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Christian Duval, Serge Lazzarini

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On the Schrödinger-Newton equation and its symmetries: a geometric view

C. DUVAL[‡] and S. LAZZARINI[§]

Centre de Physique Théorique,
Aix Marseille Université & Université de Toulon & CNRS UMR 7332,
Case 907, 13288 Marseille, France.

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Abstract

The Schrödinger-Newton (SN) equation is recast on purely geometrical grounds, namely in terms of Bargmann structures over $(n+1)$-dimensional Newton-Cartan (NC) spacetimes. Its maximal group of invariance, which we call the SN group, is determined as the group of conformal Bargmann automorphisms that preserve the coupled Schrödinger and NC gravitational field equations. Canonical unitary representations of the SN group are worked out, helping us recover, in particular, a very specific occurrence of dilations with dynamical exponent $z = (n+2)/3$.

[‡]mailto: duval-at-cpt.univ-mrs.fr
[§]mailto: lazzarini-at-cpt.univ-mrs.fr

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1 Introduction

The Schrödinger-Newton (SN) equation for a quantum non-relativistic particle of mass m in Euclidean space of dimension $n > 2$ reads

$$i\hbar\frac{\partial\psi}{\partial t}(\mathbf{x},t) = \left(-\frac{\hbar^2}{2m}\Delta_{\mathbf{R}^n} - 4\pi G m^2 C_n \int_{\mathbf{R}^n\setminus\{\mathbf{x}\}} \frac{|\psi(\mathbf{y},t)|^2}{\|\mathbf{x}-\mathbf{y}\|^{n-2}}\, dy^1\cdots dy^n\right)\psi(\mathbf{x},t) \tag{1.1}$$

where $C_n = \Gamma(n/2)/(2\pi^{n/2}(n-2))$. It has originally been introduced as a special case, $N = 1$, of the Schrödinger equation for a N-particle system in mutual gravitational interaction [11]. The non-local and non-linear differential equation (1.1) is in fact obtained, as explained below, by a specific coupling between the Schrödinger equation and the non-relativistic gravitational field equation. In the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t}(\mathbf{x},t) = \left(-\frac{\hbar^2}{2m}\Delta_{\mathbf{R}^n} + V(\mathbf{x},t)\right)\psi(\mathbf{x},t) \tag{1.2}$$

the potential energy $V(\mathbf{x},t) = m\,U(\mathbf{x},t)$ is identified to the Newtonian gravitational potential, solution of the Poisson equation[1]

$$\Delta_{\mathbf{R}^n} U(\mathbf{x},t) = 4\pi G\varrho(\mathbf{x},t) \tag{1.3}$$

where G stands for Newton's constant; the mass density of the sources is given here in terms of the quantum probability density, namely by

$$\varrho(\mathbf{x},t) = m\,|\psi(\mathbf{x},t)|^2 \tag{1.4}$$

where wave-function, ψ, is tacitly assumed to be normalized, $\int_{\mathbf{R}^n}|\psi(\mathbf{x},t)|^2\,dx^1\cdots dx^n = 1$.

Let us emphasize that, apart from its own mathematical interest as a highly non-trivial system (1.2), (1.3) and (1.4) of coupled PDE, the Schrödinger-Newton equation may, more significantly, be considered as instrumental to support Penrose's proposal for the "Gravitization of Quantum Mechanics" [39, 40]. A special class of stationary solutions can be found in, e.g., [37, 43]. The Schrödinger-Newton equation is also advocated for a dynamical interpretation of the process of reduction of wave packets in quantum mechanics, leading to a radical modification of the spreading of wave packets around and above a critical mass of the system [28, 29]. See nevertheless [1] where the limitations to the quantum mechanical description of physical phenomena by the Schrödinger-Newton equation are lucidly examined. This is however not the route that we are about to follow.

[1] The Green function of the n-dimensional Poisson equation (vanishing asymptotically) is given by $G(\mathbf{x},\mathbf{y}) = -C_n/\|\mathbf{x}-\mathbf{y}\|^{n-2}$, and reduces to the familiar expression $G(\mathbf{x},\mathbf{y}) = -1/(4\pi\|\mathbf{x}-\mathbf{y}\|)$ for $n = 3$.

We will rather be interested in the *symmetries* of this self-coupled quantum/classical system where the Schrödinger equation represents the quantum edge while the Newton field equation is granted a purely classical status within this coupled system.

We will therefore aim at determining the maximal symmetry group of the SN equation. To some extent, this will be done in the wake and the spirit of earlier work of Niederer [38] and Hagen [31] related to the maximal symmetries of the (free) Schrödinger equation, after the discovery by S. Lie of those of the heat equation.[2] In doing so, we will find it worthwhile to generalize this coupled system (1.2), (1.3) and (1.4), of partial differential equations so as to include non-inertial forces in addition to the Newtonian gravitational force, as well as a possibly curved Riemannian spatial background. This will be best achieved in geometrical terms by lifting this system of PDE to the Bargmann extended spacetime over the initially chosen Newton-Cartan (NC) spacetime; see [14] and references therein for an introduction to the subject. The inverse problem of lifting a NC structure to what we call now a Bargmann structure has been pioneered by Eisenhart [26]. Since then, the formalism has been systematically utilized in various contexts ranging (highly non exhaustively) from the analysis of the fundamental interplay between the NC and (conformal) Lorentz geometries [14, 17, 21] to the study of hidden symmetries in quantum field theory [34, 19, 20] and classical Hamiltonian dynamics systems [8, 9].

The main results of this article consist in (i) the generalization of the SN equation (1.1) as given by (3.3), (3.4) and (3.5) by means of conformal Bargmann structures, and (ii) the determination of its maximal group of invariance, coined the Schrödinger-Newton group (5.15). This vantage point enables us to treat in geometrical terms the rather complicated partial differential equation (1.1) and its natural generalization at one stroke. It also helps us produce the canonical unitary representations (5.47) and (5.52) of the SN group on the set of solutions of the SN equation. As a special outcome, we recover the representation of the specific dilation group with dynamical exponent $z = 5/3$ discovered originally in [28] in the case $n = 3$. Let us stress that in the case of spatial dimension $n = 4$, the SN group gains one more dimension (time inversions are hence recovered), and is isomorphic to the well-known Schrödinger group (5.16) of 6-dimensional Bargmann structures.

The article is organized as follows.

Section 2 is devoted to a survey of Bargmann structures over nonrelativistic Newton-Cartan spacetime on which the rest of the article heavily relies.

Then, Section 3 establishes the geometric transcription of the Schrödinger-Newton equation as a set of PDE on Bargmann extended spacetime which we call the generalized SN equations. Let us emphasize the central rôle granted to the Yamabe operator (or conformally-invariant Laplace-Beltrami operator). The SN equation (1.1) is duly recov-

[2] The so-called "Schrödinger group" has actually been discovered in a classical context by Jacobi in 1842; see, e.g., [18] for a review.

ered as a special case of the latter.

We introduce in Section 4 the maximal group of symmetries of the (generalized) Schrödinger-Newton equations, by taking advantage of the maximality of the conformal group as a group of symmetry of the Yamabe operator. We highlight, *en passant*, the new expression (4.7) relating the Schwarzian derivative to the conformal transformation of the Ricci tensor. It is shown that mass-dilation occurs naturally in this context, leading to the fundamental constraint (4.22) between the conformal dilation factor, λ, and the dilation factor, ν, of the Bargmann fundamental vector field.

In Section 5, we define in full generality the so-called "SN group" of symmetry of our Schrödinger-Newton system of equations. Explicit representations of this group are found in some particular instances, namely of the form (5.28) for spatially flat Bargmann structures. The special case $n = 4$ is investigated along the same lines; see Footnote 12 for a comment about this exceptional spatial dimension. We end this section by the quantum representations of these groups on the set of solutions, ψ, of the SN equation.

At last, Section 6 help us summarize the main results of our work and announce new directions which constitute work in progress along these lines.

2 Bargmann structures: a compendium

2.1 Main definitions

A Bargmann structure is a triple (M, g, ξ) where M is a smooth connected and orientable manifold of dimension $N = n + 2$ with $n > 0$, endowed with a Lorentz metric g, and a complete, nowhere vanishing vector field ξ which is light-like , $\mathrm{g}(\xi, \xi) = 0$, and covariantly constant with respect to the Levi-Civita connection, $\nabla\xi = 0$ [14]. Thus ξ can be viewed as the fundamental vector field of a free action on M of a 1-dimensional Lie group, say $(\mathbf{R}, +)$. The quotient $\mathcal{M} = M/(\mathbf{R}\xi)$ has been shown to be canonically endowed with a Newton-Cartan (NC) structure;[3] it will be interpreted as non-relativistic spacetime. The quotient $T = M/\xi^{\perp}$ is endowed with the structure of a smooth (1-dimensional) manifold (either topology $T \cong \mathbf{R}$ or $T \cong S^1$ is envisageable, see [12, 18]); we will denote by $\tau : \mathcal{M} \to T$ the surjection over the *time-axis* T.

[3] A NC structure is defined as a quadruple $(\mathcal{M}, h, \vartheta, \nabla^{\mathcal{M}})$ where $\mathcal{M}$ is a smooth $(n+1)$-dimensional manifold, and h a twice-contravariant tensor field whose kernel is generated by the nowhere vanishing closed 1-form ϑ; moreover the torsionfree affine connection $\nabla^{\mathcal{M}}$ parallel-transports h and ϑ. Let us recall that NC structures [10, 44, 35] (see also [18] and references therein) arise effectively from Bargmannian ones: if $\pi : M \to \mathcal{M}$ is the submersion associated with a given Bargmann structure, then $h = \pi_* \mathrm{g}^{-1}$, also $\theta = \pi^* \vartheta$, and $\nabla^{\mathcal{M}}$ is the projection on $\mathcal{M}$ of the Levi-Civita connection ∇ of (M, g).

The most general Bargmann structure is given locally by the pair[4]

$$\mathrm{g} = \mathrm{g}_{\Sigma_t} + dt \otimes \omega + \omega \otimes dt \qquad \& \qquad \xi = \frac{\partial}{\partial s} \tag{2.1}$$

where

$$\mathrm{g}_{\Sigma_t} = \mathrm{g}_{ij}(x,t) dx^i \otimes dx^j \tag{2.2}$$

is a preferred Riemannian metric on each time-slice $\Sigma_t = \tau^{-1}(\{t\})$ of $\mathcal{M}$, and where

$$\omega = \omega_i(x,t) dx^i - U(x,t) dt + ds \tag{2.3}$$

is a connection form on the principal $(\mathbf{R},+)$-bundle $\pi : M \to \mathcal{M}$.The metric g in (2.1) is known as a *Brinkmann metric* [4] or a generalized pp-wave [25]. The spacetime function U (for example, the profile of the gravitational wave whose wave-vector ξ is null and parallel) is interpreted in the present context as the Newtonian gravitational potential on NC spacetime, $\mathcal{M}$.

The 1-form $\theta = \mathrm{g}(\xi)$ associated with ξ is covariantly constant, hence closed, and verifies $\theta(\xi) = 0$; it thus descends to the time-axis T as the "clock" of the structure, locally $\theta = dt$ (with some ruthless abuse of notation).

• Let us recall that the *Bargmann group* (see, e.g., [14]), i.e., the group of strict automorphisms of a Bargmann structure, is defined by

$$\mathrm{Barg}(M, \mathrm{g}, \xi) = \{\Phi \in \mathrm{Diff}(M) \,|\, \Phi^* \mathrm{g} = \mathrm{g}, \Phi^* \xi = \xi\} \tag{2.4}$$

where Φ^* denotes the pull-back operation by the diffeomorphism Φ of M.

The *Schrödinger group* (see, e.g., [17]) is the group of conformal automorphisms of a Bargmann structure, namely

$$\mathrm{Sch}(M, \mathrm{g}, \xi) = \{\Phi \in \mathrm{Diff}(M) \,|\, \Phi^* \mathrm{g} = \lambda_\Phi \mathrm{g}, \Phi^* \xi = \xi\} \tag{2.5}$$

where $\lambda_\Phi \in C^\infty(M, \mathbf{R}_+^*)$.

• The canonical *flat Bargmann structure* on $\mathbf{R}^{n+2}$ is given by[5]

$$\mathrm{g}_0 = \delta_{ij}\, dx^i \otimes dx^j + dt \otimes ds + ds \otimes dt \qquad \& \qquad \xi = \frac{\partial}{\partial s}. \tag{2.6}$$

[4]In the chosen coordinate system, $(x^1, \ldots, x^n, t, s)$, the n-uple $x = (x^1, \ldots, x^n)$ provides coordinates on the intrinsically defined space at time t, denoted by Σ_t, while s is a fiberwise coordinate of $\pi : M \to \mathcal{M}$.

[5]The group (2.4) of automorphisms of the flat Bargmann structure (2.6) is the group $\mathrm{Barg}(\mathbf{R}^{n+1,1})$, coined *Bargmann group* by Souriau [42], generated by the transformations (5.17), (5.18) and (5.19) where $d = g = 1$ and $f = 0$; see also the early reference [30]. If $n > 2$, it is, up to equivalence, the unique $(\mathbf{R},+)$-central extension of the Galilei group of flat $(n+1)$-dimensional NC spacetime as first shown by Bargmann [2]. The Schrödinger group, $\mathrm{Sch}(\mathbf{R}^{n+1,1})$, i.e., the maximal symmetry group of the free Schrödinger equation [33, 38, 31], is again generated by the transformations (5.17), (5.18) and (5.19) where $\nu = dg - ef = 1$. The *Schrödinger group* is, if $n > 2$, the unique $(\mathbf{R},+)$-central extension of the (centerless) Schrödinger group of flat $(n+1)$-dimensional NC spacetime.

2.2 Newton-Cartan gravitational field equations

Let us recall, at this stage, that the NC gravitational field equations have been geometrically reformulated as [14]

$$\mathrm{Ric}(\mathrm{g}) = 4\pi G\varrho\, \theta \otimes \theta \tag{2.7}$$

where θ is the clock of the Bargmann structure, and ϱ is the mass-density of the sources. We note that this implies that the scalar curvature vanishes identically, $R(\mathrm{g}) = 0$.

Let us mention that for the already interesting example of spatially-flat Bargmann structures

$$\mathrm{g} = \delta_{ij}\, dx^i \otimes dx^j + dt \otimes \omega + \omega \otimes dt \qquad \& \qquad \xi = \frac{\partial}{\partial s}. \tag{2.8}$$

with ω as in (2.3), the Newton field equations (2.7) are of the form

$$\delta \mathbf{\Omega} = 0 \qquad \& \qquad \Delta_{\mathbf{R}^n} U + \frac{\partial}{\partial t}\delta \boldsymbol{\omega} + \frac{1}{2}\|\mathbf{\Omega}\|^2 = 4\pi G\varrho \tag{2.9}$$

where $\boldsymbol{\omega} = \omega_i(\mathbf{x}, t) dx^i|_{\Sigma_t}$, and $\mathbf{\Omega} = d\boldsymbol{\omega}$ is the "Coriolis" 2-form in the chosen (rotating) coordinates; we denote by δ the codifferential acting on differential forms of Euclidean space $\Sigma_t \cong \mathbf{R}^n$. (In (2.9), we have used the shorthand notation $\|\mathbf{\Omega}\|^2 = \frac{1}{2}\delta^{ik}\delta^{j\ell}\Omega_{ij}\Omega_{k\ell}$.) See also [17] for the field equations in the general case given by Equations (2.1) and (2.2).

3 The Schrödinger-Newton equation

3.1 The Bargmann lift of the Schrödinger-Newton equation

The Schrödinger-Newton (SN) field equations will be recast as follows on our Bargmann extension (M, g, ξ) of NC spacetime.

Consider first the Schrödinger equation. Let us declare "wave functions" to be complex-valued densities Ψ of M with weight

$$w = \frac{N-2}{2N}. \tag{3.1}$$

Recall that a w-density (with w a complex number) is locally of the form $\Psi = \psi(x)|\mathrm{Vol}|^w$ where ψ is a smooth complex-valued function and Vol some volume element of M; in our setting, we naturally choose the canonical volume element, $\mathrm{Vol}(\mathrm{g})$, on (M, g), so that

$$\Psi = \psi(x)|\mathrm{Vol}(\mathrm{g})|^w. \tag{3.2}$$

• The *Schrödinger equation* [14, 17] then takes, along with the mass-constraint (3.4), the form of the wave equation

$$\boxed{\Delta_Y(\mathrm{g})\Psi = 0} \tag{3.3}$$

where $\Delta_Y(\mathrm{g}) = \Delta(\mathrm{g}) - (N-2)/(4(N-1))R(\mathrm{g})$ stands for the conformally-invariant Yamabe operator [3, 24] sending w-densities to $(1-w)$-densities of M, with w necessarily as in (3.1); see (A.2). Here, $\Delta(\mathrm{g})$ is the Laplace-Beltrami operator, and $R(\mathrm{g})$ the scalar curvature of (M, g). The wave functions, Ψ, are further assumed to be eigenvectors of the *mass operator*, namely[6]

$$\boxed{\frac{\hbar}{i} L_\xi \Psi = m\Psi} \tag{3.4}$$

where $m > 0$ stands for the mass of the quantum system; see Equation (A.4). (This fixed eigenvalue will, later on, be licensed to undergo rigid dilations.)

• We are ready to formulate the last ingredient of the (generalized) *Schrödinger-Newton equations* by deciding that ϱ be in fact the quantum mass-density of the system, see Equation (1.4). In view of the *NC field equations* (2.7), we will therefore posit

$$\boxed{\mathrm{Ric}(\mathrm{g}) = 4\pi G\, m\, |\widetilde{\Psi}|^2\, \theta \otimes \theta} \tag{3.5}$$

where

$$\widetilde{\Psi} = \frac{\Psi}{\|\Psi\|_{\mathrm{g}}} \qquad \& \qquad \|\Psi\|_{\mathrm{g}}^2 = \int_{\Sigma_t} |\psi|^2 \,\mathrm{vol}(h) < +\infty \tag{3.6}$$

is the corresponding normalized wave-function; here $\mathrm{vol}(h)$ stands for the canonical volume form of Σ_t,[7] and $\psi(\mathbf{x}, t) = \psi_t(\mathbf{x})$ — with $\psi_t \in L^2(\Sigma_t, \mathrm{vol}(h))$ — is as in (3.2).

3.2 Recovering the SN equation in its standard guise

We now claim that the coupled system of PDE (3.3), (3.4),(3.5) constitutes an intrinsic generalization of the Bargmann lift of the (non-local) Schrödinger-Newton equation (1.1). Let us justify this statement in the special case where $M = \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}$, the spatial metric being Euclidean, and the connection form (2.3) given by $\omega = -U(\mathbf{x}, t)dt + ds$. In this case, the Bargmann structure (2.1) reads

$$\mathrm{g} = \delta_{ij}\, dx^i \otimes dx^j + dt \otimes ds + ds \otimes dt - 2U(\mathbf{x}, t) dt \otimes dt \qquad \& \qquad \xi = \frac{\partial}{\partial s}. \tag{3.7}$$

[6] The LHS of (3.4) features the Lie derivative of a w-density, Ψ, of M with respect to the vector field ξ.

[7] This volume form can be defined intrinsically. Indeed, call $\eta = \mathrm{g}^{-1}(\omega)$ the vector field associated with the connection form ω given by (2.3); one checks that η is null and ω-horizontal. Then $\mathrm{Vol}(\mathrm{g})(\xi, \eta)$ flows down to NC spacetime, $\mathcal{M}$; once pulled-back to Σ_t, it canonically defines the volume n-form $\mathrm{vol}(h)$. The latter admits the following local expression, namely $\mathrm{vol}(h) = \sqrt{\det(\mathrm{g}_{ij}(x,t))}\, dx^1 \wedge \cdots \wedge dx^n$, where $h = h^{ij}(x,t)\, \partial/\partial x^i \otimes \partial/\partial x^j$ and $(h^{ij}) = (\mathrm{g}_{ij})^{-1}$.

We find $\mathrm{Ric}(\mathrm{g}) = (\Delta_{\mathbf{R}^n} U) dt \otimes dt$, and the Newtonian gravitational field equations (2.7) reduce to the ordinary Poisson equation (1.3); see also (2.9) with $\boldsymbol{\omega} = 0$. Also does Equation (3.4) imply that

$$\Psi(\mathbf{x}, t, s) = e^{ims/\hbar}\, \psi(\mathbf{x}, t) |\mathrm{Vol}(\mathrm{g})|^{\frac{n}{2n+4}}. \tag{3.8}$$

The wave-function Ψ being Yamabe-harmonic (see (3.3)), we get $\Delta_{\mathbf{R}^n}\Psi + 2\partial/\partial t(\partial/\partial s \Psi) + 2U(\partial/\partial s)^2\Psi = 0$, and hence $-\hbar^2/(2m)\Delta_{\mathbf{R}^n}\psi - i\hbar\, \partial\psi/\partial t + V\psi = 0$, where $V = mU$ is the potential energy in the Schrödinger equation (1.2). At last, putting $V = \Delta^{-1}_{\mathbf{R}^n}(4\pi G\, m\varrho)$ with ϱ as in (1.4), enables us to recover the Schrödinger-Newton equation (1.1) in its original form.

3.3 The generalized Schrödinger-Newton equation

Let us now take advantage of the geometrical form (3.3), (3.4), and (3.5) of the Schrödinger-Newton equation to work out the brand new form of the SN equation in the spatially flat Bargmann structure (2.8), defined in terms of the Coriolis (co)vector potential $\boldsymbol{\omega}$ and the Newtonian potential U.

Easy calculation yields the non-trivial components of the inverse metric, g^{-1}; they read as $\mathrm{g}^{ij} = \delta^{ij}$, $\mathrm{g}^{is} = -\omega_i$, $\mathrm{g}^{ts} = 1$, and $\mathrm{g}^{ss} = 2U + \delta^{ij}\omega_i\omega_j$, for all $i, j = 1, \ldots, n$. With the same sloppy notation, we express the non-zero Christoffel symbols as $\Gamma^i_{jt} = -\frac{1}{2}\Omega_{ij}$, $\Gamma^i_{tt} = \partial_i U + \partial_t \omega_i$, $\Gamma^s_{ij} = \partial_{(i}\omega_{j)}$, $\Gamma^s_{it} = -\partial_i U - \frac{1}{2}\Omega_{ij}\omega^j$, and $\Gamma^s_{tt} = -\partial_t U - \omega^i(\partial_i U + \partial_t \omega_i)$.

With these data, the wave equation (3.3) together with (3.4) — or (3.8) — leads to the Schrödinger equation

$$\boxed{-\frac{\hbar^2}{2m}\Delta_{\mathbf{R}^n}\psi - \frac{\hbar}{2i}\left[\omega^j \circ \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^j} \circ \omega^j\right]\psi + \frac{\hbar}{i}\frac{\partial\psi}{\partial t} + m\left[U + \frac{1}{2}\|\boldsymbol{\omega}\|^2\right]\psi = 0.} \tag{3.9}$$

This equation and the coupled NC field equations (3.5) & (2.9), namely

$$\boxed{\delta\boldsymbol{\Omega} = 0 \qquad \& \qquad \Delta_{\mathbf{R}^n} U + \frac{\partial}{\partial t}\delta\boldsymbol{\omega} + \frac{1}{2}\|\boldsymbol{\Omega}\|^2 = 4\pi G\, m\, |\widetilde{\psi}|^2} \tag{3.10}$$

constitute the *generalized Schrödinger-Newton equations* resulting naturally from our choice of a Bargmann framework.

4 Scaling symmetries of the SN equation

This section is devoted to the search of the symmetries of the SN equation under dilations using the intrinsic framework provided by Bargmann structures over NC spacetimes.

4.1 Conformally related Bargmann structures

In view of Equation (3.3), the symmetries of the previous coupled system will be naturally sought inside the conformal transformations of (M, g), namely inside the pseudo-group of those local diffeomorphisms Φ of M such that

$$\Phi^* \mathrm{g} = \lambda_\Phi \, \mathrm{g} \tag{4.1}$$

with $\lambda_\Phi > 0$ a smooth function of M.

Moreover, we want that these transformations, Φ, on our Bargmann manifold do actually project as *bona fide* transformations of NC spacetime (on which the original SN equation was formulated). This implies that those Φ be mere automorphisms of the fibration $\pi : M \to \mathcal{M}$, i.e., be such that

$$\Phi^* \xi = \nu_\Phi \, \xi \tag{4.2}$$

with $\nu_\Phi \neq 0$ a smooth function of M. (Note that fibre-bundle automorphisms should require $\nu_\Phi = 1$.)

One will show below (see also [7]) that, under these circumstances, λ_Φ is the pull-back of a (positive) smooth function of the time-axis, T, and ν_Φ a (nonzero) constant, viz.,

$$d\lambda_\Phi \wedge \theta = 0 \qquad \& \qquad d\nu_\Phi = 0. \tag{4.3}$$

Indeed, if $\widehat{\mathrm{g}} = \Phi^* \mathrm{g}$ and $\widehat{\xi} = \Phi^* \xi$ are as in (4.1) and (4.2) respectively, we demand that $(M, \widehat{\mathrm{g}}, \widehat{\xi})$ be again Bargmann. The vector field $\widehat{\xi}$ being automatically $\widehat{\mathrm{g}}$-null, it remains to find under which conditions it is parallel transported by the Levi-Civita connection $\widehat{\nabla}$ of $\widehat{\mathrm{g}}$. We find that $\widehat{\nabla}\widehat{\xi} = 0$ iff $d\nu_\Phi \otimes \theta + \nu_\Phi/(2\lambda_\Phi)\,[\xi(\lambda_\Phi)\,\mathrm{g} + d\lambda_\Phi \wedge \theta] = 0$. Straightforward calculation shows that $d\nu_\Phi = (\nu_\Phi/\lambda_\Phi) d\lambda_\Phi + f\theta$ for some function f, as well as $\xi(\lambda_\Phi) = 0$. This, in turn, implies $d\lambda_\Phi \wedge \theta = 0$, hence $d\nu_\Phi = 0$. We have just proved (4.3).

From now on, and to avoid clutter, we will write λ (resp. ν) instead of λ_Φ (resp. ν_Φ) if no confusion can occur.

4.2 A Schwarzian intermezzo

Consider, on a pseudo-Riemannian manifold (M, g), a metric $\widehat{\mathrm{g}} = \lambda \, \mathrm{g}$ conformally related to g. We will denote by Ric the mapping from the set of pseudo-Riemannian metrics of M to the space of twice covariant symmetric tensor fields of M defined by the Ricci tensor.

The conformal transformation law of the Ricci tensor is well-known and reads

$$\mathrm{Ric}(\widehat{\mathrm{g}}) = \mathrm{Ric}(\mathrm{g}) - \frac{(N-2)}{2}\left(\frac{\nabla d\lambda}{\lambda} - \frac{3}{2}\,\frac{d\lambda \otimes d\lambda}{\lambda^2}\right) - \frac{1}{2}\left(\frac{\Delta\lambda}{\lambda} + \frac{(N-4)}{2}\,\frac{|d\lambda|^2}{\lambda^2}\right)\mathrm{g} \tag{4.4}$$

where $\Delta\lambda = \mathrm{g}^{\alpha\beta}\nabla_\alpha\partial_\beta\lambda$ and $|d\lambda|^2 = \mathrm{g}^{\alpha\beta}\partial_\alpha\lambda\partial_\beta\lambda$, in a coordinate system $(x^\alpha)_{\alpha=1,\ldots,N}$ of M. The scalar curvature then transforms as

$$R(\widehat{\mathrm{g}}) = \frac{R(\mathrm{g})}{\lambda} - (N-1)\left(\frac{\Delta\lambda}{\lambda^2} + \frac{(N-6)}{4}\frac{|d\lambda|^2}{\lambda^3}\right). \tag{4.5}$$

Specialize then considerations to Bargmann structures (M, g, ξ). It has just been proved, see also [21], that if $(M, \widehat{\mathrm{g}}, \widehat{\xi})$ where $\widehat{\mathrm{g}} = \lambda\mathrm{g}$ and $\widehat{\xi} = \nu\xi$ (as in (4.1) and (4.2)) is again Bargmann structure, then $\lambda : t \mapsto \lambda(t)$ is (the pull-back of) a function of the time-axis T. We then have $d\lambda = \lambda'\theta$, and hence $R(\widehat{\mathrm{g}}) = R(\mathrm{g})/\lambda$ since $\Delta\lambda = \lambda''\mathrm{g}^{\alpha\beta}\theta_\alpha\theta_\beta = 0$ and $|d\lambda|^2 = (\lambda')^2\mathrm{g}^{\alpha\beta}\theta_\alpha\theta_\beta = 0$; cf. Equation (4.5).

Introduce now $\varphi \in \mathrm{Diff}_+(T)$ via $\lambda(t) = \varphi'(t)$; then Equation (4.4) can be cast into the quite remarkable form, viz., $\mathrm{Ric}(\widehat{\mathrm{g}}) = \mathrm{Ric}(\mathrm{g}) - \frac{1}{2}(N-2)\, S(\varphi)\, \theta\otimes\theta$, where

$$S(\varphi) = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 \tag{4.6}$$

is the *Schwarzian derivative* of φ. Notice that the latter naturally shows up as a *quadratic differential* of T, namely $\mathcal{S}(\varphi) = S(\varphi)\,\theta\otimes\theta$ where

$$\mathcal{S}(\varphi) = \frac{-2}{N-2}\left(\mathrm{Ric}(\widehat{\mathrm{g}}) - \mathrm{Ric}(\mathrm{g})\right). \tag{4.7}$$

4.3 Rescalings in action

Apart from the natural actions of the group of diffeomorphisms on our geometric objects (see Section 5.1), we will first need the transformation of the latter under rescaling. Those will ultimately enter the definition of the group of Bargmann conformal transformations, leading to the group of SN symmetries under study.

- It is a classical result (see, e.g., [36]) that under a conformal rescaling

$$\mathrm{g} \mapsto \widehat{\mathrm{g}} = \lambda\mathrm{g} \qquad \& \qquad \lambda \in C^\infty(M, \mathbf{R}_+^*) \tag{4.8}$$

of the metric, the Yamabe operator enjoys the following invariance property, viz.,

$$\boxed{\Delta_Y(\widehat{\mathrm{g}}) = \Delta_Y(\mathrm{g})} \tag{4.9}$$

being clearly understood that the Yamabe operator, $\Delta_Y(\mathrm{g})$, maps here w-densities to $(1-w)$-densities of M, where the weight w is as in (3.1).[8]

[8] Let us recall that we have

$$\Delta_Y(\widehat{\mathrm{g}}) = \lambda^{-\frac{N+2}{4}} \circ \Delta_Y(\mathrm{g}) \circ \lambda^{\frac{N-2}{4}} \tag{4.10}$$

if the Yamabe operator is rather viewed as an operator acting on 0-densities, i.e., smooth functions of M.

• We will now prove that the normalized wave function $\widetilde{\Psi}$ transforms as

$$(\widetilde{\Psi})\widehat{\ } = \lambda^{-\frac{n}{4}}\widetilde{\Psi} \tag{4.11}$$

and that the latter holds true for any w-density Ψ. Indeed, in view of (3.6), we find that $(\widetilde{\Psi})\widehat{\ } = \lambda^{\frac{Nw}{2}}\Psi/\|\lambda^{\frac{Nw}{2}}\Psi\|_{\widehat{\mathrm{g}}} = \Psi/\|\Psi\|_{\widehat{\mathrm{g}}}$ since λ is a function of T. Now, in Equation (3.6) we have $\mathrm{vol}(\widehat{h}) = \lambda^{\frac{n}{2}}\mathrm{vol}(h)$ so that $\|\Psi\|_{\widehat{\mathrm{g}}} = \lambda^{\frac{n}{4}}\|\Psi\|_{\mathrm{g}}$; this ends the proof of Equation (4.11).

• Under a rescaling $\xi \mapsto \widehat{\xi} = \nu\xi$, we clearly have

$$L_{\widehat{\xi}} = \nu L_{\xi}. \tag{4.12}$$

• Let us recall that, in Section 4.2, we have already shown that

$$\boxed{\mathrm{Ric}(\widehat{\mathrm{g}}) = \mathrm{Ric}(\mathrm{g}) - \frac{(N-2)}{2}\, S(\varphi)\, \theta \otimes \theta} \tag{4.13}$$

holds true whenever (M, g, ξ) and $(M, \widehat{\mathrm{g}}, \widehat{\xi})$ are Bargmann manifolds (we remember that $\lambda(t) = \varphi'(t) > 0$ for all $t \in T$).

4.4 The dilation-invariance of the SN equation

Let us first remind that, for the wave equation (3.3), pure conformal rescalings (4.8) of the metric translate as follows: if (g, Ψ) is a solution of the wave equation $\Delta_Y(\mathrm{g})\Psi = 0$, then Equation (4.9) implies that the same is true for $(\widehat{\mathrm{g}}, \Psi)$, namely that $\Delta_Y(\widehat{\mathrm{g}})\Psi = 0$.

Likewise, in the case under study, we wish to determine under which conditions $(\widehat{\mathrm{g}}, \widehat{\xi}, \Psi)$ is a solution of the SN equations (3.3)–(3.5) if (g, ξ, Ψ) is such a solution.

• Using the above argument about the fundamental property (4.9) of the Yamabe operator, we readily get

$$\Delta_Y(\mathrm{g})\Psi = 0 \qquad \Longrightarrow \qquad \Delta_Y(\widehat{\mathrm{g}})\Psi = 0 \tag{4.14}$$

confirming that the wave equation (3.3) is indeed conformally invariant.

• As to the behavior of the mass under a dilation $\xi \mapsto \widehat{\xi} = \nu\xi$, we claim that

$$\frac{\hbar}{i} L_{\xi}\, \Psi = m\Psi \qquad \Longrightarrow \qquad \frac{\hbar}{i} L_{\widehat{\xi}}\, \Psi = \widehat{m}\Psi \tag{4.15}$$

and thus prove that the mass (as defined by (3.4)) gets dilated according to [1, 28]

$$\boxed{\widehat{m} = \nu\, m} \tag{4.16}$$

in coherence with Equation (A.5) in Appendix A.2.

• Let us write the Newton-Cartan equation (3.5) in terms of the dilated objects. Using Equations (4.16), (4.11), and the fact that dilations act on the clock as

$$\widehat{\theta} = \lambda\nu\,\theta \tag{4.17}$$

we readily find that

$$\begin{aligned} 0 &= \mathrm{Ric}(\widehat{\mathrm{g}}) - 4\pi G\widehat{m}\left|(\widetilde{\Psi})\widehat{\ }\right|^2\widehat{\theta}\otimes\widehat{\theta} \\ &= \mathrm{Ric}(\widehat{\mathrm{g}}) - 4\pi Gm\,\nu^3\lambda^{2-\frac{n}{2}}\left|\widetilde{\Psi}\right|^2\theta\otimes\theta \\ &= \mathrm{Ric}(\mathrm{g}) - 4\pi Gm\left|\widetilde{\Psi}\right|^2\theta\otimes\theta \qquad (4.18) \\ &\quad - \left[\frac{n}{2}\,S(\varphi) + 4\pi Gm\,(\nu^3\lambda^{2-\frac{n}{2}}-1)\left|\widetilde{\Psi}\right|^2\right]\theta\otimes\theta \qquad (4.19) \end{aligned}$$

in view of (4.13).

We can thus immediately conclude that

$$\mathrm{Ric}(\mathrm{g}) = 4\pi Gm\left|\widetilde{\Psi}\right|^2\theta\otimes\theta \qquad \Longrightarrow \qquad \mathrm{Ric}(\widehat{\mathrm{g}}) = 4\pi G\widehat{m}\left|(\widetilde{\Psi})\widehat{\ }\right|^2\widehat{\theta}\otimes\widehat{\theta} \tag{4.20}$$

provided, on the one hand,[9]

$$\boxed{S(\varphi) = 0} \tag{4.21}$$

since φ is an orientation-preserving diffeomorphism of the time-axis, T, while $\widetilde{\Psi}$ is defined on the whole Bargmann manifold, M, and on the other hand,

$$\boxed{\lambda^{2-\frac{n}{2}}\,\nu^3 = 1.} \tag{4.22}$$

• The fundamental constraint (4.22) thus implies that

$$\lambda = \nu^{-\frac{6}{4-n}} \tag{4.23}$$

is a (strictly positive) constant,[10] and that φ is actually an orientation-preserving *affine diffeomorphism* of T, namely

$$\varphi(t) = \lambda t + \mu \qquad \& \qquad n \neq 4 \tag{4.24}$$

with $\lambda \in \mathbf{R}_+^*$, and $\mu \in \mathbf{R}$. Inversions are lost in this case

[9] We thus get that $\varphi \in \mathrm{PGL}(2,\mathbf{R})$.

[10] We discover that, thanks to (4.16), these dilations preserve the *positivity of the mass* since $\nu \in \mathbf{R}_+^*$.

• If $n = 4$, we see that (4.22) yields $\nu = 1$ (trivial mass-dilation (4.16)); this proves that, in this case, we have

$$\varphi(t) = \frac{dt + e}{ft + g} \qquad \& \qquad dg - ef = 1 \tag{4.25}$$

or, equivalently, $\varphi \in \mathrm{PSL}(2, \mathbf{R})$.

5 Maximal group of invariance of the SN equation

5.1 Naturality relationships

Let us recall that, on any (pseudo-)Riemannian manifold (M, g) we have the *naturality* relationship of the Ricci mapping, viz.,

$$\mathrm{Ric}(\Phi^*\mathrm{g}) = \Phi^*(\mathrm{Ric}(\mathrm{g})) \tag{5.1}$$

for all $\Phi \in \mathrm{Diff}(M)$; see Equation (5.13) in [3].

Likewise, we know that

$$\Delta_Y(\Phi^*\mathrm{g}) = \Phi^*(\Delta_Y(\mathrm{g})) \tag{5.2}$$

for all $\Phi \in \mathrm{Diff}(M)$; see, e.g, [36] and references therein.

At last, the following holds true

$$L_{\Phi^*\xi} = \Phi^*(L_\xi) \tag{5.3}$$

for all $\Phi \in \mathrm{Diff}(M)$.

5.2 Schrödinger-Newton symmetries

We are now ready to determine in geometric terms the symmetries of the Schrödinger-Newton system. In contradistinction to the full Schrödinger symmetry of another system of coupled PDE, namely the Schrödinger-Chern Simons system in flat $(2+1)$-dimensional Galilei spacetime [34, 19], the symmetry of the present system turns out to be more subtle, and intricate to work out, since the system is in fact gravitationally "self-coupled".

As usual, we call $\mathrm{Conf}(M, \mathrm{g})$ the "group" of (signature-preserving) conformal transformations of the metric g of M, the *maximal invariance group* of the wave equation (3.3).

Let us recall that the chronoprojective "group" [12, 13, 5, 6] of a Bargmann structure is generated by conformal diffeomorphisms of (M, g) that are also automorphisms of the fibration $\pi : M \to \mathcal{M} = M/(\mathbf{R}\xi)$, i.e.,

$$\begin{aligned} \mathrm{Chr}(M, \mathrm{g}, \xi) &= \mathrm{Conf}(M, \mathrm{g}) \cap \mathrm{Aut}(M \to \mathcal{M}) && (5.4) \\ &= \{\Phi \in \mathrm{Diff}(M) \,|\, \Phi^*\mathrm{g} = \lambda \mathrm{g}, \Phi^*\xi = \nu\xi\} && (5.5) \end{aligned}$$

where necessarily (see (4.3)) $\lambda \in C^\infty(T, \mathbf{R}^*_+)$, and $\nu \in \mathbf{R}^*$.

Our aim is to determine in full generality the subgroup, $\mathrm{SN}(M, \mathrm{g}, \xi)$, of $\mathrm{Chr}(M, \mathrm{g}, \xi)$ that permutes the solutions, (g, ξ, Ψ), of the Schrödinger-Newton equations (3.3)–(3.5), i.e., that maps solutions to solutions of this system.

• Assuming Ψ satisfies $\Delta_Y(\mathrm{g})\Psi = 0$, we find that Equations (5.2) entails

$$\begin{aligned} 0 &= \Phi^*(\Delta_Y(\mathrm{g})\Psi) \\ &= \Delta_Y(\Phi^*\mathrm{g})\Phi^*\Psi \end{aligned} \tag{5.6}$$

which clearly confirms the $\mathrm{Diff}(M)$-naturality of the wave equation (3.3).

Moreover its turns out that (4.9) implies

$$\Delta_Y(\mathrm{g})\Phi^*\Psi = 0 \tag{5.7}$$

hence that if Ψ is Yamabe-harmonic, so is $\Phi^*\Psi$ for all $\Phi \in \mathrm{Conf}(M, \mathrm{g})$.

• Likewise, using Equations (5.3) and (A.6), we find that

$$\begin{aligned} 0 &= \Phi^* \left(-i\hbar\, L_\xi \Psi - m\, \Psi\right) \\ &= (-i\hbar\, L_{\Phi^*\xi})\Phi^*\Psi - \widehat{m}\, \Phi^*\Psi \end{aligned} \tag{5.8}$$

which guarantees the $\mathrm{Diff}(M)$-naturality of the mass equation (3.4).

Together with (4.2) and (4.16), Equation (5.8) leads to

$$\frac{\hbar}{i}\, L_\xi(\Phi^*\Psi) = m\, \Phi^*\Psi \tag{5.9}$$

proving that $\Phi^*\Psi$ has also mass m for all $\Phi \in \mathrm{Chr}(M, \mathrm{g}, \xi)$.

• We now turn to the classical NC field equation coupled to the Schrödinger equation, namely Equations (3.3) and (3.4).

Let (g, ξ, Ψ) be one of the solutions of (3.5). With the help of (5.1) we find that

$$\begin{aligned} 0 &= \Phi^*\big(\mathrm{Ric}(\mathrm{g}) - 4\pi G\, m\, |\widetilde{\Psi}|^2\, \theta \otimes \theta\big) \\ &= \Phi^*\big(\mathrm{Ric}(\mathrm{g})\big) - 4\pi G\, \widehat{m}\, |\Phi^*\widetilde{\Psi}|^2\, \Phi^*\theta \otimes \Phi^*\theta \\ &= \mathrm{Ric}(\Phi^*\mathrm{g}) - 4\pi G\, \widehat{m}\, |\Phi^*\widetilde{\Psi}|^2\, \Phi^*\theta \otimes \Phi^*\theta \end{aligned} \tag{5.10}$$

which expresses the $\mathrm{Diff}(M)$-naturality of the NC field equations (3.5).

Besides, it is an easy matter to show that

$$\Phi^*\widetilde{\Psi} = \lambda^{-\frac{n}{4}}\, \widetilde{\Phi^*\Psi} \tag{5.11}$$

for all $\Phi \in \mathrm{Chr}(M, \mathrm{g}, \xi)$. Indeed, from (3.2), and putting $(\hat{x}, \hat{t}, \hat{s}) = \Phi(x, t, s)$, we get[11]

$$\begin{aligned} \|\Phi^*\Psi\|_{\mathrm{g}}^2 &= \int_{\Sigma_t} |\lambda^{\frac{n}{4}}\, \Phi^*\psi|^2 \,\mathrm{vol}(h) \\ &= \int_{\Sigma_t} |\psi(\hat{x}, \hat{t}, \hat{s})|^2\, \lambda^{\frac{n}{2}} \sqrt{\det(\mathrm{g}_{ij}(x,t))}\, dx^1 \ldots dx^n \\ &= \int_{\Sigma_{\hat{t}}} |\psi(\hat{x}, \hat{t}, \hat{s})|^2 \sqrt{\det(\mathrm{g}_{ij}(\hat{x}, \hat{t}))}\, d\hat{x}^1 \ldots d\hat{x}^n \\ &= \|\Psi\|_{\mathrm{g}}^2. \end{aligned} \tag{5.13}$$

According to (3.6), we find $\Phi^*\widetilde{\Psi} = \Phi^*\Psi/\|\Phi^*\Psi\|_{\Phi^*\mathrm{g}} = \Phi^*\Psi/(\lambda^{\frac{n}{4}}\|\Psi\|_{\mathrm{g}})$, by using again (5.12). With the help of (5.13), we then write $\Phi^*\widetilde{\Psi} = \lambda^{-\frac{n}{4}}\, \Phi^*\Psi/\|\Phi^*\Psi\|_{\mathrm{g}}$, proving Equation (5.11).

Equations (5.10) and (5.11), together with (4.13) and (4.21) then yield in a straightforward manner

$$\mathrm{Ric}(\mathrm{g}) = 4\pi G\, m\, |\widetilde{\Phi^*\Psi}|^2\, \theta \otimes \theta \tag{5.14}$$

where (4.22) has been duly taken into account.

We stress that $\Phi^*\Psi$ thus also provides a solution of the NC gravitational field equation (3.5), for any Schrödinger-Newton diffeomorphism $\Psi \in \mathrm{SN}(M, \mathrm{g}, \xi)$ of the Bargmann manifold, i.e., any $\Phi \in \mathrm{Chr}(M, \mathrm{g}, \xi)$ for which (4.22) holds true.

5.3 The general definition of the Schrödinger-Newton group

Taking advantage of both the definition (5.5) of the chronoprojective group, and the constraint (4.22), we can now define in quite general terms the Schrödinger-Newton group of a Bargmann structure as

$$\boxed{\mathrm{SN}(M, \mathrm{g}, \xi) = \{\Phi \in \mathrm{Diff}(M) \,|\, \Phi^*\mathrm{g} = \lambda \mathrm{g}, \Phi^*\xi = \nu\xi,\ \lambda^{2-\frac{n}{2}}\, \nu^3 = 1\}.} \tag{5.15}$$

We notice that this definition makes sense if $\lambda > 0$, hence $\nu > 0$.

If $n = 4$, Equation (5.15) yields $\nu = 1$; the Schrödinger-Newton group is, in this case, isomorphic to the Schrödinger group (2.5), namely

$$\mathrm{SN}(M, \mathrm{g}, \xi) = \mathrm{Sch}(M, \mathrm{g}, \xi) \qquad \text{if} \qquad n = 4. \tag{5.16}$$

[11] We use systematically the fact that, in the chosen local coordinate system M, we have

$$\sqrt{\det(\mathrm{g}_{ij}(\hat{x}, \hat{t}))}\, \det\left(\frac{\partial \hat{x}^i}{\partial x^j}\right) = \lambda^{\frac{n}{2}} \sqrt{\det(\mathrm{g}_{ij}(x,t))}. \tag{5.12}$$

If $n \neq 4$ the full Schrödinger symmetry group (2.5) is thus broken to the Schrödinger-Newton group (5.15).[12]

5.4 The Schrödinger-Newton group for the spatially-flat model

Let us now determine explicitly the SN group (5.15) of invariance of the Schrödinger-Newton equation (1.1). We will choose, however, to deal with the most general spatially-flat Bargmann metric (2.8) and the associated generalized Schrödinger-Newton equation given by Equations (3.3), (3.4) and (3.5); as shown in Section 3.2, the case of the standard Schrödinger-Newton equation will merely follow as a special case of the generalized SN equation given by (3.9) and (3.10).

5.4.1 The chronoprojective group of the flat Bargmann structure

The general solution of Equations (4.1) and (4.2) for the flat Bargmann structure (2.6) is given by the so-called "chronoprojective group"[13] denoted $\mathrm{Chr}(\mathbf{R}^{n+1,1}) = \mathrm{Chr}(\mathbf{R}^{n+2}, \mathrm{g}_0, \xi)$ [12, 13, 5, 6, 7].

Its projective action $\Phi : (\mathbf{x}, t, s) \mapsto (\hat{\mathbf{x}}, \hat{t}, \hat{s})$ on $\mathbf{R}^{n+2}$ reads[14]

$$\hat{\mathbf{x}} = \frac{A\mathbf{x} + \mathbf{b}t + \mathbf{c}}{ft + g} \tag{5.17}$$

$$\hat{t} = \frac{dt + e}{ft + g} \tag{5.18}$$

$$\hat{s} = \frac{1}{\nu}\left[s + \frac{f}{2}\frac{\|A\mathbf{x} + \mathbf{b}t + \mathbf{c}\|^2}{ft + g} - \langle \mathbf{b}, A\mathbf{x}\rangle - \frac{1}{2}\|\mathbf{b}\|^2 t + h\right] \tag{5.19}$$

where $A \in \mathrm{O}(n)$, $(\mathbf{b}, \mathbf{c}) \in \mathbf{R}^n \times \mathbf{R}^n$ (boosts & space-translations), also

$$D = \begin{pmatrix} d & e \\ f & g \end{pmatrix} \in \mathrm{GL}(2, \mathbf{R}) \tag{5.20}$$

represents the projective group of the time-axis (whence the name *chronoprojective* transformation for (5.17)–(5.19)), and $h \in \mathbf{R}$ (extension parameter).

[12] The strange, singular, dimension $n = 4$ plays a rôle akin to that, N, of a relativistic spacetime (M, g) for which the Maxwell Lagrangian density $L(F, \mathrm{g}) = \frac{1}{4}\mathrm{g}^{\alpha\beta}\mathrm{g}^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta}\,|\mathrm{Vol}(\mathrm{g})|$ is conformally invariant. Indeed, $L(F, \lambda\mathrm{g}) = \lambda^{-(2-\frac{N}{2})}L(F, \mathrm{g})$ implies $L(F, \lambda\mathrm{g}) = L(F, \mathrm{g})$ for all $\lambda \in C^\infty(M, \mathbf{R}^*_+)$ if $N = 4$. Here, it is the conformal symmetry which is broken whenever $N \neq 4$.

[13] This group is in fact the canonical $(\mathbf{R}, +)$-extension of the projected "chronoprojective group" acting on NC spacetime via (5.17) and (5.18). One often uses the same term for both groups. Formula (5.19) is a generalization to the case $\nu \neq 1$ of the corresponding one, for $\nu = 1$, in [20].

[14] Below, $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the Euclidean scalar product of $\mathbf{R}^n$, and $\|\,\cdot\,\|$ for the associated norm.

We note that the factors

$$\lambda = \frac{1}{(ft+g)^2} \qquad \& \qquad \nu = dg - ef \tag{5.21}$$

are duly expressed in terms of the group parameters. In particular, the coefficient ν in (5.19) is given by (5.21).

5.4.2 The SN group as a subgroup of the chronoprojective group

Let us first determine the chronoprojective group $\mathrm{Chr}(\mathbf{R}^{n+2}, \mathrm{g}, \xi)$, where g and ξ are as in (2.8), namely

$$\mathrm{g} = \mathrm{g}_0 + dt \otimes \omega_i(\mathbf{x}, t)\, dx^i + \omega_i(\mathbf{x}, t)\, dx^i \otimes dt - 2U(\mathbf{x}, t)\, dt \otimes dt \qquad \& \qquad \xi = \frac{\partial}{\partial s} \tag{5.22}$$

where g_0 is the flat Bargmann metric (2.6).

We start by characterizing all $\Phi \in \mathrm{Conf}(\mathbf{R}^{n+2}, \mathrm{g})$, i.e., such that

$$\begin{aligned} \Phi^*\mathrm{g} &= \Phi^*\mathrm{g}_0 + d\hat{t} \otimes \widehat{\omega}_i\, d\hat{x}^i + \widehat{\omega}_i\, d\hat{x}^i \otimes d\hat{t} - 2\widehat{U}\, d\hat{t} \otimes d\hat{t} \\ &= \lambda \mathrm{g}_0 + \lambda \left(dt \otimes \omega_i\, dx^i + \omega_i\, dx^i \otimes dt - 2U\, dt \otimes dt \right) \end{aligned} \tag{5.23}$$

for some $\lambda \in C^\infty(\mathbf{R}^{n+2}, \mathbf{R}^*_+)$. Now we observe that the metric g_0 contains no terms of the form $dt \otimes dx^i$ (for $i = 1, \ldots, n$) or even $dt \otimes dt$. From this, we conclude that Equation (5.23) insures that $\Phi^*\mathrm{g}_0 = \lambda \mathrm{g}_0$, i.e., that $\Phi \in \mathrm{Conf}(\mathbf{R}^{n+2}, \mathrm{g}_0)$.

We furthermore want that these diffeomorphisms, Φ, satisfy Equation (4.2), namely $\Phi^*\xi = \nu\xi$; this entails that the sought SN group is in fact a (proper) subgroup of the chronoprojective group of the flat Bargmann structure,

$$\mathrm{SN}(\mathbf{R}^{n+2}, \mathrm{g}, \xi) \subset \mathrm{Chr}(\mathbf{R}^{n+2}, \mathrm{g}_0, \xi). \tag{5.24}$$

These (local) diffeomorphisms $\Phi : (\mathbf{x}, t, s) \mapsto (\hat{\mathbf{x}}, \hat{t}, \hat{s})$ are hence explicitly given by Equations (5.17), (5.18) and (5.19).

• If $n \neq 4$, we have proved in (4.24), and in quite general terms, that $\hat{t} = \lambda\, t + \mu$, where $\lambda \in \mathbf{R}^*_+$ and $\mu \in \mathbf{R}$. This entails that

$$f = 0 \tag{5.25}$$

in (5.21), so that

$$\lambda = \frac{1}{g^2} \qquad \& \qquad \nu = d\, g. \tag{5.26}$$

Returning to the fundamental constraint (4.22), we find that

$$d = \nu^{\frac{n-1}{n-4}} \qquad \& \qquad g = \nu^{-\frac{3}{n-4}}. \tag{5.27}$$

We thus claim that the Schrödinger-Newton group is, in this case, isomorphic to the multiplicative group of those matrices of the form

$$\begin{pmatrix} A & \mathbf{b} & 0 & \mathbf{c} \\ 0 & d & 0 & e \\ -\dfrac{\mathbf{b}^t A}{d} & -\dfrac{\|\mathbf{b}\|^2}{2d} & \dfrac{1}{d} & \dfrac{h}{d} \\ 0 & 0 & 0 & g \end{pmatrix} \in \mathrm{SN}(\mathbf{R}^{n+2}, \mathrm{g}, \xi) \tag{5.28}$$

where $A \in \mathrm{O}(n)$, $(\mathbf{b}, \mathbf{c}) \in \mathbf{R}^n \times \mathbf{R}^n$, $(e, h) \in \mathbf{R}^2$, and $\nu \in \mathbf{R}^*_+$, with d and g as in (5.27).

Considering the only subgroup of dilations, we find that

$$\hat{\mathbf{x}} = \nu^{\frac{3}{n-4}} \mathbf{x} \tag{5.29}$$

$$\hat{t} = \nu^{\frac{n+2}{n-4}} t \tag{5.30}$$

$$\hat{s} = \nu^{-1} s \tag{5.31}$$

with $\nu \in \mathbf{R}^*_+$, in full agreement with the claim in Reference [28] in the case $n = 3$.

We can, at this stage, compute the *dynamical exponent*, z, of the SN group. Recall that it is defined by $\hat{t} = \alpha^z t$ if $\hat{\mathbf{x}} = \alpha \mathbf{x}$ for a dilation $\alpha \in \mathbf{R}^*_+ \subset \mathrm{GL}(2, \mathbf{R})$; it measures how much is time dilated as compared to space.[15] Using (5.29) and (5.30), we readily find

$$\boxed{z = \frac{n+2}{3}.} \tag{5.32}$$

• If $n = 4$, we have already shown that the Schrödinger-Newton group is in fact defined by the constraint $\nu = 1$, i.e.,

$$dg - ef = 1 \tag{5.33}$$

in Equations (5.20) and (5.21). This confirms that, in this case, the Schrödinger-Newton group is isomorphic to the Schrödinger group, i.e.,

$$\mathrm{SN}(\mathbf{R}^{5,1}) \cong \mathrm{Sch}(\mathbf{R}^{5,1}). \tag{5.34}$$

We know that the dynamical exponent of the Schrödinger group, $\mathrm{Sch}(\mathbf{R}^{n+1,1})$ is $z = 2$, in perfect accordance with our general expression (5.32) in the case $n = 4$.

[15] The notion of dynamical exponent is specific to non-relativistic theories; see, e.g., [32]. It has clearly no relativistic analogue.

5.4.3 The SN group as group of invariance of the NC equation

Although we have already provided an intrinsic proof that Condition (4.22) is the very constraint defining the SN group, we will nonetheless, and as an alternative, return to this proof using the specific NC field equations (2.9) for the chosen spatially-flat Bargmann metric.

To this end, let us compute explicitly the expression (5.23) of the transformed metric, $\Phi^*\mathrm{g}$, using the general form of the chronoprojective transformations $(\mathbf{x}, t) \mapsto (\hat{\mathbf{x}}, \hat{t})$ given by (5.17) and (5.18). In doing so, for any spatial dimension n, we get the new Coriolis form, $\widehat{\boldsymbol{\omega}}$, and Newtonian potential, $\widehat{U}$; owing to Equations (5.25) and (5.26), we find

$$\widehat{\boldsymbol{\omega}} = \lambda^{-\frac{1}{2}}\nu^{-1}\boldsymbol{\omega}.A^{-1} \qquad \& \qquad \widehat{U} = \lambda^{-1}\nu^{-2}\left(U + \boldsymbol{\omega}.A^{-1}\mathbf{b}\right). \tag{5.35}$$

Moreover, the expression (5.17) of $\hat{\mathbf{x}}$, yields[16]

$$\hat{\partial}_i F = \lambda^{-\frac{1}{2}}(A^{-1})_i^j\,\partial_j F \qquad \& \qquad \hat{\partial}_t F = \lambda^{-1}\nu^{-1}\left(\partial_t F - \mathbf{grad}F\cdot A^{-1}\mathbf{b}\right) \tag{5.36}$$

for all $i = 1, \ldots, n$, and all function $F \in C^\infty(\mathbf{R}^{n+1}, \mathbf{R})$ defined on spacetime.

Remembering that the codifferential (the divergence operator) of the 1-form $\boldsymbol{\omega}$ is $\delta\boldsymbol{\omega} = \delta^{ij}\partial_i\omega_j$, we easily find (via (5.35) and (5.36)) that

$$\widehat{\delta}\,\widehat{\boldsymbol{\omega}} = \lambda^{-1}\nu^{-1}\delta\boldsymbol{\omega}. \tag{5.37}$$

Easy calculation, using the fact that $\Delta_{\mathbf{R}^n}U = \delta^{ij}\partial_i\partial_j U$, the field equation $\delta\boldsymbol{\Omega} = 0$, and Equation (5.35) also yields

$$\widehat{\Delta}_{\mathbf{R}^n}\widehat{U} = \lambda^{-2}\nu^{-2}\left[\Delta_{\mathbf{R}^n}U + \mathbf{grad}(\delta\boldsymbol{\omega}).A^{-1}\mathbf{b}\right] \tag{5.38}$$

where $\mathbf{b} \in \mathbf{R}^n$ is a Galilean boost.

Let us furthermore mention that $\Omega_{ij} = \lambda^{-1}\nu^{-1}(A^{-1})_i^k(A^{-1})_j^\ell\,\Omega_{k\ell}$ for all $i, j = 1, \ldots, d$, which leads to

$$\|\widehat{\boldsymbol{\Omega}}\|^2 = \lambda^{-2}\nu^{-2}\|\boldsymbol{\Omega}\|^2. \tag{5.39}$$

It is now possible to determine under which condition do the NC (alias Newton-Coriolis) equations (2.9) remain invariant under a chronoprojective transformation.

- Firstly, in view of the previous preparation, we clearly have

$$\widehat{\delta}\,\widehat{\boldsymbol{\Omega}} = \delta\,\boldsymbol{\Omega} = 0. \tag{5.40}$$

[16] To alleviate the notation, we will put $\hat{\partial}_i = \partial/\partial\hat{x}^i$. We will also write $\mathbf{grad}F = \delta^{ij}\partial_i F\,\partial_j$.

• Secondly, making use of Equations (5.36), (5.38), (5.37), (5.18), (5.39), (4.16) and (4.11), we discover that

$$\widehat{\Delta}_{\mathbf{R}^n}\widehat{U} + \frac{\partial}{\partial\hat{t}}\,\widehat{\delta\boldsymbol{\omega}} + \frac{1}{2}\|\widehat{\boldsymbol{\Omega}}\|^2 - 4\pi G\,\widehat{m}\,|(\widetilde{\Psi})\widehat{\ }|^2 = \lambda^{-2}\nu^{-2}\Big[\Delta_{\mathbf{R}^n}U + \frac{\partial}{\partial t}\delta\boldsymbol{\omega} + \frac{1}{2}\|\boldsymbol{\Omega}\|^2 - 4\pi G\,m\,|\widetilde{\Psi}|^2\,\nu^3\lambda^{2-\frac{n}{2}}\Big]. \quad (5.41)$$

Equations (5.40) and (5.41) finally enable us to confirm that the NC field equations (2.9) are indeed invariant under those chronoprojective transformations verifying $\lambda^{2-\frac{n}{2}}\,\nu^3 = 1$, as already revealed in (4.22).

5.5 The quantum representation of the SN group

Let us finish this study by working out explicitly the quantum representation of the SN group in the cases $n \neq 4$ and $n = 4$ respectively.

We claim that the mapping

$$\rho(\Phi) : \Psi \mapsto \Phi_*\Psi \quad (5.42)$$

provides a *unitary* representation, ρ, of $\mathrm{SN}(M, \mathrm{g}, \xi)$ on the *set of solutions,* Ψ*, of the Schrödinger-Newton equations* (3.3), (3.4) and (3.5). Indeed, the push-forward mapping, $\Psi \mapsto \Phi_*\Psi$, is a group homomorphism, hence ρ is a unitary representation in view of (5.13).

Let us work out the explicit form of the previously defined representation for the Schrödinger-Newton group, $\mathrm{SN}(\mathbf{R}^{n+2}, \mathrm{g}, \xi)$, of the spatially flat Bargmann structure (5.22).

• We start with the generic case $n \neq 4$. The group to represent is defined by Equations (5.17), (5.18) and (5.19) with $f = 0$ (see (5.25)) and d & g as in (5.27). In order to implement (5.42), i.e. $\rho(\Phi)\Psi = (\Phi^{-1})^*\Psi$, we need to compute $(\mathbf{x}^*, t^*, s^*) = \Phi^{-1}(\mathbf{x}, t, s)$. We readily find

$$\mathbf{x}^* = A^{-1}\left[g\,\mathbf{x} - \frac{(g\,t - e)}{d}\,\mathbf{b} - \mathbf{c}\right] \quad (5.43)$$

$$t^* = \frac{g\,t - e}{d} \quad (5.44)$$

$$s^* = \nu\,s + g\langle\mathbf{b}, \mathbf{x}\rangle - \frac{g}{2d}\|\mathbf{b}\|^2 t + \frac{e}{2d}\|\mathbf{b}\|^2 - \langle\mathbf{b}, \mathbf{c}\rangle - h \quad (5.45)$$

where, again, $A \in \mathrm{O}(n)$, $(\mathbf{b}, \mathbf{c}) \in \mathbf{R}^n \times \mathbf{R}^n$, $(e, h) \in \mathbf{R}^2$, and $\nu \in \mathbf{R}^*_+$, with d and g given by (5.27).

Moreover, referring to Equations (3.8) and (4.16), we have

$$\begin{aligned}\Phi_*\Psi &= \Phi_*\left(e^{ims/\hbar}\,\psi(\mathbf{x},t)\,|\mathrm{Vol(g)}|^{\frac{n}{2n+4}}\right)\\ &= e^{i(m/\nu)s^*/\hbar}\,(\pi(\Phi)_*\psi)(\mathbf{x},t)\,|\mathrm{Vol}(\Phi_*\mathrm{g})|^{\frac{n}{2n+4}}\\ &= e^{i(m/\nu)s^*/\hbar}\,\psi(\mathbf{x}^*,t^*)\,\lambda^{-\frac{n}{4}}|\mathrm{Vol(g)}|^{\frac{n}{2n+4}} \qquad (5.46)\end{aligned}$$

and, using (5.26) together with a slight abuse of notation, we obtain

$$\boxed{\begin{aligned}[\rho(\Phi)\psi]\,(\mathbf{x},t) &= g^{\frac{n}{2}}\,e^{\frac{im}{\nu\hbar}\left(g\langle\mathbf{b},\mathbf{x}\rangle-\frac{g}{2d}\|\mathbf{b}\|^2t+\frac{e}{2d}\|\mathbf{b}\|^2-\langle\mathbf{b},\mathbf{c}\rangle-h\right)}\times\\ &\quad\psi\left(A^{-1}\left[g\,\mathbf{x}-d^{-1}(g\,t-e)\,\mathbf{b}-\mathbf{c}\right],d^{-1}(g\,t-e)\right)\end{aligned}} \qquad (5.47)$$

for all $\Phi=(A,\mathbf{b},\mathbf{c},e,h,\nu)\in\mathrm{SN}(\mathbf{R}^{n+2},\mathrm{g},\xi)$, with d and g as in (5.27).

For the subgroup of dilations generated by $\nu\in\mathbf{R}^*_+$, we find

$$\boxed{[\rho(\nu)]\,\psi(\mathbf{x},t)=\nu^{-\frac{3n}{2(n-4)}}\,\psi\left(\nu^{-\frac{3}{n-4}}\mathbf{x},\nu^{-\frac{n+2}{n-4}}t\right)} \qquad (5.48)$$

which reduces, in the special case $n=3$, to $[\rho(\nu)\psi]\,(\mathbf{x},t)=\nu^{\frac{9}{2}}\,\psi\,(\nu^3\mathbf{x},\nu^5t)$, in total accordance with the conclusion of [28].

• Let us consider now the special case $n=4$, characterized by $\nu=1$ in (5.17), (5.18) and (5.19). We first find the useful expression of $(\mathbf{x}^*,t^*,s^*)=\Phi^{-1}(\mathbf{x},t,s)$, namely

$$\mathbf{x}^* = A^{-1}\left[\frac{\mathbf{x}-\mathbf{b}(gt-e)}{-ft+d}-\mathbf{c}\right] \qquad (5.49)$$

$$t^* = \frac{g\,t-e}{-ft+d} \qquad (5.50)$$

$$s^* = s-\frac{f}{2}\frac{\|\mathbf{x}\|^2}{-ft+d}+\frac{\langle\mathbf{b},\mathbf{x}-\mathbf{b}(gt-e)\rangle}{-ft+d}-\langle\mathbf{b},\mathbf{c}\rangle+\frac{1}{2}\|\mathbf{b}\|^2\frac{gt-e}{-ft+d}-h \qquad (5.51)$$

where, again, $A\in\mathrm{O}(4)$, $(\mathbf{b},\mathbf{c})\in\mathbf{R}^4\times\mathbf{R}^4$, $(d,e,f,g,h)\in\mathbf{R}^5$, and $dg-ef=1$.

Skipping details, and with the same calculation as before, we end up with the following projective representation of the Schrödinger group, namely

$$\boxed{\begin{aligned}[\rho(\Phi)\psi]\,(\mathbf{x},t) &= e^{\frac{im}{\hbar}\left(-\frac{1}{2}\frac{f\|\mathbf{x}\|^2}{-ft+d}+\frac{\langle\mathbf{b},\mathbf{x}-\mathbf{b}(gt-e)\rangle}{-ft+d}-\langle\mathbf{b},\mathbf{c}\rangle+\frac{1}{2}\|\mathbf{b}\|^2\frac{gt-e}{-ft+d}-h\right)}\times\\ &\quad\frac{1}{(-ft+d)^{\frac{n}{2}}}\,\psi\left(A^{-1}\left[\frac{\mathbf{x}-\mathbf{b}(gt-e)}{-ft+d}-\mathbf{c}\right],\frac{gt-e}{-ft+d}\right)\end{aligned}} \qquad (5.52)$$

for all $\Phi=(A,\mathbf{b},\mathbf{c},d,e,f,g,h)\in\mathrm{Sch}(\mathbf{R}^{5,1})$.

6 Conclusion & outlook

We have adopted, in this article, a geometric standpoint enabling us to propose, in terms of Bargmann structures over $(n+1)$-dimensional Newton-Cartan structures, an intrinsic generalization of the Schrödinger-Newton equation.

This allowed for a characterization of the maximal symmetry group of this generalized Schrödinger-Newton equation, which we have named the "Schrödinger-Newton group". The special case of spatially flat structures has been explicitly worked out for purpose of comparison with earlier work on the subject in dimension $n = 3$. In doing so, we point out the special case $n = 4$ where the Schrödinger-Newton group becomes isomorphic to the full Schrödinger group featuring, hence, inversions of the time-axis.

It should be, however, noticed that the presence of the cosmological constant, Λ, in the NC field equations (2.7), viz., $\mathrm{Ric}(\mathrm{g}) = (4\pi G\varrho + \Lambda)\,\theta \otimes \theta$ with ϱ as in (3.5), *breaks* the Schrödinger-Newton group down to the Bargmann group. Indeed one should have $\lambda^{2-\frac{n}{2}}\nu^3 = \lambda^2\nu^2 = 1$ & $n > 0$, hence $\lambda = \nu = 1$.

Let us mention that the maximal symmetries of the Lévy-Leblond-Newton equation can be uncovered using the same geometrical techniques; this is indeed the subject of a companion paper in preparation.

Finally, we would like to underline that there exists a notion of *generalized symmetry* of the Schrödinger equation associated with the infinite-dimensional *Schrödinger-Virasoro* group [32, 45]. This symmetry is being fully geometrized in [22] where the Schrödinger-Newton equation is studied from this perspective; the brand new "Schrödinger-Virasoro-Newton" group is hence naturally introduced as the subgroup of the extended Schrödinger-Virasoro group for which the relations (4.21) and (4.22) hold. This work in progress is an extension of [27] and highlights, in particular, the rôle of the Schrödinger-Virasoro-Newton group in the notable article [41] about the Lie point symmetries of the Schrödinger-Newton equation (1.1).

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A Appendix

A.1 Conformally equivariant quantization

It has been proved in [24, 23] that the Yamabe operator, see (3.3), can be obtained via Conformally Equivariant Quantization (CEQ) of the quadratic Hamiltonian function

$$H = \mathrm{g}^{\alpha\beta}(x) p_\alpha p_\beta \tag{A.1}$$

on the cotangent bundle, T^*M, of a conformally flat pseudo-Riemannian manifold (M, g) of signature (n_+, n_-), with $N = n_+ + n_-$. This quantization scheme establishes, for generic values of weights w, w', a unique isomorphism, $\mathcal{Q}_{w,w'}$, of $\mathrm{O}(n_+ + 1, n_- + 1)$-modules between the space of fiberwise polynomial functions of T^*M and that of differential operators sending w-densities to w'-densities of M. For some "resonant" values of w, w', uniqueness of CEQ is no longer guaranteed. However, imposing that the resulting differential operators be self-adjoint may restore uniqueness of CEQ. For example, in the case of the resonant values $w, w' = 1 - w$ with w as in (3.1), we obtain the self-adjoint Yamabe operator

$$\mathcal{Q}_{\frac{N-2}{2N}, \frac{N+2}{2N}}(H) = -\hbar^2 \Delta_Y(\mathrm{g}) \tag{A.2}$$

which we straightaway extend to any pseudo-Riemannian manifold.

For Bargmann structures, the fundamental vector field ξ (of the $(\mathbf{R}, +)$-principal bundle $\pi : M \to \mathcal{M}$) gives rise to a special Hamiltonian function on T^*M endowed with its canonical 1-form ϖ. Let us denote by $\xi^\sharp$ its canonical lift to T^*M, so that $L_{\xi^\sharp}\varpi = (d\varpi)(\xi^\sharp) + dm = 0$, where $m = \varpi(\xi^\sharp)$, i.e.,

$$m = p_\alpha \xi^\alpha \tag{A.3}$$

is the *momentum mapping* of the Hamiltonian $(\mathbf{R}, +)$-action on the symplectic manifold $(T^*M, d\varpi)$. The function (A.3) is interpreted as the *mass* of the classical system [14]. Then, CEQ applied to first-order fiberwise polynomials readily leads, in our case, to

$$\mathcal{Q}_{\frac{N-2}{2N}, \frac{N-2}{2N}}(m) = \frac{\hbar}{i} L_\xi. \tag{A.4}$$

A.2 Mass dilation

Under a rescaling $\xi \mapsto \widehat{\xi} = \nu\xi$, the mass (A.3) thus transforms as

$$\widehat{m} = \nu\, m. \tag{A.5}$$

Consider now a conformal Bargmann diffeomorphism Φ (verifying (4.1) and (4.2)) and its canonical lift, $\Phi^\sharp$, to T^*M. Straightforward calculation yields then $(\Phi^\sharp)^*m = (\Phi^\sharp)^*(\alpha(\xi^\sharp)) = \alpha((\Phi^\sharp)^*(\xi^\sharp)) = \alpha((\Phi^*\xi)^\sharp)) = \alpha(\nu\xi^\sharp)$, hence

$$(\Phi^\sharp)^*m = \nu\, m. \tag{A.6}$$

Again, this shows that the mass is rigidly dilated by conformal Bargmann transformations.

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