Primal and Dual Variational Formulation of a Frictional Contact Problem

Mircea Sofonea, David Danan, Cong Zheng

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Mircea Sofonea[1], *David Danan*[1] *and Cong Zheng*[2]

[1] Laboratoire de Mathématiques et Physique
Université de Perpignan Via Domitia
52 Avenue Paul Alduy, 66860 Perpignan, France

[2] Departement of Mathematics
Zhejiang University
310027 Hangzhou, China

Abstract

This paper represents a continuation of [4]. There, a mathematical model which describes the frictional contact between an elastic body and a foundation was considered. The variational and numerical analysis of the problem was provided by considering a weak formulation in terms of displacements, the so-called primal variational formulation. The aim of the current paper is to study the problem by using a weak formulation of in terms of the stress, the so-called dual variational formulation. We start by presenting the model, the assumption on the data and some preliminary results. Then we state and prove an equivalence result, Theorem 4.1. We proceed with an existence and uniqueness results, Theorem 5.1. The proofs are based on arguments of monotonicity, convexity and lower semicontinuity.

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1 Introduction

Modelling and analysis of contact process is an important topic which, currently, is still under investigation, as illustred in the recent literature. The reason is that contact phenomena abound in industry and everyday life and lead to interesting and non standard mathematical models. Here and below by a mathematical model of contact we mean a system of partial differential equations, associated to boundary value conditions and, eventually, to initial conditions, which describe a specific contact process.

The mathematical models of contact are constructed by considering various constitutive laws (i.e. materials) associated to different contact and frictional conditions. One of the most popular contact condition is the so-called the Signorini condition, introduced in [18] to describe the contact with a rigid foundation. The normal compliance contact condition represents a regularization of the Signorini contact condition and is used to describe the contact with an elastic foundation. It was introduced in [15], and used in a large number of papers, see [6, 9, 10, 12, 17] and the references therein. A more general contact condition, called the normal compliance condition with unilateral constraint, was introduced in [8]. It contains as particular cases both the Signorini contact condition and the normal compliance condition, and models the contact with an elastic-rigid foundation. On the other hand, friction is usually modelled with the Coulomb law of dry friction and its versions or regularizations.

The analysis of mathematical models of contact is based on the weak variational formulation of the corresponding models. In most of the cases it is given by a variational or hemivariational inequality in which the unknown is the displacement or the velocity field, as illustrated in [5, 6, 14, 16, 17, 19], for intance. There, the unique solvability of the models was obtained by using arguments of monotonicity, convexity and fixed point and, in the nonconvex case, by using the properties of the Clarke subdifferential. The progress made in the variational analysis of contact models in the last period was impressive, as it results from the references above. Nevertheless, many open problems still remain to be investigated and resolved. The reason is that, owing to inherent complexity, contact phenomena are modelled by nonlinear problems that are difficult to analyze and, therefore, new and nonstandard mathematical tools are required.

Two mathematical models of contact with normal compliance and unilateral constraint were recently considered in [3, 4]. In [3] the material's bahavior was described with a linear elastic constitutive law and friction was modeled with a slip-dependent version of Coulomb's law; the weak solvability of the model was proved by using arguments on pseudomonotone operators followed by a passage to the limit procedure; a convergence result was proved and its numerical validation was also provided; the question of the uniqueness of the solution was left open. In contrast, in [4] the material's bahavior was described with a nonlinear elastic constitutive law and friction was

modeled with a new condition, able to describe the transition from the Coulomb'law to Tresca's friction law. In addition, besides the existence of the weak solution to the model, its uniqueness was proved, a finite element approximation of the problem was considered and error estimates have been derived. In both [3] and [4] the analysis of the models was carried out based on variational formulations in terms of displacement, the so-called the primal variational formulation of the corresponding contact problems.

This paper represents a continuation of [4]. Its aim is to study the mathematical model in [4] by using a weak formulation in terms of the stress, the so-called dual variational formulation. Such formulations are used in the literature at least for three reasons. First, in many contact problems the main interest lies to the contact stress, since the behavior of the system and especially the surface integrity and wear depend on it. Second, such formulations can be handled numerically, save error-prone postprocessing and, therefore, lead to a better prediction to the real contact situation. And, finally, in most of the cases, they lead to interesting and non standard mathematical models. References concerning dual variational formulations in the study of various contact problems include [1, 2, 7, 11, 13, 19, 20].

The rest of the paper is structured as follows. In Section 2 we introduce the contact problem and describe the frictional contact conditions. Next, in Section 3 we introduce the notation and some preliminary material, list the assumptions on the data and state the primal and the dual variational formulation of the problem. In Section 4 we present an equivalence result, Theorem 4.1. It states the equivalence between the primal and dual variational formulation of the contact problem. Its proof is based on arguments of monotonicity and convexity. Then, in Section 5, we provide an existence and uniqueness results, Theorem 5.1. Its proof is based on arguments on quasivariational inequalities.

2 The model

Everywhere below we denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i\ , \qquad \|\boldsymbol{v}\|=(\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,\\ &\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d. \end{aligned}$$

Let $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ be a bounded domain. We denote by Γ its boundary, assumed to be Lipschitz continuous. We use the notation $\boldsymbol{x}=(x_i)$ for a typical point in $\Omega\cup\Gamma$ and we denote by $\boldsymbol{\nu}=(\nu_i)$ the outward unit normal at Γ. Here and below the indices i and j run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma

represents the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$, e.g. $u_{i,j} = \partial u_i / \partial x_j$.

The physical setting we consider in this paper is as follows. An elastic body occupies the domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz continuous boundary divided into three measurable parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. The body is in equilibrium under the action of body forces of density $\boldsymbol{f}_0$ and surface tractions of density $\boldsymbol{f}_2$ which act on Γ_2. We also assume that the body is fixed on Γ_1 and, on Γ_3, it is in frictional contact with an obstacle, the so-called foundation. The classical formulation of the contact problem we consider in this paper is as follows.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in} \quad \Omega, \tag{2.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on} \quad \Gamma_2, \tag{2.4}$$

$$\left.\begin{aligned} &u_\nu \le g, \quad \sigma_\nu + p(u_\nu) \le 0, \\ &(u_\nu - g)(\sigma_\nu + p(u_\nu)) = 0 \end{aligned}\right\} \qquad \text{on} \quad \Gamma_3, \tag{2.5}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau\| \le \mu\, p(u_\nu), \\ &-\boldsymbol{\sigma}_\tau = \mu\, p(u_\nu) \frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \quad \text{if} \quad \boldsymbol{u}_\tau \neq \boldsymbol{0} \end{aligned}\right\} \qquad \text{on} \quad \Gamma_3. \tag{2.6}$$

We now provide a short description of the equations and conditions (2.1)–(2.6) in which, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$. For more details concerning the mechanical assumptions which lead to the construction of this contact model we send the reader to [4].

First, Equation (2.1) represents the elastic constitutive law of the material in which $\mathcal{F}$ is a (possible nonlinear) constitutive function while equation (2.2) is the equation of equilibrium; we use it here since we assume that process is static. Here and below $\boldsymbol{\varepsilon}$ and Div represent the deformation and the divergence operators, respectively, i.e.

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Conditions (2.3), (2.4) represent the displacement and traction boundary conditions, respectively. Finally, conditions (2.5) and (2.6) represent the frictional contact condition with normal compliance and unilateral constraint in which u_ν, $\boldsymbol{u}_\tau$ are the normal and tangential components of $\boldsymbol{u}$ on Γ given by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$ and σ_ν,

$\boldsymbol{\sigma}_\tau$ represent the normal and the tangential stress on Γ, that is $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

Note that the contact condition (2.5) models the contact with a foundation is made by a rigid body covered by a layer made by of elastic material, say asperities. It shows that the penetration is restricted, since $u_\nu \leq g$ where $g > 0$ represents the thickness of the elastic layer. Also, combined with the friction law (2.6), condition (2.5) shows that when there is penetration, as far as the normal displacement does not reach the bound g, the contact is described with a normal compliance condition associated to the Coulomb's law of dry friction. Here p represents a positive normal compliance function such that $p(0) = 0$ and μ denotes the coefficient of friction. When the normal displacement reaches the bound g, then the normal stress is larger than the given positive value $\mu p(g)$ and, moreover, friction follows the Tresca law with the friction bound $\mu p(g)$. To conclude, this model describes a natural transition from the Coulomb law (which is valid as far as $0 < u_\nu < g$) to the Tresca frcition law (which is valid when $u_\nu = g$).

3 Variational Formulation

In the study of Problem $\mathcal{P}$ we need further notation and preliminary material that we introduce in what follows. Everywhere in this paper we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. In addition, we consider the spaces

$$Q = \left\{ \boldsymbol{\sigma} = (\sigma_{ij}) \ : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\}, \qquad H_1 = \left\{ \boldsymbol{u} = (u_i) \ : \ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in Q \right\}.$$

The spaces Q and H_1 are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij} \tau_{ij} dx, \qquad (\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q.$$

The associated norms are denoted by $\|\cdot\|_Q$ and $\|\cdot\|_{H_1}$, respectively. Also, recall that $H_1 = H^1(\Omega)^d$ algebraically and topologically.

For an element $\boldsymbol{v} \in H_1$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Also, for a regular stress function $\boldsymbol{\sigma}$ the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, dx + \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \boldsymbol{v} \, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v} \, da \quad \text{for all } \boldsymbol{v} \in H_1. \tag{3.1}$$

Consider the space

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega)^d \ : \ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \right\}.$$

It is well known that V is a real Hilbert space endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx,$$

and the associated norm $\|\cdot\|_V$, respectively. Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality. We also recall that there exists $c_0 > 0$ which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0 \|\boldsymbol{v}\|_V \quad \text{for all} \ \ \boldsymbol{v} \in V. \tag{3.2}$$

Inequality (3.2) represents a consequence of the Sobolev trace theorem.

In the study of the mechanical problem (2.1)–(2.6) we assume that the elasticity operator $\mathcal{F}$ satisfies the following conditions.

$$\begin{cases} \text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \quad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \ \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \quad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \quad\ \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \mathbf{0}_{\mathbb{S}^d}) \text{ belongs to } Q. \end{cases} \tag{3.3}$$

We also assume that the body forces and tractions densities have the regularity

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \qquad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{3.4}$$

The normal compliance function and the coefficient of friction satisfy

$$\begin{cases} \text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \quad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \le L_p |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))\,(r_1 - r_2) \ge 0 \quad \forall\, r_1,\, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \ \text{for any } r \in \mathbb{R}. \\ \text{(e) } p(\boldsymbol{x}, r) = 0 \quad \text{for all} \quad r \le 0, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.5}$$

$$\mu \in L^\infty(\Gamma_3), \qquad \mu(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \tag{3.6}$$

Note that assumptions (3.3)–(3.6) are used in the analysis of the primal variational formulation of the contact model $\mathcal{P}$. Examples of constitutive functions $\mathcal{F}$ and normal compliance functions p which satisfy assumptions (3.3) and (3.5), respectively, can be found in [6, 19], for instance. Now, to derive a dual variational of the contact model $\mathcal{P}$ we need an additional assumption on the geometry of the problem. Thus, we assume that

$$\text{there exists } \boldsymbol{\theta} \in V \text{ such that } \boldsymbol{\theta} = \boldsymbol{\nu} \ \text{ on } \Gamma_3. \tag{3.7}$$

Examples of domains Ω together with subsets $\Gamma_1,\ \Gamma_3 \subset \Gamma$ for which assumption (3.7) holds are presented below, both in the two and three dimensional case.

Example 1. *Let*

$$\Omega = \{\, \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 < 1 \,\},$$
$$\Gamma_1 = \{\, \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 = 1, \quad x_1 \le 0, \quad x_2 \le 0 \,\},$$
$$\Gamma_3 = \{\, \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 = 1, \quad x_1 \ge 0, \quad x_2 \ge 0 \,\}$$

and consider the function $\boldsymbol{\theta} : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$\boldsymbol{\theta}(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if} \quad x_1 \ge 0, \quad x_2 \ge 0, \\ (0, x_2) & \text{if} \quad x_1 \le 0, \quad x_2 \ge 0, \\ (0, 0) & \text{if} \quad x_1 \le 0, \quad x_2 \le 0, \\ (x_1, 0) & \text{if} \quad x_1 \ge 0, \quad x_2 \le 0. \end{cases}$$

Then, it is easy to see that $\boldsymbol{\theta} \in H^1(\Omega)^2$, $\boldsymbol{\theta} = \mathbf{0}$ *on* Γ_1 *and* $\boldsymbol{\theta} = \boldsymbol{\nu}$ *on* Γ_3, *i.e.* $\boldsymbol{\theta}$ *satisfies assumption* (3.7).

Example 2. *Let* a, b, c *be positive constants and let*

$$\Omega = \{\, \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ 0 < x_1 < a, \quad 0 < x_2 < b, \quad 0 < x_3 < c \,\},$$
$$\Gamma_1 = \{\, \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ 0 \le x_1 \le a, \quad 0 \le x_2 \le b, \quad x_3 = c \,\},$$
$$\Gamma_3 = \{\, \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ 0 \le x_1 \le a, \quad 0 \le x_2 \le b, \quad x_3 = 0 \,\}.$$

Consider the function $\boldsymbol{\theta} : \mathbb{R}^3 \to \mathbb{R}$ *defined by*

$$\boldsymbol{\theta}(x_1, x_2, x_3) = \Big(0, 0, \frac{x_3 - c}{c}\Big).$$

Then, it is easy to see that $\boldsymbol{\theta} \in H^1(\Omega)^2$, $\boldsymbol{\theta} = \mathbf{0}$ *on* Γ_1 *and* $\boldsymbol{\theta} = \boldsymbol{\nu}$ *on* Γ_3, *i.e.* $\boldsymbol{\theta}$ *satisfies assumption* (3.7).

We now turn to the variational formulation of Problem $\mathcal{P}$. To this end we introduce the set of admissible displacements U, the function $j : V \times V \to \mathbb{R}$ and the element $\boldsymbol{f} \in V$ defined by

$$U = \{\, \boldsymbol{v} \in V \ : \ v_\nu \le g \ \text{ a.e. on } \Gamma_3 \,\}, \tag{3.8}$$

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p(u_\nu) v_\nu \, da + \int_{\Gamma_3} \mu \, p(u_\nu) \|\boldsymbol{v}_\tau\| \, da \qquad \forall \, \boldsymbol{u} \in V, \ \boldsymbol{v} \in V, \tag{3.9}$$

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \, dx, + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v} \, da \qquad \forall \, \boldsymbol{v} \in V. \tag{3.10}$$

Assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (2.1)–(2.6) and let $\boldsymbol{v} \in U$. We use Green's formula (3.1) and definition (3.10) to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q = (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V + \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu) \, da \tag{3.11}$$

$$+ \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \, da.$$

Next, we use the frictional contact conditions (2.5), (2.6) and the definition (3.8) to obtain

$$\int_{\Gamma_3} \sigma_\nu(v_\nu - u_\nu)\, da \geq -\int_{\Gamma_3} p(u_\nu)(v_\nu - u_\nu)\, da, \tag{3.12}$$

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da \geq \int_{\Gamma_3} \mu\, p(u_\nu)\|\boldsymbol{u}_\tau\|\, da \tag{3.13}$$

$$-\int_{\Gamma_3} \mu\, p(u_\nu)\|\boldsymbol{v}_\tau\|\, da.$$

Finally, we combine equality (3.11) with inequalities (3.12) and (3.13), then we use the constitutive law (2.1) and the definitions (3.8)–(3.9). As a result we obtain the following variational formulation of the frictional contact problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. *Find a displacement field* $\boldsymbol{u}$ *such that*

$$\boldsymbol{u} \in U, \quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall\, \boldsymbol{v} \in U. \tag{3.14}$$

Note that Problem $\mathcal{P}_V$ is formulated in terms of the displacement field and, for this reason, we refer to it as the primal variational formulation of Problem $\mathcal{P}$. Once the displacement field is known, the stress field can be easily obtained by using the constitutive law (2.1).

Our aim in what follows is to derive a second variational formulation of the contact problem $\mathcal{P}$, in terms of stress, the so-called dual variational formulation. To this end we note that, since $meas\,(\Gamma_1) > 0$, the range of the deformation operator $\boldsymbol{\varepsilon} : V \to Q$, denoted $\boldsymbol{\varepsilon}(V)$, is a closed subspace of Q. A proof of this preliminary result can be found in [19] page 87. It is a direct consequence of the equality

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \qquad \forall\, \boldsymbol{v} \in V. \tag{3.15}$$

Denote by $P : Q \to \boldsymbol{\varepsilon}(V)$ the projection operator on $\boldsymbol{\varepsilon}(V) \subset Q$ and note that equality (3.15) shows that $\boldsymbol{\varepsilon} : V \to \boldsymbol{\varepsilon}(V)$ is an invertible operator. We denote in what follows by $\boldsymbol{\varepsilon}^{-1} : \boldsymbol{\varepsilon}(V) \to V$ the inverse of $\boldsymbol{\varepsilon}$. Also, we recall that assumption (3.3) implies that $\mathcal{F} : Q \to Q$ is a strongly monotone Lipschitz continuous operator and, therefore, using Proposition 1.25 in [19] it follows that it is invertible. Moreover, the inverse of $\mathcal{F}$, denoted $\mathcal{F}^{-1}$, is still a strongly monotone Lipschitz continuous operator. The ingredients above allow us to define the operator $\Lambda : Q \to V$ defined by

$$\Lambda\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^{-1} P \mathcal{F}^{-1} \boldsymbol{\sigma} \qquad \forall\, \boldsymbol{\sigma} \in Q. \tag{3.16}$$

The importance of this operator arises in the fact that it helps to inverse the elastic constitutive law, as it results from the following elementary result.

Lemma 3.1. *Assume* (3.3) *and let* $\boldsymbol{\sigma} \in Q$, $\boldsymbol{v} \in V$ *be such that* $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$. *Then* $\boldsymbol{u} = \boldsymbol{\Lambda}\boldsymbol{\sigma}$.

Proof. Equality $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ shows that $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathcal{F}^{-1}\boldsymbol{\sigma}$ and, therefore, since $\boldsymbol{\varepsilon}(\boldsymbol{u}) \in \boldsymbol{\varepsilon}(V)$ we have $\boldsymbol{\varepsilon}(\boldsymbol{u}) = P\boldsymbol{\varepsilon}(\boldsymbol{u}) = P\mathcal{F}^{-1}\boldsymbol{\sigma}$ which implies that $\boldsymbol{u} = \boldsymbol{\varepsilon}^{-1}P\mathcal{F}^{-1}\boldsymbol{\sigma}$. We combine this equality with the definition of (3.16) to see that $\boldsymbol{u} = \boldsymbol{\Lambda}\boldsymbol{\sigma}$, which concludes the proof. □

Besides the operator Λ we consider the element $\widetilde{\boldsymbol{g}} \in V$ given by

$$\widetilde{\boldsymbol{g}} = g\boldsymbol{\theta} \ \in V \tag{3.17}$$

where, recall, g is a positive constant. In addition, for every $\boldsymbol{\eta} \in V$ we consider the set $\Sigma(\boldsymbol{\eta}) \subset Q$ defined by

$$\Sigma(\boldsymbol{\eta}) = \{\, \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{\eta}, \boldsymbol{v} - \widetilde{\boldsymbol{g}}) \geq (f, \boldsymbol{v} - \widetilde{\boldsymbol{g}})_V \ \ \forall\, \boldsymbol{v} \in U \,\}. \tag{3.18}$$

Then we consider the following variational problem.

Problem $\mathcal{P}_V^D$. *Find a stress field* $\boldsymbol{\sigma}$ *such that*

$$\boldsymbol{\sigma} \in \Sigma(\Lambda\boldsymbol{\sigma}), \qquad (\mathcal{F}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma})_Q \geq (\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}), \boldsymbol{\tau} - \boldsymbol{\sigma})_Q \quad \forall\, \boldsymbol{\tau} \in \Sigma(\Lambda\boldsymbol{\sigma}). \tag{3.19}$$

We refer in what follows to $\mathcal{P}_V^D$ as the dual formulation of problem $\mathcal{P}$. Its link with the primal variational formulation, $\mathcal{P}_V$, together with its unique solvability, will be studied in the next two sections. Here we restrict ourselves to note that both problem $\mathcal{P}_V$ and $\mathcal{P}_V^D$ are expressed in term of elliptic quasivariational inequalities in which the unknown is the dispalcement and the stress field, rerspectively. In the case of Problem $\mathcal{P}_V$ the set of constraints is given but the problem is governed by the functionl j which depends on the solution. In contrast, Problem $\mathcal{P}_V^D$ there is no such functional but, there, the set of constraints depends of the solution.

We end this section with two additional preliminary results.

Lemma 3.2. *Assume* (3.5), (3.6) *and* (3.7). *Then, the element* $\widetilde{\boldsymbol{g}}$ *defined by* (3.17) *satisfies the following properties.*

$$\begin{cases} \text{(a) } \widetilde{\boldsymbol{g}} \in U. \\ \text{(b) } 2\boldsymbol{v} - \widetilde{\boldsymbol{g}} \in U \qquad \forall\, \boldsymbol{v} \in U. \\ \text{(c) } \lambda(\boldsymbol{v} - \widetilde{\boldsymbol{g}}) \in U \qquad \forall\, \lambda \geq 0, \quad \forall\, \boldsymbol{v} \in U. \\ \text{(d) } j(\boldsymbol{v}, 2\boldsymbol{v} - \widetilde{\boldsymbol{g}}) + j(\boldsymbol{v}, \widetilde{\boldsymbol{g}}) = 2j(\boldsymbol{v}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in U. \\ \text{(e) } j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \widetilde{\boldsymbol{g}}) = j(\boldsymbol{u}, \boldsymbol{v} - \widetilde{\boldsymbol{g}}) \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in U. \end{cases} \tag{3.20}$$

Proof. We use (3.7) to see that $\theta_\nu = 1$ on Γ_3 and, therefore, (3.17) implies that $\widetilde{g}_\nu = g$ on Γ_3, which shows that $\widetilde{\boldsymbol{g}} \in U$. Let $\boldsymbol{v} \in V$ and $\lambda \geq 0$. Then $2v_\nu - \widetilde{g}_\nu \leq g$ and $\lambda(v_\nu - \widetilde{g}_\nu) \leq 0$ on Γ_3 which imply that $2\boldsymbol{v} - \widetilde{\boldsymbol{g}} \in U$ and $\lambda(\boldsymbol{v} - \widetilde{\boldsymbol{g}}) \in U$, respectively. This proves the inclusions (3.20)(a)–(c).

Next, we use (3.17) to see that $\widetilde{\boldsymbol{g}}_\tau = \boldsymbol{0}$ on Γ_3 which implies that

$$\|2\boldsymbol{v}_\tau - \widetilde{\boldsymbol{g}}_\tau\| = 2\,\|\boldsymbol{v}_\tau\| - \|\widetilde{\boldsymbol{g}}_\tau\|, \quad \|\boldsymbol{v}_\tau - \widetilde{\boldsymbol{g}}_\tau\| = \|\boldsymbol{v}_\tau\| - \|\widetilde{\boldsymbol{g}}_\tau\| \quad \text{on } \Gamma_3,\ \forall\, \boldsymbol{v} \in V. \tag{3.21}$$

We now combine the definition (3.9) with equalities (3.21) to see that (3.20) (d)–(e) hold, which concludes the proof. □

Lemma 3.3. *Assume* (3.5) *and* (3.6) *and let j be the functional given by* (3.9)*. Then, for each $\boldsymbol{u} \in V$ there exists $\boldsymbol{\xi}(\boldsymbol{u}) \in V$ such that*

$$j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{\xi}(\boldsymbol{u}), \boldsymbol{v} - \boldsymbol{u}) \qquad \forall\, \boldsymbol{v} \in V \tag{3.22}$$

Proof. Let $\boldsymbol{u} \in V$ and denote by $\widetilde{\boldsymbol{\xi}}(\boldsymbol{u})$ the function defined by

$$\widetilde{\boldsymbol{\xi}}(\boldsymbol{u}) = \begin{cases} \frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} & \text{if } \boldsymbol{u}_\tau \neq \boldsymbol{0} \\ \boldsymbol{0} & \text{if } \boldsymbol{u}_\tau = \boldsymbol{0} \end{cases} \qquad \text{a.e. on } \Gamma_3.$$

Then, it is easy to see that for all $\boldsymbol{v} \in V$ we have

$$\widetilde{\boldsymbol{\xi}}(\boldsymbol{u}) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \leq \|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\| \qquad \text{a.e. on } \Gamma_3.$$

Therefore, we deduce that

$$\int_{\Gamma_3} \mu\, p(u_\nu)(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\, da \geq \int_{\Gamma_3} \mu\, p(u_\nu)\widetilde{\boldsymbol{\xi}}(\boldsymbol{u}) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da \quad \forall\, \boldsymbol{v} \in V$$

and, using (3.9) yields

$$\begin{aligned} &j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \\ &\quad \geq \int_{\Gamma_3} p(u_\nu)(v_\nu - u_\nu)\, da + \int_{\Gamma_3} \mu\, p(u_\nu)\widetilde{\boldsymbol{\xi}}(\boldsymbol{u}) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da \quad \forall\, \boldsymbol{v} \in V. \end{aligned} \tag{3.23}$$

Next we apply the Riesz representation theorem to obtain that there exists a unique element $\boldsymbol{\xi}(\boldsymbol{u}) \in V$ such that

$$(\boldsymbol{\xi}(\boldsymbol{u}), \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu\, da + \int_{\Gamma_3} \mu\, p(u_\nu)\widetilde{\boldsymbol{\xi}}(\boldsymbol{u}) \cdot \boldsymbol{v}_\tau\, da \qquad \forall\, \boldsymbol{v} \in V. \tag{3.24}$$

We combine now (3.23) and (3.24) to obtain (3.22) which concludes the proof. □

Note that Lemma 3.2 states the subdifferentiability of the function $\boldsymbol{v} \mapsto j(\boldsymbol{u}, \boldsymbol{v})$ in the point $\boldsymbol{u}$, for each $\boldsymbol{u} \in V$. Its statement follows from a classical result concerning the subdifferentiability of convex lower semicontinuous functions. Nevertheless, for the convenience of the reader, we decided to present above a constructive proof of this result.

4 Equivalence results

We now study the link between the variational problems Problem $\mathcal{P}_V$ and $\mathcal{P}_V^D$. Our main result in this section is the following equivalence result.

Theorem 4.1. *Assume* (3.3) − (3.7). *Then, the following statements hold:*

1) *If* $\boldsymbol{u}$ *is a solution to Problem* $\mathcal{P}_V$ *and* $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$, *then* $\boldsymbol{\sigma}$ *is a solution to Problem* $\mathcal{P}_V^D$.

2) *Conversely, if* $\boldsymbol{\sigma}$ *is a solution to Problem* $\mathcal{P}_V^D$, *then there exists a unique* $\boldsymbol{u} \in V$ *such that* $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ *and, moreover,* $\boldsymbol{u}$ *is a solution to Problem* $\mathcal{P}_V$.

Proof. 1) Let $\boldsymbol{u}$ be a solution to Problem $\mathcal{P}_V$ and let $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$. Then Lemma 3.1 implies that

$$\boldsymbol{u} = \Lambda\boldsymbol{\sigma} \tag{4.1}$$

and, moreover, (3.14) yields

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall\, \boldsymbol{v} \in U. \tag{4.2}$$

We take $\boldsymbol{v} = 2\boldsymbol{u} - \widetilde{\boldsymbol{g}}$ and $\boldsymbol{v} = \widetilde{\boldsymbol{g}}$ in (4.2) and use the equalities (3.20)(d), (e) to obtain

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{u}, \boldsymbol{u} - \widetilde{\boldsymbol{g}}) = (\boldsymbol{f}, \boldsymbol{u} - \widetilde{\boldsymbol{g}})_V. \tag{4.3}$$

Then, we add (4.2) and (4.3) and use (3.20)(e) to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{u}, \boldsymbol{v} - \widetilde{\boldsymbol{g}}) \geq (\boldsymbol{f}, \boldsymbol{v} - \widetilde{\boldsymbol{g}})_V \quad \forall\, \boldsymbol{v} \in U. \tag{4.4}$$

We now combine (4.4) and the definition (3.18) to deduce that

$$\boldsymbol{\sigma} \in \Sigma(\boldsymbol{u}). \tag{4.5}$$

Moreover, (3.18) and (4.3) imply that

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q \geq 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma(\boldsymbol{u})$$

and, since $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathcal{F}^{-1}\boldsymbol{\sigma}$, we deduce that

$$(\mathcal{F}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma})_Q \geq (\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}), \boldsymbol{\tau} - \boldsymbol{\sigma})_Q \geq 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma(\boldsymbol{u}). \tag{4.6}$$

We now combine (4.1), (4.5) and (4.6) to deduce that $\boldsymbol{\sigma}$ is a solution to $\mathcal{P}_V^D$, which concludes the first part of the proof.

2) Assume now that $\boldsymbol{\sigma}$ satisfies (3.19) and consider an element $\boldsymbol{z} \in Q$ such that

$$(\boldsymbol{z}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = 0 \qquad \forall\, \boldsymbol{v} \in V \tag{4.7}$$

Then, using (3.18) we deduce that $\boldsymbol{\tau} = \boldsymbol{\sigma} \pm \boldsymbol{z} \in \Sigma(\Lambda\boldsymbol{\sigma})$ and, testing in (3.19) with $\boldsymbol{\tau} = \boldsymbol{\sigma} \pm \boldsymbol{z} \in \Sigma(\Lambda\boldsymbol{\sigma})$, we deduce that

$$(\mathcal{F}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}), \boldsymbol{z})_Q = 0. \tag{4.8}$$

On the other hand, since $\boldsymbol{\varepsilon}(V)$ is a closed subspace of Q we deduce that

$$\boldsymbol{\varepsilon}(V)^{\perp\perp} = \boldsymbol{\varepsilon}(V) \tag{4.9}$$

where the superscript $\perp$ indicates the orthogonal complement in Q. Using (4.7) and (4.8) we see that $\mathcal{F}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}) \perp \boldsymbol{z}$ for all $\boldsymbol{z} \in \boldsymbol{\varepsilon}(V)^{\perp}$ and, by (4.9) it follows that $\mathcal{F}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}) \in \boldsymbol{\varepsilon}(V)$. Therefore, there exists $\widetilde{\boldsymbol{u}} \in V$ such that $\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}) = \mathcal{F}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}})$. Let $\boldsymbol{u} = \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{g}}$. Then $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathcal{F}^{-1}\boldsymbol{\sigma}$, i.e. $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$. This proves the existence part. The uniqueness of $\boldsymbol{u}$ follows from equality (3.15). In addition, Lemma 3.1 shows that

$$\Lambda\boldsymbol{\sigma} = \boldsymbol{u}. \tag{4.10}$$

Next, we prove that $\boldsymbol{u}$ is a solution to Problem $\mathcal{P}_V$ and, to this end, we start proving that $\boldsymbol{u} \in U$. Let $P_U : V \to U$ be the projector on the nonempty closed convex subset $U \subset V$. Arguing by contradiction, we assume in what follows that

$$\boldsymbol{u} \notin U. \tag{4.11}$$

Then $\boldsymbol{u} \neq P_U\boldsymbol{u}$, i.e.

$$(P_U\boldsymbol{u} - \boldsymbol{u}, P_U\boldsymbol{u} - \boldsymbol{u})_V > 0. \tag{4.12}$$

Next, using (4.12) and the variational characterization of the projection we have

$$(P_U\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v})_V \geq (P_U\boldsymbol{u} - \boldsymbol{u}, P_U\boldsymbol{u})_V > (P_U\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u})_V \qquad \forall\, \boldsymbol{v} \in U.$$

These inequalities imply that there exists $\alpha \in \mathbb{R}$ such that

$$(P_U\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v})_V > \alpha > (P_U\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u})_V \qquad \forall\, \boldsymbol{v} \in U$$

and, using (3.20)(c) with $\lambda = 1$ it follows that

$$(P_U\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v} - \widetilde{\boldsymbol{g}})_V > \alpha > (P_U\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u})_V \qquad \forall\, \boldsymbol{v} \in U. \tag{4.13}$$

Let $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\varepsilon}(P_U\boldsymbol{u} - \boldsymbol{u}) \in Q$. Then (4.13) yields

$$(\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q > \alpha > (\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q \qquad \forall\, \boldsymbol{v} \in U \tag{4.14}$$

and, taking $\boldsymbol{v} = \widetilde{\boldsymbol{g}}$ in the previous inequality, we obtain that

$$\alpha < 0. \tag{4.15}$$

Assume now that there exists $\widetilde{\boldsymbol{v}} \in U$ such that

$$(\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q < 0. \tag{4.16}$$

We use property (3.20)(c) and test with $\boldsymbol{v} = \lambda(\widetilde{\boldsymbol{v}} - \widetilde{\boldsymbol{g}})$ in (4.14), where $\lambda \geq 0$. As a result we obtain

$$\lambda(\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q > \alpha + (\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q \qquad \forall\, \lambda > 0.$$

Therefore, passing to the limit as $\lambda \to \infty$, and using (4.16) we deduce that

$$-\infty \geq \alpha + (\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q$$

which contradicts $\alpha \in \mathbb{R}$. We conclude from above that

$$(\widetilde{\boldsymbol{\tau}}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q \geq 0 \qquad \forall\, \boldsymbol{v} \in U. \tag{4.17}$$

Next, equality (4.10) combined with the regularity $\boldsymbol{\sigma} \in \Sigma(\Lambda\boldsymbol{\sigma})$ and the definition (3.18) imply that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{u}, \boldsymbol{v} - \widetilde{\boldsymbol{g}}) \geq (f, \boldsymbol{v} - \widetilde{\boldsymbol{g}})_V \qquad \forall\, \boldsymbol{v} \in U.$$

Therefore using (4.17) we deduce that $\widetilde{\boldsymbol{\tau}} + \boldsymbol{\sigma} \in \Sigma(\boldsymbol{u})$. This allows to test in (3.19) with $\boldsymbol{\tau} = \widetilde{\boldsymbol{\tau}} + \boldsymbol{\sigma}$ and, as a result, we find that

$$(\mathcal{F}^{-1}\boldsymbol{\sigma}, \widetilde{\boldsymbol{\tau}})_Q \geq (\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}), \widetilde{\boldsymbol{\tau}})_Q.$$

We now use equality $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ to obtain

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}), \widetilde{\boldsymbol{\tau}})_Q \geq (\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}), \widetilde{\boldsymbol{\tau}})_Q$$

and, since $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\varepsilon}(P_U\boldsymbol{u} - \boldsymbol{u}) \in Q$, we find that

$$(P_U\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u})_V \geq (P_U\boldsymbol{u} - \boldsymbol{u}, \widetilde{\boldsymbol{g}})_V. \tag{4.18}$$

On the other hand, taking $\boldsymbol{v} = \widetilde{\boldsymbol{g}}$ in (4.13), we obtain

$$(P_U\boldsymbol{u} - \boldsymbol{u}, \widetilde{\boldsymbol{g}})_V > (P_U\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u})_V \tag{4.19}$$

The inequalities (4.19) and (4.18) lead to a contradiction. Therefore, we deduce that assumption (4.11) is not valid, and we conclude that

$$\boldsymbol{u} \in U. \tag{4.20}$$

Next, we denote by

$$\boldsymbol{\tau}(\boldsymbol{u}) = \boldsymbol{\varepsilon}(\boldsymbol{f} - \boldsymbol{\xi}(\boldsymbol{u})) \tag{4.21}$$

where $\boldsymbol{\xi}(\boldsymbol{u})$ is the element defined by the Lemma 3.3. Then, using (3.22), we have

$$(\boldsymbol{\tau}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}))_Q \geq (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V + j(\boldsymbol{u},\boldsymbol{u}) - j(\boldsymbol{u},\boldsymbol{v}) \qquad \forall\, \boldsymbol{v}\in V. \tag{4.22}$$

We note that (3.20)(a), (b) allow to take $\boldsymbol{v}=2\boldsymbol{u}-\widetilde{\boldsymbol{g}}$ and $\boldsymbol{v}=\widetilde{\boldsymbol{g}}$ in (4.22) . Therefore, using equality (3.20)(d) we obtain

$$(\boldsymbol{\tau}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{u},\boldsymbol{u}-\widetilde{\boldsymbol{g}}) = (\boldsymbol{f},\boldsymbol{u}-\widetilde{\boldsymbol{g}})_V. \tag{4.23}$$

Then, we combine (4.22) and (4.23) and use (3.20)(e) to see that

$$(\boldsymbol{\tau}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{u},\boldsymbol{v}-\widetilde{\boldsymbol{g}}) \geq (\boldsymbol{f},\boldsymbol{v}-\widetilde{\boldsymbol{g}})_V \qquad \forall\, \boldsymbol{v}\in U. \tag{4.24}$$

This inequality implies that $\boldsymbol{\tau}(\boldsymbol{u})\in\Sigma(\boldsymbol{u})$ and, using (4.10), it shows that $\boldsymbol{\tau}(\boldsymbol{u})\in\Sigma(\Lambda\boldsymbol{\sigma})$. Therefore, testing in (3.19) with $\boldsymbol{\tau}=\boldsymbol{\tau}(\boldsymbol{u})$ we have

$$(\mathcal{F}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau}(\boldsymbol{u})-\boldsymbol{\sigma})_Q \geq (\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}),\boldsymbol{\tau}(\boldsymbol{u})-\boldsymbol{\sigma})_Q$$

and, since $\mathcal{F}^{-1}\boldsymbol{\sigma}=\boldsymbol{\varepsilon}(\boldsymbol{u})$, it follows that

$$(\boldsymbol{\tau}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q \geq (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q. \tag{4.25}$$

Next, we combine (4.23) and (4.25) to deduce that

$$(\boldsymbol{f},\boldsymbol{u}-\widetilde{\boldsymbol{g}})_V \geq (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{u},\boldsymbol{u}-\widetilde{\boldsymbol{g}}). \tag{4.26}$$

Finally, since $\boldsymbol{\sigma}\in\Sigma(\Lambda\boldsymbol{\sigma})$, (4.10) implies that $\boldsymbol{\sigma}\in\Sigma(\boldsymbol{u})$ and, therefore, since $\boldsymbol{u}\in U$, (3.18) shows that

$$(\boldsymbol{f},\boldsymbol{u}-\widetilde{\boldsymbol{g}})_V \leq (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{u},\boldsymbol{u}-\widetilde{\boldsymbol{g}}). \tag{4.27}$$

Inequalities (4.26) and (4.27) yield

$$(\boldsymbol{f},\boldsymbol{u}-\widetilde{\boldsymbol{g}})_V = (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}}))_Q + j(\boldsymbol{u},\boldsymbol{u}-\widetilde{\boldsymbol{g}}),$$

i.e.

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}})-\boldsymbol{\varepsilon}(\boldsymbol{u}))_Q - j(\boldsymbol{u},\boldsymbol{u}-\widetilde{\boldsymbol{g}}) = (\boldsymbol{f},\widetilde{\boldsymbol{g}}-\boldsymbol{u})_V. \tag{4.28}$$

On the other hand, since $\boldsymbol{\sigma}\in\Sigma(\boldsymbol{u})$ we have

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{g}})_Q + j(\boldsymbol{u},\boldsymbol{v}-\widetilde{\boldsymbol{g}}) \geq (\boldsymbol{f},\boldsymbol{v}-\widetilde{\boldsymbol{g}})_V \quad \forall\, \boldsymbol{v}\in U, \tag{4.29}$$

We now add (4.28) and (4.29) to obtain

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u})_Q + j(\boldsymbol{u},\boldsymbol{v}-\widetilde{\boldsymbol{g}}) - j(\boldsymbol{u},\boldsymbol{u}-\widetilde{\boldsymbol{g}}) \geq (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V \quad \forall\, \boldsymbol{v}\in U.$$

and, using (3.20)(e) yields

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{u},\boldsymbol{v}) - j(\boldsymbol{u},\boldsymbol{u}) \geq (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V \qquad \forall\, \boldsymbol{v}\in U.$$

Finally, since $\boldsymbol{\sigma}=\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$, we find that that

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{u},\boldsymbol{v}) - j(\boldsymbol{u},\boldsymbol{u}) \geq (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V \qquad \forall\, \boldsymbol{v}\in U. \tag{4.30}$$

We combine now inequality (4.30) with (4.20) to see that $\boldsymbol{u}$ is a solution to Problem $\mathcal{P}_V$, which completes the proof. □

5 Existence and Uniqueness results

We now proceed with the study of the existence and uniqueness of the solution for problems $\mathcal{P}_V$ and $\mathcal{P}_V^D$. Our main results in this section is the following.

Theorem 5.1. *Assume* (3.3)–(3.6). *Then the following statements hold.*

1) *Problem* $\mathcal{P}_V$ *has at least a solution.*

2) *There exist a constant* α_0 *which depends only on* Ω, Γ_1, Γ_3 *and* $\mathcal{F}$ *such that the solution of Problem* $\mathcal{P}_V$ *is unique, if*

$$L_p\|\mu\|_{L^\infty(\Gamma_3)} < \alpha_0. \tag{5.1}$$

3) *If* (3.7) *holds then Problem* $\mathcal{P}_V^D$ *has at least a solution.*

4) *If* (3.7) *and* (5.1) *hold then Problem* $\mathcal{P}_V^D$ *has a unique solution.*

5) *If* (3.7) *and* (5.1) *hold, then the solution of Problem* $\mathcal{P}_V$ *and* $\mathcal{P}_V^D$, *obtained at points* 2) *and* 4), *respectively, are related by the elastic constitutive law* $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$.

Proof. 1) The proof of this point follows from arguments of elliptic quasivariational inequalities, as shown in [4]. Nevertheless, for the convenience of the reader, we resume below its main steps.

First, we consider the operator $A : V \to V$ defined by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{5.2}$$

We use assumption (3.3) to see that A is a strongly monotone Lipschitz continuous operator on the space V. Moreover, we use assumptions (3.5), (3.6) and the property of the trace map to see that the functional j defined by (3.9) satisfies the following conditions:

$$\begin{cases} \text{For all } \boldsymbol{\eta} \in U, \text{ the function } \boldsymbol{v} \mapsto j(\boldsymbol{\eta}, \boldsymbol{v}) : U \to \mathbb{R} \\ \text{is convex, } j(\boldsymbol{\eta}, \boldsymbol{v}) \geq 0 \text{ for all } \boldsymbol{v} \in U \text{ and } j(\boldsymbol{\eta}, \mathbf{0}_X) = 0. \end{cases} \tag{5.3}$$

$$\begin{cases} \text{For all sequences } \{\boldsymbol{\eta}_n\} \subset U \text{ and } \{\boldsymbol{u}_n\} \subset U \text{ such that} \\ \boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \text{ in } V,\ \boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \text{ in } X \text{ and for all } \boldsymbol{v} \in U, \\ \text{the inequality below holds:} \\ \limsup_{n\to\infty} [j(\boldsymbol{\eta}_n, \boldsymbol{v}) - j(\boldsymbol{\eta}_n, \boldsymbol{u}_n)] \leq j(\boldsymbol{\eta}, \boldsymbol{v}) - j(\boldsymbol{\eta}, \boldsymbol{u}). \end{cases} \tag{5.4}$$

Finally, we recall that U is a closed convex subset of V such that $0_X \in U$. The existence of a solution of Problem $\mathcal{P}_V$ follows now from a classical argument of quasivariational inequalities, see for instance [19, p. 51].

2) In order to prove the uniqueness part, let $\boldsymbol{u}_1,\ \boldsymbol{u}_2 \in V$ be two solution to Problem $\mathcal{P}_V$. Then, using (3.14) and (3.3), after a standard computation we find that

$$m_{\mathcal{F}} \left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V^2 \leq j(\boldsymbol{u}_1, \boldsymbol{u}_2) - j(\boldsymbol{u}_1, \boldsymbol{u}_1) + j(\boldsymbol{u}_2, \boldsymbol{u}_1) - j(\boldsymbol{u}_2, \boldsymbol{u}_2). \tag{5.5}$$

Moreover, using the definition (3.9) of the function j, the properties (3.5) of the function p and the trace inequality (3.2) it turns that

$$\begin{aligned} j(\boldsymbol{u}_1, \boldsymbol{u}_2) - j(\boldsymbol{u}_1, \boldsymbol{u}_1) + j(\boldsymbol{u}_2, \boldsymbol{u}_1) - j(\boldsymbol{u}_2, \boldsymbol{u}_2) \\ \leq c_0^2 L_p \left\|\mu\right\|_{L^\infty(\Gamma_3)} \left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V^2. \end{aligned} \tag{5.6}$$

Let

$$\alpha_0 = \frac{m_{\mathcal{F}}}{c_0^2}\,. \tag{5.7}$$

and note that, clearly, α_0 depends only on Ω, Γ_1, Γ_3 and $\mathcal{F}$. Assume that (5.1) holds. Then it follows from (5.5)–(5.7) that $\boldsymbol{u}_1 = \boldsymbol{u}_2$, which concludes the proof.

3) Assume (3.7) holds. Denote by $\boldsymbol{u}$ a solution to Problem $\mathcal{P}_V$ whose existence is guaranteed by 1) and let $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$. Then, using Theorem 4.1 1) it follows that $\boldsymbol{\sigma}$ is a solution to Problem $\mathcal{P}_V^D$ which ends the proof.

4) Assume now that (3.7) and (5.1) hold and let $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ be two solutions to Problem $\mathcal{P}_V^D$. Then, it follow from Theorem 4.1 1) that there exists two elements $\boldsymbol{u}_1, \boldsymbol{u}_2 \in V$ such that

$$\boldsymbol{\sigma}_1 = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \qquad \boldsymbol{\sigma}_2 = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_2). \tag{5.8}$$

Moreover, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are solutions to Problem $\mathcal{P}_V$. Using now the smallness assumption (5.1) we deduce by 2) that

$$\boldsymbol{u}_1 = \boldsymbol{u}_2. \tag{5.9}$$

We combine (5.8) and (5.9) to see that $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$, which concludes the proof.

5) Let $\boldsymbol{u}$ be a solution to the Problem $\mathcal{P}_V$ and let $\boldsymbol{\sigma}$ be a solution to the Problem $\mathcal{P}_V^D$. Denote

$$\widetilde{\boldsymbol{\sigma}} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{5.10}$$

Then, Theorem 4.1 1) guarantees that $\widetilde{\boldsymbol{\sigma}}$ is a solution to Problem $\mathcal{P}_V^D$. It follows now from the uniqueness of the solution in 4) that

$$\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}. \tag{5.11}$$

Equalities (5.10) and (5.11) imply that $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$. □

Assume in what follows that (3.3)–(3.7) and (5.1) hold. Then Theorem 4.1 provides the unique solvability of both the primal variational formulation $\mathcal{P}$ and the dual variational formulation $\mathcal{P}_V^D$. Moreover, it shows that if the displacement field $\boldsymbol{u}$ is the solution of the primal variational formulation $\mathcal{P}$ and the stress field $\boldsymbol{\sigma}$ in the solution of the variational formulation $\mathcal{P}_V^D$, then $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ are connected by the elastic

constitutive law $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$. For this reason we refear to the pair $(\boldsymbol{u}, \boldsymbol{\sigma})$ as a *weak solution* to the frictional contact problem $\mathcal{P}$ and we conclude that this problem has a unique weak solution. Note that condition (5.1) represents a smallness inequality for the coefficient of friction. Whether this condition represents an intrinsic feature of this frictional contact problem $\mathcal{P}$ or it is only a limitation of our mathematical tools remains an open question. Clearly, this question deserves more investigation in the future.

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