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# TVA on American Options

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## Abstract

This paper presents a full study of the Total Valuation Adjustment (TVA) simulation on American options. It starts from the formulation of the problem under a general BSDE framework that includes the funding issue and the default of both counterparties. It finishes by giving a benchmark Nested Monte Carlo algorithm and discusses an appropriate implementation that provides accurate results within a minute simulation on Graphic Processing Units (GPUs). From a theoretical point of view, this paper can be considered as the extension to American options of the work presented on both papers Crépey (2012a) and Crépey (2012b). Regarding the algorithmic part, our study uses convergence rates developed in Newey (1997) as well as similar ideas to the one presented in Gordy and Juneja (2010) and it goes beyond the square Monte Carlo algorithm detailed in Abbas-Turki et al. (2014) for European options.

**Keywords:** TVA, American options, BSDE, Nested Monte Carlo, GPU.

# 1 Introduction

After the 2007 economic crisis and the new Basel agreements that include the calculation of the CVA (Credit Valuation Adjustment) as an important part of the prudential rules, a large number of papers and books have been published on the CVA and the counterparty risk. For a comprehensive and detailed presentation of the subject, we refer the reader to Brigo et al. (2013) and to Crépey (2012a,b). The first reference is a book that provides an in-depth overview of the subject with a large number of compelling practical examples. When the second is a double paper that presents the mathematical intuition and details of the subject including a hedging framework. We refer also the reader to Brigo et al. (2013) and to Crépey (2012a,b) for other citations that concern the counterparty risk but that are not closely related to our work.

Although there is a large number of practitioner papers on the CVA as well as some important mathematical work that explains the problem, few research has been dedicated to develop a trustable numerical procedure that can be used to perform the computations. The book Cesari et al. (2009) is one of the oldest references that presents the industry practices in computing CVA. Among the research papers, maybe the most devoted to the computing part are: P. Henry-Labordère (2012), Abbas-Turki et al. (2014a) and M. Fujii and A. Takahashi (2015). However, none of these papers develop a procedure that works for TVA for any portfolio of contracts and prove the convergence of the procedure with a reasonable error upper bound. The main reason for this is the mathematical complexity of the problem that makes the computational part very challenging.

Also due to both mathematical and computational complexity, to our knowledge, there is no paper that deals with the TVA problem when American contracts are involved. However, this point is really important especially in markets where American or Bermudan options are widely exchanged like in fixed income and equity markets. Then, the first goal of this paper is to address this lack of theoretical development concerning American options. We not only derive the TVA BSDE on one American option, we extend it on a portfolio of American

contracts.

The second goal of this paper is to propose a robust method to compute the TVA on an exposure of American, European path-dependent and path-independent contracts. Consequently, this method not only provides accurate results when the exposure is explicit in terms of the underlying assets, it also works very well when the exposure must be also simulated. This method is based on a nested Monte Carlo and it is studied in two situations: When the funding constraints can be neglected and when they must be taken into account. For both situations, we express the upper bound of the Mean Square Error (MSE) in terms of the number of simulated trajectories. In particular, this allows us to establish an asymptotic relation between the number of trajectories  $M_0$  simulated in the outer stage, the number of trajectories  $\{M_j\}_{j=1,\dots,N-1}$  simulated in the inner stages and the number of time steps  $N$ .

The asymptotic relation between  $\{M_j\}_{j=1,\dots,N-1}$ ,  $M_0$  and  $N$  is useful to decrease the execution time of the simulation. Indeed, although the implementation is performed on a GPU with a high number of computing units that run the program in parallel, choosing good values of  $M_j$  as a function of  $M_0$  and  $N$  is necessary to perform the computations within a minute of simulation. Thus, it is necessary to point out that the structure of the proposed method, based on nested Monte Carlo, allows us to compute an upper bound for the MSE and subsequently establish the wanted relation between  $\{M_j\}_{j=1,\dots,N-1}$ ,  $M_0$  and  $N$ . Moreover, we will show that the resulting algorithm provides quite accurate values as the variance and the bias of the estimator are sufficiently small.

This paper is divided into four sections: In Section 2, we give the formulation of the TVA BSDE and the pre-default TVA BSDE when the involved contracts are American. In Section 3, we detail the simulation algorithms and we express an upper bound for the MSE according to the simulation parameters. Section 4 explains the implementation and contains some numerical results. Section 5 is dedicated to the proof of theorems 3.1 and 3.2.

## 2 TVA BSDE on American options

To simplify the presentation, we start by considering the TVA on only one American then extend it in Section 2.2 to a more general case.

### 2.1 TVA with an optimal stopping time

We consider two defaultable counterparties: the bank with a default time  $\tau^b$  and the client with a default time  $\tau^c$ . After selling to the client at time 0 an American contract that should generate a cumulative dividend  $D$  till the maturity  $T > 0$ , the bank sets-up a hedging strategy including collateralization and funding portfolio given by a price-and-hedge pair  $(\Pi, \pi)$ . The promised dividend stream  $dD_t$  is effective only if none of the parties defaulted by time  $t$ . We call “funder” of the bank a third party insuring the bank’s strategy funding. Let  $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, T]}, P)$  be a filtrated probability space satisfying the usual conditions. We set  $\mathcal{G}_t = \mathcal{G}_t^{\tau^b, \tau^c} \vee \mathcal{F}_t$  where  $\mathcal{G}_t^{\tau^b, \tau^c} = \sigma(\tau^b \wedge t, \tau^c \wedge t)$  and  $\mathcal{F}$  is generated by the underlying assets  $S$  such that  $D$  is  $\mathcal{F}$ -adapted. We define also  $\bar{\mathcal{G}}_t = \sigma(\tau \wedge t)$  with  $\tau = \tau^b \wedge \tau^c$ . We assume that all the considered processes are  $\mathcal{G}$ -adapted as well as integrable and  $P$  is a risk neutral probability. The price of the TVA contract is computed as the difference between a reference clean price and a price that takes into account the counterparty risk and the funding adjustment. We refer to Crépey (2012a) and Crépey (2012b) for more details on the market conventions and the meaning of a risk neutral probability in this context.

Assuming that the Azéma supermartingale associated with  $\tau$  is positive continuous and non-increasing process, we can admit Lemma 2.1 which will be useful in the sequel.

**Lemma 2.1** *i. For any  $\mathcal{G}$ -measurable random variable  $Y$  and any  $t \geq 0$*

$$E_t(Y \mathbf{1}_{\tau > t}) = \frac{\mathbf{1}_{\tau > t}}{\mathbb{E}_t(\mathbf{1}_{\tau > t})} \mathbb{E}_t(Y \mathbf{1}_{\tau > t}),$$

*where  $E_t$  and  $\mathbb{E}_t$  denote respectively the conditional  $P$ -expectation given  $\mathcal{G}_t$  and  $\mathcal{F}_t$ .*

*ii. An  $\mathcal{F}$ -martingale stopped at  $\tau$  is a  $\bar{\mathcal{G}}$ -martingale and a  $\bar{\mathcal{G}}$ -martingale is a  $\mathcal{G}$  martingale.*

iii. An  $\mathcal{F}$ -adapted càdlàg process cannot jump at  $\tau$ .

The first point of this lemma can be found in Dellacherie-Meyer (1980) and a proof of the other two points is given in Crépey (2012b).

In the case of a European derivative, it is shown in Crépey (2012a) that the pair price-and-hedge satisfies a BSDE on  $[0, \tau \wedge T]$  under the risk neutral assumption. In the case of an American derivative, the situation is exactly the same until the optimal stopping time  $\tau^* \in [0, T]$  after which the contract is exercised and the counterparty risk disappears. Denote  $\bar{\tau} = \tau \wedge \tau^*$  the effective maturity and by analogy with Crépey (2012a) we characterize the bank portfolio as follows.

**Definition 2.1** *We call a price-and-hedge  $(\Pi, \pi)$  the pair, made of a  $\mathcal{G}$ -semimartingale  $\Pi$  and a hedge  $\pi$ , that satisfies the following BSDE on  $[0, \bar{\tau}]$ :*

$$d\Pi_t + \mathbf{1}_{t < \tau} \mathbf{1}_{t \leq \tau^*} dD_t - (r_t \Pi_t + g_t(\Pi_t, \pi_t)) dt = dm_t^\pi$$

*with the final condition  $\Pi_{\bar{\tau}} = \mathbf{1}_{\{\tau = \bar{\tau}\}} R$*

where  $m^\pi$  is a  $\mathcal{G}$ -martingale null at time 0,  $g$  is  $\mathcal{F}$ -progressively measurable function and  $R$  is a  $\mathcal{G}_{\bar{\tau}}$ -measurable recovery.

Notice that, since the  $\mathcal{F}$ -adapted process  $D$  does not jump at time  $\tau$  from Lemma 2.1 and by discounting, the previous BSDE becomes

$$(2.1) \quad d\beta_t \Pi_t + \beta_t dD_t - \beta_t g_t(\Pi_t, \pi_t) dt = \beta_t dm_t^\pi, \quad \forall t \in [0, \bar{\tau}].$$

where  $\beta = e^{-\int_0^\cdot r_s ds}$  is the risk-free discounting asset with an interest rate  $r$  generally considered as the overnight indexed swap rate. In an integral form, the last BSDE is equivalent to the following

$$(2.2) \quad \beta_t \Pi_t = E_t \left( \int_t^{\bar{\tau}} \beta_s dD_s - \int_t^{\bar{\tau}} \beta_s g_s(\Pi_s, \pi_s) ds + \beta_{\bar{\tau}} \mathbf{1}_{\{\tau = \bar{\tau}\}} R \right), \quad \forall t \in [0, \bar{\tau}].$$

When ignoring counterparty risk and assuming a risk-free funding rate, we define  $p^a$  as the discounted cumulative clean price of an American contract with an optimal stopping time  $\tau^*$ . Under the risk neutral measure  $P$ , it is known that  $(\beta_{t \wedge \tau^*} p_{t \wedge \tau^*}^a)_{0 \leq t \leq T}$  is an  $\mathcal{F}$ -martingale and  $p^a$  is given by

$$\begin{aligned} \beta_t p_t^a &= \mathbb{E}_t \left( \int_0^{\tau^*} \beta_s dD_s \right) \\ &= \int_0^t \beta_s dD_s + \mathbb{E}_t \left( \int_t^{\tau^*} \beta_s dD_s \right) \\ &=: \int_0^t \beta_s dD_s + \beta_t P_t^a, \quad \forall t \in [0, \tau^*], \end{aligned}$$

where  $P^a$  represents the price of the future cash flows of the contract which we call the clean price like in Crépey (2012b). Remark that the dividend  $D$  can be either seen as a native swap or as a virtual swap via a repo market. In the examples considered in this paper, we simplify  $D$  and make it only meaningful at  $\tau^*$ . For instance, if the bank sells a put option on an asset  $S$  with a strike  $K$  then  $D_s = (K - S_{\tau^*})_+ 1_{s \geq \tau^*}$  and thus  $P^a$  would be given by  $\beta_t P_t^a = \mathbb{E}_t (\beta_{\tau^*} (D_{\tau^*} - D_{\tau^*-})) = \mathbb{E}_t (\beta_{\tau^*} (K - S_{\tau^*})_+)$  for each  $t \in [0, \tau^*]$ .

By substituting the time  $t$  by  $\tau \wedge \tau^* = \bar{\tau}$  in the last equality and conditioning with respect to  $\mathcal{G}_t$  we get

$$E_t (\beta_{\bar{\tau}} p_{\bar{\tau}}^a) = E_t \left( \int_0^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} P_{\bar{\tau}}^a \right), \quad \forall t \in [0, \bar{\tau}].$$

Using Lemma 2.1, the  $\mathcal{F}$ -martingale  $(\beta_{t \wedge \tau^*} p_{t \wedge \tau^*}^a)_{0 \leq t \leq T}$  stopped at  $\tau$  is a  $\mathcal{G}$ -martingale, we then deduce the following representation of  $P^a$

$$(2.3) \quad \beta_t P_t^a = E_t \left( \int_t^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} P_{\bar{\tau}}^a \right), \quad \forall t \in [0, \bar{\tau}].$$

The clean price can be seen also as the solution of the following BSDE

$$(2.4) \quad d\beta_t P_t^a + \beta_t dD_t = \beta_t dm_t^a, \quad \forall t \in [0, \bar{\tau}], \quad P_{\bar{\tau}}^a = 1_{\bar{\tau}=\tau} P_{\tau}^a$$

where, under usual assumptions on  $r$ ,  $m^a$  is the  $\mathcal{G}$ -martingale null at time 0 defined by  $m_t^a = \int_0^{t \wedge \bar{\tau}} \beta_s^{-1} d(\beta_s p_s^a)$  for  $t \in [0, T]$ .

The price of the TVA contract is defined by the process  $\Theta = P^a - \Pi$ . From (2.2) and (2.3), we deduce the integral form for the TVA

$$(2.5) \quad \beta_t \Theta_t = E_t \left( \beta_{\bar{\tau}} \Theta_{\bar{\tau}} + \int_t^{\bar{\tau}} \beta_s g_s (P_s^a - \Theta_s, \pi_s) ds \right), \quad \forall t \in [0, \bar{\tau}],$$

where  $\Theta_{\bar{\tau}} = P_{\bar{\tau}}^a - \mathbf{1}_{\{\tau = \bar{\tau}\}} R$

Moreover, the TVA can be also seen as the solution of the following BSDE that combines both (2.1) and (2.4)

$$(2.6) \quad d\beta_t \Theta_t + \beta_t g_t (P_t^a - \Theta_t, \pi_t) dt = \beta_t dm_t, \quad \forall t \in [0, \bar{\tau}],$$

with  $\Theta_{\bar{\tau}} = P_{\bar{\tau}}^a - \mathbf{1}_{\{\tau = \bar{\tau}\}} R$ ,

where  $m$  is the  $\mathcal{G}$ -martingale null at time 0 defined by

$$(2.7) \quad m = m^a - m^\pi.$$

We assume now that the Azéma supermartingale of  $\tau$  is time differentiable and we denote  $\gamma_t = -\frac{d \ln(G_t)}{dt}$  the hazard intensity and  $\alpha_t = e^{-\int_0^t \gamma_s ds}$ . Denote  $\xi_t := P_t^a - E_t(R \mathbf{1}_{\{t = \bar{\tau}\}})$ , and  $\tilde{\xi}_t := \frac{1}{P(\tau > t | \mathcal{F}_t)} \mathbb{E}_t(\xi_t \mathbf{1}_{\{\tau > t\}})$ . Notice that  $\xi_\tau = \Theta_{\bar{\tau}}$  and using Lemma 2.1 one can verify that  $\tilde{\xi}_t \mathbf{1}_{\{\tau > t\}} = \xi_t \mathbf{1}_{\{\tau > t\}}$ . Performing a filtration reduction, we introduce as in Crépey (2012b), a new process called the pre-default TVA.

**Definition 2.2** *We call the pre-default TVA the solution of the following  $\mathcal{F}$ -BSDE.*

$$(2.8) \quad d\tilde{\beta}_t \tilde{\Theta}_t + \tilde{\beta}_t \tilde{g}_t (P_t^a - \tilde{\Theta}_t, \pi_t) dt = \tilde{\beta}_t d\tilde{m}_t, \quad \forall t \in [0, \tau^*], \quad \text{with } \tilde{\Theta}_{\tau^*} = 0,$$

where  $\tilde{\beta} = \alpha \beta$ ,  $\tilde{m}$  is an  $\mathcal{F}$ -martingale and  $\tilde{g}$  is the  $\mathcal{F}$ -progressively measurable function defined



by

$$\tilde{g}_t(P_t^a - \tilde{\Theta}_t, \pi_t) = g_t(P_t^a - \tilde{\Theta}_t, \pi_t) + \gamma_t \tilde{\xi}_t, \quad \forall t \in [0, \tau^*],$$

In an integral form, the pre-default TVA is given by

$$(2.9) \quad \tilde{\beta}_t \tilde{\Theta}_t = \mathbb{E}_t \left( \int_t^{\tau^*} \tilde{\beta}_s \tilde{g}_s(P_s^a - \tilde{\Theta}_s, \pi_s) + \gamma_t \tilde{\xi}_s(P_s^a - \tilde{\Theta}_s, \pi_s) ds \right), \quad \forall t \in [0, \tau^*].$$

Let  $M$  be the  $\bar{\mathcal{G}}$ -martingale defined by  $M_t := \mathbf{1}_{\{\tau > t\}} + \int_0^{t \wedge \tau} \gamma_s ds$  and assume that the locale  $\mathcal{G}$ -martingale  $\int_0^{\cdot \wedge \bar{\tau}} (\tilde{\Theta}_s - \xi_s) dM_s$  is a martingale.

**Proposition 2.1** *Let the pair price-and-hedge  $(\Pi, \pi)$  be as in Definition 2.1 with a  $\mathcal{G}$ -martingale component  $m^\pi$  defined by  $m^\pi := m_{\cdot, \wedge \bar{\tau}}^a - \tilde{m}_{\cdot, \wedge \bar{\tau}} - \int_0^{\cdot \wedge \bar{\tau}} (\tilde{\Theta}_s - \xi_s) dM_s$ . Then, the TVA price is given by  $\Theta = \tilde{\Theta}J + (1 - J)\xi_\tau$ , where  $J_t = \mathbf{1}_{\{\tau > t\}}$ .*

**Proof.** Let  $\bar{\Theta} =: \tilde{\Theta}J + (1 - J)\xi_\tau$ . Using (2.8) and (2.7), we have for  $t \in [0, \bar{\tau}]$

$$\begin{aligned} d\beta_t \bar{\Theta}_t &= dJ_t \beta_t \tilde{\Theta}_t + d(1 - J) \beta_t \xi_\tau = d\beta_{t \wedge \tau} \tilde{\Theta}_{t \wedge \tau} + \beta_t \tilde{\Theta}_t dJ_t - \beta_t \xi_t dJ_t \\ &= \frac{1}{\alpha_t} d\tilde{\beta}_t \tilde{\Theta}_t + \gamma_t \beta_t \tilde{\Theta}_t dt + \beta_t (\tilde{\Theta}_t - \xi_t) dJ_t \\ &= -\beta_t (g_t(P_t^a - \tilde{\Theta}_t, \pi_t) + \gamma_t \tilde{\xi}_t) dt + \beta_t d\tilde{m}_t + \beta_t [(\tilde{\Theta}_t - \xi_t) dJ_t + \gamma_t \tilde{\Theta}_t dt] \\ &= -\beta_t g_t(P_t^a - \tilde{\Theta}_t, \pi_t) dt + \beta_t d\tilde{m}_t + \beta_t [(\tilde{\Theta}_t - \xi_t) dJ_t + \gamma_t (\tilde{\Theta}_t - \tilde{\xi}_t) dt] \\ &= -\beta_t g_t(P_t^a - \tilde{\Theta}_t, \pi_t) dt + \beta_t (d\tilde{m}_t + (\tilde{\Theta}_t - \xi_t) dM_t) \\ &= -\beta_t g_t(P_t^a - \tilde{\Theta}_t, \pi_t) dt + \beta_t (dm_t^a - dm_t^\pi) \\ &= -\beta_t g_t(P_t^a - \tilde{\Theta}_t, \pi_t) dt + \beta_t dm_t. \end{aligned}$$

Then,  $\bar{\Theta}$  satisfies the BSDE as  $\Theta$  with the same limit condition, this ends the proof.

## 2.2 Multiple optimal stopping times

Let us consider now  $n$  American contracts with different maturities  $T_i > 0$ , different optimal stopping times  $\tau_i^* \in [0, T_i]$  and different dividend streams  $dD_t^i$ . In that case, the counterparty risk vanishes after  $\tau^* = \max_{0 \leq i \leq n} (\tau_i^*)$  and the effective maturity time becomes  $\bar{\tau} = \tau \wedge \tau^*$ . From (2.3), the clean price  $P^{a,i}$  of each contract is given by

$$\beta_t P_t^{a,i} = E_t \left( \int_t^{\bar{\tau}^i} \beta_s dD_s^i + \beta_{\bar{\tau}^i} P_{\bar{\tau}^i}^{a,i} \right), \quad \forall t \in [0, \bar{\tau}^i]$$

or equivalently  $\beta_{t \wedge \bar{\tau}^i} P_{t \wedge \bar{\tau}^i}^{a,i} = E_t \left( \int_t^{\bar{\tau}} \beta_s dD_{s \wedge \bar{\tau}^i}^i + \beta_{\bar{\tau}^i} P_{\bar{\tau}^i}^{a,i} \right), \quad \forall t \in [0, \bar{\tau}]$ ,

where  $\bar{\tau}^i = \tau \wedge \tau_i^*$ . We define the overall clean price of all contracts at time  $t \in [0, \bar{\tau}]$  by

$$P_t^a =: \frac{1}{\beta_t} \sum_{i=1}^n \beta_{t \wedge \bar{\tau}^i} P_{t \wedge \bar{\tau}^i}^{a,i}.$$

This yields to the following BSDE for  $P^a$

$$\begin{aligned} \beta_t P_t^a &= E_t \left( \int_t^{\bar{\tau}} \beta_s dD_s + \sum_{i=1}^n \beta_{\bar{\tau}^i} P_{\bar{\tau}^i}^{a,i} \right) \\ &= E_t \left( \int_t^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} P_{\bar{\tau}}^a \right), \quad \forall t \in [0, \bar{\tau}], \end{aligned}$$

where  $D$  is the global dividend stream defined by  $dD_t := \sum_{i=1}^n dD_{t \wedge \bar{\tau}^i}^i$ . By analogy to the previous section, the price-and-hedge pair  $(\Pi, \pi)$  satisfies a BSDE given, in an integral form, by

$$\beta_t \Pi_t = E_t \left( \int_t^{\bar{\tau}} \beta_s dD_s - \int_t^{\bar{\tau}} \beta_s g_s(\Pi_s, \pi_s) ds + \beta_{\bar{\tau}} \mathbf{1}_{\{\tau = \bar{\tau}\}} R \right), \quad \forall t \in [0, \bar{\tau}].$$

Thus, we deduce the integral form of the TVA BSDE

$$\beta_t \Theta_t = E_t \left( \beta_{\bar{\tau}} \Theta_{\bar{\tau}} + \int_t^{\bar{\tau}} \beta_s g_s(P_s^a - \Theta_s, \pi_s) ds \right), \quad \forall t \in [0, \bar{\tau}],$$

where  $\Theta_{\bar{\tau}} = P_{\bar{\tau}}^a + \mathbf{1}_{\{\tau = \bar{\tau}\}} R$ .

Consequently, Proposition 2.1 remains true in this setup with

$$m_t^a = \sum_{i=1}^n m_t^{a,i} = \sum_{i=1}^n \int_0^{t \wedge \bar{\tau}^i} \beta_s^{-1} d\beta_s p_s^{a,i}, \text{ where } \beta_t p_t^{a,i} = \mathbb{E}_t \left( \int_0^{\tau_i^*} \beta_s dD_s^i \right).$$

### 3 Benchmark simulation using nested Monte Carlo

We present here an overview of the overall algorithm. We also discuss the differences between Section 3.2 where we consider the funding issues and Section 3.1 where we study essentially an extension of the square Monte Carlo explained in Abbas-Turki et al. (2014). In the following, depending on the context, we will use  $P_t$  either for the clean exposure of a portfolio or for the clean exposure of only one contract.

Since the paper by Brigo and Pallavicini (2008), the (CVA) Credit Valuation Adjustment can be viewed as an option on the clean exposure called Contingent Credit Default Swap (CCDS). When the clean exposure  $P_t$  is computed on a basket of contracts that are priced by closed expressions, the CVA and, more generally, the TVA can be calculated thanks to one stage simulation using either PDE discretization or Monte Carlo as in Crépey et al. (2014). However, when the underlying contracts have to be simulated as in the case of American options, it is more reasonable to perform a two stage simulation: The outer stage for the TVA and the inner stages to compute the underlying contracts. Indeed, the TVA can be considered as a corrective value on  $P_t$  and mispricing the latter could produce important errors on the former. Thus, using one stage simulation with the same set of trajectories for both TVA and the clean exposure would be a bad idea when implementing global methods like regressions or when the default is strongly dependent on the exposure  $P_t$ .

Our purpose is to develop a method that can be considered as a benchmark not only when the exposure involves American options but also European options that are not expressed by closed formulas. Although the proposed two stage Monte Carlo is quite heavy to implement on a CPU, it will run much faster on a GPU. Moreover, in order to decrease the execution

time and run the simulation within a minute, we propose in sections 3.1 and 3.2 a judicious procedure to choose the appropriate number of trajectories that have to be drawn in the inner stages.

For both sections 3.1 and 3.2, we fix the number of time steps used for the SDE discretization of the underlying asset  $S = (S^1, \dots, S^d)$  to be a multiple of  $N$ :  $N_{sde} = q * N$ . Using  $N_{sde}$  time steps, we simulate  $M_0$  outer stage trajectories from  $t_0 = 0$  to  $t_{N_{sde}} = T$ . From each value  $S_{t_{q*k}}$ , with  $k \in \{1, \dots, N\}$ , we simulate  $M_k$  inner stage trajectories that ends at  $t_{N_{sde}} = T$ .

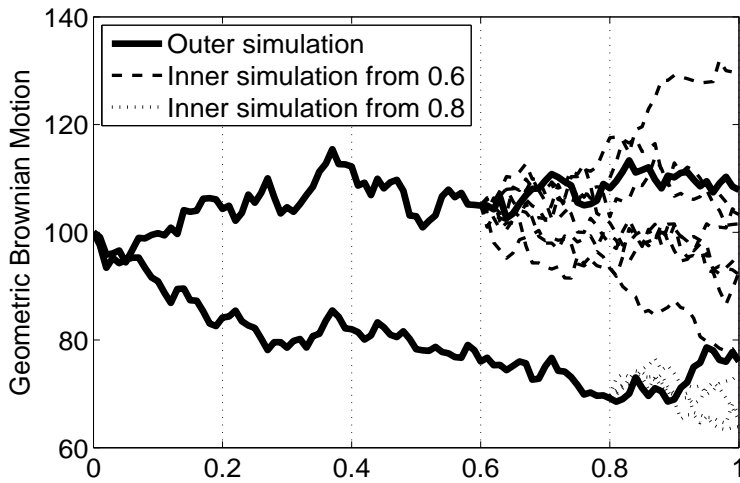


Figure 1: An example of a two stage simulation with  $M_0 = 2$ ,  $M_6 = 8$  and  $M_8 = 4$ .

In Figure 1, we illustrate only two inner simulations starting at different times. Because the inner simulations are used to compute the clean exposure  $P_t$ , it is conceivable to draw fewer trajectories when  $t$  approaches  $T$ . Indeed, as  $t \rightarrow T$  the simulation variance  $V_t$  associated to  $P_t$  decreases to 0. In addition,  $V_t$  produces a bias on the outer simulation of the TVA which gets smaller as  $V_t \rightarrow 0$ . We point out also that the bias of the estimator of  $P_t$  vanishes when the exposure involves only European contracts. For American contracts, the bias produced by Longstaff-Schwartz algorithm is generally small and will be neglected. We refer the reader to Glasserman (2003) for more details on the bias of American options estimators.

For fixed and sufficiently big values of  $M_0$  and  $N$ , our purpose is to study the effect of  $V_t$  on the bias of the TVA estimator and thus on the choice of  $M_1 \dots M_{N-1}$ . We will do it in Section 3.1 for

$$(3.1) \quad \text{CVA}_{0,T} = \sum_{k=0}^{N-1} E \left( P_{t_{q^*(k+1)}}^+ 1_{\tau \in (t_{q^*k}, t_{q^*(k+1)}]} \right)$$

when we neglect the funding constraints and in Section 3.2 for

$$(3.2) \quad \Theta_{t_{q^*k}} = E_{t_{q^*k}} \left( \Theta_{t_{q^*(k+1)}} + hg(t_{q^*(k+1)}, P_{t_{q^*(k+1)}}, \Theta_{t_{q^*(k+1)}}) \right), \quad \Theta_{t_{q^*N}} = 0$$

where  $\Theta_t$  is the pre-default TVA process and  $h = T/N$ . In this section,  $P$  replaces  $P^a$  used in the previous section,  $\text{CVA}_{0,T}$  represents  $\Theta_0$  given in (2.5) when we set  $g$  and  $R$  to zero and the  $\Theta$  of (3.2) is used for the pre-default TVA  $\tilde{\Theta}_t$  introduced in (2.8). Also to simplify notations, we assume that  $T = 1$  then  $h = 1/N$ . In standard applications when  $T \neq 1$ , one should increase or decrease linearly the value of  $N$  depending on whether  $T > 1$  or  $T < 1$ .

Like for (3.2), we consider an intensity model for the default time  $\tau$  of equation (3.1) and we assume that its hazard rate is a function of the exposition  $P_t$ . When compared to structural models or to intensity models with a hazard rate expressed as a function of the underlying assets, assuming that the hazard rate depends on  $P_t$  is more difficult to study as one must take into account the effect of  $V_t$  on simulating  $\tau$ . Admitting that we are dealing with this complex setting, one can extend the obtained results bellow to simpler situations.

Unlike (3.1), (3.2) involves the computation of some conditional expectations during a backward induction. Consequently, although both (3.1) and (3.2) are based on an intensity model and on a two stage simulation like the one illustrated in Figure 1, they are implemented differently. Below, we detail the implementation of each expression and calculate asymptotically its MSE. In the following, we replace the  $t_{q^*k}$  index by  $k$  and we denote by  $\hat{P}_1, \dots, \hat{P}_{N-1}$  the simulated expositions using the inner trajectories. Then, we define  $\Delta_k^P = \sqrt{M_k} (\hat{P}_k - P_k)$ ,  $\Delta_k^\Theta = (\hat{\Theta}_k - \Theta_k)$  where  $\hat{\Theta}$  is the simulated value of  $\Theta$  and

we make the following assumption that is an extension of Assumption 1 in Gordy & Juneja (2010).

**Assumption 3.1** *Defining  $\varphi_{M_0, \dots, M_{N-1}}(p_1, \dots, p_{N-1}, \theta_1, \dots, \theta_{N-1}, \delta_1^p, \dots, \delta_{N-1}^p, \delta_1^\theta, \dots, \delta_{N-1}^\theta)$  as the density of the random vector  $(P_1, \dots, P_{N-1}, \Theta_1, \dots, \Theta_{N-1}, \Delta_1^P, \dots, \Delta_{N-1}^P, \Delta_1^\Theta, \dots, \Delta_{N-1}^\Theta)$  with respect to the Lebesgue measure, we assume that its partial derivatives*

$$\partial_{u_k} \varphi_{M_0, \dots, M_{N-1}}, \quad \partial_{u_k, u_l}^2 \varphi_{M_0, \dots, M_{N-1}}, \quad u_k = p_k \text{ or } \theta_k \text{ and } u_l = p_l \text{ or } \theta_l \text{ with } k, l = 1, \dots, N-1$$

*exist and are continuous for each  $(M_0, \dots, M_{N-1})$ . With  $u_k, u_l$  as before and  $u_i = p_i$  or  $\theta_i$ ,*

$$\text{for } i = 1, \dots, N-1 \quad \lim_{|u_i| \rightarrow \infty} \varphi_{M_0, \dots, M_{N-1}} = 0, \quad \lim_{|u_i| \rightarrow \infty} \partial_{u_k} \varphi_{M_0, \dots, M_{N-1}} = 0, \quad \lim_{|u_i| \rightarrow \infty} \partial_{u_k, u_l}^2 \varphi_{M_0, \dots, M_{N-1}} = 0$$

*uniformly on all the variables except  $u_i$  and uniformly on  $M_0, \dots, M_{N-1}$ . Moreover, for each  $(M_0, \dots, M_{N-1})$ , there exist nonnegative functions  $\varphi_{M_0, \dots, M_{N-1}}^0, \varphi_{M_0, \dots, M_{N-1}}^1$  and  $\varphi_{M_0, \dots, M_{N-1}}^2$*

$$\text{such that } \varphi_{M_0, \dots, M_{N-1}} \leq \varphi_{M_0, \dots, M_{N-1}}^0(\delta_1^p, \dots, \delta_{N-1}^p, \delta_1^\theta, \dots, \delta_{N-1}^\theta),$$

$$|\partial_{u_k} \varphi_{M_0, \dots, M_{N-1}}| \leq \varphi_{M_0, \dots, M_{N-1}}^1(\delta_1^p, \dots, \delta_{N-1}^p, \delta_1^\theta, \dots, \delta_{N-1}^\theta),$$

$$|\partial_{u_k, u_l}^2 \varphi_{M_0, \dots, M_{N-1}}| \leq \varphi_{M_0, \dots, M_{N-1}}^2(\delta_1^p, \dots, \delta_{N-1}^p, \delta_1^\theta, \dots, \delta_{N-1}^\theta)$$

*for all  $(p_1, \dots, p_{N-1}, \theta_1, \dots, \theta_{N-1}, \delta_1^p, \dots, \delta_{N-1}^p, \delta_1^\theta, \dots, \delta_{N-1}^\theta)$  with*

$$\sup_{M_0, \dots, M_{N-1}} \int_{\mathbb{R}^{2N-2}} |\delta_k^p|^{r_1} |\delta_l^\theta|^{r_2} \varphi_{M_0, \dots, M_{N-1}}^i(\delta_1^p, \dots, \delta_{N-1}^p, \delta_1^\theta, \dots, \delta_{N-1}^\theta) d\delta_1^p \dots d\delta_{N-1}^p d\delta_1^\theta \dots d\delta_{N-1}^\theta < \infty$$

*for  $k, l = 1, \dots, N-1, i = 0, 1, 2, r_1 \geq 0, r_2 \geq 0$  and  $0 \leq r_1 + r_2 \leq 4$ .*

This assumption is needed to justify the Taylor expansion performed in sections 3.1, 3.2 and ensures that one can ignore the higher order terms. In what follows, we assume also that the underlying asset  $S$  is a truncation of a positive Lévy process. The truncation should be performed such that the support of  $S$  is a Cartesian product of compact connected intervals on which the density of  $S$  is bounded away from zero.

### 3.1 TVA without funding constraint

We present an optimized version of the square Monte Carlo simulation MC2 taken as a benchmark algorithm in Abbas-Turki et al. (2014). Because MC2 was not the main subject of the latter paper, the authors implemented a simple version of MC2 with  $M_0 = M_1 = \dots = M_{N-1}$ . However, here we would like to express  $M_1, \dots, M_{N-1}$  as functions of  $(M_0, N)$  that have to be sufficiently big. The other difference with the MC2 presented in Abbas-Turki et al. (2014) is the possibility here to simulate American options using the inner trajectories. This will be performed thanks to  $N * M_0$  local dynamic programming inductions that are explained at the end of this subsection and its implementation is detailed in Section 4.

We introduce new functions  $F_{k+1}^1(x_1, \dots, x_{k+1})$ ,  $F_{k+1}^2(x_1, \dots, x_{k+1})$ ,  $F_{k+1}^3(x_1, \dots, x_{k+1})$  and  $F_{k+1}^4(x_1, \dots, x_l, y)$  with

$$(3.3) \quad \left\{ \begin{array}{l} F_{k+1}^1(x_1, \dots, x_{k+1}) = E \left( 1_{\tau \in (k*h, (k+1)*h]} | P_1 = x_1, \dots, P_{k+1} = x_{k+1} \right), \\ F_{k+1}^2(x_1, \dots, x_{k+1}) = (x_{k+1})^+ F_{k+1}^1(x_1, \dots, x_{k+1}), \\ F_{k+1}^3(x_1, \dots, x_{k+1}) = (x_{k+1})^+ E \left( 1_{\tau > (k+1)*h} | P_1 = x_1, \dots, P_{k+1} = x_{k+1} \right) \text{ and} \\ F_{k+1}^4(x_1, \dots, x_j, y) = E \left( F_{k+1}^2(x_1, \dots, x_j, P_{j+1}, \dots, P_{k+1}) | S_j = y \right). \end{array} \right.$$

Thus, the simulated value  $\widehat{\text{CVA}}_{0,T}$  of (3.1) is given by

$$(3.4) \quad \widehat{\text{CVA}}_{0,T} = \sum_{k=0}^{N-1} \frac{1}{M_0} \sum_{i=1}^{M_0} F_{k+1}^2 \left( \widehat{P}_1(S_1^i), \dots, \widehat{P}_{k+1}(S_{k+1}^i) \right)$$

where  $\{S^i\}_{i \in \{1, \dots, M_0\}}$  are independent copies of the underlying asset  $S$  that are generated in the outer simulation.

As said previously, we take the hazard rate of  $\tau$  to be a function of the exposition  $P_t$ . This function is assumed constant by parts and can be decomposed using a family  $\{f_k\}_{1 \leq k \leq N}$  of

twice differentiable functions such that

$$(3.5) \quad P(\tau > k * h | P_1 = x_1, \dots, P_k = x_k) = \exp\left(-\frac{1}{N} \sum_{i=1}^k f_i(x_i)\right).$$

Before announcing the main theorem of this section, we should consider an additional constraint on  $M_1, \dots, M_{N-1}$  that makes possible dealing with American contracts using regression methods. Indeed, an approximation of an American contract by a Bermudan leads to

$$(3.6) \quad P_k(x) = \sup_{\theta \in \mathcal{T}_{k,N}} \mathbb{E}(\Phi_{k,\theta}(S_\theta) | S_k = x),$$

where  $\Phi_{s,t}(x) = \beta_t \Upsilon(x) / \beta_s$  with  $\Upsilon$  is the payoff and  $\beta$  is the risk-free discounting asset. Besides,  $\mathcal{T}_{k,N}$  represents the set of stopping times that take their values in  $\{k, \dots, N\}$ . The computation of (3.6) is performed using the Longstaff-Schwartz algorithm introduced in Longstaff and Schwartz (2001) and well detailed in Clément & al. (2002). We set  $P_k(x) = \mathbb{E}(\Phi_{k,\tau_k}(S_{\tau_k}) | S_k = x)$  with

$$(3.7) \quad \begin{aligned} \tau_N &= N, \\ \forall k \in \{N-1, \dots, 0\}, \quad \tau_k &= k1_{A_k} + \tau_{k+1}1_{A_k^c}, \end{aligned}$$

where  $A_k = \left\{ \Phi_{k+1,k}(S_k) > E\left(P_{k+1}(S_{k+1}) | S_k\right) \right\}$ . The conditional expectation involved in  $A_k$  is approximated using a regression on a basis of monomial functions where  $K_k$  is its cardinal. This regression uses the inner trajectories and thus allows us to approximate  $P_k(x)$  by

$$(3.8) \quad \widehat{P}_k(x) = \frac{1}{M_k} \sum_{i=1}^{M_k} \Phi_{k,\widehat{\tau}_k^i} \left( S_{\widehat{\tau}_k^i}^i | S_k^i = x \right)$$

and the dependence on the inner trajectories can be seen from fixing  $S_k^i$  to be equal to  $x$ . In (3.8),  $\widehat{\tau}_k^i$  is simulated thanks to a similar induction to (3.7) in which we replace the



conditional expectation involved in  $A_k$  by a regression.

Then, we are able to announce the following result.

**Theorem 3.1** *As long as Assumption 3.1 is fulfilled and  $K_j^3/M_j \rightarrow 0$  for each  $j$  when American contracts are involved, we get*

$$\begin{aligned}
MSE\left(\widehat{CVA}_{0,T} - CVA_{0,T}\right) &= E\left[\left(\widehat{CVA}_{0,T} - CVA_{0,T}\right)^2\right] \\
&\leq \frac{N^2}{M_0} \max_{k \in \{0, \dots, N-1\}} \text{Var}\left(F_{k+1}^2\left(\widehat{P}_1(S_1^i), \dots, \widehat{P}_{k+1}(S_{k+1}^i)\right)\right) \\
&\quad + \sum_{j=1}^N \frac{1}{4NM_j^2} \left(E\left[V_j(S_j^i) f_j''(P_j(S_j^i)) F_j^3(P_1(S_1^i), \dots, P_j(S_j^i))\right]\right)^2 \\
&\quad + \sum_{j=1}^N \frac{1}{4NM_j^2} \left(E\left[V_j(S_j^i) f_j''(P_j(S_j^i)) \sum_{k=j}^{N-1} F_{k+1}^4(P_1(S_1^i), \dots, P_j(S_j^i), S_j^i)\right]\right)^2 \\
&\quad + \sum_{j=1}^N \frac{N}{4M_j^2} \left(E\left[V_j(S_j^i) F_j^1(P_1(S_1^i), \dots, P_j(S_j^i)) | P_j(S_j^i) = 0\right] \varphi_j(0)\right)^2 \\
&\quad + N \sum_{j=1}^N (N-j+1)^2 O\left(\frac{1}{M_j^4}\right)
\end{aligned}$$

where  $\varphi_j$  is the density of  $P_j(S_j^i)$  and  $V_j(x) = \text{Var}\left(\sqrt{M_j}\left(\widehat{P}_j(x) - P_j(x)\right)\right)$ .

Using this result, it is natural to take  $M_j \sim \sqrt{M_0}/N$  when  $\varphi_j(0)$  vanishes or generally when  $\varphi_j(0)$  is small enough, otherwise, it is sufficient to take  $M_j \sim \sqrt{M_0}$ . However, we point out that with both choices, it is necessary to make sure that  $N$  is not too big in order to control the variance term in the previous inequality. Moreover, when American options are involved,  $K_j^3/M_j$  must be small enough.

Theorem 3.1 allows to establish a condition on the value of  $M_j$  for all  $j \in \{1, \dots, N-1\}$ . However, it is interesting to see how  $M_j$  should decrease with respect to  $j$  as  $V_j(x)$  also decreases. It is easy to express a relationship between  $M_j$  and  $M_{j+1}$  when the exposition involves only European contracts and the underlying asset  $S$  is a truncation of a log-Normal process. In this case and assuming a sufficient regularity on the function  $\Phi_{0,N}$ ,  $V_j(x) =$

$\text{Var} \left( \Phi_{0,N} \left( \sqrt{(N-j)/N} G \right) \right) \approx (N-j)/N \Phi'_{0,N}(0) + O((N-j)^2/N^2)$  where  $G$  is a truncation of a standard Gaussian variable. Thus, one can set

$$(3.9) \quad M_j = \frac{N-j}{N-1} M_1 \text{ with either } M_1 = \frac{\sqrt{M_0}}{N} \text{ or } M_1 = \sqrt{M_0}.$$

When American options are involved, we will see that the choice (3.9) is good numerically as long as  $M_0$  is big enough. Indeed, this ensures that the value of  $M_{N-1}$ , which is the smallest among all the  $M_{j>0}$ , makes  $K_{N-1}^3/M_{N-1}$  small enough.

Even though we presented only asymptotic choices of  $M_j$ , we will make them quantitative in Section 4. In particular, we will see that  $K_j^3/M_j \leq 1$  is quite sufficient because of the induction (3.7) robustness that is used for the dynamic programming.

### 3.2 Pre-default TVA BSDE

As said previously, although both the approximation of (3.1) and the approximation of (3.2) are based on an outer stage and on an inner stage simulation, the conditional expectation involved in (3.2) requires a more advanced implementation. In particular, we need to simulate two independent sets  $\{S^i\}_{i \in \{1, \dots, M_0\}}$  and  $\{\tilde{S}^i\}_{i \in \{1, \dots, M_0\}}$  of the underlying asset  $S$ . Where  $\{S^i\}_{i \in \{1, \dots, M_0\}}$  are used in the outer simulation and  $\{\tilde{S}^i\}_{i \in \{1, \dots, M_0\}}$  are used to compute the regression matrix  $\Psi_k$  that is given for each  $k$  by

$$(3.10) \quad \Psi_k = \mathfrak{T} \left( \frac{1}{M_0} \sum_{i=0}^{M_0} \psi(\tilde{S}_k^i)^t \psi(\tilde{S}_k^i) \right)$$

and  $\psi$  is a basis of monomial functions where  $K$  is its cardinal.  $\mathfrak{T}$  is an operator that must satisfy assumption 3.2. Then, the simulated value  $\hat{\Theta}_k(S_k^i)$  of (3.2) is defined thanks to the

following induction

$$(3.11) \left\{ \begin{array}{l} \text{For } k = 1, \dots, N - 1 \\ \widehat{\Theta}_k(x) = {}^t\psi(x)\Psi_k^{-1} \left[ \frac{1}{M_0} \sum_{j=1}^{M_0} \psi(S_k^j) \left( \widehat{\Theta}_{k+1}(S_{k+1}^j) + \frac{1}{N} g\left(k+1, \widehat{\Theta}_{k+1}(S_{k+1}^j), \widehat{P}_{k+1}(S_{k+1}^j)\right) \right) \right] \\ \text{and } \widehat{\Theta}_N(x) = 0, \quad \widehat{\Theta}_0(S_0) = \frac{1}{M_0} \sum_{j=1}^{M_0} \left( \widehat{\Theta}_1(S_1^j) + \frac{1}{N} g\left(1, \widehat{\Theta}_1(S_1^j), \widehat{P}_1(S_1^j)\right) \right). \end{array} \right.$$

The proof of Theorem 3.2 involves an intermediary random function  $\widetilde{\Theta}_k(x)$  that satisfies

$$(3.12) \left\{ \begin{array}{l} \text{For } k = 1, \dots, N - 1 \\ \widetilde{\Theta}_k(x) = {}^t\psi(x)\Psi_k^{-1} \left[ \frac{1}{M_0} \sum_{j=1}^{M_0} \psi(S_k^j) \left( \Theta_{k+1}(S_{k+1}^j) + \frac{1}{N} g\left(k+1, \Theta_{k+1}(S_{k+1}^j), P_{k+1}(S_{k+1}^j)\right) \right) \right] \\ \text{and } \widetilde{\Theta}_N(x) = 0, \quad \widetilde{\Theta}_0(S_0) = \frac{1}{M_0} \sum_{j=1}^{M_0} \left( \Theta_1(S_1^j) + \frac{1}{N} g\left(1, \Theta_1(S_1^j), P_1(S_1^j)\right) \right). \end{array} \right.$$

$\mathfrak{T}$  used in (3.10) denotes a set of transformations that makes  $\Psi_k$  satisfy:

**Assumption 3.2**  $\Psi_k \rightarrow_{a.s.} E\left(\psi(\widetilde{S}_k^i) {}^t\psi(\widetilde{S}_k^i)\right)$  as  $M_0 \rightarrow \infty$  and  $\Psi_k$  remains symmetric. For a fixed  $M_0$ , denoting by  $\{\chi_{k,i}\}_{i=1,\dots,K}$  the eigenvalues of  $\Psi_k$  we have also the property

$$\max_{i=1,\dots,K} E\left(\left|\chi_{k,i}^{-1}\right|^4\right) < \infty \text{ and setting } \Sigma_k = E(\Psi_k^{-1})$$

we assume that for any  $K \times K$  deterministic matrix  $A$  there exists a positive function  $h_k^A(M_0, K)$  that vanishes as  $K^3/M_0 \rightarrow 0$  such that  $|E(\Psi_k^{-1}A\Psi_k^{-1}) - \Sigma_k A \Sigma_k| \leq h_k^A(M_0, K)$ .

Basically, Assumption 3.2 announces a fact that a programmer must check when implementing the regression. It is even natural to assume that  $\mathfrak{T}$  allows to have  $\{\chi_{k,i}^{-1}\}_{i=1,\dots,K} \in L^\infty$ , when this fact is true one can compute the function  $h_k^A$  that fulfills the previous inequality. Now, we are ready to announce the theorem.

**Theorem 3.2** As long as assumptions 3.1 and 3.2 are fulfilled and  $K_j^3/M_j \rightarrow 0$  for each

$j$  when American contracts are involved, if  $\{\Theta_i(x)\}_{0 \leq i \leq N-1}$  is of class  $\mathcal{C}^s$  on the support of  $S \in \mathbb{R}^d$  then there exists a positive constant  $C$  such that for each  $0 \leq k \leq N-1$

$$E \left[ \left( \widehat{\Theta}_k(S_k^i) - \Theta_k(S_k^i) \right)^2 \right] \leq \frac{CK^{N-1}}{N^2} \sum_{l=k}^{N-1} \left( E \left[ \frac{V_{l+1}(S_{l+1}^j) \partial_P^2 g(l+1, \Theta_{l+1}(S_{l+1}^j), P_{l+1}(S_{l+1}^j))}{2M_l} \right] \right)^2 + O \left( \frac{K}{M_0} + \frac{K}{N^4 M_l^2} + \frac{K^2}{N^2 M_0} + \frac{K^{1-2s/d}}{N^2} + K^{-2s/d} \right).$$

The  $s$ -continuous differentiability assumed in Theorem 3.2 can be gotten either from the regularity of  $g$  or from the regularity of the transition density of  $S$  using (3.2).

From this result, a reasonable choice of the number of inner trajectories is  $M_j \sim \sqrt{M_0/N}$  for  $j \in \{1, \dots, N-1\}$ . Moreover, using the same arguments as the one presented in Section 3.1, when the exposition involves only European contracts and the underlying asset  $S$  is a truncation of a log-Normal process, we can set

$$(3.13) \quad M_j = \frac{N-j}{N-1} M_1 \text{ with } M_1 = \sqrt{\frac{M_0}{N}}.$$

Also, when American options are involved, we will see that the choice (3.13) is good numerically as long as  $M_0$  is big enough.

Although the proof of Theorem 3.2 is detailed in the last section, we should make few remarks on the key tools that are needed for the proof. The first point that is really essential is the independence between  $\Psi_k$  and  $\{S^i\}_{i \in \{1, \dots, M_0\}}$ , otherwise the computation of the square error would be much more difficult. This is why  $\Psi_k$  is computed thanks to a new set of outer trajectories  $\{\widetilde{S}^i\}_{i \in \{1, \dots, M_0\}}$ . The other important point is the use of Theorem 4 in Newey (1997) which announces the following.

**Theorem 3.3** *We denote  $f(X) = E(Y|X)$  and  $\widehat{f}$  the approximation of  $f$  thanks to a regression on a basis of  $K$  monomials. We assume that the support of  $X \in \mathbb{R}^d$  is a cartesian product of compact connected intervals on which the density of  $X$  is bounded away from zero. More-*

over, we compute  $\hat{f}$  thanks to a regression that involves independent copies  $(X_i, Y_i)_{i \in \{1, \dots, M_0\}}$  of the couple  $(X, Y)$ . If  $f$  is of class  $\mathcal{C}^s$  on the support of  $X$  and if  $K^3/M_0 \rightarrow 0$ , then

$$\int [f(x) - \hat{f}(x)]^2 dF_0(x) = O_p(K/M_0 + K^{-2s/d})$$

where  $F_0$  is the cumulative distribution of  $X$ .

Theorem 3.3 with a  $\Psi_k$  that satisfies Assumption 3.2 allows to conduct all the computations in order to get Theorem 3.2.

## 4 Simulation framework and results

This section starts by presenting additional details on how the simulations are implemented and computing the complexity induced by this implementation. Then, it finishes by some compelling numerical results that shows the robustness of the method presented in this paper and that illustrates the theoretical asymptotic result established in the previous section.

### 4.1 Implementation and complexity

As already explained in Section 3, the proposed algorithm is based on nested (square) Monte Carlo simulation. As Monte Carlo is well suited to parallel architecture, we perform the implementation of our algorithm on the GPU “NVIDIA geforce 980 GTX” which includes 2048 parallel processing units. This massive computing power allows to reduce the execution time of the overall solution to make it quite interesting to use for real applications in the industry. This fact is true despite the high complexity of our method that has however the property to be very accurate.

The parallelization of this method is performed according to both the outer and the inner trajectories. First, one has to simulate the outer trajectories and to store them on the GPU if enough memory is available, otherwise to store them on the machine RAM (Random Access

Memory). Then, at each time step  $j \in \{1, \dots, N\}$  of  $m$  outer trajectories with  $m = M_0/m_0$ , we simulate the  $M_j$  inner trajectories. The reason of considering the outer trajectories per a group of  $m$  realizations is due to the limitation of the memory space available on GPU which makes impossible dealing with  $M_0 \times M_j$  memory space occupation when  $M_0$  and  $M_j$  are sufficiently big. Consequently, for each  $j \in \{1, \dots, N\}$ , we find ourselves obliged to repeat  $m_0$  times the same operation but on different  $m \times M_j$  data.

When the exposure involves only European contracts, the  $m_0$  repeated operations compose a common Monte Carlo simulation for the  $m \times M_j$  inner trajectories. However, when the exposure includes also American contracts, one has also to perform a lot of regressions in addition to the standard Monte Carlo operations. Formally speaking, if the exposure includes also American contracts, one has to perform  $N - j - 1$  regressions at each time step  $j \in \{1, \dots, N - 1\}$  and for each outer trajectory using  $M_j$  inner trajectories involved in a projection on  $K_j$  monomials. The complexity of each regression is proportional to  $M_j \times K_j^3$  and then the computations performed per each time step are of the order of  $O((N - j - 1)M_0M_j(d + K_j^3))$  where  $d$  is the number of underlying assets  $S$ . To explain exactly how the different operations are performed and optimized on the GPU, we are preparing a paper that will be submitted within a couple of months to a computing journal.

Let us now study the overall complexity of the algorithm assuming that  $K_j = K_0$  does not change according to  $j$  with  $M_j \gtrsim K_0^3$  and that  $M_0$  is much bigger than  $N^2$ . The complexity induced by the computation of (3.4) and of (3.11) are almost the same and one can get the following orders:

- For  $M_j = \frac{N - j}{N - 1}M_1$  with  $M_1 = \frac{\sqrt{M_0}}{N}$ , then the complexity  $\sim O((d + K_0)NM_0^{3/2})$ .
- For  $M_j = \frac{N - j}{N - 1}M_1$  with  $M_1 = \frac{\sqrt{M_0}}{\sqrt{N}}$ , then the complexity  $\sim O((d + K_0)(NM_0)^{3/2})$ .
- For  $M_j = \frac{N - j}{N - 1}M_1$  with  $M_1 = \sqrt{M_0}$ , then the complexity  $\sim O((d + K_0)N^2M_0^{3/2})$ .

These orders justify the necessity to employ the GPU computing power to reduce the execution time and make the overall solution usable by the banking industry.

## 4.2 Numerical results

In this section, we give some numerical results associated to three different expositions. For these examples we use the same three dimensional Black & Scholes model for the underlying assets  $S = (S^1, S^2, S^3)$

$$(4.1) \quad dS_t^i = rdt + \sigma_i \sum_{j=1}^i \varrho_{ij} dW_t^j, \quad i = 1, 2, 3,$$

with  $r =$  the risk neutral interest rate  $= \ln(1.1)$ ,  $S_0^i = 100$ ,  $\sigma_i = 0.2$  and  $\varrho = \{\varrho_{ij}\}_{1 \leq i, j \leq d}$  comes from the Cholesky decomposition of the correlation matrix  $(\delta_{i-j} + 0.5(1 - \delta_{i-j}))_{i, j=1, 2, 3}$  where  $\delta$  is the Kronecker symbol. We point that our method is quite robust according to the dimension of the problem and when increasing the number of random factors one has to increase smoothly the number of outer trajectories  $M_0$  to have the same accuracy order. Studying more quantitatively the affect of the dimension and the choice of the model will be done in a future work that deals with various practical examples.

For the following simulation, we take  $T = 1$ ,  $M_0 = 13 \times 10^4$  trajectories,  $N = 10$  and  $N_{sde} = 50$ . Although one can simulate exactly (4.1) without discretization, we use here  $N_{sde} > N$  because we need sufficient number of time steps in order to simulate  $\bar{S}_T^3 = \sup_{0 \leq t \leq T} S_t^3$  by  $\sup_{0 \leq k \leq N_{eds}} S_{t_k}^3$  that will be involved in two European path-dependent examples of tables 1 & 2.

Table 1: European exposition with a payoff  $\Upsilon(S_T) = \left(\frac{S_T^1}{2} + \frac{S_T^2}{2} - \bar{S}_T^3\right)_+$  and  $f_i(x_i) = 0.01 + 0.01(x_i)_+$ ,  $g(k, p, \theta) = 0.01p(p - \theta)_+$

$M_1$	$\Theta_0$	$\Theta_0$ std	$\text{CVA}_{0,T}$	$\text{CVA}_{0,T}$ std
$\frac{\sqrt{M_0}}{N}$	0.01364	$4 \times 10^{-5}$	0.0296	$2 \times 10^{-4}$
$\frac{\sqrt{M_0}}{\sqrt{N}}$	0.01307	$4 \times 10^{-5}$	0.0294	$2 \times 10^{-4}$
$\sqrt{M_0}$	0.01265	$3 \times 10^{-5}$	0.0291	$2 \times 10^{-4}$

In tables 1, 2 and 3, we study the changes in the values of the estimators of  $\text{CVA}_{0,T}$  and of the pre-default TVA  $\Theta_0$  when  $M_1$  increases with  $M_j = \frac{N-j}{N-1}M_1$ . In all tables, the values

postfixed by std represent the empirical standard deviation computed on 16 realizations of the estimator. From the values of these standard deviations, we conclude that  $M_0 = 130K$  trajectories is sufficiently big as the variance of the simulations is quite small.

Table 2: European exposition with a payoff:  $\Upsilon(S_T) = \left(\frac{3S_T^1}{10} + \frac{7S_T^2}{10} - \overline{S}_T^3\right)_+ - \left(\frac{7S_T^1}{10} + \frac{3S_T^2}{10} - \overline{S}_T^3\right)_+$   
and  $f_i(x_i) = 0.01 + 0.01(x_i)_+$ ,  $g(k, p, \theta) = 0.05(p - \theta)_+$

$M_1$	$\Theta_0$	$\Theta_0$ std	$\text{CVA}_{0,T}$	$\text{CVA}_{0,T}$ std
$\frac{\sqrt{M_0}}{N}$	$2.72 \times 10^{-3}$	$10^{-5}$	0.0365	$8 \times 10^{-4}$
$\frac{\sqrt{M_0}}{\sqrt{N}}$	$2.44 \times 10^{-3}$	$10^{-5}$	0.0453	$8 \times 10^{-4}$
$\sqrt{M_0}$	$2.28 \times 10^{-3}$	$10^{-5}$	0.0520	$8 \times 10^{-4}$

Table 3: American exposition with a payoff:  $\Upsilon(S_T) = \left(\kappa - \frac{S_T^1}{3} - \frac{S_T^2}{3} - \frac{S_T^3}{3}\right)_+$ ,  $\kappa = 100$   
and  $f_i(x_i) = 0.01 + 0.01(x_i)_+$ ,  $g(k, p, \theta) = 0.01p(p - \theta)_+$

$M_1$	$\Theta_0$	$\Theta_0$ std	$\text{CVA}_{0,T}$	$\text{CVA}_{0,T}$ std
$\frac{\sqrt{M_0}}{\sqrt{N}}$	0.0242	$10^{-4}$	0.0356	$2 \times 10^{-4}$
$\sqrt{M_0}$	0.0229	$10^{-4}$	0.0351	$2 \times 10^{-4}$

Regarding the value of  $M_1$  that affects the bias of the estimators, we see in table 1 that  $M_1 = \frac{\sqrt{M_0}}{N}$  is quite sufficient to compute  $\text{CVA}_{0,T}$  and taking  $M_1 = \frac{\sqrt{M_0}}{\sqrt{N}}$  produces a very small bias on  $\Theta_0$  when compared to  $\sqrt{M_0}$ . We have also the same case for  $\Theta_0$  in table 2, however  $\text{CVA}_{0,T}$  has a big bias if we take  $M_1$  too small which is due to the fact that the density of the exposition has a significant value at zero. As far as Table 3 is concerned, we do not show the simulations for  $M_1 = \frac{\sqrt{M_0}}{N}$  since this number is not sufficient to perform the regressions needed for the Longstaff-Schwartz algorithm required by our condition  $M_j \gtrsim K_0^3$  (here  $K_0 = 4$ ). Nevertheless, we see that  $M_1 = \frac{\sqrt{M_0}}{\sqrt{N}}$  is quite sufficient to have very accurate results of  $\Theta_0$  and  $\text{CVA}_{0,T}$ .



## 5 Proof of theorems 3.1 and 3.2

**Proof of Theorem 3.1:** The condition  $K_j^3/M_j \rightarrow 0$  is related to the convergence of the Longstaff-Schwartz algorithm studied in Stentoft (2004) which uses the results presented in Newey (1997) and Clément & al. (2002). Given (3.5), (3.3) becomes

$$(5.1) \quad \left\{ \begin{array}{l} F_{k+1}^1(x_1, \dots, x_{k+1}) = e^{-\frac{1}{N} \sum_{i=1}^k f_i(x_i)} \left( 1 - e^{-\frac{1}{N} f_{k+1}(x_{k+1})} \right), \\ F_{k+1}^2(x_1, \dots, x_{k+1}) = (x_{k+1})^+ F_{k+1}^1(x_1, \dots, x_{k+1}), \\ F_{k+1}^3(x_1, \dots, x_{k+1}) = (x_{k+1})^+ e^{-\frac{1}{N} \sum_{i=1}^{k+1} f_i(x_i)} \text{ and} \\ F_{k+1}^4(x_1, \dots, x_j, y) = e^{-\frac{1}{N} \sum_{i=1}^j f_i(x_i)} E \left( e^{-\frac{1}{N} \sum_{i=j+1}^{k+1} f_i(P_i(S_i))} \middle| S_j = y \right). \end{array} \right.$$

Besides,  $\text{MSE}(\widehat{\text{CVA}}_{0,T} - \text{CVA}_{0,T})$  can be decomposed into two terms: a variance term  $E \left[ \left( \widehat{\text{CVA}}_{0,T} - E \left[ \widehat{\text{CVA}}_{0,T} \right] \right)^2 \right]$  and a square bias term  $\left( E \left[ \widehat{\text{CVA}}_{0,T} - \text{CVA}_{0,T} \right] \right)^2$ . Regarding the variance, one gets easily

$$E \left[ \left( \widehat{\text{CVA}}_{0,T} - E \left[ \widehat{\text{CVA}}_{0,T} \right] \right)^2 \right] \leq \frac{N^2}{M_0} \max_{k \in \{0, \dots, N-1\}} \text{Var} \left( F_{k+1}^2 \left( \widehat{P}_1(S_1^i), \dots, \widehat{P}_{k+1}(S_{k+1}^i) \right) \right)$$

As for the square bias term one has to perform a second order Taylor expansion according to  $P$ . Doing the computations, we obtain

$$\begin{aligned} E \left[ \widehat{\text{CVA}}_{0,T} - \text{CVA}_{0,T} \right] &= \frac{1}{2} \sum_{j=1}^N \frac{1}{NM_j} E \left[ (\Delta_j^p)^2 f_j''(P_j(S_j^i)) F_j^3(P_1(S_1^i), \dots, P_j(S_j^i)) \right] \\ &\quad + \frac{1}{2} \sum_{j=1}^N \frac{1}{M_j} E \left[ (\Delta_j^p)^2 \varepsilon(P_j(S_j^i)) F_j^1(P_1(S_1^i), \dots, P_j(S_j^i)) \right] \\ &\quad - \frac{1}{2} \sum_{j=1}^N \frac{1}{NM_j} E \left[ (\Delta_j^p)^2 f_j''(P_j(S_j^i)) \sum_{k=l}^{N-1} F_{k+1}^2(P_1(S_1^i), \dots, P_{k+1}(S_{k+1}^i)) \right] \\ &\quad + \frac{1}{2} \sum_{j=1}^N (N-j+1) O \left( \frac{1}{M_j^2} \right) \end{aligned}$$

and  $\varepsilon$  is the Dirac distribution at 0. Thanks to an integration by parts using Assumption 3.1:

$$E\left[(\Delta_j^p)^2 \varepsilon(P_j(S_j^i)) F_j^1(P_1(S_1^i), \dots, P_j(S_j^i))\right] = E\left[(\Delta_j^p)^2 F_j^1(P_1(S_1^i), \dots, P_j(S_j^i)) | P_j(S_j^i) = 0\right] \varphi_j(0).$$

We point out that, conditionally to  $\sigma(\{S_l^i\}_{l=1, \dots, j})$ ,  $(\Delta_j^p)$  and  $F_{k+1}^2(P_1(S_1^i), \dots, P_{k+1}(S_{k+1}^i))$  are independent as  $(\Delta_j^p)$  involves the inner simulation which is conditionally to  $\sigma(\{S_l^i\}_{l=1, \dots, j})$  independent from the outer simulation and thus independent from  $\sigma(\{S_l^i\}_{l=j+1, \dots, k+1})$ .

Finally, conditioning according to  $\sigma(\{S_l^i\}_{l=1, \dots, j})$  in each expectation involved in the value of  $E\left[\widehat{\text{CVA}}_{0,T} - \text{CVA}_{0,T}\right]$  and using the Markov property, then taking the square of it and using the inequality  $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$ , we get the required result. ■

**Proof of Theorem 3.2:** Like in the proof of Theorem 3.1, the condition  $K_j^3/M_j \rightarrow 0$  is needed for the convergence of the Longstaff-Schwartz algorithm. In this proof, we study separately the variance  $E\left[\left(\widehat{\Theta}_k(x) - E\left[\widehat{\Theta}_k(x)\right]\right)^2\right]$  and the bias  $E\left[\widehat{\Theta}_k(x) - \Theta_k(x)\right]$  involved in  $E\left[\left(\widehat{\Theta}_k(x) - \Theta_k(x)\right)^2\right]$ . It is quite heavy to develop each term involved in the variance. Because we are interested by the term that decreases the least as  $M_0 \rightarrow \infty$ , we assume that  $\widehat{\Theta}_k(x)$  is independent from  $S_k^j$  as the terms  $\{S_i^j\}_{k+1 \leq i \leq N}$  are involved only once (weighted by  $1/M_0$ ) when computing backwardly  $\widehat{\Theta}_k(x)$ .

Denoting

$$\bar{\Theta}_{k+1}(S_{k+1}^j) = \widehat{\Theta}_{k+1}(S_{k+1}^j) + \frac{1}{N} g(k+1, \widehat{\Theta}_{k+1}(S_{k+1}^j), \widehat{P}_{k+1}(S_{k+1}^j)),$$

then  $\widehat{\Theta}_k(x) = {}^t\psi(x) \Psi_k^{-1} \left[ \frac{1}{M_0} \sum_{j=1}^{M_0} \psi(S_k^j) \bar{\Theta}_{k+1}(S_{k+1}^j) \right]$  and using Assumption 3.2

$$E\left[\left(\widehat{\Theta}_k(x) - E\left[\widehat{\Theta}_k(x)\right]\right)^2\right] = E\left[tr\left(A_k(x) \begin{bmatrix} \frac{1}{M_0^2} \sum_{i,j} \psi(S_{k+1}^i) {}^t\psi(S_{k+1}^j) \bar{\Theta}_{k+1}(S_{k+1}^i) \bar{\Theta}_{k+1}(S_{k+1}^j) \\ -m_{k+1} {}^t m_{k+1} \end{bmatrix}\right) + h_k^{\psi(x) {}^t\psi(x)}(M_0, K) O\left(\frac{K}{M_0}\right)\right]$$

with  $A_k(x) = \Sigma_k \psi(x) {}^t\psi(x) \Sigma_k$ ,  $m_{k+1} = E(\psi(S_{k+1}^i) \bar{\Theta}_{k+1}(S_{k+1}^i))$  and  $tr$  is the trace operator.

Thanks to the asymptotic independence of  $\widehat{\Theta}_k(x)$  and  $S_k^i$ , we obtain

$$E \left[ \left( \widehat{\Theta}_k(S_k^i) - E \left[ \widehat{\Theta}_k(S_k^i) \right] \right)^2 \right] = O \left( \frac{K}{M_0} \right).$$

Regarding the bias, if we denote  $B_k(x) = E \left[ \widehat{\Theta}_k(x) - \widetilde{\Theta}_k(x) \right]$  then

$$\begin{aligned} \left( E \left[ \widehat{\Theta}_k(S_k^i) - \Theta_k(S_k^i) \right] \right)^2 &\leq 2 \left( E \left[ B_k(S_k^i) \right] \right)^2 + 2 \left( E \left[ \widetilde{\Theta}_k(S_k^i) - \Theta_k(S_k^i) \right] \right)^2 \\ &\leq 2 \left( E \left[ B_k(S_k^i) \right] \right)^2 + 2E \left( \left[ \widetilde{\Theta}_k(S_k^i) - \Theta_k(S_k^i) \right]^2 \right) \end{aligned}$$

and using Theorem 4 in Newey (1997) as well as the boundedness part in Assumption 3.2, we have

$$E \left( \left[ \widetilde{\Theta}_k(S_k^i) - \Theta_k(S_k^i) \right]^2 \right) = O \left( \frac{K}{M_0} + K^{-2s/d} \right).$$

Besides

$$B_k(x) = {}^t \psi(x) \Sigma_k E \left( \psi(S_k^j) \left[ \begin{array}{c} \widehat{\Theta}_{k+1}(S_{k+1}^j) - \Theta_{k+1}(S_{k+1}^j) + \frac{1}{N} g(k+1, \widehat{\Theta}_{k+1}(S_{k+1}^j), \widehat{P}_{k+1}(S_{k+1}^j)) \\ - \frac{1}{N} g(k+1, \Theta_{k+1}(S_{k+1}^j), P_{k+1}(S_{k+1}^j)) \end{array} \right] \right)$$

and performing a second order Taylor expansion according to  $\Theta_{k+1}(S_{k+1}^j)$  and  $P_{k+1}(S_{k+1}^j)$

$$B_k(x) = {}^t \psi(x) \Sigma_k E \left( \psi(S_k^j) \left[ \begin{array}{c} \Delta_{k+1}^\Theta(S_{k+1}^j) + \frac{1}{N} \Delta_{k+1}^\Theta(S_{k+1}^j) \partial_{\Theta} g(k+1, \Theta_{k+1}(S_{k+1}^j), P_{k+1}(S_{k+1}^j)) \\ + \frac{1}{2NM_k} [\Delta_{k+1}^P(S_{k+1}^j)]^2 \partial_{P^2}^2 g(k+1, \Theta_{k+1}(S_{k+1}^j), P_{k+1}(S_{k+1}^j)) \\ + \frac{1}{2N} [\Delta_{k+1}^\Theta(S_{k+1}^j)]^2 \partial_{\Theta^2}^2 g(k+1, \Theta_{k+1}(S_{k+1}^j), P_{k+1}(S_{k+1}^j)) \\ + o \left( \left| \Delta_{k+1}(S_{k+1}^j) \right|_2^2 \right) \end{array} \right] \right)$$

where  $\Delta_{k+1}(S_{k+1}^j) = (\Delta_{k+1}^\Theta(S_{k+1}^j), \Delta_{k+1}^P(S_{k+1}^j))$  and  $|\cdot|_2$  is the Euclidean norm. In the previous equality, the first order term in  $\Delta_{k+1}^P(S_{k+1}^j)$  vanishes as  $E \left( \Delta_{k+1}^P(S_{k+1}^j) | S_{k+1}^j \right) = 0$  that removes also the term in  $\Delta_{k+1}^P(S_{k+1}^j) \Delta_{k+1}^\Theta(S_{k+1}^j)$  as the random functions  $\Delta_{k+1}^P(x)$  and  $\Delta_{k+1}^\Theta(x)$  are independent because the first one is simulated using the inner trajectories and

the other one using the outer ones. Besides, one can only keep the first and the third term and ignore all the others and obtain

$$B_k(x) = {}^t\psi(x) \Sigma_k E \left( \psi(S_k^j) \left[ \Delta_{k+1}^\Theta(S_{k+1}^j) + \frac{[\Delta_{k+1}^P(S_{k+1}^j)]^2}{2NM_k} \partial_P^2 g(k+1, \Theta_{k+1}(S_{k+1}^j), P_{k+1}(S_{k+1}^j)) \right] + O \left( \frac{1}{N^2 M_l} + \frac{\sqrt{K}}{N\sqrt{M_0}} + \frac{K^{-s/d}}{N} \right) \right).$$

After conditioning according to  $S_{k+1}^j$  and ignoring  $E \left( \tilde{\Theta}_{k+1}(S_{k+1}^j) - \Theta_{k+1}(S_{k+1}^j) | S_{k+1}^j \right)$ , we obtain the following induction

$$B_k(x) = {}^t\psi(x) \Sigma_k E \left( \psi(S_k^j) \left[ B_{k+1}(S_{k+1}^j) + \frac{V_{k+1}(S_{k+1}^j)}{2NM_k} \partial_P^2 g(k+1, \Theta_{k+1}(S_{k+1}^j), P_{k+1}(S_{k+1}^j)) \right] + O \left( \frac{1}{N^2 M_l} + \frac{\sqrt{K}}{N\sqrt{M_0}} + \frac{K^{-s/d}}{N} \right) \right)$$

and because  $B_N(x) = 0$ , one can establish for each  $0 < k \leq N-1$

$$E \left[ B_k(S_k^j) \right] = E \left( {}^t\psi(S_k^j) \right) \Sigma_k \sum_{l=k}^{N-1} \left\{ \prod_{i=k+1}^l \Phi_i \Sigma_i E \left( \psi(S_l^j) \left[ \frac{V_{l+1}(S_{l+1}^j) \partial_P^2 g(l+1, \Theta_{l+1}(S_{l+1}^j), P_{l+1}(S_{l+1}^j))}{2NM_l} + O \left( \frac{1}{N^2 M_l} + \frac{\sqrt{K}}{N\sqrt{M_0}} + \frac{K^{-s/d}}{N} \right) \right] \right) \right\}$$

where  $\Phi_i = E \left( \psi(S_{i-1}^j) {}^t\psi(S_i^j) \right)$ . For  $k=0$ ,  $E \left[ B_0(S_0^j) \right]$  differs from the other  $E \left[ B_k(S_k^j) \right]_{k>0}$  by a last term ( $l=0$ ) which does not involve  $\psi(S_0^j)$  or  $\Sigma_0$ .

There exists then a positive constant  $C$  such that for each  $0 \leq k \leq N-1$

$$\left( E \left[ B_k(S_k^j) \right] \right)^2 \leq \frac{CK}{N^2} \sum_{l=k}^{N-1} \left( E \left[ \frac{V_{l+1}(S_{l+1}^j) \partial_P^2 g(l+1, \Theta_{l+1}(S_{l+1}^j), P_{l+1}(S_{l+1}^j))}{2M_l} \right] \right)^2 + O \left( \frac{K}{N^4 M_l^2} + \frac{K^2}{N^2 M_0} + \frac{K^{1-2s/d}}{N^2} \right).$$

Using this final expression as well as the asymptotic behavior of  $E \left( \left[ \tilde{\Theta}_k(S_k^j) - \Theta_k(S_k^j) \right]^2 \right)$  and of the variance term, we get the required result. ■

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