibrium of a priori smooth solutions for both Boltzmann and Landau equations for general initial data.

In a close-to-equilibrium framework, Guo in [9] has developed a theory of perturbative solutions in a space with a weight prescribed by the equilibrium of type $H^N_{x,v}(\mu^{-1/2})$, for any $N \geq 8$, and for all cases $\gamma \in [-3, 1]$, using an energy method. Later, for $\gamma \in [-2, 1]$, Mouhot-Neumann [15] improve this result to $H^N_{x,v}(\mu^{-1/2})$, for any $N \geq 4$.

Let us underline the fact that Theorem 1.1 largely improves previous results on the Cauchy theory associated to the Landau equation in a perturbative setting. Indeed, we considerably have enlarged the space in which the Cauchy theory has been developed in two ways: the weight of our space is much less restrictive (it can be a polynomial or stretched exponential weight instead of the inverse Maxwellian equilibrium) and we also require less assumptions on the derivatives, in particular no derivatives in the velocity variable.

Moreover, we also deal with the problem of the decay to equilibrium of the solutions that we construct. This problem has been considered in several papers by Guo and Strain in [17, 18] first for Coulombian interactions ($\gamma = -3$) for which they proved an almost exponential decay and then, they have improved this result dealing with very soft potentials ($\gamma \in [-3, -2)$) and proving a decay to equilibrium with a rate of type $e^{-\lambda t^p}$ with $p \in (0, 1)$. In the case $\gamma \in [-2, 1]$, Yu [25] has proved an exponential decay in $H^N_{x,v}(\mu^{-1/2})$, for any $N \geq 8$, and Mouhot-Neumann [15] in $H^N_{x,v}(\mu^{-1/2})$, for any $N \geq 4$.

We here emphasize that our strategy to prove Theorem 1.1 is completely different from the one of Guo in [9]. Indeed, he uses an energy method and his strategy is purely nonlinear, he directly derives energy estimates for the nonlinear problem while the first step of our proof is the study of the linearized equation and more precisely the study of its spectral properties. Then, we go back to the nonlinear problem combining the new spectral estimates obtained on the linearized equation with some bilinear estimates on the collision operator. Thanks to this method, we are able to develop a Cauchy theory in a space which is much larger than the one from the previous paper [9]. Moreover, we obtain the convergence of solutions towards the equilibrium with an explicit exponential rate.

Our strategy is thus based on the study of the linearized equation. And then, we go back to the fully nonlinear problem. This is a standard strategy to develop a Cauchy theory in a close-to-equilibrium regime. However, we have to emphasize here that our study of the nonlinear problem is very tricky. Indeed, usually (for example in the case of the non-homogeneous Boltzmann equation for hard spheres in [8]), the gain induced by the linear part of the equation allows directly to control the nonlinear part of the equation so that the linear part is dominant and we can use the decay of the semigroup of the linearized equation. In our case, it is more difficult because the gain induced by the linear part is anisotropic and it is not possible to conclude using only natural estimates on the bilinear Landau operator. As a consequence, we establish some new very accurate estimates on the Landau operator to be able to deal with this problem.

Since the study of the linearized equation is the cornerstone of the proof of our main result, we here present the result that we obtain on it and briefly remind previous results.

1.3.2. *The linearized equation.* We remind the definition of the linearized operator at first order around the equilibrium:

$$\Lambda f = Q(\mu, f) + Q(f, \mu) - v \cdot \nabla_x f.$$

We study spectral properties of the linearized operator Λ in various weighted Sobolev spaces $W^{n,p}_x W^{\ell,p}_v$. Let us state our main result on the linearized operator (see Theorem 2.1 for a precise statement), which widely generalizes previous results since we are able to deal with a more general class of spaces.

Theorem 1.2. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)** *defined in Subsection 2.1 and a weight function m. Let $\mathcal{E}$ be one of the admissible spaces defined in* (2.2)*. Then, there exist explicit constants $\lambda_1 > 0$ and $C > 0$ such that*

$$\forall\, t \geq 0, \quad \forall\, f \in \mathcal{E}, \quad \|S_\Lambda(t) f - \Pi_0 f\|_{\mathcal{E}} \leq C\, e^{-\lambda_1 t}\, \|f - \Pi_0 f\|_{\mathcal{E}},$$

where $S_\Lambda(t)$ is the semigroup associated to Λ and Π_0 the projector onto the null space of Λ by (1.16).

We first make a brief review on known results on spectral gap properties of the homogeneous linearized operator $\mathcal{L}$ defined in (1.8). On the Hilbert space $L^2_v(\mu^{-1/2})$, a simple computation gives that $\mathcal{L}$ is self-adjoint and $\langle \mathcal{L}h, h\rangle_{L^2_v(\mu^{-1/2})} \leq 0$, which implies that the spectrum of $\mathcal{L}$ on $L^2_v(\mu^{-1})$ is included in $\mathbb{R}^-$. Moreover, the nullspace is given by

$$N(\mathcal{L}) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}.$$

We can now state the existing results on the spectral gap of $\mathcal{L}$ on $L^2_v(\mu^{-1/2})$. Summarising results of Degond and Lemou [5], Guo [9], Baranger and Mouhot [2], Mouhot [13], Mouhot and Strain [16] for all cases $\gamma \in [-3, 1]$, we have: there is a constructive constant $\lambda_0 > 0$ (spectral gap) such that

$$\langle -\mathcal{L}h, h\rangle_{L^2_v(\mu^{-1/2})} \geq \lambda_0 \|h\|^2_{H^1_{v,**}(\mu^{-1/2})}, \quad \forall\, h \in N(\mathcal{L})^\perp, \tag{1.14}$$

where the anisotropic norm $\|\cdot\|_{H^1_{v,**}(\mu^{-1/2})}$ is defined by

$$\begin{aligned}\|h\|^2_{H^1_{v,**}(\mu^{-1/2})} := {} & \|\langle v\rangle^{\gamma/2} P_v \nabla h\|^2_{L^2_v(\mu^{-1/2})} + \|\langle v\rangle^{(\gamma+2)/2}(I - P_v)\nabla h\|^2_{L^2_v(\mu^{-1/2})} \\ & + \|\langle v\rangle^{(\gamma+2)/2} h\|^2_{L^2_v(\mu^{-1/2})},\end{aligned}$$

where P_v denotes the projection onto the v-direction, more precisely $P_v g = \left(\frac{v}{|v|} \cdot g\right) \frac{v}{|v|}$. We also have from [9] the reverse inequality, which implies a spectral gap for $\mathcal{L}$ in $L^2_v(\mu^{-1/2})$ if and only if $\gamma + 2 \geq 0$.

Let us now mention the works which have studied spectral properties of the full linearized operator $\Lambda = \mathcal{L} - v \cdot \nabla_x$. Mouhot and Neumann [15] prove explicit coercivity estimates for hard and moderately soft potentials ($\gamma \in [-2, 1]$) in $H^\ell_{x,v}(\mu^{-1/2})$ for $\ell \geq 1$, using the known spectral estimate for $\mathcal{L}$ in (1.14). It is worth mentioning that the third author has obtained in [23] an exponential decay to equilibrium for the full linearized equation in $L^2_{x,v}(\mu^{-1/2})$ by a different method, and the decay rate depends on the size of the domain. Let us summarize results that we will use in the remainder of the paper in the following theorem.

Theorem 1.3 ([15])**.** *Consider $\ell_0 \geq 1$ and $E := H^{\ell_0}_{x,v}(\mu^{-1/2})$. Then, there exists a constructive constant $\lambda_0 > 0$ (spectral gap) such that Λ satisfies on E:*

(i) the spectrum $\Sigma(\Lambda) \subset \{z \in \mathbb{C} : \Re e z \leq -\lambda_0\} \cup \{0\}$;
(ii) the null space $N(\Lambda)$ is given by

$$N(\Lambda) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}, \tag{1.15}$$

and the projection Π_0 *onto* $N(\Lambda)$ *by*

$$\begin{aligned}\Pi_0 f = &\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f\,dx\,dv\right)\mu + \sum_{i=1}^{3}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} v_i f\,dx\,dv\right) v_i\mu \\ &+ \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} \frac{|v|^2-3}{6}\,f\,dx\,dv\right)\frac{(|v|^2-3)}{6}\,\mu;\end{aligned} \tag{1.16}$$

(iii) Λ *is the generator of a strongly continuous semigroup* $S_\Lambda(t)$ *that satisfies*

$$\forall\, t\geq 0,\ \forall\, f\in E,\quad \|S_\Lambda(t)f-\Pi_0 f\|_E \leq e^{-\lambda_0 t}\|f-\Pi_0 f\|_E. \tag{1.17}$$

To prove Theorem 1.2, our strategy follows the one initiated by Mouhot in [14] for the homogeneous Boltzmann equation for hard potentials with cut-off. The latter theory has then been developed and extend in an abstract setting by Gualdani, Mischler and Mouhot [8], and Mischler and Mouhot [11]. They have applied it to Fokker-Planck equations and the spatially inhomogeneous Boltzmann equation for hard spheres. This strategy has also been used for the homogeneous Landau equation for hard and moderately soft potentials by the first author in [3, 4] and by the second author for the fractional Fokker-Planck equation and the homogeneous Boltzmann equation for hard potentials without cut-off in [19, 20] (see also [12] for related works).

Let us describe in more details this strategy. We want to apply the abstract theorem of enlargement of the space of semigroup decay from [8, 11] to our linearized operator Λ. We shall deduce the spectral/semigroup estimates of Theorem 1.2 on "large spaces" $\mathcal{E}$ using the already known spectral gap estimates for Λ on $H^\ell_{x,v}(\mu^{-1/2})$, for $\ell\geq 1$, described in Theorem 1.3. Roughly speaking, to do that, we have to find a splitting of Λ into two operators $\Lambda=\mathcal{A}+\mathcal{B}$ which satisfy some properties. The first part $\mathcal{A}$ has to be bounded, the second one $\mathcal{B}$ has to have some dissipativity properties, and also the semigroup $(\mathcal{A}\mathcal{S}_{\mathcal{B}}(t))$ is required to have some regularization properties.

We end this introduction by describing the organization of the paper. In Section 2 we consider the linearized equation and prove a precise version of Theorem 1.2. In Section 3 we come back to the nonlinear equation and prove our main result Theorem 1.1.

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2. The linearized equation

2.1. **Functional spaces.** Let us now make our assumptions on the different potentials γ and weight functions $m=m(v)$:

(H1) Hard potentials $\gamma\in(0,1]$**.** For $p\in[1,\infty]$ we consider the following cases

- *(i) Polynomial weight*: let $m=\langle v\rangle^k$ with $k>\gamma+2+3(1-1/p)$, and define the abscissa $\lambda_{m,p}:=\infty$.
- *(ii) Stretched exponential weight*: let $m=e^{r\langle v\rangle^s}$ with $r>0$ and $s\in(0,2)$, and define the abscissa $\lambda_{m,p}:=\infty$.
- *(iii) Exponential weight*: let $m=e^{r\langle v\rangle^2}$ with $r\in(0,1/2)$ and define the abscissa $\lambda_{m,p}:=\infty$.

(H2) Maxwellian molecules $\gamma = 0$**.** For $p \in [1, \infty]$ we consider the following cases

(*i*) *Polynomial weight*: let $m = \langle v \rangle^k$ with $k > \gamma + 2 + 3(1 - 1/p)$, and define the abscissa $\lambda_{m,p} := 2[k - (\gamma + 3)(1 - 1/p)]$.

(*ii*) *Stretched exponential weight*: let $m = e^{r\langle v \rangle^s}$ with $r > 0$ and $s \in (0, 2)$, and define the abscissa $\lambda_{m,p} := \infty$.

(*iii*) *Exponential weight*: let $m = e^{r\langle v \rangle^2}$ with $r \in (0, 1/2)$ and define the abscissa $\lambda_{m,p} := \infty$.

(H3) Moderately soft potentials $\gamma \in [-2, 0)$**.** For $p \in [1, \infty]$ we consider the following cases

(*i*) *Stretched exponential weight for* $\gamma \in (-2, 0)$: let $m = e^{r\langle v \rangle^s}$ with $r > 0$, $s \in (0, 2)$ and $s + \gamma > 0$, and define the abscissa $\lambda_{m,p} := \infty$.

(*ii*) *Exponential weight for* $\gamma \in (-2, 0)$: let $m = e^{r\langle v \rangle^2}$ with $r \in (0, 1/2)$ and define the abscissa $\lambda_{m,p} := \infty$.

(*iii*) *Exponential weight for* $\gamma = -2$: let $m = e^{r\langle v \rangle^2}$ with $r \in (0, 1/2)$, and define the abscissa $\lambda_{m,p} := 4r(1 - 2r)$.

Under these hypothesis, we shall use the following notation for the functional spaces:

$$E := H^{\ell_0}_{x,v}(\mu^{-1/2}), \quad \ell_0 \geq 1, \tag{2.1}$$

in which space we already know that the linearized operator Λ has a spectral gap (Theorem 1.3), and also, under hypotheses **(H1)**, **(H2)** or **(H3)**,

$$\mathcal{E} := \begin{cases} L^p_{x,v}(m), & \forall\, p \in [1, \infty]; \\ W^{n,p}_x W^{\ell,p}_v(m), & \forall\, p \in [1, 2],\ n \in \mathbb{N}^*,\ \ell \in \mathbb{N}; \end{cases} \tag{2.2}$$

and for each space we define the associated abscissa $\lambda_{\mathcal{E}} = \lambda_{m,p}$.

The main result of this section, which is a precise version of Theorem 1.2, reads

Theorem 2.1. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)** *with some weight* m*, and let* $\mathcal{E}$ *be one of the admissible spaces defined in* (2.2)*.*

Then, for any $\lambda < \lambda_{\mathcal{E}}$ *and any* $\lambda_1 \leq \min\{\lambda_0, \lambda\}$*, where we recall that* $\lambda_0 > 0$ *is the spectral gap of* Λ *on* E *(see* (1.17)*), there is a constructive constant* $C > 0$ *such that the operator* Λ *satisfies on* $\mathcal{E}$*:*

(i) $\Sigma(\Lambda) \subset \{z \in \mathbb{C} \mid \Re z \leq -\lambda_1\} \cup \{0\}$*;*

(ii) the null-space $N(\Lambda)$ *is given by* (1.15) *and the projection* Π_0 *onto* $N(\Lambda)$ *by* (1.16)*;*

(iii) Λ *is the generator of a strongly continuous semigroup* $\mathcal{S}_\Lambda(t)$ *that verifies*

$$\forall\, t \geq 0,\ \forall\, f \in \mathcal{E}, \quad \|\mathcal{S}_\Lambda(t) f - \Pi_0 f\|_{\mathcal{E}} \leq C\, e^{-\lambda_1 t}\, \|f - \Pi_0 f\|_{\mathcal{E}}.$$

Remark 2.2. (1) Observe that:

- Cases **(H1)**, **(H2)-(ii)-(iii)** or **(H3)-(i)-(ii)**: we can recover the optimal estimate $\lambda_1 = \lambda_0$ since $\lambda_{m,p} = +\infty$.
- Case **(H2)-(i)**: in this case we have $m = \langle v \rangle^k$, and we can recover the optimal estimate $\lambda_1 = \lambda_0$ if $k > 0$ is large enough such that $\lambda_{m,p} = 2k - 6(1 - 1/p) > \lambda_0$. Otherwise, we obtain $\lambda_1 < 2k - 6(1 - 1/p)$.
- Case **(H3)-(iii)**: in this case we have $\gamma = -2$, $m = e^{r\langle v \rangle^2}$ and $\lambda_{m,p} = 4r(1 - 2r)$ and the condition $0 < r < 1/2$.

(2) This theorem also holds for other choices of space, namely for a space $\mathcal{E}$ that is an interpolation space of two admissible spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ in (2.2).

The proof of Theorem 2.1 uses the fact that the properties (i)-(ii)-(iii) with $\lambda_1 = \lambda_0$ hold on the small space E (Theorem 1.3) and the strategy described in section 1.3.2.

In a similar way we shall obtain Theorem 2.1, we shall also deduce a regularity estimate on the semigroup $\mathcal{S}_\Lambda$ that will be of crucial importance in the study of the nonlinear equation in Section 3. For the sake of simplicity, and because it is the case that we shall use for the nonlinear equation, we only present this result for the particular case of $p = 2$ and $\ell = 0$ in (2.2).

Define the space $H^1_{v,*}(m)$, associated to the norm

$$\|f\|^2_{H^1_{v,*}(m)} = \|f\|^2_{L^2_{x,v}(m\langle v\rangle^{(\gamma+\sigma)/2})} + \|P_v\nabla f\|^2_{L^2(m\langle v\rangle^{\gamma/2})} + \|(I-P_v)\nabla f\|^2_{L^2(m\langle v\rangle^{(\gamma+2)/2})},$$

as well as the space $H^n_x(H^1_{v,*}(m))$, with $n \in \mathbb{N}$, by

$$\|f\|^2_{H^n_x(H^1_{v,*}(m))} = \sum_{0\le j\le n} \|\nabla^j_x f\|^2_{L^2_x(H^1_{v,*}(m))} = \sum_{0\le j\le n} \int_{\mathbb{T}^3_x} \|\nabla^j_x f\|^2_{H^1_{v,*}(m)}. \tag{2.3}$$

We hence define the negative Sobolev space $H^n_x(H^{-1}_{v,*}(m))$ by duality in the following way

$$\|f\|_{H^n_x(H^{-1}_{v,*}(m))} := \sup_{\|\phi\|_{H^n_x(H^1_{v,*}(m))}\le 1} \langle f,\phi\rangle_{H^n_x L^2_v(m)}. \tag{2.4}$$

Theorem 2.3. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)** *with some weight m. Let $\mathcal{E} = H^n_x L^2_v(m)$ and $\mathcal{E}_{-1} = H^n_x(H^{-1}_{v,*}(m))$. Then, for any $\lambda < \lambda_1$, the following regularity estimate holds*

$$\int_0^\infty e^{2\lambda t}\,\|\mathcal{S}_\Lambda(t)(I-\Pi_0)f\|^2_{\mathcal{E}}\,dt \le C\|(I-\Pi_0)f\|^2_{\mathcal{E}_{-1}}, \tag{2.5}$$

for some constant $C > 0$.

2.2. **Splitting of the linearized operator.** We decompose the linearized Landau operator $\mathcal{L}$ defined in (1.8) as $\mathcal{L} = \mathcal{A}_0 + \mathcal{B}_0$, where we define

$$\mathcal{A}_0 f := (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu, \qquad \mathcal{B}_0 f := (a_{ij} * \mu)\partial_{ij} f - (c * \mu) f. \tag{2.6}$$

Consider a smooth positive function $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^3_v)$ such that $0 \le \chi(v) \le 1$, $\chi(v) \equiv 1$ for $|v| \le 1$ and $\chi(v) \equiv 0$ for $|v| > 2$. For any $R \ge 1$ we define $\chi_R(v) := \chi(R^{-1}v)$ and in the sequel we shall consider the function $M\chi_R$, for some constant $M > 0$.

Then, we make the final decomposition of the operator Λ as $\Lambda = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A} := \mathcal{A}_0 + M\chi_R, \qquad \mathcal{B} := \mathcal{B}_0 - v\cdot\nabla_x - M\chi_R, \tag{2.7}$$

where $M > 0$ and $R > 0$ will be chosen later (see Lemma 2.7).

2.3. **Preliminaries.** We have the following results concerning the matrix $\bar{a}_{ij}(v)$.

Lemma 2.4. *The following properties hold:*

(a) The matrix $\bar{a}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace $v^\perp$. Moreover, when $|v| \to +\infty$ we have

$$\ell_1(v) \sim 2\langle v\rangle^\gamma \quad \text{and} \quad \ell_2(v) \sim \langle v\rangle^{\gamma+2}.$$

(b) The function $\bar{a}_{ij}$ is smooth, for any multi-index $\beta \in \mathbb{N}^3$

$$|\partial^\beta \bar{a}_{ij}(v)| \le C_\beta \langle v\rangle^{\gamma+2-|\beta|}$$

and

$$\begin{aligned}\bar{a}_{ij}(v)\xi_i\xi_j &= \ell_1(v)|P_v\xi|^2 + \ell_2(v)|(I-P_v)\xi|^2\\ &\ge c_0\big\{\langle v\rangle^\gamma|P_v\xi|^2 + \langle v\rangle^{\gamma+2}|(I-P_v)\xi|^2\big\},\end{aligned}$$

for some constant $c_0 > 0$ and where P_v is the projection on v, i.e. $P_v\xi_i = \left(\xi\cdot\frac{v}{|v|}\right)\frac{v_i}{|v|}$.

(c) We have

$$\bar{a}_{ii}(v) = \mathrm{tr}(\bar{a}(v)) = \ell_1(v) + 2\ell_2(v) = 2\int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\mu(v_*)\,dv_* \qquad \text{and} \qquad \bar{b}_i(v) = -\ell_1(v)\,v_i.$$

(d) If $|v| > 1$, we have

$$|\partial^\beta \ell_1(v)| \le C_\beta \langle v\rangle^{\gamma-|\beta|} \qquad \text{and} \qquad |\partial^\beta \ell_2(v)| \le C_\beta \langle v\rangle^{\gamma+2-|\beta|}.$$

Proof. We just give the proof of item (d) since (a) comes from [5, Propositions 2.3 and 2.4, Corollary 2.5], (b) is [9, Lemma 3] and (c) is evident. For item (d), the estimate of $|\partial^\beta \ell_2(v)|$ directly comes from (a) and [9, Lemma 2]. For $\ell_1(v)$, using (b) and (c),

$$\partial_v \bar{b}_i(v) = \partial_v\big(-\ell_1(v)v_i\big),$$

and hence

$$|\partial_v \ell_1(v)||v| \le C\big(|\ell_1(v)| + |\partial_v \bar{b}_i(v)|\big) \le C\langle v\rangle^\gamma,$$

note that $|v| > 1$, we thus have

$$|\partial_v \ell_1(v)| \le C|v|^{-1}\langle v\rangle^\gamma \le C\langle v\rangle^{\gamma-1}.$$

The high order estimate is similar and hence we omit the details. □

The following elementary lemma will be useful in the sequel (see [3, Lemma 2.5] and [4, Lemma 2.3]).

Lemma 2.5. *Let $J_\alpha(v) := \int_{\mathbb{R}^3} |v-w|^\alpha \mu(w)\,dw$, for $-3 < \alpha \le 3$. Then it holds:*
(a) If $2 < \alpha \le 3$, then $J_\alpha(v) \le |v|^\alpha + C_\alpha|v|^{\alpha/2} + C_\alpha$, for some constant $C_\alpha > 0$.
(b) If $0 \le \alpha \le 2$, then $J_\alpha(v) \le |v|^\alpha + C_\alpha$, for some constant $C_\alpha > 0$.
(c) If $-3 < \alpha < 0$, then $J_\alpha(v) \le C\langle v\rangle^\alpha$ for some constant $C > 0$.

We define the function $\varphi_{m,p}$ as

$$\varphi_{m,p}(v) := \bar{a}_{ij}(v)\frac{\partial_{ij} m}{m} + (p-1)\bar{a}_{ij}(v)\frac{\partial_i m}{m}\frac{\partial_j m}{m} + 2\bar{b}_i(v)\frac{\partial_i m}{m} + \left(\frac{1}{p} - 1\right)\bar{c}(v), \tag{2.8}$$

and also the function $\tilde{\varphi}_{m,p}$ given by

$$\begin{aligned}\tilde{\varphi}_{m,p}(v) := &\left(\frac{2}{p} - 1\right)\bar{a}_{ij}(v)\frac{\partial_{ij} m}{m} + \left(2 - \frac{2}{p}\right)\bar{a}_{ij}(v)\frac{\partial_i m}{m}\frac{\partial_j m}{m} \\ &+ \frac{2}{p}\,\bar{b}_i(v)\frac{\partial_i m}{m} + \left(\frac{1}{p} - 1\right)\bar{c}(v),\end{aligned} \tag{2.9}$$

and hereafter, in order to treat both weight functions at the same time, we remind the notation: $\sigma = 0$ when $m = \langle v\rangle^k$ and $\sigma = s$ when $m = e^{r\langle v\rangle^s}$.

We prove the following result concerning $\varphi_{m,p}$ and $\tilde{\varphi}_{m,p}$.

Lemma 2.6. *Consider* **(H1)**, **(H2)** *or* **(H3)**, *and let $\varphi_{m,p}$ and $\tilde{\varphi}_{m,p}$ be defined in* (2.8) *and* (2.9) *respectively. Then we have:*

- *Assume $\sigma \in [0, 2)$:*

(1) For all positive $\lambda < \lambda_{m,p}$ and $\delta \in (0, \lambda_{m,p} - \lambda)$ we can choose M and R large enough such that

$$\varphi_{m,p}(v) - M\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$
$$\tilde{\varphi}_{m,p}(v) - M\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$

(2) For all positive $\lambda < \lambda_{m,p}$ and $\delta \in (0, \lambda_{m,p} - \lambda)$ we can choose M and R large enough such that

$$\varphi_{m,p}(v) - M\chi_R(v) + M\partial_j\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$
$$\tilde{\varphi}_{m,p}(v) - M\chi_R(v) + M\partial_j\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.$$

• *Assume $\sigma = 2$: The same conclusion as before holds for $\tilde{\varphi}_{m,p}$. Moreover, concerning $\varphi_{m,p}$, the previous estimates also hold if we restrict $r \in (0, 1/(2p))$ in assumptions* **(H1)-(iii), (H2)-(iii), (H3)-(ii)**, *and also modifying the value of the abscissa $\lambda_{m,p} = 4r(1-2rp)$ in* **(H3)-(iii)**.

Proof of Lemma 2.6. Step 1. Polynomial weight. Consider $m = \langle v\rangle^k$ under hypothesis **(H1)** or **(H2)**. On the one hand, we have

$$\frac{\partial_i m}{m} = k v_i \langle v\rangle^{-2}, \qquad \frac{\partial_i m}{m}\,\frac{\partial_j m}{m} = k^2 v_i v_j \langle v\rangle^{-4},$$
$$\frac{\partial_{ij} m}{m} = \delta_{ij}\, k\langle v\rangle^{-2} + k(k-2) v_i v_j \langle v\rangle^{-4}.$$

Hence, from definitions (1.4)-(1.9) and Lemma 2.4 we obtain

$$\bar{a}_{ij}\,\frac{\partial_{ij} m}{m} = (\delta_{ij}\bar{a}_{ij})\, k\langle v\rangle^{-2} + (\bar{a}_{ij} v_i v_j)\, k(k-2)\langle v\rangle^{-4} = \bar{a}_{ii}\, k\langle v\rangle^{-2} + \ell_1(v)\, k(k-2)|v|^2\langle v\rangle^{-4},$$

where we recall that the eigenvalue $\ell_1(v) > 0$ is defined in Lemma 2.4. Moreover, arguing exactly as above we obtain

$$\bar{a}_{ij}\,\frac{\partial_i m}{m}\,\frac{\partial_j m}{m} = (\bar{a}_{ij} v_i v_j)\, k^2\langle v\rangle^{-4} = \ell_1(v)\, k^2 |v|^2 \langle v\rangle^{-4}$$

and also, using the fact that $\bar{b}_i(v) = -\ell_1(v) v_i$ from Lemma 2.4,

$$\bar{b}_i\,\frac{\partial_i m}{m} = -\ell_1(v) v_i\, k v_i \langle v\rangle^{-2} = -\ell_1(v)\, k|v|^2\langle v\rangle^{-2}.$$

On the other hand, from item (c) of Lemma 2.4 and definitions (1.4)-(1.9) we obtain that

$$\bar{a}_{ii}(v) = \ell_1(v) + 2\ell_2(v) \quad \text{and} \quad \bar{c}(v) = -2(\gamma+3) J_\gamma(v),$$

where J_α is defined in Lemma 2.5. It follows that

$$\begin{aligned}\varphi_{m,p}(v) &= 2k\ell_2(v)(v)\langle v\rangle^{-2} + k\ell_1(v)\langle v\rangle^{-2} + k(k-2)\,\ell_1(v)\,|v|^2\langle v\rangle^{-4} \\ &\quad + (p-1)k^2\,\ell_1(v)\,|v|^2\langle v\rangle^{-4} - 2k\,\ell_1(v)\,|v|^2\langle v\rangle^{-2} + 2(\gamma+3)\left(1-\frac{1}{p}\right) J_\gamma(v).\end{aligned} \tag{2.10}$$

Since $\ell_1(v) \sim 2\langle v\rangle^\gamma$, $\ell_2(v) \sim \langle v\rangle^{\gamma+2}$ and $\ell_1(v)|v|^2 \sim 2\ell_2(v)$ when $|v| \to +\infty$ thanks to Lemma 2.4, and also $J_\gamma(v) \sim \langle v\rangle^\gamma$ from Lemma 2.5 (since in this case we have $\gamma \geq 0$), the dominant terms in (2.10) are the first, fifth and sixth ones, all of order $\langle v\rangle^\gamma$. Then we obtain

$$\limsup_{|v|\to+\infty} \varphi_{m,p}(v) \leq -2\,[k - (\gamma+3)(1-1/p)]\,\langle v\rangle^\gamma, \tag{2.11}$$

and recall that $k > (\gamma+3)(1-1/p)$. Doing the same kind of computations, we obtain the same asymptotic for $\tilde{\varphi}_{m,p}$,

$$\limsup_{|v|\to+\infty} \tilde{\varphi}_{m,p}(v) \leq -2[k - (\gamma+3)(1-1/p)]\langle v\rangle^\gamma. \tag{2.12}$$

Step 2. Stretched exponential weight. We consider now $m = \exp(r\langle v\rangle^s)$ satisfying **(H1)**, **(H2)** or **(H3)**. In this case we have

$$\frac{\partial_i m}{m} = rs v_i \langle v\rangle^{s-2}, \quad \frac{\partial_i m}{m}\frac{\partial_j m}{m} = r^2 s^2 v_i v_j \langle v\rangle^{2s-4},$$
$$\frac{\partial_{ij} m}{m} = rs\langle v\rangle^{s-2}\delta_{ij} + rs(s-2) v_i v_j \langle v\rangle^{s-4} + r^2 s^2 v_i v_j \langle v\rangle^{2s-4}.$$

Then we obtain

$$\begin{aligned}\varphi_{m,p}(v) &= 2rs\,\ell_2(v)\langle v\rangle^{s-2} + rs\,\ell_1(v)\langle v\rangle^{s-2} + rs(s-2)\,\ell_1(v)|v|^2\langle v\rangle^{s-4} \\ &\quad + pr^2 s^2\,\ell_1(v)|v|^2\langle v\rangle^{2s-4} - 2rs\,\ell_1(v)|v|^2\langle v\rangle^{s-2} + 2(\gamma+3)\left(1-\frac{1}{p}\right) J_\gamma(v)\end{aligned} \tag{2.13}$$

In the case $0 < s < 2$, arguing as in step 1, the dominant terms in (2.13) when $|v| \to +\infty$ are the first and fifth one, both of order $\langle v\rangle^{\gamma+s}$. Then we obtain

$$\limsup_{|v|\to+\infty} \varphi_{m,p}(v) \le -2rs\langle v\rangle^{\gamma+s}, \tag{2.14}$$

and recall that $s + \gamma > 0$. In the same way we obtain

$$\limsup_{|v|\to+\infty} \tilde{\varphi}_{m,p}(v) \le -2rs\langle v\rangle^{\gamma+s}. \tag{2.15}$$

In the case $s = 2$, the dominant terms in (2.13) when $|v| \to +\infty$ are the first, fourth and fifth ones, all of order $\langle v\rangle^{\gamma+2}$. Hence we get

$$\limsup_{|v|\to+\infty} \varphi_{m,p}(v) \le -4r(1-2pr)\langle v\rangle^{\gamma+2}. \tag{2.16}$$

However, a similar computation gives

$$\limsup_{|v|\to+\infty} \tilde{\varphi}_{m,p}(v) \le -4r(1-2r)\langle v\rangle^{\gamma+2}, \tag{2.17}$$

which is better than the asymptotic of $\varphi_{m,p}$. Thus we need the condition $r < 1/2$ for $\tilde{\varphi}_{m,p}$ (which is better than the condition $r < 1/(2p)$ for $\varphi_{m,p}$).

Step 3. Conclusion. Finally, thanks to the asymptotic behaviour in (2.11), (2.14) and (2.16), for any $\lambda < \lambda_{m,p}$ we can choose M and R large enough such that $\varphi_{m,p}(v) - M\chi_R(v) \le -\lambda - \delta\langle v\rangle^{\gamma+\sigma}$ for some $\delta > 0$ small enough, which gives us point (1) of the lemma.

For the point (2) we use $\partial_j \chi_R(v) = R^{-1}\partial_j\chi(v/R)$ and write

$$\varphi_{m,p}(v) - M\chi_R(v) + M\partial_j\chi_R(v) \le \varphi_{m,p}(v) - M\chi_R(v) + M\frac{C_\chi}{R}\mathbf{1}_{R\le|v|\le 2R} =: \Phi(v).$$

We fix some $\bar{\lambda} \in (\lambda, \lambda_{m,p})$. First we choose R_1 large enough such that, for all $|v| \ge R_1$, we have

$$\varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} \le -\bar{\lambda}$$

for some $\delta > 0$ small enough, which implies that, for any $|v| \ge 2R_1$,

$$\Phi(v) + \delta\langle v\rangle^{\gamma+\sigma} = \varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} \le -\bar{\lambda}.$$

Then we choose $M > 0$ large enough such that, for all $|v| \le R_1$,

$$\Phi(v) + \delta\langle v\rangle^{\gamma+\sigma} = \varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} - M\chi_{R_1}(v) \le -\bar{\lambda}.$$

Finally, we choose $R \ge R_1$ large enough such that, for any $R \le |v| \le 2R$,

$$\Phi(v) + \delta\langle v\rangle^{\gamma+\sigma} \le \varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} + M\frac{C_\chi}{R} \le -\bar{\lambda} + M\frac{C_\chi}{R} \le -\lambda,$$

and we easily observe that now for $R_1 \le |v| \le R$ we have

$$\Phi(v) + \delta\langle v\rangle^{\gamma+\sigma} = \varphi_{m,p}(v) + \delta\langle v\rangle^{\gamma+\sigma} - M\chi_R(v) \le -\bar{\lambda} - M \le -\lambda,$$

which concludes the proof for $\varphi_{m,p}$. Concerning $\tilde{\varphi}_{m,p}$, in the same way, inequalities (2.12), (2.15) and (2.17) yield the result. □

2.4. **Hypodissipativity.** In this subsection we prove hypodissipativity properties for the operator $\mathcal{B}$ on the admissible spaces $\mathcal{E}$ defined in (2.2).

Hereafter we define the space $W^{1,p}_{v,*}(m)$, with $1<p<\infty$, associated to the norm

$$\|f\|^p_{W^{1,p}_{v,*}(m)} = \|f\|^p_{L^p_{x,v}(m\langle v\rangle^{(\gamma+\sigma)/p})} + \|P_v\nabla f^{p/2}\|^2_{L^2(m^{p/2}\langle v\rangle^{\gamma/2})} + \|(I-P_v)\nabla f^{p/2}\|^2_{L^2(m^{p/2}\langle v\rangle^{(\gamma+2)/2})},$$

as well as the space $W^{n,p}_x(W^{1,p}_{v,*}(m))$, with $n\in\mathbb{N}$, by

$$\|f\|^p_{W^{n,p}_x(W^{1,p}_{v,*}(m))} = \sum_{0\le j\le n}\|\nabla^j_x f\|^p_{L^p_x(W^{1,p}_{v,*}(m))} = \sum_{0\le j\le n}\int_{\mathbb{T}^3_x}\|\nabla^j_x f\|^p_{W^{1,p}_{v,*}(m)}. \tag{2.18}$$

Lemma 2.7. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**. *Let $p\in[1,+\infty]$ and $n\in\mathbb{N}$. Then, for any $\lambda<\lambda_{m,p}$ we can choose $M>0$ and $R>0$ large enough such that the operator $(\mathcal{B}+\lambda)$ is dissipative in $W^{n,p}_xL^p_v(m)$, in the sense that*

$$\forall\, t\ge 0,\quad \|\mathcal{S}_{\mathcal{B}}(t)\|_{\mathscr{B}(W^{n,p}_xL^p_v(m))}\le Ce^{-\lambda t}. \tag{2.19}$$

Moreover there hold: if $p=1$

$$\int_0^\infty e^{\lambda t}\,\|\mathcal{S}_{\mathcal{B}}(t)\|_{\mathscr{B}(W^{n,1}_xL^1_v(m),W^{n,1}_xL^1_v(m\langle v\rangle^{\gamma+\sigma}))}\,dt<\infty, \tag{2.20}$$

and if $1<p<\infty$

$$\int_0^\infty e^{\lambda pt}\,\|\mathcal{S}_{\mathcal{B}}(t)\|^p_{\mathscr{B}(W^{n,p}_xL^p_v(m),W^{n,p}_x(W^{1,p}_{v,*}(m)))}\,dt<\infty, \tag{2.21}$$

Proof of Lemma 2.7. We only consider the case $n=0$, the general case being treated in the sama way since ∇_x commutes with $\mathcal{B}$.

Let us denote $\Phi'(z)=|z|^{p-1}\mathrm{sign}(z)$ and consider the equation

$$\partial_t f=\mathcal{B}f=\mathcal{B}_0 f-v\cdot\nabla_x f-M\chi_R f.$$

For all $p\in[1,+\infty)$, we have

$$\frac1p\frac{d}{dt}\|f\|^p_{L^p_{x,v}(m)}=\int(\mathcal{B}f)\Phi'(f)\,m^p.$$

From (1.5) and (2.6), last integral is equal to

$$\begin{aligned}&\int\bar a_{ij}(v)\partial_{ij}f(x,v)\Phi'(f)m^p-\int\bar c(v)f(x,v)\Phi'(f)m^p\\&\quad-\int v\cdot\nabla_x f(x,v)\Phi'(f)m^p-\int M\chi_R(v)f(x,v)\Phi'(f)m^p\\&=:T_1+T_2+T_3+T_4.\end{aligned}$$

The term T_3 vanishes thanks to its divergence structure and terms T_2 and T_4 are easily computed, giving

$$T_2=-\int\bar c(v)|f(x,v)|^pm^p\quad\text{and}\quad T_4=-\int M\chi_R(v)|f(x,v)|^pm^p.$$

Let us compute then the term T_1. Using that $\partial_{ij}f\Phi'(f)=p^{-1}\partial_{ij}(|f|^p)-(p-1)\partial_i f\partial_j f|f|^{p-2}$ we obtain

$$T_1=\frac1p\int\bar a_{ij}(v)\partial_{ij}(|f|^p)m^p-(p-1)\int\bar a_{ij}(v)\partial_i f\partial_j f\,|f|^{p-2}m^p.$$

Performing two integrations by parts on the first integral of T_1 it yields

$$\begin{aligned}\int(\mathcal{B}f)\Phi'(f)\,m^p=&-\frac{4}{p^2}(p-1)\int\bar a_{ij}(v)\,\partial_i(f^{p/2})\,\partial_j(f^{p/2})\,m^p\\&+\int\{\varphi_{m,p}(v)-M\chi_R(v)\}\,|f|^p\,m^p,\end{aligned}$$

where $\varphi_{m,p}$ is defined in (2.8). We can also get, by a similar computation,

$$\int (\mathcal{B}f)\Phi'(f)\, m^p = -\frac{4}{p^2}(p-1)\int \bar{a}_{ij}(v)\,\partial_i(mf^{p/2})\,\partial_j(mf^{p/2})\, m^{p-2} + \int \{\tilde{\varphi}_{m,p}(v) - M\chi_R(v)\}\,|f|^p\, m^p.$$

Thanks to Lemma 2.6, for any $\lambda < \lambda_{m,p}$ we can choose M and R large enough such that $\varphi_{m,p}(v) - M\chi_R(v) \le -\lambda + \delta\langle v\rangle^{\gamma+\sigma}$. Hence it follows, using Lemma 2.4,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|f\|^p_{L^p(m)} \le -c_0(p-1)\int \{\langle v\rangle^\gamma |P_v\nabla_v(f^{p/2})|^2 + \langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v(f^{p/2})|^2\}\, m^p \\ - \lambda\|f\|^p_{L^p(m)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{p}} f\|^p_{L^p(m)}.\end{aligned} \tag{2.22}$$

or

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|f\|^p_{L^p(m)} \le -c_0(p-1)\int \{\langle v\rangle^\gamma |P_v\nabla_v(mf^{p/2})|^2 + \langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v(mf^{p/2})|^2\}\, m^{p-2} \\ - \lambda\|f\|^p_{L^p(m)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{p}} f\|^p_{L^p(m)},\end{aligned} \tag{2.23}$$

from which we easily obtain (2.19) for any $1 \le p < \infty$. For $p = \infty$, let $g = mf$, it is easy to check that g satisfies the equation

$$\partial_t g + v\cdot\nabla_x g = \bar{a}_{ij}(v)\partial_{ij}g - 2\bar{a}_{ij}(v)\frac{\partial_i m}{m}\partial_j g + \tilde{\varphi}_{m,\infty}(v)g - M\chi_R(v)g\,,$$

by the standard maximum principle argument (for example, see [24]), we have

$$\|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^\infty_{x,v}(m)} \le e^{-\lambda t}\|f\|_{L^\infty_{x,v}(m)}.$$

This completes the proof of (2.19).

The proof of (2.20) and (2.21) follows easily from (2.22) by keeping all the terms at the right-hand side and integrating in time. □

Define the operator $\mathcal{B}_m$ by $\mathcal{B}_m h := m\mathcal{B}(m^{-1}h)$, more precisely

$$\begin{aligned}\mathcal{B}_m h &= \bar{a}_{ij}\partial_{ij}h - 2\bar{a}_{ij}\frac{\partial_i m}{m}\partial_j h + \left\{2\bar{a}_{ij}\frac{\partial_i m}{m}\frac{\partial_j m}{m} - \bar{a}_{ij}\frac{\partial_{ij}m}{m} - \bar{c} - M\chi_R\right\}h - v\cdot\nabla_x h \\ &=: \bar{a}_{ij}\partial_{ij}h + \beta_j\partial_j h + (\zeta - M\chi_R)h - v\cdot\nabla_x h.\end{aligned} \tag{2.24}$$

Observe that if f satisfies $\partial_t f = \mathcal{B}f$, then $h := mf$ satisfies $\partial_t h = \mathcal{B}_m h$. We then define the operator $\mathcal{B}^*_m$, the (formal) adjoint operator of $\mathcal{B}_m$ with respect to the usual scalar product $L^2_{x,v}$, by

$$\begin{aligned}\mathcal{B}^*_m\phi &= \bar{a}_{ij}\partial_{ij}\phi + \left\{2\bar{b}_j + 2\bar{a}_{ij}\frac{\partial_i m}{m}\right\}\partial_j\phi + \left\{\bar{a}_{ij}\frac{\partial_{ij}m}{m} + 2\bar{b}_j\frac{\partial_j m}{m} - M\chi_R\right\}\phi + v\cdot\nabla_x\phi \\ &=: \bar{a}_{ij}\partial_{ij}\phi + \beta^*_j\partial_i\phi + (\zeta^* - M\chi_R)\phi + v\cdot\nabla_x\phi.\end{aligned} \tag{2.25}$$

Remark that, denoting $h_t := \mathcal{S}_{\mathcal{B}_m}(t)h_0$ and $\phi_t := \mathcal{S}_{\mathcal{B}^*_m}(t)\phi_0$, which verify the equations $\partial_t h_t = \mathcal{B}_m h_t$ and $\partial_t\phi_t = \mathcal{B}^*_m\phi_t$, there holds

$$\langle h_t, \phi_0\rangle_{H^n_x L^2_v} = \langle h_0, \phi_t\rangle_{H^n_x L^2_v}.$$

Lemma 2.8. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, *and let* $n \in \mathbb{N}$. *Then, for any* $\lambda < \lambda_{m,2}$, *we can choose* M *and* R *large enough such that the operator* $(\mathcal{B}^*_m + \lambda)$ *is hypo-dissipative in* $H^n_x L^2_v$, *in the sense that*

$$\forall\, t \ge 0, \quad \|S_{\mathcal{B}^*_m}(t)\|_{\mathscr{B}(H^n_x L^2_v)} \le Ce^{-\lambda t}. \tag{2.26}$$

Moreover there holds

$$\int_0^\infty e^{2\lambda t}\,\|S_{\mathcal{B}}(t)\|^2_{\mathscr{B}(H^n_x(H^{-1}_{v,*}(m)),H^n_xL^2_v(m))}\,dt\le\infty, \tag{2.27}$$

where we recall that $H^n_x(H^{-1}_{v,*}(m))$ *is defined in* (2.4).

Proof. We consider the case $n=0$, the others being the same because ∇_x commutes with $\mathcal{B}^*_m$.

Let $\partial_t\phi=\mathcal{B}^*_m\phi$, where we recall that $\mathcal{B}^*_m$ is defined in (2.25). We have

$$\begin{aligned}\int(\mathcal{B}^*_m\phi)\,\phi&=\int\left(\bar a_{ij}\frac{\partial_{ij}m}{m}+2\bar b_j\frac{\partial_j m}{m}-M\,\chi_R\right)\phi^2\\&\quad+\int\left(\bar a_{ij}\frac{\partial_j m}{m}+\bar b_i\right)\partial_i(\phi^2)+\int v\cdot\nabla_x\phi\,\phi+\int\bar a_{ij}\partial_{ij}\phi\,\phi\\&=:T_1+T_2+T_3+T_4.\end{aligned}$$

Performing one integration by parts, we obtain

$$T_2=\int\left(-\bar a_{ij}\frac{\partial_{ij}m}{m}+\bar a_{ij}\frac{\partial_i m}{m}\frac{\partial_j m}{m}-\bar b_j\frac{\partial_j m}{m}-\bar c\right)\phi^2.$$

The term T_3 gives no contribution thanks to its divergence structure in x. Moreover we deal with T_4 using that $\partial_{ij}\phi\,\phi=\frac12\partial_{ij}(\phi^2)-\partial_i\phi\partial_j\phi$, which implies

$$T_4=-\int\bar a_{ij}\partial_i\phi\partial_j\phi+\frac12\int\bar c\,\phi^2.$$

Finally, we obtain that

$$\begin{aligned}\int(\mathcal{B}^*_m\phi)\,\phi&=-\int\bar a_{ij}\partial_i\phi\partial_j\phi+\int\{\tilde\varphi_{m,2}-M\chi_R\}\,\phi^2\\&\le-c_0\int\left\{\langle v\rangle^\gamma|P_v\nabla_v\phi|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v\phi|^2\right\}+\int\{\tilde\varphi_{m,2}-M\chi_R\}\,\phi^2.\end{aligned} \tag{2.28}$$

where we recall that $\tilde\varphi_{m,2}$ is defined in (2.9).

Thanks to Lemma 2.6, for any positive $\lambda<\lambda_{m,2}$ and $\delta\in(0,\lambda_{m,2}-\lambda)$, we can thus find $M,R>0$ large enough such that $\tilde\varphi_{m,2}(v)-M\chi_R\le-\lambda-\delta\langle v\rangle^{\gamma+\sigma}$. We can conclude that

$$\begin{aligned}\frac12\frac{d}{dt}\|\phi\|^2_{L^2}&\le-\lambda\|\phi\|^2_{L^2}-\delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\phi\|^2_{L^2}\\&\quad-c_0\left\{\|\langle v\rangle^{\frac\gamma2}P_v\nabla_v\phi\|^2_{L^2(m)}+\|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v\phi\|^2_{L^2(m)}\right\}.\end{aligned}$$

From this inequality we easily obtain (2.26) and also the regularity estimate

$$\int_0^\infty e^{2\lambda t}\,\|\mathcal{S}_{\mathcal{B}^*_m}(t)\phi\|^2_{L^2_x(H^1_{v,*})}\,dt\lesssim\|\phi\|^2_{L^2_xL^2_v}.$$

Consider now the function h that satisfies $\partial_t h=\mathcal{B}_m h$. Using that $\langle\mathcal{S}_{\mathcal{B}_m}(t)h,\phi\rangle_{H^n_xL^2_v}=\langle h,\mathcal{S}_{\mathcal{B}^*_m}(t)\phi\rangle_{H^n_xL^2_v}$, this last estimate implies by duality (see (2.4))

$$\int_0^\infty e^{2\lambda t}\,\|\mathcal{S}_{\mathcal{B}_m}(t)h\|^2_{L^2_xL^2_v}\,dt\lesssim\|h\|^2_{L^2_x(H^{-1}_{v,*})}.$$

Finally we deduce (2.27) by using the fact that $S_{\mathcal{B}_m}(t)h=mS_{\mathcal{B}}(t)f$. □

We now investigate hypodissipative properties of $\mathcal{B}$ in high-order velocity spaces.

Lemma 2.9. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, $\ell\in\mathbb{N}$ *and* $n\in\mathbb{N}^*$. *Then, for any* $\lambda<\lambda_{m,1}$, *we can choose* $M>0$ *and* $R>0$ *large enough such that the operator* $\mathcal{B}+\lambda$ *is hypo-dissipative in* $W^{n,1}_xW^{\ell,1}_v(m)$, *in the sense that*

$$\forall\,t\ge0,\quad\|\mathcal{S}_{\mathcal{B}}(t)\|_{\mathscr{B}(W^{n,1}_xW^{\ell,1}_v(m))}\le Ce^{-\lambda t}.$$

Proof of Lemma 2.9. Consider the equation

$$\partial_t f = \mathcal{B} f = \mathcal{B}_0 f - v \cdot \nabla_x f - M\chi_R f.$$

Remind that $\mathcal{B}_0 f = Q(\mu, f)$ and remark that x-derivatives commute with the operator $\mathcal{B}$, thus for any multi-index $\alpha, \beta \in \mathbb{N}^3$, we have

$$\partial_v^\alpha \partial_x^\beta (\mathcal{B} f) = \partial_v^\alpha (\mathcal{B} \partial_x^\beta f)$$

and

$$\partial_v^\alpha \mathcal{B}_0 f = \partial_v^\alpha Q(\mu, f) = \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2} Q(\partial_v^{\alpha_1} \mu, \partial_v^{\alpha_2} f)$$

and, writing $v \cdot \nabla_x f = v_i \partial_{x_i} f$,

$$\partial_v^\alpha \mathcal{B} f = \mathcal{B} \partial_v^\alpha f + \sum_{\alpha_1 + \alpha_2 = \alpha, |\alpha_1| \geq 1} C_{\alpha_1, \alpha_2} \big\{ Q(\partial_v^{\alpha_1} \mu, \partial_v^{\alpha_2} f) - (\partial_v^{\alpha_1} v_i) \partial_{x_i} (\partial_v^{\alpha_2} f) - M(\partial_v^{\alpha_1} \chi_R)(\partial_v^{\alpha_2} f) \big\}$$

finally

$$\begin{aligned} \partial_v^\alpha \partial_x^\beta \mathcal{B} f &= \mathcal{B}(\partial_v^\alpha \partial_x^\beta f) \\ &+ \sum_{\alpha_1 + \alpha_2 = \alpha, |\alpha_1| = 1} C_{\alpha_1, \alpha_2} \big\{ Q(\partial_v^{\alpha_1} \mu, \partial_v^{\alpha_2} \partial_x^\beta f) - (\partial_v^{\alpha_1} v_i) \partial_{x_i} (\partial_v^{\alpha_2} \partial_x^\beta f) - M(\partial_v^{\alpha_1} \chi_R)(\partial_v^{\alpha_2} \partial_x^\beta f) \big\} \\ &+ \sum_{\alpha_1 + \alpha_2 = \alpha, |\alpha_1| \geq 2} C_{\alpha_1, \alpha_2} \big\{ Q(\partial_v^{\alpha_1} \mu, \partial_v^{\alpha_2} \partial_x^\beta f) - M(\partial_v^{\alpha_1} \chi_R)(\partial_v^{\alpha_2} \partial_x^\beta f) \big\}. \end{aligned}$$

We shall treat in full details the case $\ell = n = 1$, the others $\ell, n \geq 2$ being treated in the same way.

Case $\ell = n = 1$: *Step 1. Derivatives in x.* First, using the computation (2.22) for $p = 1$, we have

$$\frac{d}{dt} \|f\|_{L^1_{x,v}(m)} = \int \{\varphi_{m,1}(v) - M\chi_R(v)\} \, |f| \, m. \tag{2.29}$$

As explained before, the x-derivatives commute with the operator $\mathcal{B}$, so for any multi-index $\beta \in \mathbb{N}^3$ we get from (2.22) that

$$\frac{d}{dt} \|\partial_x^\beta f\|_{L^1_{x,v}(m)} = \int \{\varphi_{m,1}(v) - M\chi_R(v)\} |\partial_x^\beta f| \, m. \tag{2.30}$$

Step 2. Derivatives in v. We now consider the derivatives in v. For any $\alpha \in \mathbb{N}^3$ with $|\alpha| = 1$, we compute the evolution of v-derivatives:

$$\partial_t (\partial_v^\alpha f) = \mathcal{B}(\partial_v^\alpha f) + Q(\partial_v^\alpha \mu, f) - (\partial_v^\alpha v_i) \partial_{x_i} f - M(\partial_v^\alpha \chi_R) f.$$

From the previous equation we deduce that

$$\begin{aligned} \frac{d}{dt} \|\partial_v^\alpha f\|_{L^1_{x,v}(m)} &= \int \big\{ \mathcal{B}(\partial_v^\alpha f) + Q(\partial_v^\alpha \mu, f) - (\partial_v^\alpha v_i) \partial_{x_i} f - M(\partial_v^\alpha \chi_R) f \big\} \mathrm{sign}(\partial_v^\alpha f) \, m \\ &=: T_1 + T_2 + T_3 + T_4 + T_5, \end{aligned}$$

where

$$T_1 = \int \mathcal{B}(\partial_v^\alpha f)\,\mathrm{sign}(\partial_v^\alpha f)\,m$$

$$T_2 = \int (\partial_v^\alpha \bar{a}_{ij})\,\partial_{ij} f\,\mathrm{sign}(\partial_v^\alpha f)\,m$$

$$T_3 = -\int (\partial_v^\alpha \bar{c})\,f\,\mathrm{sign}(\partial_v^\alpha f)\,m$$

$$T_4 = -\int (\partial_v^\alpha v_i)\partial_{x_i} f\,\mathrm{sign}(\partial_v^\alpha f)\,m = 0$$

$$T_5 = -\int M(\partial_v^\alpha \chi_R) f\,\mathrm{sign}(\partial_v^\alpha f)\,m.$$

Again using the computation (2.22) of Lemma 2.7 for $p = 1$, we have

$$T_1 = \int \{\varphi_{m,1}(v) - M\chi_R(v)\}|\partial_v^\alpha f|\,m.$$

Concerning T_5, we use the following fact on the derivative of χ_R:

$$|\partial_v^\alpha \chi_R(v)| = \frac{1}{R}\left|\partial_v^\alpha \chi\left(\frac{v}{R}\right)\right| \le \frac{C}{R}\mathbf{1}_{R\le|v|\le 2R},$$

which implies that

$$T_5 \le M\frac{C}{R}\|\mathbf{1}_{R\le|v|\le 2R} f\|_{L^1_{x,v}(m)}.$$

Performing integration by parts, we get

$$T_2 + T_3 = -\int \partial_v^\alpha \bar{a}_{ij}\,\partial_i f\,\partial_j m\,\mathrm{sign}(\partial_v^\alpha f) + \int \partial_v^\alpha \bar{b}_j\,\partial_j m\, f\,\mathrm{sign}(\partial_v^\alpha f) =: A + B.$$

When m is a polynomial weight $m = \langle v\rangle^k$, we can easily estimate $T_2 + T_3$, thanks to another integration by parts, by

$$T_2 + T_3 = \int \{(\partial_v^\alpha \bar{a}_{ij})\,\partial_{ij} m + 2(\partial_v^\alpha \bar{b}_j)\,\partial_j m\}\, f\,\mathrm{sign}(\partial_v^\alpha f) \lesssim \|\langle v\rangle^{\gamma-1} f\|_{L^1_{x,v}(m)},$$

where we have used $|\partial_v^\alpha \bar{a}_{ij}| \le C\langle v\rangle^{\gamma+1}$, $|\partial_v^\alpha \bar{b}_j| \le \langle v\rangle^\gamma$, $|\partial_j m| \le C\langle v\rangle^{-1} m$ and $|\partial_{ij} m| \le C\langle v\rangle^{-2} m$.

We now investigate the case of (stretched) exponential weight $m = e^{r\langle v\rangle^s}$. First, we can easily estimate the term B, since $\partial_j m = Cv_j\langle v\rangle^{\sigma-2} m$, as

$$B \lesssim \|\langle v\rangle^{\gamma+s-1} f\|_{L^1_{x,v}(m)}.$$

For the other term, integrating by parts again (first with respect to the ∂_v^α-derivative then to the ∂_i-derivative), gives us

$$A = -\int \left\{\bar{a}_{ij}\frac{\partial_{ij} m}{m} + \bar{b}_j\frac{\partial_j m}{m}\right\}|\partial_v^\alpha f|\,m + \int \bar{a}_{ij}\,\partial_i(\partial_v^\alpha m)\,\partial_j f\,\mathrm{sign}(\partial_v^\alpha f),$$

and we investigate the last term in the right-hand side. Recall that

$$\bar{a}_{ij}\xi_i\xi_j = \ell_1(v)|P_v\xi|^2 + \ell_2(v)|(I - P_v)\xi|^2,$$

we decompose $\partial_j f = P_v\partial_j f + (I - P_v)\partial_j f$ and similarly for $\partial_j(\partial_v^\alpha m)$, then a tedious but straightforward computation yields

$$\begin{aligned}
&\int \bar{a}_{ij}\,\partial_i(\partial_v^\alpha m)\,\partial_j f\,\mathrm{sign}(\partial_v^\alpha f)\\
&\quad= \int \left\{rs\ell_1(v)\langle v\rangle^{s-2} + rs(s-2)\ell_1(v)|v|^2\langle v\rangle^{s-4} + r^2s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4}\right\} P_v\partial_v^\alpha f\,\mathrm{sign}(\partial_v^\alpha f)\,m\\
&\qquad+ \int rs\ell_2(v)\langle v\rangle^{s-2}\,(I - P_v)\partial_v^\alpha f\,\mathrm{sign}(\partial_v^\alpha f)\,m.
\end{aligned}$$

Recall that $\varphi_{m,1}(v) = \bar{a}_{ij}\frac{\partial_{ij}m}{m} + 2\bar{b}_j\frac{\partial_j m}{m}$ (see eq. (2.8)), hence we obtain

$$T_1 + A \leq \int \{\psi_{m,1}(v) - M\chi_R(v)\}\, |\partial_v^\alpha f|\, m$$

with

$$\begin{aligned}\psi_{m,1}(v) := \bar{b}_j\frac{\partial_j m}{m} &+ rs\ell_2(v)\langle v\rangle^{s-2} + rs\ell_1(v)\langle v\rangle^{s-2} \\ &+ rs(s-2)\ell_1(v)|v|^2\langle v\rangle^{s-4} + r^2s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4}.\end{aligned}$$

Thanks to the asymptotic behaviour of $\ell_1(v)$ and $\ell_2(v)$ in Lemma 2.4 and arguing as in Lemma 2.6, we obtain first that

$$\begin{cases} \limsup_{|v|\to+\infty} \psi_{m,1}(v) \leq -rs\langle v\rangle^{\gamma+s}, & \text{if } 0 < s < 2; \\ \limsup_{|v|\to+\infty} \psi_{m,1}(v) \leq -2r(1-4r), & \text{if } s = 2; \end{cases} \tag{2.31}$$

and then for any positive $\lambda < \lambda_{m,1}$ and $\delta \in (0, \lambda_{m,1} - \lambda)$ we can choose M, R large enough such that $\psi_{m,1}(v) - M\chi_R(v) \leq -\lambda - \delta\langle v\rangle^{\gamma+\sigma}$.

Putting together all the previous estimates of this step, and denoting $\varphi^\sigma(v) = \varphi_{m,1}(v)$ when $m = \langle v\rangle^k$ and $\varphi^\sigma(v) = \psi_{m,1}(v)$ when $m = e^{r\langle v\rangle^s}$, we obtain

$$\frac{d}{dt}\|\partial_v^\alpha f\|_{L^1_{x,v}(m)} \leq \int \{\varphi^\sigma(v) - M\chi_R(v)\}\, |\partial_v^\alpha f|\, m + \int \left\{ C\langle v\rangle^{\gamma+\sigma-1} + C\frac{M}{R}\mathbf{1}_{R\leq|v|\leq 2R} \right\} |f|\, m. \tag{2.32}$$

Step 3. Conclusion. Consider the standard norm on $W^{1,1}_{x,v}(m)$

$$\|f\|_{W^{1,1}_{x,v}(m)} = \|f\|_{L^1_{x,v}(m)} + \|\nabla_x f\|_{L^1_{x,v}(m)} + \|\nabla_v f\|_{L^1_{x,v}(m)}.$$

Gathering the previous estimates (2.29), (2.30) and (2.32), we finally obtain

$$\begin{aligned}\frac{d}{dt}\|f\|_{W^{1,1}_{x,v}(m)} \leq \int \left\{ \varphi_{m,1}(v) + C\langle v\rangle^{\gamma+\sigma-1} + M\,\frac{C}{R}\mathbf{1}_{R\leq|v|\leq 2R} - M\chi_R \right\} |f|\, m \\ + \int \{\varphi_{m,1}(v) - M\chi_R\}|\nabla_x f|\, m + \int \{\varphi^\sigma(v) - M\chi_R\}|\nabla_v f|\, m.\end{aligned}$$

Remark that, since $\sigma \in [0,2]$, the function $\phi^0_m(v) := \varphi_{m,1}(v) + C\langle v\rangle^{\gamma+\sigma-1}$ has the same asymptotic behaviour of $\varphi_{m,1}(v)$ (see eq. (2.11) and eq. (2.14)). Then, arguing as in Lemma 2.6 (and (2.31)), for any positive $\lambda < \lambda_{m,1}$ and $\delta \in (0, \lambda_{m,1} - \lambda)$, one may find $M > 0$ and $R > 0$ large enough such that

$$\begin{aligned}\varphi_{m,1}(v) + C\langle v\rangle^{\gamma+\sigma-1} + M\,\frac{C}{R}\mathbf{1}_{R\leq|v|\leq 2R} - M\chi_R &\leq -\lambda - \delta\langle v\rangle^{\gamma+\sigma}, \\ \varphi_{m,1}(v) - M\chi_R &\leq -\lambda - \delta\langle v\rangle^{\gamma+\sigma}, \\ \varphi^\sigma(v) - M\chi_R &\leq -\lambda - \delta\langle v\rangle^{\gamma+\sigma}.\end{aligned}$$

This implies that

$$\frac{d}{dt}\|f\|_{W^{1,1}_{x,v}(m)} \leq -\lambda\|f\|_{W^{1,1}_{x,v}(m)} - \delta\|f\|_{W^{1,1}_{x,v}(m\langle v\rangle^{\gamma+\sigma})},$$

which concludes the proof in the case $\ell = 1$.

Case $\ell \geq 2$: The higher order derivatives are treated in the same way, so we omit the proof. □

Lemma 2.10. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, $\ell \in \mathbb{N}$ *and* $n \in \mathbb{N}^*$. *Then, for any* $\lambda < \lambda_{m,2}$, *we can choose* $M > 0$ *and* $R > 0$ *large enough such that the operator* $\mathcal{B} + \lambda$ *is hypo-dissipative in* $H^n_x H^\ell_v(m)$, *in the sense that*

$$\forall\, t \geq 0, \quad \|\mathcal{S}_{\mathcal{B}}(t)\|_{\mathscr{B}(H^n_x H^\ell_v(m))} \leq Ce^{-\lambda t}.$$

Proof of Lemma 2.10. Let us consider the equation $\partial_t f = \mathcal{B} f = \mathcal{B}_0 f - M\chi_R f$. Again we treat the case $\ell = 1$ in full details, the others $\ell \geq 2$ being the same.

Case $\ell = n = 1$: *Step 1. L^2 estimate.* The $L^2_{x,v}(m)$ estimate is a special case of Lemma 2.7, from which we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_{x,v}(m)} \leq -c_0 \int \{\langle v\rangle^\gamma |P_v \nabla_v f|^2 + \langle v\rangle^{\gamma+2} |(I-P_v)\nabla_v f|^2\}\, m^2 \\ + \int \{\varphi_{m,2}(v) - M\chi_R(v)\} f^2\, m^2.\end{aligned} \tag{2.33}$$

Step 2. x-derivatives. Recall that the x-derivatives commute with the equation, so for any $\beta \in \mathbb{N}^3$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x^\beta f\|^2_{L^2_{x,v}(m)} \leq -c_0 \int \{\langle v\rangle^\gamma |P_v \nabla_v (\partial_x^\beta f)|^2 + \langle v\rangle^{\gamma+2} |(I-P_v)\nabla_v (\partial_x^\beta f)|^2\}\, m^2 \\ + \int \{\varphi_{m,2}(v) - M\chi_R(v)\} |\partial_x^\beta f|^2\, m^2.\end{aligned} \tag{2.34}$$

Step 3. v-derivatives. Let $\alpha \in \mathbb{N}^3$ with $|\alpha| = 1$. We recall the equation satisfied by $\partial_v^\alpha f$

$$\partial_t \partial_v^\alpha f = \mathcal{B}(\partial_v^\alpha f) + Q(\partial_v^\alpha \mu, f) - (\partial_v^\alpha v_i)\, \partial_{x_i} f - M(\partial_v^\alpha \chi_R) f.$$

From last equation we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_v^\alpha f\|^2_{L^2_{x,v}(m)} &= \int \{\mathcal{B}(\partial_v^\alpha f) + Q(\partial_v^\alpha \mu, f) - (\partial_v^\alpha v_i)\, \partial_{x_i} f - M(\partial_v^\alpha \chi_R) f\}\ \partial_v^\alpha f\, m^2 \\ &=: T_1 + T_2 + T_3 + T_4 + T_5,\end{aligned}$$

where

$$\begin{aligned}T_1 &= \int \mathcal{B}(\partial_v^\alpha f)\, \partial_v^\alpha f\, m^2 \\ T_2 &= \int (\partial_v^\alpha \bar{a}_{ij})\, \partial_{ij} f\, \partial_v^\alpha f\, m^2 \\ T_3 &= -\int (\partial_v^\alpha \bar{c})\, f\, \partial_v^\alpha f\, m^2 \\ T_4 &= -\int (\partial_v^\alpha v_i)\, \partial_{x_i} f\, \partial_v^\alpha f\, m^2 \\ T_5 &= -\int M(\partial_v^\alpha \chi_R) f\, \partial_v^\alpha f\, m^2.\end{aligned}$$

We have from Lemma 2.7

$$\begin{aligned}T_1 \leq -c_0 \int \{\langle v\rangle^\gamma |P_v \nabla_v (\partial_v^\alpha f)|^2 + \langle v\rangle^{\gamma+2} |(I-P_v)\nabla_v (\partial_v^\alpha f)|^2\}\, m^2 \\ + \int \{\varphi_{m,2}(v) - M\chi_R(v)\} |\partial_v^\alpha f|^2\, m^2.\end{aligned} \tag{2.35}$$

The terms T_3, T_4 and T_5 are easy to estimate: for any $\varepsilon > 0$ we get

$$T_4 \leq \varepsilon \|\partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\partial_x^\alpha f\|^2_{L^2_{x,v}(m)}, \tag{2.36}$$

$$T_5 \leq M\frac{C}{R}\|\mathbf{1}_{R\leq|v|\leq 2R}\, \partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + M\frac{C}{R}\|\mathbf{1}_{R\leq|v|\leq 2R}\, f\|^2_{L^2_{x,v}(m)}, \tag{2.37}$$

and using Lemma 2.4-(b),

$$
\begin{aligned}
T_3 &\le C\int \langle v\rangle^{\gamma-1}\,|f|\,|\partial_v^\alpha f|\,m^2\\
&\le C\|\langle v\rangle^{\frac{\gamma-1}{2}}\partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + C\|\langle v\rangle^{\frac{\gamma-1}{2}} f\|^2_{L^2_{x,v}(m)}.
\end{aligned}
\tag{2.38}
$$

Let us now deal with the part T_2. Performing integrations by parts, we have:

$$
\begin{aligned}
T_2 &= \int (\partial_v^\alpha \bar a_{ij})\,\partial_{ij} f\,\partial_v^\alpha f\,m^2\\
&= -\int (\partial_v^\alpha \bar b_j)\,\partial_j f\,\partial_v^\alpha f\,m^2 - \int (\partial_v^\alpha \bar a_{ij})\,\partial_j f\,\partial_i(\partial_v^\alpha f)\,m^2 - \int (\partial_v^\alpha \bar a_{ij})\,\partial_j f\,\partial_v^\alpha f\,\partial_i m^2\\
&=: -\left(T_{21}+T_{22}+T_{23}\right).
\end{aligned}
$$

We first deal with T_{21}. Using Lemma 2.4, we have

$$
\begin{aligned}
T_{21} &\le C\int \langle v\rangle^{\gamma}\,|\partial_j f|\,|\partial_v^\alpha f|\,m^2\\
&\le C\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v f\|^2_{L^2_{x,v}(m)} = C\|\langle v\rangle^{\frac{\gamma}{2}} P_v\nabla_v f\|^2_{L^2_{x,v}(m)} + C\|\langle v\rangle^{\frac{\gamma}{2}}(I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)}.
\end{aligned}
\tag{2.39}
$$

As far as T_{22} is concerned, the integration by parts gives,

$$
\begin{aligned}
T_{22} &= -\int \partial_v^\alpha\left[(1-\chi)m^2\right]\bar a_{ij}\,\partial_j f\,\partial_i(\partial_v^\alpha f) - \int (1-\chi)m^2\,\overline{a}_{ij}\,\partial_j(\partial_v^\alpha f)\,\partial_i(\partial_v^\alpha f)\\
&\quad - \int (1-\chi)m^2\,\overline{a}_{ij}\,\partial_j f\,\partial_i(\partial_v^\alpha\partial_v^\alpha f) - \int (\partial_v^\alpha \bar a_{ij})\,\partial_j f\,\partial_i(\partial_v^\alpha f)\,\chi m^2\\
&=: -\left(\widetilde{T}_{221}+\widetilde{T}_{222}+\widetilde{T}_{223}\right) + T_{220}.
\end{aligned}
$$

Let us estimate $\widetilde{T}_{222}+\widetilde{T}_{223}$, using integration by parts,

$$
\begin{aligned}
&\widetilde{T}_{222}+\widetilde{T}_{223}\\
&= \int (1-\chi)m^2\Big[\ell_1(v)\,P_v\nabla_v(\partial_v^\alpha\partial_v^\alpha f)\cdot P_v\nabla_v f + \ell_2(v)\,(I-P_v)\nabla_v(\partial_v^\alpha\partial_v^\alpha f)\cdot(I-P_v)\nabla_v f\Big]\\
&\quad + \int (1-\chi)m^2\Big[\ell_1(v)\,P_v\nabla_v(\partial_v^\alpha f)\cdot P_v\nabla_v(\partial_v^\alpha f) + \ell_2(v)\,(I-P_v)\nabla_v(\partial_v^\alpha f)\cdot(I-P_v)\nabla_v(\partial_v^\alpha f)\Big]\\
&= -\widetilde{T}_{221} - \int (\partial_v^\alpha \ell_1(v))\,P_v\nabla_v(\partial_v^\alpha f)\cdot P_v\nabla_v f\,(1-\chi)m^2\\
&\quad - \int (\partial_v^\alpha \ell_2(v))\,(I-P_v)\nabla_v(\partial_v^\alpha f)\cdot(I-P_v)\nabla_v f\,(1-\chi)m^2\\
&\quad - \int \left[\ell_1(v)-\ell_2(v)\right](I-P_v)\partial_v^\alpha(\partial_v^\alpha f)\,\frac{v\cdot\nabla_v f}{|v|^2}\,(1-\chi)m^2\\
&\quad - \int \left[\ell_1(v)-\ell_2(v)\right](I-P_v)\nabla_v\partial_v^\alpha f\,\frac{v\cdot\nabla_v g}{|v|^2}\,(1-\chi)m^2\\
&=: -\widetilde{T}_{221} + T_{221} + ... + T_{224}\,.
\end{aligned}
$$

This means $T_{22} = T_{220} + T_{221} + ... + T_{224}$. In order to estimate T_{22}, we need to estimate T_{22i} for $i = 0, \dots, 4$ (lemma 2.4 plays an important role in those estimates). First of all, we obtain

$$
\begin{aligned}
T_{220} &\le C\int_{|v|\le 2} \langle v\rangle^{\gamma+1}|\nabla_v f|\,|\nabla_v(\partial_v^\alpha f)|\,|\chi|m^2\\
&\le \varepsilon\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v f\|^2_{L^2_{x,v}(m)}
\end{aligned}
$$

For T_{221}, we have

$$\begin{aligned} T_{221} &\le C \int_{|v|\ge 1} \langle v\rangle^{\gamma-1} |P_v\nabla_v f|\, |P_v\nabla_v(\partial_v^\alpha f)|\, m^2 \\ &\le \varepsilon \|\langle v\rangle^{\frac{\gamma-1}{2}} P_v\nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma-1}{2}} P_v\nabla_v f\|^2_{L^2_{x,v}(m)}\,. \end{aligned}$$

For T_{222}, we have

$$\begin{aligned} T_{222} &\le C \int_{|v|\ge 1} \langle v\rangle^{\gamma+1} |(I-P_v)\nabla_v f|\, |(I-P_v)\nabla_v(\partial_v^\alpha f)|\, m^2 \\ &\le \varepsilon \|\langle v\rangle^{\frac{\gamma+1}{2}} (I-P_v)\nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+1}{2}} (I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)}. \end{aligned}$$

For T_{223}, we obtain

$$\begin{aligned} T_{223} &\le C \int_{|v|\ge 1} \big(\langle v\rangle^{\gamma-1} + \langle v\rangle^{\gamma+1}\big) |\nabla_v f|\, |(I-P_v)\nabla_v(\partial_v^\alpha f)|\, m^2 \\ &\le \varepsilon \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v f\|^2_{L^2_{x,v}(m)}. \end{aligned}$$

Finally, for T_{224},

$$\begin{aligned} T_{224} &\le C \int_{|v|\ge 1} \big(\langle v\rangle^{\gamma-1} + \langle v\rangle^{\gamma+1}\big) |\nabla_v(\partial_v^\alpha f)|\, |(I-P_v)\nabla_v f|\, m^2 \\ &\le \varepsilon \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v(\partial_v^\alpha f)\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)} \end{aligned}$$

This completes the estimate of T_{22} that we write, gathering previous bounds, as

$$\begin{aligned} T_{22} \le{}& \varepsilon \|\langle v\rangle^{\frac{\gamma}{2}} P_v\nabla_v(\partial_v^\alpha f)\|_{L^2_{x,v}(m)} + \varepsilon \|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v(\partial_v^\alpha f)\|_{L^2_{x,v}(m)} \\ & C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}} P_v\nabla_v f\|_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+2}{2}} (I-P_v)\nabla_v f\|_{L^2_{x,v}(m)}. \end{aligned} \tag{2.40}$$

Concerning T_{23}, we apply the same process as T_{22}: we first write

$$\begin{aligned} T_{23} = &- \int (\partial_v^\alpha \bar a_{ij})\, \partial_j f\, \partial_i m^2\, \chi g \\ &- \int \partial_v^\alpha \ell_1(v)\, P_v\nabla_v m^2 \cdot P_v\nabla_v f\, (1-\chi)\, \partial_v^\alpha f \\ &- \int \partial_v^\alpha \ell_2(v)\, (I-P_v)\nabla_v m^2 \cdot (I-P_v)\nabla_v f\, (1-\chi)\, \partial_v^\alpha f \\ &- \int \big[\ell_1(v) - \ell_2(v)\big]\, (I-P_v)\partial_v^\alpha m^2\, \frac{v\cdot\nabla_v f}{|v|^2}\, (1-\chi)\, \partial_v^\alpha f \\ &- \int \big[\ell_1(v) - \ell_2(v)\big]\, (I-P_v)\partial_v^\alpha f\, \frac{v\cdot\nabla_v m^2}{|v|^2}\, (1-\chi)\, \partial_v^\alpha f \\ =:{}& T_{230} + ... + T_{234}. \end{aligned}$$

Note that $(I-P_v)\nabla_v m^2 = 0$, one can easily get $T_{232} = T_{233} = 0$. Let us estimate the other terms, by Lemma 2.4, we have

$$\begin{aligned} T_{230} &\le C \int_{|v|\le 2} \langle v\rangle^{\gamma+\sigma}\, |\nabla_v f|\, |\partial_v^\alpha f|\, |\chi|\, m^2 \\ &\le \varepsilon \|\langle v\rangle^{\frac{\gamma}{2}} \partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v f\|^2_{L^2_{x,v}(m)} \end{aligned}$$

also

$$\begin{aligned} T_{231} &\le C\int_{|v|>1}\langle v\rangle^{\gamma+\sigma-2}\,|P_v\nabla_v f|\,|\partial_v^\alpha f|\,m^2 \\ &\le C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v f\|^2_{L^2_{x,v}(m)} + \varepsilon\|\langle v\rangle^{\frac{\gamma+2\sigma-4}{2}}\partial_v^\alpha f\|^2_{L^2_{x,v}(m)}, \end{aligned}$$

and

$$\begin{aligned} T_{234} &\le C\int_{|v|>1}\left(\langle v\rangle^{\gamma+\sigma-2}+\langle v\rangle^{\gamma+\sigma}\right)|(I-P_v)\nabla_v f|\,|\partial_v^\alpha f|\,m^2 \\ &\le C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)} + \varepsilon\|\langle v\rangle^{\frac{\gamma+2\sigma-2}{2}}\partial_v^\alpha f\|^2_{L^2_{x,v}(m)}. \end{aligned}$$

Gathering previous inequalities we complete the estimate of T_{23}

$$\begin{aligned} T_{23} &\le \varepsilon\|\langle v\rangle^{\frac{\gamma}{2}}\partial_v^\alpha f\|^2_{L^2_{x,v}(m)} + \varepsilon\|\langle v\rangle^{\frac{\gamma+2\sigma-2}{2}}\partial_v^\alpha f\|^2_{L^2_{x,v}(m)} \\ &\quad + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v f\|^2_{L^2_{x,v}(m)} + C(\varepsilon)\|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v f\|^2_{L^2_{x,v}(m)}. \end{aligned} \tag{2.41}$$

Putting together (2.35) to (2.41) we get, using the fact that $1+\langle v\rangle^\gamma+\langle v\rangle^{\gamma+2\sigma-2}\lesssim\langle v\rangle^{\gamma+\sigma}$,

(2.42)

$$\begin{aligned} \frac12\frac{d}{dt}\|\partial_v^\alpha f\|^2_{L^2_{x,v}(m)} &\le -(c_0-\varepsilon)\int\left\{\langle v\rangle^\gamma|P_v\nabla_v(\partial_v^\alpha f)|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v(\partial_v^\alpha f)|^2\right\}m^2 \\ &\quad+\int\left\{\varphi_{m,2}(v)+\varepsilon\langle v\rangle^{\gamma+\sigma}+C\langle v\rangle^{\gamma-1}+M\frac{C}{R}\mathbf{1}_{R\le|v|\le 2R}-M\chi_R(v)\right\}|\partial_v^\alpha f|^2\,m^2 \\ &\quad+C(\varepsilon)\int\left\{\langle v\rangle^\gamma|P_v\nabla_v f|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v f|^2\right\}m^2 \\ &\quad+\int\left\{C\langle v\rangle^{\gamma-1}+M\frac{C}{R}\mathbf{1}_{R\le|v|\le 2R}\right\}|f|^2\,m^2+C(\varepsilon)\|\partial_x^\alpha f\|^2_{L^2_{x,v}(m)}. \end{aligned}$$

Step 4. Conclusion in the case $\ell=n=1$. We now introduce the following norm on $H^1_xH^1_v(m)$

$$\|f\|^2_{\widetilde{H}^1(m)} := \|f\|^2_{L^2_{x,v}(m)}+\|\nabla_x f\|^2_{L^2_{x,v}(m)}+\eta\,\|\nabla_v f\|^2_{L^2_{x,v}(m)},$$

which is equivalent to the standard $H^1_{x,v}(m)$-norm for any $\eta>0$. Gathering estimates (2.33), (2.34) and (2.42) of previous steps, we obtain

$$\begin{aligned} \frac12\frac{d}{dt}\|f\|^2_{\widetilde{H}^1(m)} &\le (-c_0+\eta\,C(\varepsilon))\int\left\{\langle v\rangle^\gamma|P_v\nabla_v f|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v f|^2\right\}m^2 \\ &\quad+\int\left\{\psi^0_m(v)+\eta\,M\frac{C}{R}\mathbf{1}_{R\le|v|\le 2R}-M\chi_R(v)\right\}f^2\,m^2 \\ &\quad-c_0\sum_{|\beta|=1}\int\left\{\langle v\rangle^\gamma|P_v\nabla_v(\partial_x^\beta f)|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v(\partial_x^\beta f)|^2\right\}m^2 \\ &\quad+\int\left\{\psi^1_m(v)-M\chi_R(v)\right\}|\nabla_x f|^2\,m^2 \\ &\quad+\eta(-c_0+\varepsilon)\sum_{|\alpha|=1}\int\left\{\langle v\rangle^\gamma|P_v\nabla_v(\partial_v^\alpha f)|^2+\langle v\rangle^{\gamma+2}|(I-P_v)\nabla_v(\partial_v^\alpha f)|^2\right\}m^2 \\ &\quad+\eta\int\left\{\psi^2_m(v)+M\frac{C}{R}\mathbf{1}_{R\le|v|\le 2R}-M\chi_R(v)\right\}|\nabla_v f|^2\,m^2. \end{aligned}$$

where we have defined

$$\begin{aligned} \psi^0_m(v) &:= \varphi_{m,2}(v)+C\eta\langle v\rangle^{\gamma-1}, \\ \psi^1_m(v) &:= \varphi_{m,2}(v)+\eta C(\varepsilon), \\ \psi^2_m(v) &:= \varphi_{m,2}(v)+\varepsilon\langle v\rangle^{\gamma+\sigma}+C\langle v\rangle^{\gamma-1}. \end{aligned}$$

Let us fix any $\lambda < \lambda_{m,2}$. We first choose $\varepsilon > 0$ small enough so that $-c_0 + \varepsilon < 0$ and $-\lambda_{m,2} + \varepsilon < -\lambda$. Then we choose $\eta > 0$ small enough such that $-c_0 + \eta\, C(\varepsilon) \leq 0$ and $-\lambda_{m,2} + \eta C(\varepsilon) < -\lambda$. Hence the functions ψ_m^i have the same asymptotic behaviour than $\varphi_{m,2}$ (see (2.11), (2.14) and (2.16)). Then, using Lemma 2.6, for any $\lambda < \lambda_{m,2}$ and $\delta \in (0, \lambda_{m,2} - \lambda)$, one may find $M > 0$ and $R > 0$ large enough such that

$$\begin{aligned}
\psi_m^0(v) + \eta\, M \frac{C}{R} \mathbf{1}_{R \leq |v| \leq 2R} - M\chi_R(v) &\leq -\lambda - \delta \langle v \rangle^{\gamma+\sigma}, \\
\psi_m^1(v) - M\chi_R(v) &\leq -\lambda - \delta \langle v \rangle^{\gamma+\sigma}, \\
\psi_m^2(v) + M \frac{C}{R} \mathbf{1}_{R \leq |v| \leq 2R} - M\chi_R(v) &\leq -\lambda - \delta \langle v \rangle^{\gamma+\sigma}.
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|f\|^2_{\widetilde{H}^1(m)} \leq & -\lambda \|f\|^2_{\widetilde{H}^1(m)} - \delta \|f\|^2_{\widetilde{H}^1(m\langle v\rangle^{(\gamma+\sigma)/2})} \\
& - K\Big\{ \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2(m)} + \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v) \nabla_v f\|^2_{L^2(m)} \Big\} \\
& - K\Big\{ \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v (\nabla_x f)\|^2_{L^2(m)} + \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v) \nabla_v (\nabla_x f)\|^2_{L^2(m)} \Big\} \\
& - K\Big\{ \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v (\nabla_v f)\|^2_{L^2(m)} + \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v) \nabla_v (\nabla_v f)\|^2_{L^2(m)} \Big\},
\end{aligned}$$

and then

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{H^1_{x,v}(m)} \leq C e^{-\lambda t} \|f\|_{H^1_{x,v}(m)}.$$

This concludes the proof of the hypodissipativity of $\mathcal{B} + \lambda$ in $H^1_{x,v}(m)$.

Case $\ell \geq 2$: The higher order derivatives are treated in the same way, introducing the (equivalent) norm on $H^n_x H^\ell_v(m)$

$$\|f\|^2_{\widetilde{H^n_x H^\ell_v(m)}} = \|f\|^2_{L^2(m)} + \sum_{1 \leq |\alpha|+|\beta| \leq \max(\ell,n); |\alpha| \leq \ell; |\beta| \leq n} \eta^{|\alpha|} \|\partial_v^\alpha \partial_x^\beta f\|^2_{L^2(m)},$$

and choosing $\eta > 0$ small enough as in the case $\ell = 1$. □

Lemma 2.11. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, $\ell \in \mathbb{N}$ *and* $n \in \mathbb{N}^*$, *and* $p \in [1, 2]$. *Then, for any* $\lambda < \lambda_{m,p}$, *we can choose* $M > 0$ *and* $R > 0$ *large enough such that the operator* $\mathcal{B} + \lambda$ *is hypo-dissipative in* $W^{n,p}_x W^{\ell,p}_v(m)$, *in the sense that*

$$\forall\, t \geq 0, \quad \|\mathcal{S}_{\mathcal{B}}(t)\|_{\mathscr{B}(W^{n,p}_x W^{\ell,p}_v(m))} \leq C e^{-\lambda t}.$$

Proof. It is a consequence of Lemmas 2.9 and 2.10, together with the Riesz-Thorin interpolation theorem. □

2.5. **Regularization.** We now turn to the boundedness of $\mathcal{A}$ as well as regularization properties of $\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)$. We recall the operator $\mathcal{A}$ defined in (2.7)

$$\mathcal{A} f = \mathcal{A}_0 f + M\chi_R f = (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu + M\chi_R f,$$

for M and R large enough chosen before. Thanks to the smooth cut-off function χ_R, for any $q \in [1, +\infty]$, $p \geq q$ and any weight function m under the hypotheses **(H1)**-**(H2)**-**(H3)**, we easily obtain

$$\|M\chi_R f\|_{L^q_{x,v}(\mu^{-1/2})} \lesssim \|f\|_{L^q_x L^p_v(m)}.$$

Taking derivatives we get an analogous estimate, for any $n, \ell \in \mathbb{N}$,

$$\|M\chi_R f\|_{W^{n,q}_x W^{\ell,q}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,q}_x W^{\ell,p}_v(m)},$$

Arguing by duality we also have

$$\|M\chi_R f\|_{H^n_x H^{-1}_v(\mu^{-1/2})} \lesssim \|f\|_{H^n_x H^{-1}_v(m)}.$$

Finally we obtain

$$M\chi_R \in \begin{cases} \mathscr{B}\left(L^p_{x,v}(m), L^p_{x,v}(\mu^{-1/2})\right), & \forall\, p \in [1,\infty]; \\ \mathscr{B}\left(W^{n,p}_x W^{\ell,p}_v(m), W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})\right), & \forall\, p \in [1,2],\ n \in \mathbb{N}^*,\ \ell \in \mathbb{N}. \end{cases} \tag{2.43}$$

We know obtain the boundedness of $\mathcal{A}$.

Lemma 2.12. *Consider* **(H1)**, **(H2)** *or* **(H3)** *and a weight function m.*

(i) For any $p \in [1,\infty]$, there holds

$$\mathcal{A} \in \mathscr{B}\left(L^p_{x,v}(m), L^p_{x,v}(\mu^{-1/2})\right).$$

(ii) For any $p \in [1,2]$, $n \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, there holds

$$\mathcal{A} \in \mathscr{B}\left(W^{n,p}_x W^{\ell,p}_v(m), W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})\right).$$

In particular $\mathcal{A} \in \mathscr{B}(E) \cap \mathscr{B}(\mathcal{E})$ for any admissible space $\mathcal{E}$ in (2.2).

Proof. Thanks to (2.43) we just need to consider the operator $\mathcal{A}_0$. We write

$$\mathcal{A}_0 f = (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu$$

and split the proof into several steps.

Step 1. Since $\gamma \in [-2,1]$ we have $|a_{ij}(v - v_*)| \lesssim \langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+2}$, which implies $|(a_{ij} * f)(v)| \lesssim \langle v\rangle^{\gamma+2}\|f\|_{L^1_v(\langle v\rangle^{\gamma+2})}$. Therefore, for any $p \in [1,\infty]$, we have

$$\|(a_{ij} * f)\partial_{ij}\mu\|_{L^p_v(\mu^{-1/2})} \lesssim \|f\|_{L^1_v(\langle v\rangle^{\gamma+2})},$$

from which we can also easily deduce

$$\|\partial^\alpha_v \partial^\beta_x (a_{ij} * f)\partial_{ij}\mu\|_{L^p_v(\mu^{-1/2})} \lesssim \sum_{\alpha_1 \le \alpha} \|\partial^{\alpha_1}_v \partial^\beta_x f\|_{L^1_v(\langle v\rangle^{\gamma+2})}.$$

Integrating in the x-variable, we finally get

$$\|(a_{ij} * f)\partial_{ij}\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,1}_v(\langle v\rangle^{\gamma+2})}.$$

Step 2. Assume $\gamma \in [0,1]$. In that case we have $|c(v - v_*)| \lesssim \langle v\rangle^\gamma \langle v_*\rangle^\gamma$ and the same argument as above gives

$$\|(c * f)\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,1}_v(\langle v\rangle^\gamma)}.$$

Step 3. Assume $\gamma \in [-2,0)$. We decompose $c = c_+ + c_-$ with $c_+ = c\mathbf{1}_{|\cdot|>1}$ and $c_- = c\mathbf{1}_{|\cdot|\le 1}$. For the non-singular term c_+ we easily get, for any $p \in [1,\infty]$,

$$\|(c_+ * f)\mu\|_{L^p_v(\mu^{-1/2})} \lesssim \|f\|_{L^1_v}$$

whence

$$\|(c_+ * f)\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,1}_v}.$$

We now investigate the singular term c_-. For any $p \in [1, 3/|\gamma|)$ we get

$$\begin{aligned} \|(c_- * f)\mu\|^p_{L^p_v(\mu^{-1/2})} = \|(c_- * f)\mu^{1/2}\|^p_{L^p_v} &\lesssim \int_v \left| \int_{v_*} |v - v_*|^\gamma \mathbf{1}_{|v-v_*|\le 1} |f(v_*)| \right|^p \mu^{1/2}(v) \\ &\lesssim \int_{v_*} |f(v_*)|^p \left\{ \int_v |v - v_*|^{\gamma p} \mathbf{1}_{|v-v_*|\le 1}\, \mu^{1/2}(v) \right\} \\ &\lesssim \|f\|^p_{L^p_v(\langle v\rangle^\gamma)}, \end{aligned}$$

where we have used that $|\gamma|p < 3$ (so that the integral in v is bounded) and Lemma 3.3. Taking derivatives and integrating in x it follows

$$\|(c_- * f)\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,p}_v(\langle v\rangle^\gamma)}, \quad \forall\, p \in [1, 3/|\gamma|).$$

Remark that by Hölder's inequality, for any $q \in (3/(3+\gamma), \infty]$ we have

$$|(c_- * f)(v)| \lesssim \int_{v_*} |v - v_*|^\gamma \, \mathbf{1}_{|v-v_*|\le 1} \, |f(v_*)| \lesssim \left(\int_{v_*} |v - v_*|^{\gamma q'} \, \mathbf{1}_{|v-v_*|\le 1} \right)^{1/q'} \|f\|_{L^q_v} \lesssim \|f\|_{L^q_v},$$

which implies

$$\|(c_- * f)\mu\|_{L^p_v(\mu^{-1/2})} \lesssim \|f\|_{L^q_v}, \quad \forall\, p \in [1, \infty],$$

and similarly

$$\|(c_- * f)\mu\|_{W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})} \lesssim \|f\|_{W^{n,p}_x W^{\ell,q}_v}, \quad \forall\, p \in [1, \infty].$$

Observe that in particular the operator $Tf = (c_- * f)\mu$ is a bounded operator from $W^{n,1}_x W^{\ell,1}_v(m)$ to $W^{n,1}_x W^{\ell,1}_v(\mu^{-1/2})$ and from $W^{n,\infty}_x W^{\ell,\infty}_v(m)$ to $W^{n,\infty}_x W^{\ell,\infty}_v(\mu^{-1/2})$, thus by interpolation also from $W^{n,p}_x W^{\ell,p}_v(m)$ to $W^{n,p}_x W^{\ell,p}_v(\mu^{-1/2})$ for any $p \in [1, \infty]$. This together with estimates of previous steps completes the proof of points (i) and (ii). □

We turn now to regularization properties of the semigroup $\mathcal{S}_\mathcal{B}$. We follow a technique introduced by Hérau [10] for Fokker-Plank equations (see also [22, Section A.21] and [11]).

Lemma 2.13. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)** *and let m_0 be some weight function with $\gamma + \sigma > 0$. Define*

$$m_1 := \begin{cases} m_0 & \text{if } \gamma \in [0,1]; \\ \langle v\rangle^{\frac{|\gamma|}{2}} m_0 & \text{if } \gamma \in [-2, 0). \end{cases} \qquad m_2 := \begin{cases} m_0 & \text{if } \gamma \in [0,1]; \\ \langle v\rangle^{4|\gamma|} m_0 & \text{if } \gamma \in [-2, 0). \end{cases}$$

Then there hold:

(1) From L^2 to H^ℓ for $\ell \ge 1$:

$$\forall\, t \in (0,1], \qquad \|\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(L^2(m_1), H^\ell(m_0))} \le C\, t^{-3\ell/2}$$

(2) From L^1 to L^2:

$$\forall\, t \in (0,1], \qquad \|\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(L^1(m_2), L^2(m_1))} \le C\, t^{-8}.$$

(3) From L^2 to L^∞:

$$\forall\, t \in (0,1], \qquad \|\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(L^2(m_2), L^\infty(m_1))} \le C\, t^{-8}.$$

Proof of Lemma 2.13. We consider the equation $\partial_t f = \mathcal{B} f$ and split the proof into three steps.

Step 1: from L^2 to H^ℓ. We only prove the case $\ell = 1$, the other cases being treated in the same way. Let us define

$$\mathcal{F}(t, f) := \|f\|^2_{L^2(m_1)} + \alpha_1\, t\, \|\nabla_v f\|^2_{L^2(m_0)} + \alpha_2\, t^2\, \langle \nabla_x f, \nabla_v f\rangle_{L^2(m_0)} + \alpha_3\, t^3\, \|\nabla_x f\|^2_{L^2(m_0)}.$$

We now choose α_i, $i = 1, 2, 3$ such that $0 < \alpha_3 \le \alpha_2 \le \alpha_1 \le 1$ and $\alpha_2^2 \le 2\alpha_1\alpha_3$. Then, there holds

$$2\mathcal{F}(t, f) \ge \alpha_3\, t^3\, \|\nabla_{x,v} f\|^2_{L^2(m_0)}.$$

Moreover, denoting $f_t = \mathcal{S}_\mathcal{B}(t) f$, we have

$$\begin{aligned} \frac{d}{dt}\, \mathcal{F}(t, f_t) = {} & \frac{d}{dt} \|f_t\|^2_{L^2(m_1)} + \alpha_1\, \|\nabla_v f_t\|^2_{L^2(m_0)} + \alpha_1\, t\, \frac{d}{dt} \|\nabla_v f_t\|^2_{L^2(m_0)} \\ & + 2\alpha_2\, t\, \langle \nabla_x f_t, \nabla_v f_t\rangle_{L^2(m_0)} + \alpha_2\, t^2\, \frac{d}{dt} \langle \nabla_x f_t, \nabla_v f_t\rangle_{L^2(m_0)} \\ & + 3\alpha_3\, t^2\, \|\nabla_x f_t\|^2_{L^2(m_0)} + \alpha_3\, t^3\, \frac{d}{dt} \|\nabla_x f_t\|^2_{L^2(m_0)}. \end{aligned}$$

We need to compute

$$\frac{d}{dt}\langle \nabla_x f, \nabla_v f\rangle_{L^2(m_0)} = \sum_{|\alpha|=1}\int \{\partial_x^\alpha(\mathcal{B}f)\,(\partial_v^\alpha f) + (\partial_x^\alpha f)\,\partial_v^\alpha(\mathcal{B}f)\}\; m_0^2.$$

Let us denote $f_x := \partial_x^\alpha f$ and $f_v := \partial_v^\alpha f$ to simplify and recall that

$$\partial_x^\alpha(\mathcal{B}f) = \bar a_{ij}\partial_{ij} f_x - \bar c f_x - v\cdot\nabla_x f_x - M\chi_R f_x,$$

and

$$\begin{aligned}\partial_v^\alpha(\mathcal{B}f) = \bar a_{ij}\partial_{ij} f_v - \bar c f_v - v\cdot\nabla_x f_v - M\chi_R f_v \\ + (\partial_v^\alpha \bar a_{ij})\partial_{ij} f - (\partial_v^\alpha \bar c) f - f_x - M(\partial_v^\alpha\chi_R) f.\end{aligned}$$

Using the same computation as in Lemma 2.10, we obtain

$$\int \{\partial_x^\alpha(\mathcal{B}f)\,(\partial_v^\alpha f) + (\partial_x^\alpha f)\,\partial_v^\alpha(\mathcal{B}f)\}\; m_0^2 = T_0 + T_1 + T_2 + T_3,$$

where

$$T_0 := -2\int \bar a_{ij}\,\partial_i f_x\,\partial_j f_v\, m_0^2,$$

$$T_1 := \int \{\varphi_{m_0,2}(v) - 2M\chi_R(v)\}\, f_x\, f_v\, m_0^2,$$

$$T_2 := -\int \left\{(\partial_v^\alpha \bar a_{ij})\,\frac{\partial_i m_0^2}{m_0^2} + \partial_v^\alpha \bar b_j\right\}\partial_j f\, f_x\, m_0^2 - \int \{\partial_v^\alpha \bar c + M(\partial_v^\alpha\chi_R)\}\, f\, f_x\, m_0^2 - \int |f_x|^2\, m_0^2$$

and

$$T_3 := -\int (\partial_v^\alpha \bar a_{ij})\,\partial_i f\,\partial_j f_x\, m_0^2.$$

For the term T_1, from the proof of Lemma 2.6 we get

$$T_1 \lesssim \int \langle v\rangle^{\gamma+\sigma}|f_x|\,|f_v|\, m_0^2 \lesssim \varepsilon t\,\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\,\partial_x^\alpha f\|^2_{L^2(m_0)} + \varepsilon^{-1}t^{-1}\,\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\,\partial_v^\alpha f\|^2_{L^2(m_0)}.$$

In a similar way, using $|\partial_v^\alpha \bar a_{ij}| \le C\langle v\rangle^{\gamma+1}$, $|\partial_v^\alpha \bar b_j| \le C\langle v\rangle^{\gamma}$ and $|\partial_i m^2| \le C\langle v\rangle^{\sigma-1}m^2$, we obtain for the second term

$$\begin{aligned}T_2 &\lesssim \int \langle v\rangle^{\gamma+\sigma}|\nabla_v f|\,|f_x|\, m_0^2 + \int\left\{\langle v\rangle^{\gamma-1} + \frac{M}{R}\mathbf{1}_{R\le|v|\le 2R}\right\}|f|\,|f_x|\, m_0^2 - \|\partial_x^\alpha f\|^2_{L^2(m_0)} \\ &\lesssim \varepsilon t\int\left\{\langle v\rangle^{\gamma+\sigma} + \langle v\rangle^{\gamma-1} + \frac{M}{R}\mathbf{1}_{R\le|v|\le 2R}\right\}|\partial_x^\alpha f|^2\, m_0^2 + \varepsilon^{-1}t^{-1}\int\left\{\langle v\rangle^{\gamma-1} + \frac{M}{R}\mathbf{1}_{R\le|v|\le 2R}\right\}|f|^2\, m_0^2 \\ &\quad + \varepsilon^{-1}t^{-1}\,\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\,\nabla_v f\|^2_{L^2(m_0)} - \|\partial_x^\alpha f\|^2_{L^2(m_0)}.\end{aligned}$$

We now investigate T_0 and, decomposing $\partial_i f_x = P_v\partial_i f_x + (I-P_v)\partial_i f_x$ and the same for $\partial_j f_v$, we easily get

$$\begin{aligned}T_0 \lesssim{}& \varepsilon t\big\{\|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v(\partial_x^\alpha f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v(\partial_x^\alpha f)\|^2_{L^2(m_0)}\big\} \\ &+ \varepsilon^{-1}t^{-1}\big\{\|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v(\partial_v^\alpha f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v(\partial_v^\alpha f)\|^2_{L^2(m_0)}\big\}.\end{aligned}$$

For the remainder term T_3, arguing as in the proof of Lemma 2.10 (term T_{22} in that lemma, see (2.40)) gives us

$$\begin{aligned}T_3 \lesssim{}& \varepsilon t\big\{\|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v(\partial_x^\alpha f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v(\partial_x^\alpha f)\|^2_{L^2(m_0)}\big\} \\ &+ \varepsilon^{-1}t^{-1}\big\{\|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v f\|^2_{L^2(m_0)}\big\}.\end{aligned}$$

Finally, putting together previous estimates we obtain

$$
\begin{aligned}
&\int \{\nabla_x(\mathcal{B}f)\nabla_v f + \nabla_x f\nabla_v(\mathcal{B}f)\}\, m_0^2 \\
&\lesssim \varepsilon t\Big\{\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_x f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v(\nabla_x f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v(\nabla_x f)\|^2_{L^2(m_0)}\Big\} \\
&\quad + C\varepsilon^{-1}t^{-1}\Big\{\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_v f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v(\nabla_v f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v(\nabla_v f)\|^2_{L^2(m_0)}\Big\} \\
&\quad + C\varepsilon^{-1}t^{-1}\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}P_v\nabla_v f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v f\|^2_{L^2(m_0)}\Big\} \\
&\quad + C\varepsilon^{-1}t^{-1}\|f\|^2_{L^2(m_0)} - \|\nabla_x f\|^2_{L^2(m_0)}.
\end{aligned}
$$

Using Cauchy-Schwarz inequality, we also write the following

$$
2\alpha_2 t\langle\nabla_x f, \nabla_v f\rangle_{L^2(m_0)} \le \alpha_2\left(\varepsilon t^2\|\nabla_x f\|^2_{L^2(m_0)} + C\varepsilon^{-1}\|\nabla_v f\|^2_{L^2(m_0)}\right).
$$

Moreover, picking up estimates of Lemma 2.10, it follows that: for any $0<\lambda<\lambda_{m,2}$ and $0<\delta<\lambda_{m,2}-\lambda$, there are $M,R>0$ large enough such that,

$$
\begin{aligned}
\int(\mathcal{B}f)f\, m_1^2 \le -c_0\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v f\|^2_{L^2(m_1)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v f\|^2_{L^2(m_1)}\Big\} \\
- \lambda\|f\|^2_{L^2(m_1)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}f\|^2_{L^2(m_1)},
\end{aligned}
$$

also, for some $\varepsilon_0>0$ to be chosen later,

$$
\begin{aligned}
\int \nabla_v(\mathcal{B}f)\nabla_v f\, m_0^2 \le -c_0\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v(\nabla_v f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v(\nabla_v f)\|^2_{L^2(m_0)}\Big\} \\
- \lambda\|\nabla_v f\|^2_{L^2(m_0)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_v f_t\|^2_{L^2(m_0)} \\
+ C\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v f\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v f\|^2_{L^2(m_0)}\Big\} \\
+ C\|f\|^2_{L^2(m_0)} + C\varepsilon_0^{-1}t^{-1}\|\nabla_v f\|^2_{L^2(m_0)} + C\varepsilon_0 t\|\nabla_x f\|^2_{L^2(m_0)},
\end{aligned}
$$

and finally

$$
\begin{aligned}
\int \nabla_x(\mathcal{B}f)\nabla_x f\, m_0^2 \le -c_0\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v(\nabla_x f)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v(\nabla_x f)\|^2_{L^2(m_0)}\Big\} \\
- \lambda\|\nabla_x f\|^2_{L^2(m_0)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_x f\|^2_{L^2(m_0)}.
\end{aligned}
$$

We choose

$$
\varepsilon_0 = \varepsilon^2, \quad \alpha_1 := \varepsilon^{5/2}, \quad \alpha_2 := \varepsilon^4, \quad \alpha_3 := \varepsilon^{9/2}.
$$

Therefore, for any $t\in[0,1]$, we can gather previous estimates to obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}(t,f_t) \\
&\le + \left(-c_0 + C\varepsilon^{1/2} + C\varepsilon^{5/2} + C\varepsilon^3\right)\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v f_t\|^2_{L^2(m_1)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v f_t\|^2_{L^2(m_1)}\Big\} \\
&\quad + t\varepsilon^{5/2}\left(-c_0 + C\varepsilon^{1/2}\right)\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v(\nabla_v f_t)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v(\nabla_v f_t)\|^2_{L^2(m_0)}\Big\} \\
&\quad + t^3\varepsilon^{9/2}\left(-c_0 + C\varepsilon^{1/2}\right)\Big\{\|\langle v\rangle^{\frac{\gamma}{2}}\, P_v\nabla_v(\nabla_x f_t)\|^2_{L^2(m_0)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}\,(I-P_v)\nabla_v(\nabla_x f_t)\|^2_{L^2(m_0)}\Big\} \\
&\quad - \lambda\|f_t\|^2_{L^2(m_1)} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}f_t\|^2_{L^2(m_1)} + Ct(\varepsilon^{5/2}+\varepsilon^3)\|f_t\|^2_{L^2(m_0)} \\
&\quad - \lambda\varepsilon^{5/2}t\|\nabla_v f_t\|^2_{L^2(m_0)} - t\varepsilon^{5/2}\left(\delta - C\varepsilon^{1/2}\right)\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_v f_t\|^2_{L^2(m_0)} \\
&\quad - t^2\left(\lambda\varepsilon^{9/2}t - C\varepsilon^{9/2} - \varepsilon^5 - C\varepsilon^{9/2}t + \varepsilon^4\right)\|\nabla_x f_t\|^2_{L^2(m_0)} - t^3\varepsilon^{9/2}\left(\delta - \varepsilon^{1/2}\right)\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}\nabla_x f_t\|^2_{L^2(m_0)}.
\end{aligned}
$$

We then choose $\varepsilon > 0$ small enough such that the following conditions are fulfilled:

$$\begin{cases} -c_0 + C\varepsilon^{1/2} + C\varepsilon^{5/2} + C\varepsilon^3 < -K < 0, \\ -c_0 + C\varepsilon^{1/2} < -K < 0, \\ -\lambda + Ct(\varepsilon^{5/2} + \varepsilon^3) < -K < 0, \\ \delta - C\varepsilon^{1/2} < -K < 0, \\ C\varepsilon^{9/2} + \varepsilon^5 + C\varepsilon^{9/2} - \varepsilon^4 < -K < 0. \end{cases}$$

We have then proved that, for any $t \in [0,1]$,

$$\frac{d}{dt}\mathcal{F}(t,f_t) \leq -K'\Big\{\|f_t\|^2_{L^2(m_1)} + \|\nabla_v f_t\|^2_{L^2(m_0)} + t^2\|\nabla_x f\|^2_{L^2(m_0)}\Big\} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)},$$

which implies

$$Ct^3\|\nabla_{x,v} f_t\|^2_{L^2(m_0)} \leq \mathcal{F}(t,f_t) \leq \mathcal{F}(0,f_0) = \|f_0\|^2_{L^2(m_1)}.$$

We deduce

$$\forall\, t \in (0,1], \quad \|\nabla_{x,v}\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(m_0)} \leq C\, t^{-3/2}\, \|f_0\|_{L^2(m_1)},$$

and the proof of point (1) for $\ell = 1$ is complete.

Step 2: from L^1 to L^2. We define,

$$\begin{aligned} \mathcal{G}(t,f_t) &:= \|f_t\|^2_{L^1(m_2)} + \alpha_0\, t^N\, \widetilde{\mathcal{F}}(t,f_t), \\ \widetilde{\mathcal{F}}(t,f_t) &:= \|f_t\|^2_{L^2(m_1)} + \alpha_1\, t^2\|\nabla_v f_t\|^2_{L^2(m_0)} \\ &\quad + \alpha_2\, t^4\langle\nabla_x f_t, \nabla_v f_t\rangle_{L^2(m_0)} + \alpha_3\, t^6\|\nabla_x f_t\|^2_{L^2(m_0)}, \end{aligned}$$

for some N to be chosen later. Thanks to Hölder and Sobolev inequalities (in $\mathbb{T}^3_x \times \mathbb{R}^3_v$), there holds

$$\|\langle v\rangle^q g\|^2_{L^2} \lesssim \|\nabla_{x,v} g\|^{3/2}_{L^2}\, \|\langle v\rangle^{4q} g\|^{1/2}_{L^1},$$

which implies that

$$\begin{aligned} \|f\|^2_{L^2(m_1)} &\lesssim \|f\|^{1/2}_{L^1(m_2)}\, \|\nabla_{x,v}(m_0 f)\|^{3/2}_{L^2} \\ &\lesssim C_\varepsilon t^{-15}\|f\|^2_{L^1(m_2)} + \varepsilon t^5\|\nabla_{x,v} f\|^2_{L^2(m_0)} + \varepsilon t^5\|\langle v\rangle^{\sigma-1} f\|^2_{L^2(m_0)} \\ &\lesssim C_\varepsilon t^{-15}\|f\|^2_{L^1(m_2)} + \varepsilon t^5\|\nabla_{x,v} f\|^2_{L^2(m_0)} + \varepsilon t^5\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f\|^2_{L^2(m_1)}, \end{aligned} \tag{2.44}$$

where we have used in last line that $\langle v\rangle^{\sigma-1} m_0 \lesssim \langle v\rangle^{\frac{\gamma+\sigma}{2}} m_1$. Arguing as in step 1, we have

$$\frac{d}{dt}\widetilde{\mathcal{F}}(t,f_t) \leq -K'\Big\{\|f_t\|^2_{L^2(m_1)} + \|\nabla_v f_t\|^2_{L^2(m_0)} + t^4\|\nabla_x f\|^2_{L^2(m_0)}\Big\} - \delta\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)}.$$

Putting together previous estimates it follows

$$\begin{aligned} \frac{d}{dt}\mathcal{G}(t,f_t) &\leq -K\|f_t\|^2_{L^1(m_2)} + \alpha_0 N t^{N-1}\widetilde{\mathcal{F}}(t,f) \\ &\quad - K'\alpha_0 t^N\Big\{\|f_t\|^2_{L^2(m_1)} + \|\nabla_v f_t\|^2_{L^2(m_0)} + t^4\|\nabla_x f\|^2_{L^2(m_0)}\Big\} - \delta\alpha_0 t^N\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)} \\ &\leq -K\|f_t\|^2_{L^1(m_2)} \\ &\quad + \alpha_0 N t^{N-1}\|f_t\|^2_{L^2(m_1)} + C\alpha_0 N t^{N+1}\|\nabla_v f_t\|^2_{L^2(m_0)} + C\alpha_0 N t^{N+5}\|\nabla_x f_t\|^2_{L^2(m_0)} \\ &\quad - K'\alpha_0 t^N\Big\{\|f_t\|^2_{L^2(m_1)} + \|\nabla_v f_t\|^2_{L^2(m_0)} + t^4\|\nabla_x f\|^2_{L^2(m_0)}\Big\} - \delta\alpha_0 t^N\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)}. \end{aligned}$$

Choose $t_* \in (0,1)$ so that $Nt^{N+1} \ll K't^N$ then, for any $t \in [0,t_*]$,

$$\frac{d}{dt}\mathcal{G}(t,f_t) \le -K\|f_t\|^2_{L^1(m_2)} + C\alpha_0 t^{N-1}\|f_t\|^2_{L^2(m_1)} - \delta\alpha_0 t^N \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)} \\ - K''\alpha_0 t^N \Big\{ \|\nabla_v f_t\|^2_{L^2(m_0)} + t^4 \|\nabla_x f\|^2_{L^2(m_0)} \Big\}.$$

Thanks to (2.44), for any $t \in [0,t_*]$, we get

$$\frac{d}{dt}\mathcal{G}(t,f_t) \le -(K - C_\varepsilon\alpha_0 t^{N-16})\|f_t\|^2_{L^1(m_2)} - \alpha_0 t^N(\delta - C\varepsilon)\|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f_t\|^2_{L^2(m_1)} \\ - \alpha_0 t^{N+4}(K'' - C\varepsilon)\|\nabla_{x,v} f\|^2_{L^2(m_0)}$$

Taking $N = 16$ and choosing $\varepsilon > 0$ small enough then $\alpha_0 > 0$ small enough, we get $\frac{d}{dt}\mathcal{G}(t,f_t) \le 0$ then

$$\forall\, t \in [0,t_*], \quad Ct^{16}\|f_t\|^2_{L^2(m_1)} \le \mathcal{G}(t,f_t) \le \mathcal{G}(0,f_0) = \|f_0\|^2_{L^1(m_2)}.$$

This ends the proof of point (2), using the fact that the norm is propagated for $t > t_*$.

Step 3: From L^2 to L^∞. Arguing by duality as in Lemma 2.8, the proof follows as in step 2. □

We define the convolution $\mathcal{S}_1 * \mathcal{S}_2$ by

$$(\mathcal{S}_1 * \mathcal{S}_2)(t) := \int_0^t \mathcal{S}_1(\tau)\,\mathcal{S}_2(t-\tau)\,d\tau,$$

and, for $n \in \mathbb{N}^*$, we define $\mathcal{S}^{(*n)}$ by $\mathcal{S}^{(n)} = \mathcal{S} * \mathcal{S}^{(*(n-1))}$ with $\mathcal{S}^{(*1)} = \mathcal{S}$.

Corollary 2.14. *Consider hypothesis* **(H1)**, **(H2)** *or* **(H3)**, *and spaces $\mathcal{E}_0, \mathcal{E}_1$ of the type E or $\mathcal{E}$ defined in* (2.1) *and* (2.2). *Then for any $\lambda' < \lambda < \lambda_{m,p}$ (where λ is defined in Lemmas 2.7, 2.8, 2.9, 2.10 or 2.11) there exists $N \in \mathbb{N}$ such that*

$$\|(\mathcal{A}\mathcal{S}_\mathcal{B})^{(*N)}(t)\|_{\mathscr{B}(\mathcal{E}_1,\mathcal{E}_0)} \le C\, e^{-\lambda' t}, \qquad \forall\, t \ge 0.$$

Proof. It is a consequence of the hypodissipativity properties of $\mathcal{B}$ (Lemmas 2.7, 2.9, 2.10 and 2.11), the boundedness of the operator $\mathcal{A}$ (Lemma 2.12), and the regularization properties in Lemma 2.13, together with [11, Lemma 2.4] and [8, Lemma 2.17]. □

2.6. **Proof of Theorem 2.1.** Thanks to the estimates proven in previous section, we can now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\mathcal{E}$ be an admissible space defined in (2.2) and consider $\ell_0 \ge 1$ large enough such that $E := H^{\ell_0}_{x,v}(\mu^{-1/2})$ defined in (2.1) satisfies $E \subset \mathcal{E}$. Recall that in the small/reference space E we already have a spectral gap in Theorem 1.3.

Then the proof of Theorem 2.1 is a consequence of the hypo-dissipative properties of $\mathcal{B}$ in Lemmas 2.7, 2.9, 2.10, 2.11, the boundedness of $\mathcal{A}$ in Lemma 2.12 and the regularizing properties of $(\mathcal{A}\mathcal{S}_\mathcal{B})^{(*N)}$ in Corollary 2.14, with which we are able to apply the "extension theorem" from [8, Theorem 2.13] (see also [11, Theorem 1.1]). □

2.7. **Proof of Theorem 2.3.** We give in this subsection a regularity estimate for the semigroup $\mathcal{S}_\mathcal{L}$.

Proof of Theorem 2.3. A key argument in the proof of [8, Theorem 2.13] in order to obtain the exponential decay (that gives point (*iii*) in Theorem 2.1) is the following factorization of the semigroup, for any $\ell \in \mathbb{N}^*$,

$$\mathcal{S}_\Lambda(t)(I-\Pi_0) = \sum_{j=0}^{\ell-1}((I-\Pi_0)\mathcal{S}_\mathcal{B} * (\mathcal{A}\mathcal{S}_\mathcal{B})^{(*j)})(t) + (\mathcal{S}_\Lambda(I-\Pi_0) * (\mathcal{A}\mathcal{S}_\mathcal{B})^{(*\ell)})(t), \tag{2.45}$$

which has been used with $\ell = N$ given by Corollary 2.14. We now turn to the proof of (2.5), and recall that $\mathcal{E} = H^n_x L^2_v(m)$ and $\mathcal{E}_{-1} = H^n_x(H^{-1}_{v,*}(m))$. For sake of simplicity, in what follows, we denote $e_\lambda(t) := e^{\lambda t}$. We write (2.45) with $\ell = N+1$

$$\mathcal{S}_\Lambda(t)(I-\Pi_0) = \sum_{j=0}^{N}((I-\Pi_0)\mathcal{S}_\mathcal{B} * (\mathcal{A}\mathcal{S}_\mathcal{B})^{(*j)})(t) + (\mathcal{S}_\Lambda(I-\Pi_0) * (\mathcal{A}\mathcal{S}_\mathcal{B})^{(*N)} * (\mathcal{A}\mathcal{S}_\mathcal{B}))(t),$$

so that, for any $\lambda_D < \lambda_{m,2}$ and any $\lambda < \lambda_1$, where $\lambda_1 \leq \min\{\lambda_0, \lambda_D\}$ is given by Theorem 2.1, we have

$$e^{\lambda t}\mathcal{S}_\Lambda(t)(I-\Pi_0) = \sum_{j=0}^{N}\mathcal{S}_j(t) + \mathcal{S}_{N+1}(t) \tag{2.46}$$

with

$$\mathcal{S}_j(t) = \left((I-\Pi_0)e_\lambda\mathcal{S}_\mathcal{B} * (e_\lambda\mathcal{A}\mathcal{S}_\mathcal{B})^{(*j)}\right)(t), \quad j = 0, \dots, N,$$

and

$$\mathcal{S}_{N+1}(t) = \left(e_\lambda\mathcal{S}_\Lambda(I-\Pi_0) * (e_\lambda\mathcal{A}\mathcal{S}_\mathcal{B})^{(*N)} * (e_\lambda\mathcal{A}\mathcal{S}_\mathcal{B})\right)(t).$$

We now prove that $\|e^{\lambda t}\mathcal{S}_\Lambda(t)(I-\Pi_0)\|_{\mathscr{B}(\mathcal{E}_{-1},\mathcal{E})} \in L^2_t(\mathbb{R}_+)$ by evaluating each term in (2.46), which in turn completes the proof of (2.5). Using Lemma 2.12, we easily observe that thanks to Lemmas 2.7 and 2.8 there hold

$$\|e^{\lambda t}\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})} \leq C\|e^{\lambda t}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})},$$

and also

$$\|e^{\lambda t}\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E},\mathcal{E})} \leq C\|e^{\lambda t}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E},\mathcal{E})} \leq Ce^{-(\lambda_D-\lambda)t},$$

from which we first obtain

$$\|e^{\lambda t}\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})} \in L^2_t(\mathbb{R}_+), \quad \|e^{\lambda t}\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E},\mathcal{E})} \in L^1_t(\mathbb{R}_+).$$

Therefore we deduce

$$\|\mathcal{S}_0(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})} = \|e^{\lambda t}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})} \in L^2_t(\mathbb{R}_+),$$

and, for $j = 1, \dots, N$,

$$\|\mathcal{S}_j(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})} \leq C\|e^{\lambda t}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E},\mathcal{E})} * \|(e_\lambda\mathcal{A}\mathcal{S}_\mathcal{B})^{(*(j-1))}(t)\|_{\mathcal{B}(\mathcal{E},\mathcal{E})} * \|e^{\lambda t}\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})},$$

which implies by induction

$$\|\mathcal{S}_j(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})} \in L^1_t(\mathbb{R}_+) * L^1_t(\mathbb{R}_+) * L^2_t(\mathbb{R}_+) \subset L^2_t(\mathbb{R}_+).$$

For the last term we first observe that, thanks to Theorem 1.3,

$$\|e^{\lambda t}\mathcal{S}_\Lambda(t)(I-\Pi_0)\|_{\mathscr{B}(E,E)} \leq Ce^{-(\lambda_0-\lambda)t} \in L^1_t(\mathbb{R}_+),$$

and also, thanks to Corollary 2.14,

$$\|(e_\lambda\mathcal{A}\mathcal{S}_\mathcal{B})^{(*N)}(t)\|_{\mathcal{B}(\mathcal{E},E)} \leq Ce^{-(\lambda_D-\lambda)t} \in L^1_t(\mathbb{R}_+).$$

These estimates finally yield

$$\begin{aligned}&\|\mathcal{S}_{N+1}(t)\|_{\mathcal{B}(\mathcal{E}_{-1},\mathcal{E})}\\ &\quad\leq C\|e^{\lambda t}\mathcal{S}_\Lambda(t)(I-\Pi_0)\|_{\mathscr{B}(E,E)} * \|(e_\lambda\mathcal{A}\mathcal{S}_\mathcal{B})^{(*N)}(t)\|_{\mathcal{B}(\mathcal{E},E)} * \|e^{\lambda t}\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(\mathcal{E}_{-1},\mathcal{E})} \in L^2_t(\mathbb{R}_+),\end{aligned}$$

which completes the proof of (2.5). □

3. The nonlinear equation

This section is devoted to the proof of Theorem 1.1. We develop a perturbative Cauchy theory for the (nonlinear) Landau equation using the estimates on the linearized operator obtained in the previous section.

3.1. **Functional spaces.** We recall the following definitions

$$\|f\|^2_{H^1_{v,*}(m)} = \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} f\|^2_{L^2_v(m)} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v\nabla_v f\|^2_{L^2_v(m)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v f\|^2_{L^2_v(m)},$$

and we also define the (stronger) norm

$$\|f\|^2_{H^1_{v,**}(m)} = \|\langle v\rangle^{\frac{\gamma+2}{2}} f\|^2_{L^2_v(m)} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v\nabla_v f\|^2_{L^2_v(m)} + \|\langle v\rangle^{\frac{\gamma+2}{2}}(I-P_v)\nabla_v f\|^2_{L^2_v(m)}.$$

Recall the space $\mathcal{H}^3_x L^2_v(m)$ defined in (1.11) associated to the norm

$$\|f\|^2_{\mathcal{H}^3_x L^2_v(m)} = \sum_{0\le j\le 3} \|\nabla^j_x f\|^2_{L^2_x L^2_v(m\langle v\rangle^{-j(1-\sigma/2)})},$$

and also the space $\mathcal{H}^3_x(H^1_{v,*}(m))$ defined in (1.13) by

$$\begin{aligned}\|f\|^2_{\mathcal{H}^3_x(H^1_{v,*}(m))} &= \sum_{0\le j\le 3} \|\nabla^j_x f\|^2_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-j(1-\sigma/2)}))}.\\ &= \sum_{0\le j\le 3} \int_{\mathbb{T}^3_x} \|\nabla^j_x f\|^2_{H^1_{v,*}(m\langle v\rangle^{-j(1-\sigma/2)})}.\end{aligned}$$

We define in a similar way the space $\mathcal{H}^3_x(H^1_{v,**}(m))$ using the norm $H^1_{v,**}(m)$ (instead of $H^1_{v,*}(m)$). We also define the negative Sobolev space $\mathcal{H}^3_x(H^{-1}_{v,*}(m))$ by duality in the following way

$$\begin{aligned}\|f\|_{\mathcal{H}^3_x(H^{-1}_{v,*}(m))} &:= \sup_{\|\phi\|_{\mathcal{H}^3_x(H^1_{v,*}(m))}\le 1} \langle f,\phi\rangle_{\mathcal{H}^3_x L^2_v(m)}\\ &:= \sup_{\|\phi\|_{\mathcal{H}^3_x(H^1_{v,*}(m))}\le 1} \sum_{0\le j\le 3} \langle \nabla^j_x f, \nabla^j_x\phi\rangle_{L^2_x L^2_v(m\langle v\rangle^{-j(1-\sigma/2)})}.\end{aligned} \tag{3.1}$$

The results on the linearized operator Λ in Theorems 2.1 and 2.3 are stated for spaces of the type $H^3_x L^2_v(m)$, but they can be easily adapted for the spaces $\mathcal{H}^3_x L^2_v(m)$ above, more precisely we have:

Corollary 3.1. *Consider hypothesis* **(H1), (H2) or (H3)** *and some weight function m, with the additional assumption $k > \gamma + 5 + 3/2$ in the case of polynomial weight $m = \langle v\rangle^k$. Then for any $\lambda < \lambda_{m,2}$ and any $\lambda_1 \le \min\{\lambda_0, \lambda\}$, there exists a constant $C > 0$ such that*

$$\forall\, t \ge 0,\ \forall f \in \mathcal{H}^3_x L^2_v(m), \quad \|\mathcal{S}_\Lambda(t)(I-\Pi_0)f\|_{\mathcal{H}^3_x L^2_v(m)} \le C\, e^{-\lambda_1 t}\, \|(I-\Pi_0)f\|_{\mathcal{H}^3_x L^2_v(m)}.$$

Moreover, for any $\lambda < \lambda_1$,

$$\int_0^\infty e^{2\lambda t}\, \|\mathcal{S}_\Lambda(t)(I-\Pi_0)f\|^2_{\mathcal{H}^3_x L^2_v(m)}\, dt \le C\|(I-\Pi_0)f\|^2_{\mathcal{H}^3_x(H^{-1}_{v,*}(m))}.$$

3.2. **Dissipative norm for the linearized equation.** We construct now a norm for which the linearized semigroup $\mathcal{S}_\Lambda(t)$ is dissipative, with a rate as close as we want to the optimal rate decay from Theorem 2.1, and also has a stronger dissipativity property.

Proposition 3.2. *Consider some weight function m satisfying* **(H0)**, *and let $X := \mathcal{H}^3_x L^2_v(m)$ and $Y := \mathcal{H}^3_x(H^1_{v,*}(m))$. Consider another weight function $\bar m$ satisfying* **(H1)-(H2)-(H3)** *with $\bar m \lesssim m\langle v\rangle^{-(1-\sigma/2)}$ and denote $\bar X := \mathcal{H}^3_x L^2_v(\bar m)$.*

Define for any $\eta > 0$ and any $\lambda_2 < \lambda_1$ (where $\lambda_1 > 0$ is the optimal rate in Theorem 2.1) the equivalent norm on X

$$|\!|\!| f |\!|\!|^2_X := \eta \|f\|^2_X + \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f\|^2_{\bar X}\, d\tau. \tag{3.2}$$

Then there is $\eta > 0$ small enough such that the solution $f_t = \mathcal{S}_\Lambda(t) f$ to the linearized equation satisfies, for any $t \geq 0$ and some constant $K > 0$,

$$\frac{1}{2}\frac{d}{dt}|\!|\!|\mathcal{S}_\Lambda(t) f|\!|\!|^2_X \leq -\lambda_2 |\!|\!|\mathcal{S}_\Lambda(t) f|\!|\!|^2_X - K\|\mathcal{S}_\Lambda(t) f\|^2_Y, \quad \forall\, f \in X,\, \Pi_0 f = 0.$$

Proof. First we remark that the norm $|\!|\!|\cdot|\!|\!|_{\mathcal{H}^3_x L^2_v(m)}$ is equivalent to the norm $\|\cdot\|_{\mathcal{H}^3_x L^2_v(m)}$ defined in (1.11) for any $\eta > 0$ and any $\lambda_2 < \lambda_1$. Indeed, using Corollary 3.1, we have

$$\begin{aligned}\eta \|f\|^2_{\mathcal{H}^3_x L^2_v(m)} \leq |\!|\!| f |\!|\!|^2_{\mathcal{H}^3_x L^2_v(m)} &= \eta \|f\|^2_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\, d\tau \\ &\leq \eta \|f\|^2_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty C^2 e^{-2(\lambda_1 - \lambda_2)\tau} \|f\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\, d\tau \leq (\eta + C)\|f\|^2_{\mathcal{H}^3_x L^2_v(m)}.\end{aligned}$$

We now compute, denoting $f_t = \mathcal{S}_\Lambda(t) f$,

$$\frac{1}{2}\frac{d}{dt}|\!|\!| f_t |\!|\!|^2_{\mathcal{H}^3_x L^2_v(m)} = \eta \langle \Lambda f_t\, f_t\rangle_{\mathcal{H}^3_x L^2_v(m)} + \frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}\|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 t} f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\, d\tau =: I_1 + I_2.$$

For I_1 we write $\Lambda = \mathcal{A} + \mathcal{B}$. Arguing exactly as in Section 2, more precisely Lemma 2.12, we first obtain that $\mathcal{A} \in \mathcal{B}(\mathcal{H}^3_x L^2_v(\bar m), \mathcal{H}^3_x L^2_v(\mu^{-1/2}))$, whence

$$\langle \mathcal{A} f_t, f_t\rangle_{\mathcal{H}^3_x L^2_v(m)} \leq C \|f_t\|_{\mathcal{H}^3_x L^2_v(\bar m)}.$$

Moreover, repeating the estimates for the hypodissipativity of $\mathcal{B}$ in Lemmas 2.7 and 2.10 we easily get, for any $\lambda_2 \leq \lambda < \lambda_{m,2}$ and some $K > 0$,

$$\langle \mathcal{B} f, f\rangle_{\mathcal{H}^3_x L^2_v(m)} \leq -\lambda \|f\|^2_{\mathcal{H}^3_x L^2_v(m)} - K\|f\|^2_{\mathcal{H}^3_x(H^1_{v,*}(m))},$$

therefore it follows

$$I_1 \leq -\lambda\eta \|f_t\|^2_{\mathcal{H}^3_x L^2_v(m)} - \eta K \|f_t\|^2_{\mathcal{H}^3_x(H^1_{v,*}(m))} + \eta C \|f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}.$$

The second term is computed exactly

$$\begin{aligned} I_2 &= \frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}\|\mathcal{S}_\Lambda(\tau + t) e^{\lambda_2 \tau} f\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\, d\tau \\ &= \frac{1}{2}\int_0^\infty \frac{\partial}{\partial \tau}\|\mathcal{S}_\Lambda(\tau + t) e^{\lambda_2 \tau} f\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\, d\tau - \lambda_2 \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\, d\tau \\ &= \frac{1}{2}\Big[\|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\Big]_{\tau=0}^{\tau=+\infty} - \lambda_2 \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\, d\tau \\ &= -\frac{1}{2}\|f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar m)} - \lambda_2 \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar m)}\, d\tau \end{aligned}$$

where we have used the semigroup decay from Corollary 3.1.

Gathering previous estimates and using that $\lambda \geq \lambda_2$ we obtain

$$I_1 + I_2 \leq -\lambda_2 \left\{ \eta \|f_t\|^2_{\mathcal{H}^3_x L^2_v(m)} + \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar{m})} \, d\tau \right\}$$
$$- \eta K \|f_t\|^2_{\mathcal{H}^3_x(H^1_{v,*}(m))} + \eta C \|f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar{m})} - \frac{1}{2} \|f_t\|^2_{\mathcal{H}^3_x L^2_v(\bar{m})}.$$

We complete the proof choosing $\eta > 0$ small enough. □

3.3. **Nonlinear estimates.** We prove in this section some estimates for the nonlinear operator Q. We will use the following auxiliary results.

Lemma 3.3. *Let $-3 < \alpha < 0$ and $\theta > 3$. Then*

$$A_\alpha(v) := \int_{\mathbb{R}^3} |v - v_*|^\alpha \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^\alpha.$$

Proof. Let $|v| \leq 1/2$, thus $|v_*| + 1/2 \leq 1 + |v - v_*|$ and we get

$$A_\alpha(v) = \int_{\mathbb{R}^3} |v_*|^\alpha \langle v - v_* \rangle^{-\theta} \, dv_* \lesssim \int_{\mathbb{R}^3} |v_*|^\alpha \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^\alpha.$$

Consider now $|v| > 1/2$ and split the integral into two regions: $|v - v_*| > \langle v \rangle/4$ and $|v - v_*| \leq \langle v \rangle/4$. For the first region we obtain

$$\int_{\mathbb{R}^3} \mathbf{1}_{|v-v_*| > \frac{\langle v \rangle}{4}} |v - v_*|^\alpha \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^\alpha \int_{\mathbb{R}^3} \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^\alpha.$$

For the second region, $|v| > 1/2$ and $|v - v_*| \leq \langle v \rangle/4$ imply $|v_*| \geq |v|/4$, hence

$$\int_{\mathbb{R}^3} \mathbf{1}_{|v-v_*| \leq \frac{\langle v \rangle}{4}} |v - v_*|^\alpha \langle v_* \rangle^{-\theta} \, dv_* \lesssim \langle v \rangle^{-\theta} \int_{\mathbb{R}^3} \mathbf{1}_{|v-v_*| \leq \frac{\langle v \rangle}{4}} |v - v_*|^\alpha \, dv_* \lesssim \langle v \rangle^{-\theta+\alpha+3} \lesssim \langle v \rangle^\alpha.$$

□

Lemma 3.4. *There holds:*

(i) For any $\theta > \gamma + 4 + 3/2$

$$|(a_{ij} * f)(v) \, v_i v_j| + |(a_{ij} * f)(v) \, v_i| + |(a_{ij} * f)(v)| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L^2_v(\langle v \rangle^\theta)}.$$

(ii) For any $\theta' > (\gamma + 1)_+ + 3/2$ (where $x_+ := \max\{x, 0\}$)

$$|(b_j * f)(v)| \lesssim \langle v \rangle^{\gamma+1} \|f\|_{L^2_v(\langle v \rangle^{\theta'})}.$$

(iii) If $\gamma \in [0, 1]$, for any $\theta'' > \gamma + 3/2$

$$|(c * f)(v)| \lesssim \langle v \rangle^\gamma \|f\|_{L^2_v(\langle v \rangle^{\theta''})}.$$

(iv) If $\gamma \in [-2, 0)$, for any $p > \frac{3}{3+\gamma}$ and $\theta'' > 3(1 - 1/p)$

$$|(c * f)(v)| \lesssim \langle v \rangle^\gamma \|f\|_{L^p_v(\langle v \rangle^{\theta''})}.$$

In particular, when $\gamma \in (-3/2, 0)$ we can choose $p = 2$ and $\theta'' > 3/2$; and when $\gamma \in [-2, -3/2]$ we can choose $p = 4$ and $\theta'' > 9/4$.

Proof. Recall that 0 is an eigenvalue of the matrix a_{ij} so that $a_{ij}(v - v_*)v_i = a_{ij}(v - v_*)v_{*i}$ and $a_{ij}(v - v_*)v_i v_j = a_{ij}(v - v_*)v_{*i}v_{*j}$. Using this we can easily obtain, for any $\theta > \gamma + 4 + 3/2$,

$$|(a_{ij} * f)(v) \, v_i v_j| = |\int_{v_*} a_{ij}(v - v_*) v_i v_j f_*| = \left| \int_{v_*} a_{ij}(v - v_*) v_{*i} v_{*j} f_* \right|$$
$$\lesssim \int_{v_*} \langle v \rangle^{\gamma+2} \langle v_* \rangle^{\gamma+4} |f_*| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L^1_v(\langle v \rangle^{\gamma+4})}$$
$$\lesssim \langle v \rangle^{\gamma+2} \|f\|_{L^2_v(\langle v \rangle^\theta)}.$$

In a similar way we get

$$|(a_{ij} * f)(v)\, v_i| \lesssim \langle v\rangle^{\gamma+2} \|f\|_{L^2_v(\langle v\rangle^{\theta-1})},$$

and we easily have, since $\gamma \in [-2,1]$,

$$|(a_{ij} * f)(v)| \lesssim \langle v\rangle^{\gamma+2} \|f\|_{L^2_v(\langle v\rangle^{\theta-2})}.$$

For the term $(b * f)$, we recall that $b_i(z) = -2|z|^\gamma z_i$ and we separate into two cases. When $\gamma \in [-1,1]$ we have, for any $\theta' > \gamma + 1 + 3/2$,

$$\begin{aligned}|(b_i * f)(v)| &\lesssim \int_{v_*} |v - v_*|^{\gamma+1} |f_*| \lesssim \int_{v_*} \langle v\rangle^{\gamma+1} \langle v_*\rangle^{\gamma+1} |f_*| \\ &\lesssim \langle v\rangle^{\gamma+1} \|f\|_{L^1_v(\langle v\rangle^{\gamma+1})} \lesssim \langle v\rangle^{\gamma+1} \|f\|_{L^2_v(\langle v\rangle^{\theta'})}.\end{aligned}$$

When $\gamma \in [-2,-1)$ we use Lemma 3.3 to obtain, for any $\theta' > 3/2$,

$$\begin{aligned}|(b_i * f)(v)| &\lesssim \int_{v_*} |v - v_*|^{\gamma+1} \langle v_*\rangle^{-\theta'} \langle v_*\rangle^{\theta'} |f_*| \lesssim \left(\int_{v_*} |v - v_*|^{2(\gamma+1)} \langle v_*\rangle^{-2\theta'} \right)^{1/2} \|f\|_{L^2_v(\langle v\rangle^{\theta'})} \\ &\lesssim \langle v\rangle^{\gamma+1} \|f\|_{L^2_v(\langle v\rangle^{\theta'})}.\end{aligned}$$

Finally for the last term $(c * f)$, recall that $c(z) = -2(\gamma+3)|z|^\gamma$ and separate into two cases. When $\gamma \in [0,1]$ then, for any $\theta'' > \gamma + 3/2$,

$$\begin{aligned}|(c * f)(v)| &\lesssim \int_{v_*} |v - v_*|^{\gamma} |f_*| \lesssim \int_{v_*} \langle v\rangle^{\gamma} \langle v_*\rangle^{\gamma} |f_*| \\ &\lesssim \langle v\rangle^{\gamma} \|f\|_{L^1_v(\langle v\rangle^{\gamma})} \lesssim \langle v\rangle^{\gamma} \|f\|_{L^2_v(\langle v\rangle^{\theta''})}.\end{aligned}$$

When $\gamma \in [-2,0)$ we use Lemma 3.3 to obtain, for any $p > \frac{3}{3+\gamma}$ and for any $\theta'' > 3(1-1/p)$,

$$\begin{aligned}|(c * f)(v)| &\lesssim \int_{v_*} |v - v_*|^{\gamma} \langle v_*\rangle^{-\theta''} \langle v_*\rangle^{\theta''} |f_*| \lesssim \left(\int_{v_*} |v - v_*|^{\gamma \frac{p}{p-1}} \langle v_*\rangle^{-\theta'' \frac{p}{p-1}} \right)^{(p-1)/p} \|f\|_{L^p_v(\langle v\rangle^{\theta''})} \\ &\lesssim \langle v\rangle^{\gamma} \|f\|_{L^2_v(\langle v\rangle^{\theta''})},\end{aligned}$$

thanks to $|\gamma| p/(p-1) < 3$. □

We now prove nonlinear estimates for the Landau operator Q.

Lemma 3.5. *Consider hypothesis* **(H1)**, **(H2)** *or* (**H3**).

(i) For any $\theta > \gamma + 4 + 3/2$, there holds

$$\langle Q(f,g), h\rangle_{L^2_v(m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\theta})} \|g\|_{H^1_{v,**}(m)} \|h\|_{H^1_{v,*}(m)}.$$

(ii) For any $\theta > \gamma + 4 + 3/2$ and $\theta' > 9/4$, there holds

$$\langle Q(f,g), g\rangle_{L^2_v(m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\theta})} \|g\|^2_{H^1_{v,*}(m)}, \quad \text{if } \gamma \in (-3/2, 1];$$

and

$$\langle Q(f,g), g\rangle_{L^2_v(m)} \lesssim \|f\|_{L^2_v(\langle v\rangle^{\theta})} \|g\|^2_{H^1_{v,*}(m)} + \|f\|_{H^1_v(\langle v\rangle^{\theta'})} \|g\|^2_{L^2_v(m)} \quad \text{if } \gamma \in [-2, -3/2].$$

Proof. We write

$$\begin{aligned}\langle Q(f,g), h\rangle_{L^2_v(m)} &= \int \partial_j \{(a_{ij} * f)\partial_i g - (b_j * f) g\}\, h\, m^2 \\ &= -\int (a_{ij} * f) \partial_i g\, \partial_j h\, m^2 - \int (a_{ij} * f) \partial_i g\, \partial_j m^2\, h \\ &\quad + \int (b_j * f) g\, \partial_j h\, m^2 + \int (b_j * f) g\, h\, \partial_j m^2 \\ &=: T_1 + T_2 + T_3 + T_4.\end{aligned}$$

Step 1. Point (i). We estimate each term separately.

Step 1.1. For the first term, since the estimate for $|v| \leq 1$ is evident, we only consider the case $|v| > 1$. We decompose $\partial_i g = P_v \partial_i g + (I - P_v)\partial_i g$ and similarly for $\partial_j h$, where we recall that $P_v \partial_i g = v_i |v|^{-2}(v \cdot \nabla_v g)$. We hence write

$$\begin{aligned} T_1 &= \int (a_{ij} * f) \left\{ P_v \partial_i g \, P_v \partial_j h + P_v \partial_i g \, (I - P_v)\partial_j h + (I - P_v)\partial_i g \, P_v \partial_j h + (I - P_v)\partial_i g \, (I - P_v)\partial_j h \right\} m^2 \\ &=: T_{11} + T_{12} + T_{13} + T_{14}. \end{aligned}$$

Therefore we have, using Lemma 3.4,

$$\begin{aligned} T_{11} &= \int (a_{ij} * f) v_i v_j \, \frac{(v \cdot \nabla_v g)}{|v|^2} \, \frac{(v \cdot \nabla_v h)}{|v|^2} \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \int \langle v \rangle^{\gamma+2} |v|^{-2} \, |\nabla_v g| \, |\nabla_v h| \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \, \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L^2_v(m)} \, \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L^2_v(m)}. \end{aligned}$$

Moreover

$$\begin{aligned} T_{12} &= \int (a_{ij} * f) v_i \, \frac{(v \cdot \nabla_v g)}{|v|^2} \left\{ (I - P_v)\partial_j h \right\} m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \int \langle v \rangle^{\gamma+2} |v|^{-1} \, |\nabla_v g| \, |(I - P_v)\nabla_v h| \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \, \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L^2_v(m)} \, \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v)\nabla_v h\|_{L^2_v(m)}, \end{aligned}$$

and similarly

$$T_{13} \lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \, \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v)\nabla_v g\|_{L^2_v(m)} \, \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L^2_v(m)}.$$

For the term T_{14} we obtain

$$\begin{aligned} T_{14} &= \int (a_{ij} * f) \left\{ (I - P_v)\partial_i g \right\} \left\{ (I - P_v)\partial_j h \right\} m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \int \langle v \rangle^{\gamma+2} |(I - P_v)\nabla_v g| \, |(I - P_v)\nabla_v h| \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \, \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v)\nabla_v g\|_{L^2_v(m)} \, \|\langle v \rangle^{\frac{\gamma+2}{2}} (I - P_v)\nabla_v h\|_{L^2_v(m)}. \end{aligned}$$

Step 1.2. Let us investigate the second term T_2, and again we only consider $|v| > 1$. Since $\partial_j m^2 = C v_j \langle v \rangle^{\sigma-2} m^2$, where we recall that $\sigma = 0$ when $m = \langle v \rangle^k$ and $\sigma = s$ when $m = e^{r\langle v \rangle^s}$, the same argument as for T_1 gives us

$$\begin{aligned} T_2 &= \int (a_{ij} * f) \left\{ P_v \partial_i g \, \partial_j m^2 + (I - P_v)\partial_i g \, \partial_j m^2 \right\} h \\ &=: T_{21} + T_{22}. \end{aligned}$$

Then we have

$$\begin{aligned} T_{21} &= C \int (a_{ij} * f) v_i v_j \langle v \rangle^{\sigma-2} \, \frac{(v \cdot \nabla_v g)}{|v|^2} \, h \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \int \langle v \rangle^{\gamma+2} \langle v \rangle^{\sigma-2} \, |v|^{-1} \, |\nabla_v g| \, |h| \, m^2 \\ &\lesssim \|f\|_{L^2_v(\langle v \rangle^\theta)} \, \|\langle v \rangle^{\frac{\gamma+\sigma-2}{2}} \nabla_v g\|_{L^2_v(m)} \, \|\langle v \rangle^{\frac{\gamma+\sigma}{2}} h\|_{L^2_v(m)}, \end{aligned}$$

and we recall that $\gamma+\sigma-2\le\gamma$. For the other term we get

$$\begin{aligned}T_{21}&=C\int(a_{ij}*f)v_j\langle v\rangle^{\sigma-2}\left\{(I-P_v)\partial_i g\right\}h\,m^2\\&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\int\langle v\rangle^{\gamma+2}\langle v\rangle^{\sigma-2}\left|(I-P_v)\nabla_v g\right|\left|h\right|m^2\\&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}(I-P_v)\nabla_v g\|_{L^2_v(m)}\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}h\|_{L^2_v(m)},\end{aligned}$$

and recall that $\gamma+\sigma\le\gamma+2$.

Step 1.3. For the term T_4,

$$\begin{aligned}T_4&=C\int(b_j*f)\,v_j\langle v\rangle^{\sigma-2}\,g\,h\,m^2\\&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\int\langle v\rangle^{\gamma+1}\langle v\rangle^{\sigma-1}\left|g\right|\left|h\right|m^2\\&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}g\|_{L^2_v(m)}\|\langle v\rangle^{\frac{\gamma+\sigma}{2}}h\|_{L^2_v(m)}.\end{aligned}$$

Remark that up to now we have obtained

$$T_1+T_2+T_4\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|g\|_{H^1_{v,*}(m)}\|h\|_{H^1_{v,*}(m)},$$

however in the estimate of the term T_3 (see below) we will get a worst estimate (with the norm $\|g\|_{H^1_{v,**}(m)}$ instead of $\|g\|_{H^1_{v,*}(m)}$).

Step 1.4. We finally investigate the term T_3 and we get

$$\begin{aligned}T_3&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\int\langle v\rangle^{\gamma+1}\left|g\right|\left|\nabla_v h\right|m^2\\&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|\langle v\rangle^{\frac{\gamma+2}{2}}g\|_{L^2_v(m)}\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v h\|_{L^2_v(m)}\\&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|g\|_{H^1_{v,**}(m)}\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_v h\|_{L^2_v(m)}.\end{aligned}$$

We complete the proof of point (i) gathering previous estimates.

Step 2. Point (ii). Arguing as in Step 1, with h replaced by g, we already have

$$T_1+T_2+T_4\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|g\|^2_{H^1_{v,*}(m)},$$

and we only estimate the term T_3. Integrating by parts we get

$$T_3=\int(b_j*f)\,g\,\partial_j g\,m^2\quad=-\frac12\int(c*f)\,g^2\,m^2-\frac12\int(b_j*f)\,\partial_j m^2\,g^2=:I+II.$$

The term II can be estimated exactly as T_4. For I, thanks to Lemma 3.4, we obtain

$$I\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|\langle v\rangle^{\frac{\gamma}{2}}g\|^2_{L^2_v(m)},\quad\text{if }\gamma\in(-3/2,1];$$

and

$$\begin{aligned}I&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|\langle v\rangle^{\frac{\gamma}{2}}g\|^2_{L^2_v(m)}+\|f\|_{L^4_v(\langle v\rangle^{\theta'})}\|\langle v\rangle^{\frac{\gamma}{2}}g\|^2_{L^2_v(m)},\quad\text{if }\gamma\in[-2,-3/2];\\&\lesssim\|f\|_{L^2_v(\langle v\rangle^\theta)}\|\langle v\rangle^{\frac{\gamma}{2}}g\|^2_{L^2_v(m)}+\|f\|_{H^1_v(\langle v\rangle^{\theta'})}\|\langle v\rangle^{\frac{\gamma}{2}}g\|^2_{L^2_v(m)}\end{aligned}$$

and that concludes the proof. □

Lemma 3.6. *Let assumption* **(H0)** *be in force.*

(i) There holds

$$\langle Q(f,g),h\rangle_{\mathcal{H}^3_xL^2_v(m)}\lesssim\|f\|_{\mathcal{H}^3_xL^2_v(m)}\|g\|_{\mathcal{H}^3_x(H^1_{v,**}(m))}\|h\|_{\mathcal{H}^3_x(H^1_{v,*}(m))},$$

therefore

$$\|Q(f,g)\|_{\mathcal{H}^3_x(H^{-1}_{v,*}(m))}\lesssim\|f\|_{\mathcal{H}^3_xL^2_v(m)}\|g\|_{\mathcal{H}^3_x(H^1_{v,**}(m))}.$$

(ii) There holds

$$\langle Q(f,g),g\rangle_{\mathcal{H}^3_x L^2_v(m)} \lesssim \|f\|_{\mathcal{H}^3_x L^2_v(m)} \|g\|^2_{\mathcal{H}^3_x(H^1_{v,*}(m))} + \|f\|_{\mathcal{H}^3_x(H^1_{v,*}(m))} \|g\|^2_{\mathcal{H}^3_x L^2_v(m)}.$$

Proof. We only prove point (ii). Point (i) can be proven in the same manner, using the estimate of Lemma 3.5-(i) instead of Lemma 3.5-(ii) as we shall do next.

We write

$$\langle Q(f,g),g\rangle_{\mathcal{H}^3_x L^2_v(m)} = \langle Q(f,g),g\rangle_{L^2_x L^2_v(m)} + \sum_{1\le|\beta|\le 3} \langle \partial^\beta_x Q(f,g), \partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-|\beta|(1-\sigma/2)})},$$

and

$$\partial^\beta_x Q(f,g) = \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2} Q(\partial^{\beta_1}_x f, \partial^{\beta_2}_x g).$$

The proof of the lemma is a consequence of Lemma 3.5 together with the following inequalities, that we shall use in the sequel when integrating in $x\in\mathbb{T}^3$,

$$\|u\|_{L^\infty(\mathbb{T}^3_x)} \lesssim \|u\|_{H^2(\mathbb{T}^3_x)}, \quad \|u\|_{L^6(\mathbb{T}^3_x)} \lesssim \|u\|_{H^1(\mathbb{T}^3_x)}, \quad \|u\|_{L^3(\mathbb{T}^3_x)} \lesssim \|u\|^{1/2}_{H^1(\mathbb{T}^3_x)} \|u\|^{1/2}_{L^2(\mathbb{T}^3_x)}. \tag{3.3}$$

Step 1. Using Lemma 3.5-(ii) and (3.3) we easily get, for $\theta > \gamma+4+3/2$ and $\theta' > 9/4$,

$$\begin{aligned}\langle Q(f,g),g\rangle_{L^2_x L^2_v(m)} &\lesssim \int_{\mathbb{T}^3_x} \|f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|^2_{H^1_{v,*}(m)} + \|f\|_{H^1_v(\langle v\rangle^{\theta'})} \|g\|^2_{L^2_v(m)} \\ &\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|g\|^2_{L^2_x(H^1_{v,*}(m))} + \|f\|_{H^2_x(H^1_v(\langle v\rangle^{\theta'}))} \|g\|^2_{L^2_x L^2_v(m)}.\end{aligned}$$

Step 2. Case $|\beta|=1$. Arguing as in the previous step, from Lemma 3.5-(ii) and (3.3), it follows

$$\begin{aligned}&\langle Q(f,\partial^\beta_x g),\partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|^2_{H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})} + \|f\|_{H^1_v(\langle v\rangle^{\theta'})} \|\nabla_x g\|^2_{L^2_v(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|^2_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)}))} + \|f\|_{H^2_x(H^1_v(\langle v\rangle^{\theta'}))} \|\nabla_x g\|^2_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})}.\end{aligned}$$

Moreover, using now Lemma 3.5-(i), we get

$$\begin{aligned}&\langle Q(\partial^\beta_x f,g),\partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|_{H^1_{v,**}(m\langle v\rangle^{-(1-\sigma/2)})} \|\nabla_x g\|_{H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)})} \\ &\quad\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|g\|_{L^2_x(H^1_{v,**}(m\langle v\rangle^{-(1-\sigma/2)}))} \|\nabla_x g\|_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-(1-\sigma/2)}))}.\end{aligned}$$

Step 3. Case $|\beta|=2$. When $\beta_2=\beta$ we have

$$\begin{aligned}&\langle Q(f,\partial^\beta_x g),\partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla^2_x g\|^2_{H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} + \|f\|_{H^1_v(\langle v\rangle^{\theta'})} \|\nabla^2_x g\|^2_{L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla^2_x g\|^2_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)}))} + \|f\|_{H^2_x(H^1_v(\langle v\rangle^{\theta'}))} \|\nabla^2_x g\|^2_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})}.\end{aligned}$$

If $|\beta_1|=|\beta_2|=1$ then we obtain

$$\begin{aligned}&\langle Q(\partial^{\beta_1}_x f,\partial^{\beta_2}_x g),\partial^\beta_x g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|_{H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)})} \|\nabla^2_x g\|_{H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})} \\ &\quad\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|_{L^2_x(H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)}))} \|\nabla^2_x g\|_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)}))}.\end{aligned}$$

Finally, when $\beta_1 = \beta$ we get

$$\begin{aligned}
&\langle Q(\partial_x^\beta f, g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-2(1-\sigma/2)})}\\
&\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|_{H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)})} \|\nabla_x^2 g\|_{H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)})}\\
&\quad\lesssim \|\nabla_x^2 f\|_{L^6_x L^2_v(\langle v\rangle^\theta)} \|g\|_{L^3_x(H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)}))} \|\nabla_x^2 g\|_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)}))}\\
&\quad\lesssim \|\nabla_x^2 f\|_{H^1_x L^2_v(\langle v\rangle^\theta)} \|g\|^{1/2}_{L^2_x(H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)}))} \|g\|^{1/2}_{H^1_x(H^1_{v,**}(m\langle v\rangle^{-2(1-\sigma/2)}))} \|\nabla_x^2 g\|_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-2(1-\sigma/2)}))}.
\end{aligned}$$

Step 4. Case $|\beta| = 3$. When $\beta_2 = \beta$ we obtain

$$\begin{aligned}
\langle Q(f, \partial_x^\beta g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})} &\lesssim \|f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla_x^3 g\|^2_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)}))}\\
&\quad + \|f\|_{H^2_x(H^1_v(\langle v\rangle^{\theta'}))} \|\nabla_x^3 g\|^2_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})}.
\end{aligned}$$

If $|\beta_1| = 1$ and $|\beta_2| = 2$ then

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})}\\
&\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla_x^2 g\|_{H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \|\nabla_x^3 g\|_{H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}\\
&\quad\lesssim \|\nabla_x f\|_{H^2_x L^2_v(\langle v\rangle^\theta)} \|\nabla_x^2 g\|_{L^2_x(H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)}))} \|\nabla_x^3 g\|_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)}))}.
\end{aligned}$$

When $|\beta_1| = 2$ and $|\beta_2| = 1$ then we get

$$\begin{aligned}
&\langle Q(\partial_x^{\beta_1} f, \partial_x^{\beta_2} g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})}\\
&\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x^2 f\|_{L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|_{H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \|\nabla_x^3 g\|_{H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}\\
&\quad\lesssim \|\nabla_x^2 f\|_{H^1_x L^2_v(\langle v\rangle^\theta)} \|\nabla_x g\|^{1/2}_{L^2_x(H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)}))} \|\nabla_x g\|^{1/2}_{H^1_x(H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)}))} \|\nabla_x^3 g\|_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)}))}.
\end{aligned}$$

Finally, when $\beta_1 = \beta$, it follows

$$\begin{aligned}
&\langle Q(\partial_x^\beta f, g), \partial_x^\beta g\rangle_{L^2_x L^2_v(m\langle v\rangle^{-3(1-\sigma/2)})}\\
&\quad\lesssim \int_{\mathbb{T}^3_x} \|\nabla_x^3 f\|_{L^2_v(\langle v\rangle^\theta)} \|g\|_{H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)})} \|\nabla_x^3 g\|_{H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)})}\\
&\quad\lesssim \|\nabla_x^3 f\|_{L^2_x L^2_v(\langle v\rangle^\theta)} \|g\|_{H^2_x(H^1_{v,**}(m\langle v\rangle^{-3(1-\sigma/2)}))} \|\nabla_x^3 g\|_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-3(1-\sigma/2)}))}.
\end{aligned}$$

Step 5. Conclusion. We can conclude the proof gathering previous estimates and remarking that, for any $n = 0, 1, 2$, there holds

$$\|\langle v\rangle^{\frac{\gamma+2}{2}} \nabla_x^n g\|_{L^2_x L^2_v(m\langle v\rangle^{-(n+1)(1-\sigma/2)})} = \|\langle v\rangle^{\frac{\gamma+\sigma}{2}} \nabla_x^n g\|_{L^2_x L^2_v(m\langle v\rangle^{-n(1-\sigma/2)})},$$

which implies

$$\|\nabla_x^n g\|_{L^2_x(H^1_{v,**}(m\langle v\rangle^{-(n+1)(1-\sigma/2)}))} \lesssim \|\nabla_x^n g\|_{L^2_x(H^1_{v,*}(m\langle v\rangle^{-n(1-\sigma/2)}))},$$

and observing also that

$$\|f\|_{H^3_x L^2_v(\langle v\rangle^\theta)} \lesssim \|f\|_{\mathcal{H}^3_x L^2_v(m)}$$

and

$$\|f\|_{H^2_x(H^1_v(\langle v\rangle^{\theta'}))} \lesssim \|f\|_{\mathcal{H}^3_x(H^1_{v,*}(m))}.$$

□

3.4. **Proof of Theorem 1.1.** We consider the Cauchy problem for the perturbation $f = F - \mu$. The equation satisfied by $f = f(t,x,v)$ is

$$\begin{cases} \partial_t f = \Lambda f + Q(f,f) \\ f_{|t=0} = f_0 = F_0 - \mu. \end{cases} \tag{3.4}$$

From the conservation laws (see (1.6) and (1.10)), for all $t > 0$, $\Pi_0 f_t = 0$ since $\Pi_0 f_0 = 0$, more precisely $\int f_t(x,v)\,dx\,dv = \int v_j f_t(x,v)\,dx\,dv = \int |v|^2 f_t(x,v)\,dx\,dv = 0$, and also $\Pi_0 Q(f_t,f_t) = 0$.

Hereafter we fix some weight function m that satisfies hypothesis **(H0)**. We also fix a weight function m_0 satisfying the assumptions of Corollary 3.1 (i.e. m_0 satisfies **(H1)**, **(H2)** or **(H3)** with the additional condition $k_0 > \gamma + 5 + 3/2$ if $m_0 = \langle v\rangle^{k_0}$) such that $m_0 \lesssim m\langle v\rangle^{-(1-\sigma/2)}$. Observe that this is always possible under the assumptions on m.

We will construct solutions on $L^\infty_t(\mathcal{H}^3_x L^2_v(m))$ under a smallness assumption on the initial data $\|f_0\|_{\mathcal{H}^3_x L^2_v(m)} \le \epsilon_0$. We introduce the notation to simplify

$$\begin{cases} X := \mathcal{H}^3_x L^2_v(m), \quad Y := \mathcal{H}^3_x(H^1_{v,*}(m)), \quad Y' := \mathcal{H}^3_x(H^{-1}_{v,*}(m)), \\ X_0 := \mathcal{H}^3_x L^2_v(m_0), \quad Y_0 := \mathcal{H}^3_x(H^1_{v,*}(m_0)), \quad Y_0' := \mathcal{H}^3_x(H^{-1}_{v,*}(m_0)), \quad Z_0 := \mathcal{H}^3_x(H^1_{v,**}(m_0)), \end{cases}$$

where we recall that these spaces are defined in (1.11)-(1.13)-(3.1), and we also remark that $\|f\|_{Z_0} \lesssim \|f\|_Y$.

We split the proof of Theorem 1.1 into three parts: Theorem 3.9, Theorem 3.10 and Theorem 3.11 below.

3.4.1. *A priori estimates.* We start proving a stability estimate.

Proposition 3.7. *Any solution $f = f_t$ to (3.4) satisfies, at least formally, the following differential inequality: for any $\lambda_2 < \lambda_1$ there holds*

$$\frac{1}{2}\frac{d}{dt}|||f|||^2_X \le -\lambda_2 |||f|||^2_X - \big(K - C|||f|||_X\big)\|f\|^2_Y,$$

for some constants $K, C > 0$.

Proof of Proposition 3.7. Recall that the norm $|||\cdot|||_X$ is defined in Proposition 3.2 and it is equivalent to the $\|\cdot\|_X$-norm. Thanks to (3.4) we write

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|||f|||^2_X &= \eta\langle f, \Lambda f\rangle_X + \int_0^\infty \langle \mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f, \mathcal{S}_\Lambda(\tau) e^{\lambda_2\tau}\Lambda f\rangle_{X_0}\,d\tau \\ &\quad + \eta\langle f, Q(f,f)\rangle_X + \int_0^\infty \langle \mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f, \mathcal{S}_\Lambda(\tau) e^{\lambda_2\tau} Q(f,f)\rangle_{X_0}\,d\tau \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For the linear part $I_1 + I_2$, we already have from Proposition 3.2 that, for any $\lambda_2 < \lambda_1$,

$$I_1 + I_2 \le -\lambda_2 |||f|||^2_X - K\|f\|^2_Y.$$

Let us investigate the nonlinear part. For the term I_3, Lemma 3.6-(ii) gives us directly

$$I_3 \lesssim \|f\|_X \|f\|^2_Y + \|f\|^2_X \|f\|_Y \lesssim |||f|||_X \|f\|^2_Y.$$

For the last term I_4, we use the fact that $\Pi_0 f_t = 0$ and $\Pi_0 Q(f_t, f_t) = 0$ for all $t \geq 0$, together with Corollary 3.1 to get

$$\begin{aligned}
&\int_0^\infty \langle \mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f, \mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} Q(f,f) \rangle_{X_0} \, d\tau \\
&\quad \leq \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f\|_{X_0} \, \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} Q(f,f)\|_{X_0} \, d\tau \\
&\quad \leq \left(\int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f\|_{X_0}^2 \, d\tau \right)^{1/2} \left(\int_0^\infty \|S_\Lambda(\tau) e^{\lambda_2 \tau} Q(f,f)\|_{X_0}^2 \, d\tau \right)^{1/2} \\
&\quad \lesssim \left(\int_0^\infty e^{-2(\lambda_1 - \lambda_2)\tau} \|f\|_{X_0}^2 \, d\tau \right)^{1/2} \left(\int_0^\infty e^{2\lambda_2 \tau} \|S_\Lambda(\tau) Q(f,f)\|_{X_0}^2 \, d\tau \right)^{1/2} \\
&\quad \lesssim \|f\|_{X_0} \, \|Q(f,f)\|_{Y_0'}.
\end{aligned}$$

From Lemma 3.6-(i) we have

$$\|Q(f,f)\|_{Y_0'} \lesssim \|f\|_{X_0} \, \|f\|_{Z_0}.$$

Therefore, using that $m_0 \lesssim m \langle v \rangle^{-(1-\sigma/2)}$ so that $\|f\|_{Z_0} \lesssim \|f\|_Y$, we obtain

$$I_4 \lesssim \|f\|_X \, \|f\|_Y^2 \lesssim |||f|||_X \, \|f\|_Y^2,$$

and the proof is complete. □

We prove now an a priori estimate on the difference of two solutions to (3.4).

Proposition 3.8. *Consider two solutions f and g to (3.4) associated to initial data f_0 and g_0, respectively. Then, at least formally, the difference $f - g$ satisfies the following differential inequality*

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} |||f - g|||_{X_0}^2 \leq -K \|f - g\|_{Y_0}^2 + C |||g|||_{X_0} \, \|f - g\|_{Y_0}^2 \\
+ C \big(\|g\|_{Y_0} + \|f\|_Y \big) \, |||f - g|||_{X_0} \, \|f - g\|_{Y_0},
\end{aligned}$$

for some constants $K, C > 0$.

Proof. We write the equation safisfied by $f - g$:

$$\begin{cases} \partial_t (f - g) = \Lambda (f - g) + Q(g, f - g) + Q(f - g, f), \\ (f - g)_{|t=0} = f_0 - g_0. \end{cases}$$

Denote $\overline{X}_0 := \mathcal{H}_x^3 L_v^2(\bar{m}_0)$ where $\bar{m}_0 \lesssim m_0 \langle v \rangle^{-(1-\sigma/2)}$ (see (3.2)). Then we compute

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} |||f_t - g_t|||_{X_0}^2 &= \eta \langle (f - g), \Lambda (f - g) \rangle_{X_0} + \int_0^\infty \langle S_\Lambda(\tau) e^{\lambda_2 \tau} (f - g), S_\Lambda(\tau) e^{\lambda_2 \tau} \Lambda (f - g) \rangle_{\overline{X}_0} \, d\tau \\
&\quad + \eta \langle (f - g), Q(g, f - g) \rangle_{X_0} + \int_0^\infty \langle S_\Lambda(\tau) e^{\lambda_2 \tau} (f - g), S_\Lambda(\tau) e^{\lambda_2 \tau} Q(g, f - g) \rangle_{\overline{X}_0} \, d\tau \\
&\quad + \eta \langle (f - g), Q(f - g, f) \rangle_{X_0} + \int_0^\infty \langle S_\Lambda(\tau) e^{\lambda_2 \tau} (f - g), S_\Lambda(\tau) e^{\lambda_2 \tau} Q(f - g, f) \rangle_{\overline{X}_0} \, d\tau \\
&=: T_1 + T_2 + T_3 + T_4 + T_5 + T_6.
\end{aligned}$$

Arguing as in Proposition 3.7 we easily obtain,

$$T_1 + T_2 \leq -K \|f - g\|_{Y_0}^2,$$

and also

$$T_3 + T_4 \lesssim |||g|||_{X_0} \, \|f - g\|_{Y_0}^2 + \|g\|_{Y_0} \, |||f - g|||_{X_0} \, \|f - g\|_{Y_0}.$$

Moreover, for the last part $T_5 + T_6$, arguing as in Proposition 3.7 and using Lemma 3.6-(i), we get

$$T_5 + T_6 \lesssim |||f - g|||_{X_0} \, \|f\|_{Z_0} \, \|f - g\|_{Y_0} \lesssim |||f - g|||_{X_0} \, \|f\|_{Y} \, \|f - g\|_{Y_0},$$

which completes the proof. □

3.4.2. *Cauchy problem in the close-to-equilibrium setting.* Thanks to the a priori estimates in Proposition 3.7 and Proposition 3.8, we are now able to construct solutions to (3.4) on $L^\infty_t(X) = L^\infty_t(\mathcal{H}^3_x L^2_v(m))$, assuming a smallness condition on the initial data.

Theorem 3.9. *There is a constant $\epsilon_0 = \epsilon_0(m) > 0$ such that, if $|||f_0|||_X \leq \epsilon_0$ then there exists a global weak solution f to (3.4) that satisfies, for some constant $C > 0$,*

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \leq C\epsilon_0.$$

Moreover, if $F_0 = \mu + f_0 \geq 0$ then $F(t) = \mu + f(t) \geq 0$.

Proof. The proof follows a standard argument by introducing an iterative scheme and using the estimates established in Propositions 3.7 and 3.8, thus we only sketch it.

For any integer $n \geq 1$ we define the iterative scheme

$$\begin{cases} \partial_t f^n = \Lambda f^n + Q(f^{n-1}, f^n) \\ f^n_{|t=0} = f_0 \end{cases} \quad \forall\, n \geq 1, \quad \text{and} \quad \begin{cases} \partial_t f^0 = \Lambda f^0 \\ f^0_{|t=0} = f_0 \end{cases}.$$

Firstly, the functions f^n are well defined on X for all $t \geq 0$ thanks to the semigroup theory in Theorem 2.1 and Corollary 3.1, and the stability estimates proven below.

Step 1. Stability of the scheme. We first prove the stability of the scheme on X. Thanks to Propositions 3.7, we prove by induction that, if $\epsilon_0 > 0$ is small enough, there holds

$$\forall\, n \geq 0, \forall\, t \geq 0, \quad A_n(t) := |||f^n_t|||^2_X + K \int_0^t \|f^n_\tau\|^2_Y \, d\tau \leq 2\epsilon_0^2. \tag{3.5}$$

Step 2. Convergence of the scheme. We now turn to the convergence of the scheme in X_0. Denote $d^n = f^{n+1} - f^n$ that satisfies

$$\begin{cases} \partial_t d^n = \Lambda d^n + Q(f^n, d^n) + Q(d^{n-1}, f^n), \quad \forall\, n \in \mathbb{N}^*; \\ \partial_t d^0 = \Lambda d^0 + Q(f^0, f^1). \end{cases}$$

Thanks to Proposition 3.7, Proposition 3.8 and estimate (3.5), we then prove by induction that, for $\epsilon_0 > 0$ small enough, it holds

$$\forall\, t \geq 0, \forall\, n \geq 0, \quad B_n(t) := |||d^n_t|||^2_{X_0} + K \int_0^t \|d^n_\tau\|^2_{Y_0} \, d\tau \leq (C'\epsilon_0)^{2n}, \tag{3.6}$$

for some constant $C' > 0$ that does not depend on ϵ_0.

Therefore the sequence $(f^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty([0,\infty); X_0) = L^\infty([0,\infty); \mathcal{H}^3_x L^2_v(m_0))$, and its limit f satisfies (3.4) in a weak sense. We then deduce that

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \leq C\epsilon_0,$$

by passing to the limit $n \to \infty$ in (3.5). Moreover, since $F_0 = \mu + f_0 \geq 0$ we easily obtain that $F(t) = \mu + f(t) \geq 0$ (see e.g. [9]). □

We can now address the problem of uniqueness.

Theorem 3.10. *There is a constant $\epsilon_0 = \epsilon_0(m) > 0$ such that, if $|||f_0|||_X \leq \epsilon_0$ then there exists a unique global weak solution $f \in L^\infty([0,\infty); X) \cap L^2([0,\infty); Y)$ to (3.4) such that*

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \leq C\epsilon_0.$$

Proof. Let f and g be two solutions to (3.4) with same initial data $g_0 = f_0$ that satisfy

$$\|f\|_{L^\infty([0,\infty);X)} + \|f\|_{L^2([0,\infty);Y)} \leq C\epsilon_0.$$

and

$$\|g\|_{L^\infty([0,\infty);X)} + \|g\|_{L^2([0,\infty);Y)} \leq C\epsilon_0.$$

The difference $f - g$ satisfies then

$$\partial_t(f-g) = \Lambda(f-g) + Q(g, f-g) + Q(f-g, f),$$

with $f_0 = g_0$. We then compute the standard $L^2_x L^2_v(m_0)$-norm of the difference $f - g$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f-g\|^2_{L^2_xL^2_v(m_0)} &= \langle \Lambda(f-g), f-g\rangle_{L^2_xL^2_v(m_0)} + \langle Q(g, f-g), f-g\rangle_{L^2_xL^2_v(m_0)} \\ &\quad + \langle Q(f-g, f), f-g\rangle_{L^2_xL^2_v(m_0)}.\end{aligned}$$

We write $\Lambda = \mathcal{A} + \mathcal{B}$ so that we obtain

$$\langle \Lambda(f-g), f-g\rangle_{L^2_xL^2_v(m_0)} \leq -K\|f-g\|^2_{L^2_x(H^1_{v,*}(m_0))} + C\|f-g\|^2_{L^2_xL^2_v(m_0)}.$$

Moreover, Lemma 3.5-(ii) together with (3.3) gives

$$\langle Q(g, f-g), f-g\rangle_{L^2_xL^2_v(m_0)} \leq C\|g\|_{H^2_xL^2_v(m_0)}\|f-g\|^2_{L^2_x(H^1_{v,*}(m_0))} + C\|g\|_{H^2_x(H^1_v(m_0))}\|f-g\|^2_{L^2_xL^2_v(m_0)},$$

whence, integrating in time,

$$\begin{aligned}&\int_0^t \langle Q(g_\tau, f_\tau - g_\tau), f_\tau - g_\tau\rangle_{L^2_xL^2_v(m_0)}\, d\tau \\ &\quad\leq C\sup_{\tau\in[0,t]}\|g_\tau\|_{H^2_xL^2_v(m_0)}\int_0^t \|f_\tau - g_\tau\|^2_{L^2_x(H^1_{v,*}(m_0))} \\ &\qquad + C\left(\int_0^t \|g_\tau\|^2_{H^2_x(H^1_v(m_0))}\right)^{1/2}\left(\sup_{\tau\in[0,t]}\|f_\tau - g_\tau\|^2_{L^2_xL^2_v(m_0)} + \int_0^t \|f_\tau - g_\tau\|^2_{L^2_xL^2_v(m_0)}\right).\end{aligned}$$

Thanks to Lemma 3.5-(i) it follows

$$\langle Q(f-g, f), f-g\rangle_{L^2_xL^2_v(m_0)} \leq C\|f-g\|_{L^2_xL^2_v(m_0)}\|f\|_{H^2_x(H^1_{v,**}(m_0))}\|f-g\|_{L^2_x(H^1_{v,*}(m_0))},$$

which integrating in time gives

$$\begin{aligned}&\int_0^t \langle Q(f_\tau - g_\tau, f_\tau), f_\tau - g_\tau\rangle_{L^2_xL^2_v(m_0)}\, d\tau \\ &\leq C\left(\sup_{\tau\in[0,t]}\|f_\tau - g_\tau\|_{L^2_xL^2_v(m_0)}\right)\int_0^t \|f_\tau\|_{H^2_x(H^1_{v,**}(m_0))}\|f_\tau - g_\tau\|_{L^2_x(H^1_{v,*}(m_0))} \\ &\leq C\left(\int_0^t \|f_\tau\|^2_{H^2_x(H^1_{v,**}(m_0))}\right)^{1/2}\left(\sup_{\tau\in[0,t]}\|f_\tau - g_\tau\|^2_{L^2_xL^2_v(m_0)} + \int_0^t \|f_\tau - g_\tau\|^2_{L^2_x(H^1_{v,*}(m_0))}\right),\end{aligned}$$

and observe that $\|f\|_{L^2_t(H^2_x(H^1_{v,**}(m_0)))} \lesssim \|f\|_{L^2_t(Y)} \leq C\epsilon_0$. Therefore

$$\begin{aligned}&\|f_t - g_t\|^2_{L^2_xL^2_v(m_0)} + K\int_0^t \|f_\tau - g_\tau\|^2_{L^2_x(H^1_{v,*}(m_0))}\, d\tau \\ &\quad\leq C\int_0^t \|f_\tau - g_\tau\|^2_{L^2_xL^2_v(m_0)}\, d\tau + C\epsilon_0\int_0^t \|f_\tau - g_\tau\|^2_{L^2_x(H^1_{v,*}(m_0))}\, d\tau \\ &\qquad + C\epsilon_0\left(\sup_{\tau\in[0,t]}\|f_\tau - g_\tau\|^2_{L^2_xL^2_v(m_0)} + \int_0^t \|f_\tau - g_\tau\|^2_{L^2_x(H^1_{v,*}(m_0))}\, d\tau\right),\end{aligned}$$

and when $\epsilon_0 > 0$ is small enough we conclude the proof of uniqueness by Gronwall's inequality. □

3.4.3. *Convergence to equilibrium in the close-to-equilibrium setting.*

Theorem 3.11. *There is a positive constant $\epsilon_1 \le \epsilon_0$ so that, if $|||f_0|||_X \le \epsilon_1$, then the unique global weak solution f to (3.4) (constructed in Theorems 3.9 and 3.10) verifies an exponential decay: for any $\lambda_2 < \lambda_1$ there exists $C > 0$ such that*

$$\forall\, t \ge 0, \quad \|f(t)\|_X \le C\, e^{-\lambda_2 t}\, \|f_0\|_X,$$

where we recall that $\lambda_1 > 0$ is the optimal rate given by the semigroup decay in Theorem 2.1.

Proof. From Theorem 3.9 we have

$$\sup_{t\ge 0} |||f(t)|||_X^2 + \int_0^t \|f(\tau)\|_Y^2\, d\tau \le C\epsilon_1^2.$$

Using Proposition 3.7 we get, if $\epsilon_1 > 0$ is small enough so that $-K + C\epsilon_1 \le -K/2$, and for any $\lambda_2 < \lambda_1$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||f|||_X^2 &\le -\lambda_2 |||f|||_X^2 - (K - C\epsilon_1)\|f\|_Y^2 \\ &\le -\lambda_2 |||f|||_X^2 - \frac{K}{2}\|f\|_Y^2,\end{aligned}$$

and then we deduce an exponential convergence

$$\forall\, t \ge 0, \qquad |||f(t)|||_X \le e^{-\lambda_2 t}\, |||f_0|||_X,$$

which implies

$$\forall\, t \ge 0, \qquad \|f(t)\|_X \le C e^{-\lambda_2 t}\, \|f_0\|_X.$$

□

References

[1] Alexandre, R., and Villani, C. On the Landau approximation in plasma physics. *Ann. Inst. H. Poincaré Anal. Non Linéaire 21*, 1 (2004), 61–95.

[2] Baranger, C., and Mouhot, C. Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials. *Rev. Matem. Iberoam. 21* (2005), 819–841.

[3] Carrapatoso, K. Exponential convergence to equilibrium for the homogeneous Landau equation with hard potentials. *Bull. Sci. math.* (2014). `http://dx.doi.org/10.1016/j.bulsci.2014.12.002`.

[4] Carrapatoso, K. On the rate of convergence to equilibrium for the homogeneous Landau equation with soft potentials. *J. Math. Pures Appl.* (2015). `http://dx.doi.org/10.1016/j.matpur.2015.02.008`.

[5] Degond, P., and Lemou, M. Dispersion relations for the linearized Fokker-Planck equation. *Arch. Ration. Mech. Anal. 138* (1997), 137–167.

[6] Desvillettes, L., and Villani, C. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math. 159*, 2 (2005), 245–316.

[7] DiPerna, R., and Lions, P.-L. On the Cauchy problem for the Boltzmann equation: Global existence and weak stability. *Ann. Math 130* (1989), 312–366.

[8] Gualdani, M., Mischler, S., and Mouhot, C. Factorization for non-symmetric operators and exponential H-Theorem. arxiv:1006.5523.

[9] Guo, Y. The Landau equation in a periodic box. *Comm. Math. Phys. 231* (2002), 391–434.

[10] Hérau, F. Short and long time behavior of the Fokker-Planck equation in a confining potential and applications. *J. Funct. Anal. 244*, 1 (2007), 95–118.

[11] Mischler, S., and Mouhot, C. Exponential stability of slowly decaying solutions to the kinetic-Fokker-Planck equation. *Preprint arxiv:1412.7487*.

[12] Mischler, S., and Scher, J. Spectral analysis of semigroups and growth-fragmentation equations. *Preprint arxiv:1310.7773*.

[13] Mouhot, C. Explicit coercivity estimates for the linearized boltzmann and landau operators. *Comm. Part. Diff Equations 261* (2006), 1321–1348.

[14] Mouhot, C. Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials. *Comm. Math. Phys. 261* (2006), 629–672.

[15] Mouhot, C., and Neumann, L. Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. *Nonlinearity 19*, 4 (2006), 969–998.
[16] Mouhot, C., and Strain, R. Spectral gap and coercivity estimates for the linearized boltzmann collision operator without angular cutoff. *J. Math. Pures Appl. 87* (2007), 515–535.
[17] Strain, R. M., and Guo, Y. Almost exponential decay near Maxwellian. *Comm. Partial Differential Equations 31*, 1-3 (2006), 417–429.
[18] Strain, R. M., and Guo, Y. Exponential decay for soft potentials near Maxwellian. *Arch. Ration. Mech. Anal. 187*, 2 (2008), 287–339.
[19] Tristani, I. Fractional fokker-planck equation. *To appear in Comm. Math. Sci.* (2013).
[20] Tristani, I. Exponential convergence to equilibrium for the homogeneous Boltzmann equation for hard potentials without cut-off. *J. Stat. Phys. 157*, 3 (2014), 474–496.
[21] Villani, C. On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Rational Mech. Anal. 143*, 3 (1998), 273–307.
[22] Villani, C. Hypocoercivity. *Mem. Amer. Math. Soc. 202* (2009), iv+141.
[23] Wu, K.-C. Exponential time decay estimates for the Landau equation on torus. arXiv:1301.0734.
[24] Wu, K.-C. Pointwise Description for the Linearized Fokker-Planck-Boltzmann Model. *J. Stat. Phys.*, 2015. DOI: 10.1007/s10955-015-1206-0.
[25] Yu, H. The exponential decay of global solutions to the generalized Landau equation near Maxwellians. *Quart. Appl. Math. 64*, 1 (2006), 29–39.

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