

# Homology Computation during an Incremental Construction of Simplicial or Cellular Structures

P. Lienhardt<sup>a</sup>, S. Peltier<sup>a</sup>, S. Alayrangués<sup>a</sup>, L. Fuchs<sup>a</sup>

<sup>a</sup>*Université de Poitiers, Laboratoire XLIM, Département SIC, CNRS 7252, Bâtiment SP2MI - Téléport 2; Boulevard Marie & Pierre Curie; BP 30179, 86962 Futuroscope-Chasseneuil Cedex; France*

---

## Abstract

Controlling the construction of geometric objects is important for many applications. Homology (groups and generators) may be useful for this control. For such incremental construction processes, it is interesting to *incrementally* compute the homology, i.e. to deduce the homological information at step  $s$  of the construction from the homological information computed at step  $s - 1$ .

Results of *effective homology* [RS06] are applied for this purpose: at each step, a correspondence between the chain complex associated with a given simplicial or "cellular" combinatorial structure is maintained with a "smaller" chain complex, from which the homology groups and homology generators can be more efficiently computed.

*Keywords:* Homology computation, Semi-simplicial sets, Incidence graphs, Combinatorial maps

---

## 1. Introduction

### *Context*

Geometric Modeling deals with the representation and the construction of geometric objects. For instance, many representations and related construction operations have been conceived for CAD/CAM applications; since a modeled geometric object can be manufactured, it is necessary to control its construction, in order to detect any problem as soon as possible. Depending on the application, homology [EH10] (i.e. homology groups and/or the

correspondence between their generators and parts of the object at each step of its construction) may be useful for controlling the construction.

Many representations and construction operations have been conceived for handling subdivided geometric objects, i.e. objects partitioned into cells of different dimensions, and many methods have been conceived and optimized for computing the homology of subdivided objects, for instance based on Smith Normal Form of incidence matrices (e.g. [Ago76, Sto96, Gie96, DSV01, DHSW03, PAFL06]), or on simplifications as elementary reductions (e.g. [MB09, GDJMR09, DKMW11]).

Such methods make it possible to compute the homology at each step of the construction of a geometric object; but the whole homological information has to be computed at each step, without taking advantage of the information known at the previous step. So, it seems interesting for such a process to *incrementally* compute the homology, i.e. to deduce the homology of the object at step  $s$  from the homology of the object at step  $s - 1$ , with respect to the operation which is applied. Such incremental computation can be done by applying results of *effective homology* [RS06, Ser]. For instance, Boltcheva *et al* [BMLH10] applied it to design the *Mayer-Vietoris (MV) algorithm*, which computes the homological information of abstract simplicial complexes from the homological information of sub-complexes and their intersections. This algorithm has been applied for the *Manifold-Connected decomposition* of abstract simplicial complexes [BCMA<sup>+</sup>11].

The main tools and results of effective homology on which this paper is based are:

- the notion of *homological equivalence*: it makes it possible to maintain an association between a chain complex  $C$  and a "small" chain complex  $C^S$ , which are homologically equivalent; so, the homological information of  $C$  can be efficiently deduced from  $C^S$ ;
- the *Short Exact Sequence Theorem* (or SES theorem: cf. theorem 1 page 12), which makes it possible to incrementally compute the homological information of a chain complex for some operations.

### *Goal*

The idea of computing the homology during an incremental construction of an object  $O$  is the following. Let  $O_i$  be the object at step  $i$ ,  $C_i$  be its associated chain complex, and  $C_i^S$  be a "small" chain complex associated with  $C_i$  by a homological equivalence. Let  $O_{i+1}$  be the object at next step, obtained

by applying a construction operation to  $O_i$ . The goal is to efficiently deduce  $C_{i+1}$  from  $C_i$  and  $C_{i+1}^S$  from  $C_i^S$  (and the corresponding homological equivalence). This approach provides a true optimization if the computational cost of both  $C_{i+1}^S$  and its homology is lower than the cost of a direct computation of the homology of  $C_{i+1}$ . This approach is here studied for several simplicial and "cellular" combinatorial structures, and related basic construction operations.

### *Simplicial Structures*

First, semi-simplicial sets [EZ50, May67] and related basic operations are studied: cf. Section 3. In particular, it is shown that a homological equivalence can be constructed:

- if  $C_{i+1}$  is the result of a cone operation, in linear time in the size of  $C_i$ : cf. theorem 2 page 20;
- if  $C_{i+1}$  is the result of the identification<sup>1</sup> of subsets of simplices (or the inverse operation), by one application of the SES theorem: cf. theorem 3 page 25. The study of complexity shows that the incremental computation of the homology is efficient compared with a direct computation of the homology of  $C_{i+1}$ : cf. remark 1 page 27.

These results can be generalized for cubical and simplicial structures [BH81, PFL09]; moreover, theorem 3 generalizes in some sense the results of [BMLH10] (cf. Section 3.4).

### *"Cellular" Structures*

Then, "cellular" combinatorial structures and related basic operations are studied: cf. Section 4. Many "cellular" combinatorial structures have been defined and studied in order to represent the structure of subdivided geometric objects, for instance for applications in Geometric Modeling, Computational Geometry, Image Processing and Analysis, etc. Most of them can be classified into two main classes: incidence graphs and combinatorial maps [DL14]. Notions related to these structures are recalled in section 4.1.

For many applications, the cells of an object are not necessarily convex; moreover, cells of "cellular" combinatorial structures exist, whose geometric

---

<sup>1</sup>Let  $\phi$  be a morphism between two subsets of simplices; identifying these subsets consists in "merging" the simplices with respect to  $\phi$ .

realizations are not homeomorphic to balls. Since it is not possible to combinatorially decide whether the geometric realization of a cell is homeomorphic or not to a ball (this comes from theorems in sections 3 and 4 of [CL06]), it is not possible in general to directly apply results about homology obtained for simplicial structures on "cellular" combinatorial structures.

A *simplicial analog*, which is a semi-simplicial set, can be associated with any "cellular" combinatorial structure [Bri93, Lie94, Ber99]. So, a *simplicial chain complex* can be associated with any "cellular" combinatorial structure: it is the chain complex associated with its simplicial analog. It is thus possible to compute the homology of a "cellular" structure by computing the homology of its simplicial analog. Since it is generally possible to know in an efficient way the modifications of its simplicial analog when an operation is applied to a cellular structure, it is also possible to apply the results presented in section 3 in order to incrementally compute the homology of a "cellular" combinatorial structure during a construction process, but *the optimization of the cellular representation is lost* (there are many more simplices in a simplicial analog than cells in the corresponding "cellular" combinatorial structure).

*It is possible to take advantage of this "cellular" optimization under some assumptions.* First, basic operations are studied, which modify whole cells, in particular the creation of a cell, and the identification of isomorphic cells (and the inverse operation). It is shown in section 4.2 that, if homological equivalences are associated with the structure and *also with each of its cells*, then it is possible to efficiently maintain homological informations at each step of a construction process, in order to optimize the computation of the homology: cf. theorem 4 page 43, theorem 5 page 47 and remark 2 page 51.

Second, a subclass of structures is studied, for which it is possible to associate a *cellular chain complex* with each "cellular" combinatorial structure, i.e. a chain complex in which each generator corresponds to a cell, such that it is homologically equivalent to the corresponding simplicial chain complex [ADLP15, ALP15, Bas10]. In this case, a better optimization can be obtained: cf. theorem 6 page 58, theorem 7 page 60 and remark 3 page 60 in section 4.3.

### *Outline*

Useful notions and results of effective homology are recalled in Section 2. The definitions of semi-simplicial sets and basic operations are recalled in

Section 3.1. The incremental computation of the homological information for the adding/removing simplices and cone operations (resp. the identification operation and its inverse) is studied in Section 3.2 (resp. Section 3.3).

Main notions related to "cellular" combinatorial structures are recalled in Section 4.1. The incremental computation of the homological information is studied in Section 4.2 for general "cellular" combinatorial structures constructed by applying cellular operations. It is studied in Section 4.3 for the subclasses with which cellular chain complexes can be associated.

Some perspectives are presented in Section 5.

## 2. Effective homology bases

This section is mainly based on the course notes of J. Rubio and F. Sergeraert [RS06]. Some of their results are also presented in [BMLH10]. Here, properties and theorems are not explained nor proved, but they can be directly checked (cf. also [AFLP15] for more precisions); in particular, all reductions related to theorem 1 are explicitly stated (cf. section 2.2).

### Definition 1. [Bou89a]

A chain complex  $(C, \partial)$  is defined by  $C$ , a graded module<sup>2</sup> of type  $\mathbb{Z}$  over the commutative graded ring<sup>3</sup>  $\mathbb{Z}$ , and  $\partial : C \mapsto C$ , a boundary operator, i.e. a graded module morphism of degree  $-1$  such that<sup>4</sup> the composition  $\partial\partial = 0$ .

For any  $q$  in  $\mathbb{Z}$ ,  $C_q$  is the group of  $q$ -chains. The image  $B_q = C_{q+1}\partial_{q+1} \subset C_q$  is the (sub)group of  $q$ -boundaries. The kernel  $Z_q = Ker(\partial_q) \subset C_q$  is the (sub)group of  $q$ -cycles. Since  $\partial_{q+1}\partial_q = 0$ ,  $B_q \subset Z_q$ ;  $H_q = Z_q/B_q$  is the  $q^{th}$  homology group<sup>5</sup> of  $C$ .

**Definition 2.** A chain complex morphism  $\Phi$  between two chain complexes  $(C^0, \partial^0)$  and  $(C^1, \partial^1)$  is a graded module morphism between  $C^0$  and  $C^1$  such that  $c\partial^0\Phi = c\Phi\partial^1$ , for any chain  $c \in C^0$ .

---

<sup>2</sup>See [Bou89b] for the definitions of basic algebraic notions, and more precisely (II.11.2 p366) for the definition of graded modules. These notions are recalled in [AFLP15] (Annex 6.1 page 48).

<sup>3</sup>The graduation is the trivial graduation, i.e.  $\mathbb{Z}_0 = \mathbb{Z}$ , and  $\mathbb{Z}_i = 0$  for  $i \neq 0$ .

<sup>4</sup>Note that  $x\delta$  (resp.  $x\delta\epsilon$ ,  $\delta\epsilon$ ) denotes  $\delta(x)$  (resp.  $\epsilon(\delta(x))$ ,  $\epsilon \circ \delta$ ).

<sup>5</sup>In fact, all these groups: chain, boundary, cycle, homology, are graded modules.

**Definition 3.** The homology  $H$  of  $(C, \partial)$  is the direct sum of its homology groups. By extension, the chain complex  $(H, 0)$  is also called the homology of  $(C, \partial)$ .

2.1. Effective homology tools

2.1.1. Reduction

A reduction associates with some chain complex a generally "smaller" one [KMM04, GDJMR09, RS06].

**Definition 4.** A reduction  $\rho = ((C^0, \partial^0), (C^1, \partial^1), h, f, g)$ , denoted  $\rho : (C^0, \partial^0) \rightrightarrows (C^1, \partial^1)$ , or  $(C^0, \partial^0) \xrightarrow{\rho} (C^1, \partial^1)$ , is a diagram:

$$\rho = \boxed{\begin{array}{ccc} & h & \\ \curvearrowright & & \leftarrow g \\ (C^0, \partial^0) & & (C^1, \partial^1) \\ & f & \rightarrow \end{array}}$$

where  $(C^0, \partial^0)$  and  $(C^1, \partial^1)$  are chain-complexes,  $f$  and  $g$  are chain-complex morphisms, and  $h$  is a homotopy operator, i.e. a graded module morphism of degree +1, which satisfy the relations:

- (a)  $gf = id_{C^1}$
- (b)  $fg + h\partial^0 + \partial^0h = id_{C^0}$
- (c)  $hf = gh = hh = 0$

Several reductions are described in Example 7 and Example 8 in Annex 6.1 pages 62 and 65, on which the definition properties a), b) and c) are explained.

**Property 1.** If a reduction exists between two chain complexes  $(C^0, \partial^0)$  and  $(C^1, \partial^1)$ , then their homologies are isomorphic.

**Definition 5.** Let  $\rho^{01} = ((C^0, \partial^0), (C^1, \partial^1), h^{01}, f^{01}, g^{01})$  and  $\rho^{12} = ((C^1, \partial^1), (C^2, \partial^2), h^{12}, f^{12}, g^{12})$  be two reductions.

The composition  $\rho^{01}\rho^{12}$  is  $((C^0, \partial^0), (C^2, \partial^2), h^{02}, f^{02}, g^{02})$ , such that:

- $h^{02} = h^{01} + f^{01}h^{12}g^{01}$
- $f^{02} = f^{01}f^{12}$
- $g^{02} = g^{12}g^{01}$

**Property 2.** The composition of two reductions is a reduction.

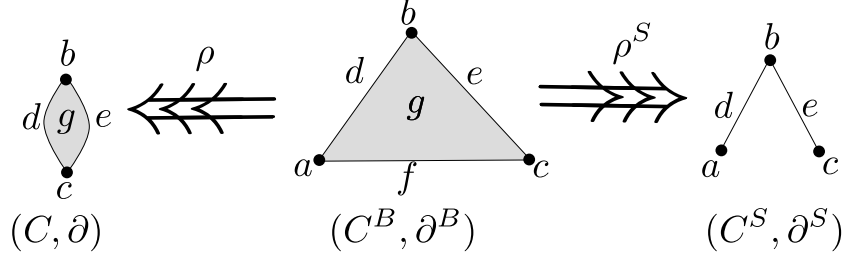


Figure 1: Homological equivalence between  $(C, \partial)$  and  $(C^S, \partial^S)$  through  $(C^B, \partial^B)$ ; reduction  $\rho$  (resp.  $\rho^S$ ) is characterized by the fact that the image of  $a$  (resp.  $f$ ) by the homotopy operator is  $f$  (resp.  $g$ ). Note that no reduction exists between  $(C, \partial)$  and  $(C^S, \partial^S)$ .

### 2.1.2. Homological equivalence

**Definition 6.** A homological equivalence  $\Upsilon$  between two chain-complexes,  $(C, \partial)$  and  $(C^S, \partial^S)$  is a pair of reductions  $\rho$  and  $\rho^S$  involving a chain complex  $(C^B, \partial^B)$ , such that  $\Upsilon : (C, \partial) \xleftarrow{\rho} (C^B, \partial^B) \xrightarrow{\rho^S} (C^S, \partial^S)$ .

$\Upsilon$  is sometimes denoted  $(C, \partial) \xleftrightarrow{\Upsilon} (C^S, \partial^S)$ . This notion allows to relate chain complexes which are not directly related by a reduction (see Fig. 1). From a practical point of view, this notion is used to associate a "small" chain complex  $(C^S, \partial^S)$  with another chain complex  $(C, \partial)$ , through a "bigger" one  $(C^B, \partial^B)$  (see Section 3). Note that, as a straightforward consequence of Definition 6 and Property 1, the homologies of all these chains complexes are isomorphic. Hence, since  $(C^S, \partial^S)$  is "smaller" than  $(C, \partial)$ , meaning  $C^S$  contains less generators than  $C$ , the homology of  $(C, \partial)$  can be efficiently computed by computing the homology of  $(C^S, \partial^S)$ .

### 2.1.3. Cone of a morphism

**Definition 7.** Let  $\phi$  be a chain complex morphism which maps  $(C^0, \partial^0)$  onto  $(C^1, \partial^1)$ . The cone of  $\phi$ ,  $Cone(\phi) = (C^\phi, \partial^\phi)$ , is the chain complex defined by:

$$\forall i, C_i^\phi = C_{i-1}^0 \oplus C_i^1 \text{ and } \partial_i^\phi = \begin{pmatrix} -\partial_{i-1}^0 & \phi_{i-1} \\ 0 & \partial_i^1 \end{pmatrix}$$

$C_i^\phi$  is the direct sum of  $C_{i-1}^0$  and  $C_i^1$ ,  $\forall i$ . Each element  $c_i = c_{i-1}^0 + c_i^1$  of  $C_i^\phi$  is denoted  $c_i = (c_{i-1}^0 \ c_i^1)$ , where  $c_{i-1}^0 \in C_{i-1}^0$  and  $c_i^1 \in C_i^1$ , and:

$$c_i \partial_i^\phi = (c_{i-1}^0 \quad c_i^1) \begin{pmatrix} -\partial_{i-1}^0 & \phi_{i-1} \\ 0 & \partial_i^1 \end{pmatrix} = (-c_{i-1}^0 \partial_{i-1}^0 \quad c_{i-1}^0 \phi_{i-1} + c_i^1 \partial_i^1)$$

**Example 1.** *This example is illustrated on Fig. 2. Let  $(C^0, \partial^0)$  and  $(C^1, \partial^1)$  be two chain complexes defined by:*

$C^0$		$b^0$	$c^0$			$f^0$	
$\partial^0$		0	0			$c^0 - b^0$	
$C^1$	$a^1$	$b^1$	$c^1$	$d^1$	$e^1$	$f^1$	$g^1$
$\partial^1$	0	0	0	0	$b^1 - a^1$	$c^1 - b^1$	$d^1 - c^1$

Let  $\phi$  be defined by:  $b^0 \rightarrow b^1$ ,  $c^0 \rightarrow c^1$ ,  $f^0 \rightarrow f^1$ . Then, for instance:

$$(0 \quad e^1) \begin{pmatrix} -\partial_0^0 & \phi_0 \\ 0 & \partial_1^1 \end{pmatrix} = (0 \quad b^1 - a^1)$$

$$(b^0 \quad 0) \begin{pmatrix} -\partial_0^0 & \phi_0 \\ 0 & \partial_1^1 \end{pmatrix} = (0 \quad b^1)$$

$$(f^0 \quad 0) \begin{pmatrix} -\partial_1^0 & \phi_1 \\ 0 & \partial_2^1 \end{pmatrix} = (-c^0 + b^0 \quad f^1)$$

and  $(C^\phi, \partial^\phi)$  is defined by:

$C^\phi$	$a^1$	$b^1$	$c^1$	$d^1$	$e^1$	$f^1$	$g^1$	$b^0$	$c^0$	$f^0$
$\partial^\phi$	0	0	0	0	$b^1 - a^1$	$c^1 - b^1$	$d^1 - c^1$	$b^1$	$c^1$	$b^0 - c^0 + f^1$

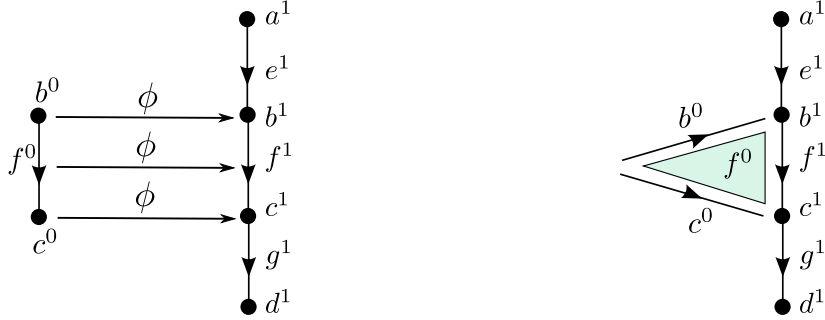
Note that the cone of a morphism is simply the representation of a chain complex morphism as a chain complex. It contains no more information.

#### 2.1.4. Perturbation and related lemmas

**Definition 8.** ([RS06] page 48) *Let  $(C, \partial)$  be a chain complex. Let  $\delta : C \rightarrow C$  be a graded module morphism of degree  $-1$ , such that  $(\partial + \delta)^2 = 0$ .  $\delta$  is called a perturbation of  $\partial$ .*

Example: let  $(C, \partial)$  be a chain complex; since  $(C, 0)$  also is a chain complex,  $\partial$  is a perturbation of the null boundary operator.





(a) Graphical representation of two chain complexes  $(C^0, \partial^0)$  and  $(C^1, \partial^1)$ , and a chain complex morphism  $\phi$  between them

(b) Graphical representation of  $(C^\phi, \partial^\phi)$

Figure 2: Example of the cone of a chain-complex morphism

**Property 3.** *Easy Perturbation Lemma ([RS06] page 48-49)*

Let  $\rho = ((C^0, \partial^0), (C^1, \partial^1), h, f, g)$  be a reduction, and let  $\delta^1$  be a perturbation of  $\partial^1$ , then  $\rho^\delta = ((C^0, \partial^0 + \delta^0), (C^1, \partial^1 + \delta^1), h, f, g)$  is a reduction, where  $\delta^0 = f\delta^1g$ .

**Property 4.** *Basic Perturbation Lemma ([RS06] page 49)*

Let  $\rho = ((C^0, \partial^0), (C^1, \partial^1), h, f, g)$  be a reduction, and let  $\delta^0$  be a perturbation of  $\partial^0$  which satisfies the nilpotency hypothesis, i.e.:

$$\forall c \in C^0, \exists i \in \mathbb{N} \mid c(\delta^0 h)^i = 0$$

Then  $\rho^\delta = ((C^0, \partial^0 + \delta^0), (C^1, \partial^1 + \delta^1), h^\delta, f^\delta, g^\delta)$  is a reduction, where:

- $\delta^1 = g\Phi\delta^0f = g\delta^0\Psi f$
- $f^\delta = \Psi f$
- $g^\delta = g\Phi$
- $h^\delta = h\Phi = \Psi h$

with<sup>6</sup>  $\Phi = \sum_{i=0}^{i=\infty} (-1)^i (\delta^0 h)^i$  and  $\Psi = \sum_{i=0}^{i=\infty} (-1)^i (h\delta^0)^i$ .

<sup>6</sup>Due to the nilpotency hypothesis, each sum contains a finite number of non null terms.

The proofs of the Easy and Basic Perturbation Lemmas are recalled in [AFLP15] (Annex 6.4 page 56).

## 2.2. SES theorem

**Definition 9.** ([RS06] page 71) *An effective short exact sequence of chain complexes is a diagram:*

$$0 \xrightarrow{0} (C^0, \partial^0) \xrightleftharpoons[i]{r} (C^1, \partial^1) \xrightleftharpoons[j]{s} (C^2, \partial^2) \xrightarrow{0} 0$$

where

- $(C^0, \partial^0), (C^1, \partial^1), (C^2, \partial^2)$  are chain complexes,
- $i, j$  are chain complex morphisms,
- $r, s$  are graded module morphisms,

which satisfy

1.  $ir = id_{C^0}$
2.  $ri + js = id_{C^1}$
3.  $sj = id_{C^2}$

Note that  $i, s$  are injective, and  $j, r$  are surjective due to properties 1 and 3. Several properties can also be deduced. For instance,  $ij = 0$  (since  $r$  is surjective and  $rij = (id_{C^1} - js)j = j - j = 0$  due to properties 2 and 3),  $sr = 0$  (for similar reasons), and  $-\partial^2 s \partial^1 r = s \partial^1 r \partial^0$ .

**Example 2.** *Let  $(C^0, \partial^0), (C^1, \partial^1), (C^2, \partial^2)$  be the chain complexes depicted on Fig. 3. Together with  $i, j, r$  and  $s$  defined below, we obtain an effective short exact sequence.*

- $i : a^0 \rightarrow a'^1 - a''^1; d^0 \rightarrow d'^1 - d''^1;$
- $j : a'^1 \rightarrow a^2; a''^1 \rightarrow a^2; d'^1 \rightarrow d^2; d''^1 \rightarrow d^2; \forall x^1 \notin \{a'^1, a''^1, d'^1, d''^1\}, x^1 \rightarrow x^2;$
- $s : a^2 \rightarrow a'^1; d^2 \rightarrow d'^1; h^2 \rightarrow h''^1; \forall x^2 \notin \{a^2, d^2, h^2\}, x^2 \rightarrow x'^1;$

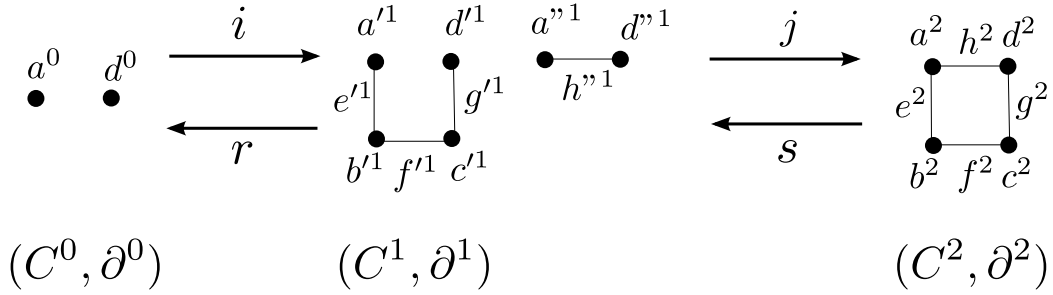


Figure 3: An Effective Short Exact Sequence

- $r : a^1 \rightarrow 0; d^1 \rightarrow 0; a^{n1} \rightarrow -a^0; d^{n1} \rightarrow -d^0; \forall x^1 \notin \{a^1, a^{n1}, d^1, d^{n1}\}, x^1 \rightarrow 0.$

**Example 3.** *Generalization of example 2.*

*More generally, consider the following construction.*

1. *Let  $(C^1, \partial^1) = (C'^1, \partial'^1) \oplus (C^{n1}, \partial^{n1})$ , where  $C'^1$  and  $C^{n1}$  are respectively decomposed into two subsets  $P^1, R^1$  and  $P^{n1}, R^{n1}$ , such that<sup>7</sup>:*
  - $(P^1, \partial^1|_{P^1})$  and  $(P^{n1}, \partial^{n1}|_{P^{n1}})$  are subcomplexes,
  - $(P^1, \partial^1|_{P^1})$  is isomorphic to  $(P^{n1}, \partial^{n1}|_{P^{n1}})$ .
2. *Let us "glue"  $(C'^1, \partial'^1)$  and  $(C^{n1}, \partial^{n1})$  along  $P^1$  and  $P^{n1}$ , producing the chain complex  $(C^2, \partial^2) = (C^1, \partial^1)/(P^1 = P^{n1})$ <sup>8</sup>. This chain complex can be decomposed into  $P^2, R^2$ , such that the subcomplex  $(P^2, \partial^2|_{P^2})$  is isomorphic to  $(P^1, \partial^1|_{P^1})$ .*
3. *Let  $(C^0, \partial^0)$  be a chain complex isomorphic to  $(P^1, \partial^1|_{P^1})$ . In order to simplify notations, if  $x^0$  belongs to  $C^0$  then  $x^1, x^{n1}$  are its images by the corresponding isomorphisms. Similar notations are also used for elements of  $P^1, P^{n1}$ , and  $P^2, R^1, R^{n1}$  and  $R^2$ .*
4. *Let  $i, j, s, r$  be defined by:*

<sup>7</sup>In example 2,  $P^1 = \{a^1, d^1\}$ ,  $R^1 = \{e^1, b^1, f^1, c^1, g^1\}$ ,  $P^{n1} = \{a^{n1}, d^{n1}\}$ ,  $R^{n1} = \{h^{n1}\}$ .

<sup>8</sup> $(C^1, \partial^1)/(P^1 = P^{n1})$  denotes the quotient of  $(C^1, \partial^1)$  by the equivalence relation  $P^1 = P^{n1}$ .  $(C^2, \partial^2)$  is thus the complex obtained by identifying the elements of  $P^1$  with the corresponding elements of  $P^{n1}$ .

- $i : \forall x^0 \in C^0, x^0 \rightarrow x'^1 - x''^1;$
- $j : \forall x'^1 \in P'^1, x'^1 \rightarrow x^2; \forall x''^1 \in P''^1, x''^1 \rightarrow x^2; \forall x^1 \in R'^1 \cup R''^1, x^1 \rightarrow x^2;$
- $s : \forall x^2 \in P^2, x^2 \rightarrow x'^1; \forall x^2 \in R^2, x^2 \rightarrow x^1, \text{ where } x^1 \in R'^1 \cup R''^1;$
- $r : \forall x'^1 \in P'^1, x'^1 \rightarrow 0; \forall x''^1 \in P''^1, x''^1 \rightarrow -x^0; \forall x^1 \in R'^1 \cup R''^1, x^1 \rightarrow 0.$

*This construction is an effective short exact sequence, since:*

- $\forall x^0 \in C^0, x^0 i r = x'^1 r - x''^1 r = 0 - (-x^0) = x^0;$
- $\forall x'^1 \in P'^1, x'^1 (r i + j s) = 0 + x^2 s = x'^1;$   
 $\forall x''^1 \in P''^1, x''^1 (r i + j s) = -x^0 i + x^2 s = -(x'^1 - x''^1) + x'^1 = x''^1;$   
 $\forall x^1 \in R'^1 \cup R''^1, x^1 (r i + j s) = 0 + x^2 s = x^1 ;$
- $\forall x^2 \in P^2, x^2 s j = x'^1 j = x^2;$   
 $\forall x^2 \in R^2, x^2 s j = x^1 j = x^2.$

We give below only the subparts of the SES theorem which are useful for our constructions: part 1 will be used for the identification operation; whereas part 2 will be used for the inverse operation (see section 3 and section 4).

**Theorem 1.** *SES Theorem ([RS06] page 71)*

*Let*

$$0 \xrightarrow{0} (C^0, \partial^0) \xrightleftharpoons[i]{r} (C^1, \partial^1) \xrightleftharpoons[j]{s} (C^2, \partial^2) \xrightarrow{0} 0$$

*be a short exact sequence;*

1. *given two homological equivalences  $\Upsilon^0 : (C^0, \partial^0) \rightleftarrows (C^{S0}, \partial^{S0})$  and  $\Upsilon^1 : (C^1, \partial^1) \rightleftarrows (C^{S1}, \partial^{S1})$ , it is possible to construct a homological equivalence  $\Upsilon^2$  relating  $(C^2, \partial^2)$  to a "small" chain complex constructed from  $(C^{S0}, \partial^{S0})$  and  $(C^{S1}, \partial^{S1})$ ;*
2. *given two homological equivalences  $\Upsilon^0 : (C^0, \partial^0) \rightleftarrows (C^{S0}, \partial^{S0})$  and  $\Upsilon^2 : (C^2, \partial^2) \rightleftarrows (C^{S2}, \partial^{S2})$ , it is possible to construct a homological equivalence  $\Upsilon^1$  relating  $(C^1, \partial^1)$  to a "small" chain complex constructed from  $(C^{S0}, \partial^{S0})$  and  $(C^{S2}, \partial^{S2})$ .*

**Sketch of proof**

- proof of 1. (see Fig. 4).

Let

$$\Upsilon^0 : (C^0, \partial^0) \xleftarrow{\rho^0} (C^{B0}, \partial^{B0}) \xrightarrow{\rho^{S0}} (C^{S0}, \partial^{S0})$$

and

$$\Upsilon^1 : (C^1, \partial^1) \xleftarrow{\rho^1} (C^{B1}, \partial^{B1}) \xrightarrow{\rho^{S1}} (C^{S1}, \partial^{S1})$$

be the two homological equivalences, with :

- $\rho^0 = ((C^{B0}, \partial^{B0}), (C^0, \partial^0), h^0, f^0, g^0)$ ,
- $\rho^{S0} = ((C^{B0}, \partial^{B0}), (C^{S0}, \partial^{S0}), h^{S0}, f^{S0}, g^{S0})$ ,
- $\rho^1 = ((C^{B1}, \partial^{B1}), (C^1, \partial^1), h^1, f^1, g^1)$ ,
- $\rho^{S1} = ((C^{B1}, \partial^{B1}), (C^{S1}, \partial^{S1}), h^{S1}, f^{S1}, g^{S1})$ .

Then two chain complex morphisms,  $i^B : (C^{B0}, \partial^{B0}) \longrightarrow (C^{B1}, \partial^{B1})$  and  $i^S : (C^{S0}, \partial^{S0}) \longrightarrow (C^{S1}, \partial^{S1})$  can be deduced from the chain complex morphism  $i$  and both homological equivalences:

- $i^B = f^0 i g^1 : (C^{B0}, \partial^{B0}) \longrightarrow (C^{B1}, \partial^{B1})$ ;
- $i^S = g^{S0} i^B f^{S1} : (C^{S0}, \partial^{S0}) \longrightarrow (C^{S1}, \partial^{S1})$ .

Moreover there exists a homological equivalence

$$\Upsilon^2 : (C^2, \partial^2) \xleftarrow{\rho^2} Cone(i^B) = (C^{B2}, \partial^{B2}) \xrightarrow{\rho^{S2}} Cone(i^S) = (C^{S2}, \partial^{S2})$$

Remember that  $C^{B2} = C^{B0} \oplus C^{B1}$  and  $C^{S2} = C^{S0} \oplus C^{S1}$ .  $\rho^2$  and  $\rho^{S2}$  are respectively defined by:

1.  $\rho^2 = ((C^{B2}, \partial^{B2}), (C^2, \partial^2), h^2, f^2, g^2)$ , with:
  - $h^2 = \begin{pmatrix} -h^0 & 0 \\ f^1 r g^0 & h^1 \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$
  - $f^2 = \begin{pmatrix} 0 \\ f^1 j \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^2$ ,
  - $g^2 = \begin{pmatrix} -s \partial^1 r g^0 & s g^1 \end{pmatrix} : C^2 \longrightarrow C^{B0} \oplus C^{B1}$ ,
2.  $\rho^{S2} = ((C^{B2}, \partial^{B2}), (C^{S2}, \partial^{S2}), h^{S2}, f^{S2}, g^{S2})$ , with:
  - $h^{S2} = \begin{pmatrix} -h^{S0} & h^{S0} i^B h^{S1} \\ 0 & h^{S1} \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$ ,

- $f^{S2} = \begin{pmatrix} f^{S0} & h^{S0}i^B f^{S1} \\ 0 & f^{S1} \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{S0} \oplus C^{S1}$ ,
- $g^{S2} = \begin{pmatrix} g^{S0} & -g^{S0}i^B h^{S1} \\ 0 & g^{S1} \end{pmatrix} : C^{S0} \oplus C^{S1} \longrightarrow C^{B0} \oplus C^{B1}$ .

It is easy to check that  $\rho^2$  and  $\rho^{S2}$  are reductions. The complete constructions and proofs are recalled in [AFLP15] (Annex 6.5 page 58).

- proof of 2.

Let

$$\Upsilon^0 : (C^0, \partial^0) \xleftarrow{\rho^0} (C^{B0}, \partial^{B0}) \xrightarrow{\rho^{S0}} (C^{S0}, \partial^{S0})$$

and

$$\Upsilon^2 : (C^2, \partial^2) \xleftarrow{\rho^2} (C^{B2}, \partial^{B2}) \xrightarrow{\rho^{S2}} (C^{S2}, \partial^{S2})$$

be the two homological equivalences, with :

- $\rho^0 = ((C^{B0}, \partial^{B0}), (C^0, \partial^0), h^0, f^0, g^0)$ ,
- $\rho^{S0} = ((C^{B0}, \partial^{B0}), (C^{S0}, \partial^{S0}), h^{S0}, f^{S0}, g^{S0})$ ,
- $\rho^2 = ((C^{B2}, \partial^{B2}), (C^2, \partial^2), h^2, f^2, g^2)$ ,
- $\rho^{S2} = ((C^{B2}, \partial^{B2}), (C^{S2}, \partial^{S2}), h^{S2}, f^{S2}, g^{S2})$ .

Let  $\chi$ ,  $\chi^B$ , and  $\chi^S$  be defined by:

- $\chi = s\partial^1 r : (C^2, \partial^2) \longrightarrow (C^0, \partial^0)$ ,
- $\chi^B = f^2 \chi g^0 : (C^{B2}, \partial^{B2}) \longrightarrow (C^{B0}, \partial^{B0})$ ,
- $\chi^S = g^{S2} \chi^B f^{S0} : (C^{S2}, \partial^{S2}) \longrightarrow (C^{S0}, \partial^{S0})$ .

Let  $\Upsilon^1 : (C^1, \partial^1) \xleftarrow{\rho^1} (C^{B1}, \partial^{B1}) \xrightarrow{\rho^{S1}} (C^{S1}, \partial^{S1})$  be defined by:

- $C^{B1} = C^{B2} \oplus C^{B0}$ ,  $C^{S1} = C^{S2} \oplus C^{S0}$ ,
- $\partial^{B1} = \begin{pmatrix} \partial^{B2} & \chi^B \\ 0 & \partial^{B0} \end{pmatrix}$ ,  $\partial^{S1} = \begin{pmatrix} \partial^{S2} & \chi^S \\ 0 & \partial^{S0} \end{pmatrix}$
- $\rho^1 = ((C^{B1}, \partial^{B1}), (C^1, \partial^1), h^1, f^1, g^1)$ , with  $h^1 = \begin{pmatrix} h^2 & 0 \\ 0 & h^0 \end{pmatrix}$ ,  
 $f^1 = \begin{pmatrix} f^2 s \\ f^0 i \end{pmatrix}$ ,  $g^1 = \begin{pmatrix} j g^2 & r g^0 \end{pmatrix}$

- $\rho^{S1} = ((C^{B1}, \partial^{B1}), (C^{S1}, \partial^{S1}), h^{S1}, f^{S1}, g^{S1})$ , with
 
$$h^{S1} = \begin{pmatrix} h^{S2} & -h^{S2}\chi^B h^{S0} \\ 0 & h^{S0} \end{pmatrix}, f^{S1} = \begin{pmatrix} f^{S2} & -h^{S2}\chi^B f^{S0} \\ 0 & f^{S0} \end{pmatrix},$$

$$g^{S1} = \begin{pmatrix} g^{S2} & -g^{S2}\chi^B h^{S0} \\ 0 & g^{S0} \end{pmatrix}$$

It can be checked that  $\Upsilon^1$  is a homological equivalence. This mainly comes from the following property: let  $c^2 \in C^2$ , then  $c^2\chi\partial^0 = -c^2\partial^2\chi$  (for more details see [RS06] p. 73).  $\square$

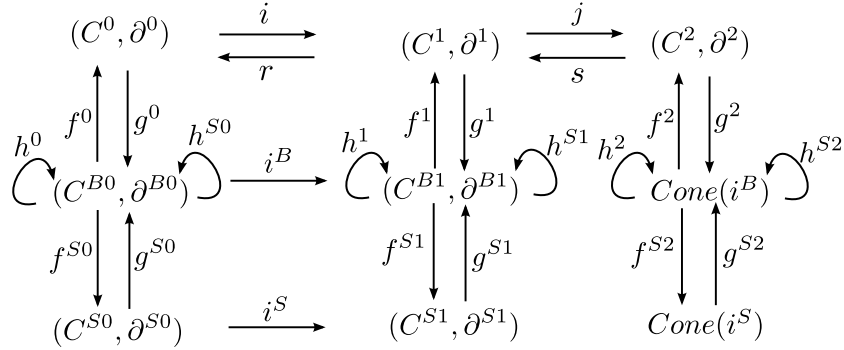


Figure 4: Effective short exact sequence theorem part 1.

**Example 4.** It is displayed on Fig. 5 and comes from Example 2.

Two homological equivalences are associated with  $(C^0, \partial^0)$  and  $(C^1, \partial^1)$ : cf. the left and middle of Fig. 5. Remember that  $a^0i = a^1 - a''^1$ ,  $d^0i = d^1 - d''^1$ .  $i^B$  can be deduced from  $i$  and reductions  $\rho^0$  and  $\rho^1$ , i.e.  $i^B: a^{B0} \rightarrow a'^{B1} - a''^{B1}$ ,  $d^{B0} \rightarrow d'^{B1} - d''^{B1}$ .  $i^S$  can then be deduced from  $i^B$  and reductions  $\rho^{S0}$  and  $\rho^{S1}$ , i.e.  $i^S: a^{S0} \rightarrow a'^{S1} - a''^{S1}$ ,  $d^{S0} \rightarrow d'^{S1} - d''^{S1}$ . Thus, by applying SES theorem, a homological equivalence involving  $(C^2, \partial^2)$ ,  $Cone(i^B)$  and  $Cone(i^S)$  can be deduced: cf. the right of Fig. 5. Note that chain complexes are represented as geometric objects: for instance, on the right of Fig. 5,  $a^{B0}$  is incident to  $a'^{B1}$  and  $a''^{B1}$ , meaning that the boundary of  $a^{B0}$  contains  $a'^{B1}$  and  $a''^{B1}$  as generators (since  $a^{B0}$  corresponds to  $a^0$ ,  $a^0i = a^1 - a''^1$ , and  $a^1$  and  $a''^1$  corresponds to  $a'^{B1}$  and  $a''^{B1}$ ).

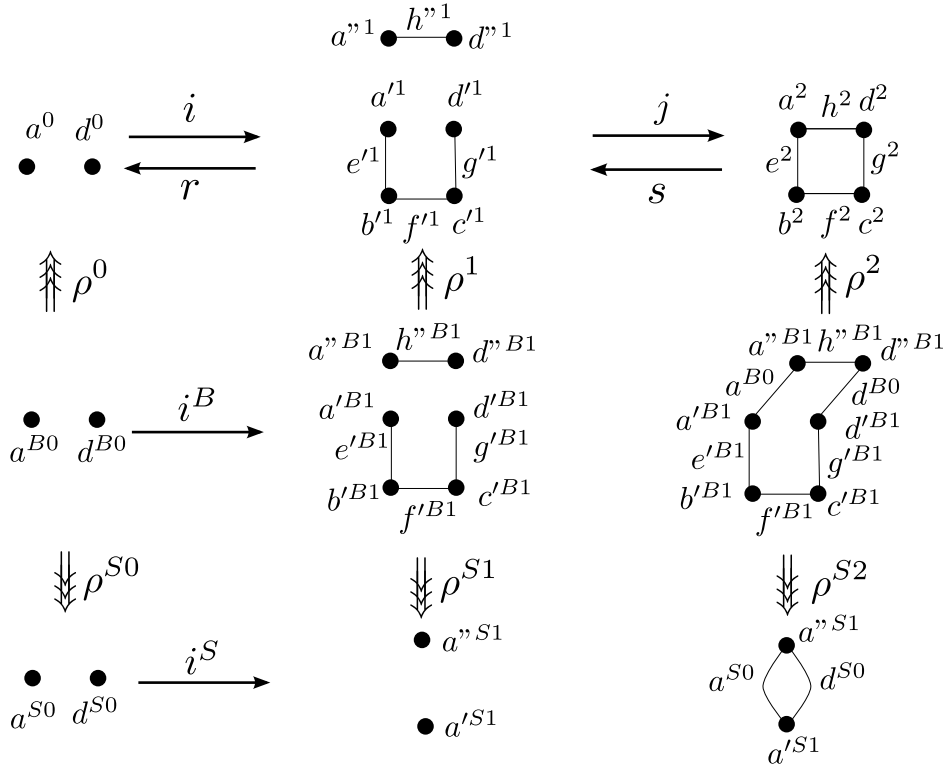


Figure 5: Illustration of SES theorem part 1.

### 3. Application to semi-simplicial sets

#### 3.1. Semi-simplicial sets

Semi-simplicial sets are well-known in Algebraic Topology [EZ50, May67]; they generalize abstract simplicial complexes, for which a similar approach has been followed in [BMLH10, BCMA<sup>+</sup>11]. Moreover semi-simplicial sets are deeply related to "cellular" combinatorial structures studied in section 4.

##### 3.1.1. Definition

**Definition 10.** An  $n$ -dimensional semi-simplicial set  $S = (K, (d_j)_{j \in [0, n]})$  is defined by (cf. figure 6):

- $K = \bigcup_{i \in [0, n]} K_i$ , where  $K_i$  is a finite set the elements of which are called  $i$ -simplices;



- $\forall j \in [0, n], d_j : K \longrightarrow K$ , called *face operator*, is s.t.:
  - $\forall i \in [1, n], \forall j \in [0, i], d_j : K_i \longrightarrow K_{i-1}$ ;  $\forall j > i$ ,  $d_j$  is undefined on  $K_i$ , and no face operator is defined on  $K_0$ ;
  - $\forall i \in [2, n], \forall j, k \in [0, i], d_j d_k = d_k d_{j-1}$  for  $k < j$ .

Let  $\sigma$  be a  $i$ -simplex. Intuitively, if  $\sigma$  is considered as a sequence  $(v_0, \dots, v_i)$  of  $i+1$  vertices, then  $\sigma d_j$  corresponds to the sequence  $(v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_i)$ . A *proper face*  $\tau$  of  $\sigma$  is the image of  $\sigma$  by a non empty composition of face operators; if  $k$  is the dimension of  $\tau$ , then  $\tau$  is an  $k$ -*face* of  $\sigma$ . A simplex which is not the proper face of any simplex is a *main simplex*. The *boundary* of  $\sigma$  is the semi-simplicial set restricted to its proper faces. The *star* of  $\sigma$  is the set of simplices from which  $\sigma$  is a proper face. The *connected component* of  $\sigma$  is the semi-simplicial set obtained from  $\sigma$  by closure of the relations "boundary" and "star". Simplex  $\sigma$  is *complete* if and only if it is incident to  $i + 1$  distinct vertices.

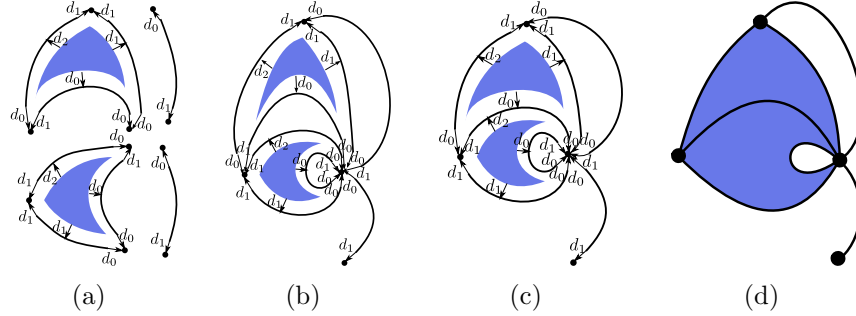


Figure 6: (c) A semi-simplicial set  $S$ . (a) Complete isolated simplices corresponding to the main simplices of  $S$ . (b) and (c),  $S$  is constructed by applying identifications of 0- and 1-simplices. (d) the corresponding simplicial "object".

### 3.1.2. Basic operations

Any semi-simplicial set can be constructed by adding isolated complete simplices (and their boundaries), and by identifying simplices (cf. Fig. 6). Any complete simplex (and its boundary) can be constructed by a sequence of cone operations.

The definitions of operations for adding or removing isolated complete simplices are straightforward.

**Definition 11.** (cf. Fig. 7)

Let  $S = (K, (d_j)_{j \in [0, n]})$  be a semi-simplicial set.  $S^c = (K^c, (d_j^c)_{j \in [0, n+1]})$ , the cone of  $S$ , is defined by:

- Let  $K'$  be a "copy" of  $K$  by a one-to-one mapping  $\phi$  which associates a  $(j + 1)$ -simplex of  $K'$  with any  $j$ -simplex of  $K$ . Let  $v$  be a 0-simplex which belongs neither to  $K$  nor to  $K'$ .

$$K^c = K \cup K' \cup \{v\}$$

- Face operators are defined by:

– Let  $\mu \in K$ ,  $\forall j$ ,  $\mu d_j^c = \mu d_j$

– Let  $\mu \in K'_i$ :

\* if  $i = 1$ , then  $\mu d_1^c = \mu \phi^{-1}$  and  $\mu d_0^c = v$ ,

\* else  $\mu d_i^c = \mu \phi^{-1}$ , and  $\forall 0 \leq j < i$ ,  $\mu d_j^c = \mu \phi^{-1} d_j \phi$ .

Intuitively, if a  $i$ -simplex  $\sigma$  of  $K$  is associated with a sequence  $(v_0, \dots, v_i)$  of 0-simplices, then its corresponding  $(i + 1)$ -simplex in  $K'$  corresponds to  $(v_0, \dots, v_i, v)$ . Note also that any complete  $i$ -simplex can be constructed from a 0-simplex by applying  $i$  cone operations.

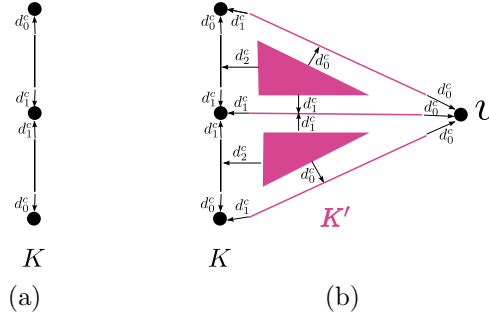


Figure 7: Cone operation applied to a semi-simplicial set  $S = (K, (d_j))$  composed of three vertices and two edges.

**Definition 12.** (cf. Fig. 6)

Let  $S = (K, (d_j)_{j \in [0, n]})$  be a semi-simplicial set, and let  $\sigma$  and  $\tau$  be two  $k$ -simplices such that  $\sigma d_j = \tau d_j$ ,  $\forall j$ .

$S^i = (K^i, (d_j^i)_{j \in [0, n]})$ , the identification of  $\sigma$  and  $\tau$  in  $S$ , is defined by:

- $K^i$  is defined by:
  - $\forall j \neq k, K_j^i = K_j,$
  - $K_k^i = K_k - \{\sigma, \tau\} \cup \{\rho\},$  where  $\rho$  is a  $k$ -simplex which does not belong to  $K_k.$
- Face operators are defined by:
  - $\forall h \notin \{k, k+1\}, \forall \mu \in K_h^i, \forall j, \mu d_j^i = \mu d_j,$
  - $\forall \mu \in K_{k+1}^i, \forall j,$  if  $\mu d_j = \sigma$  or  $\mu d_j = \tau$  then  $\mu d_j^i = \rho$  else  $\mu d_j^i = \mu d_j,$
  - $\forall \mu \in K_k^i, \forall j,$  if  $\mu \neq \rho,$   $\mu d_j^i = \mu d_j$  else  $\rho d_j^i = \sigma d_j.$

In other words, identifying  $\sigma$  and  $\tau$ , if  $\sigma$  and  $\tau$  have the same boundary, consists in replacing them by a new simplex  $\rho$  such that the boundary (resp. the star) of  $\rho$  is the boundary of  $\sigma$  (resp. the union of the stars of  $\sigma$  and  $\tau$ ). This operation can be easily extended in order to identify two simplices having different proper faces by identifying first these distinct proper faces and then the simplices themselves. It can then be extended in order to simultaneously identify a subset of simplices having the same dimension, and then again in order to identify several such subsets.

### 3.1.3. Homology computation during an incremental construction process

A chain complex can be associated with any semi-simplicial set.

**Definition 13.** *The chain complex  $(C, \partial)$  associated with a semi-simplicial set  $S = (K, (d_j)_{j \in [0, n]})$  is defined by:*

- $C = \bigcup_{i \in [0, n]} C_i$  where  $\forall i, C_i$  is the group generated by  $K_i,$
- $\forall i > 0, \partial_i$  is linearly generated by:  $\forall \sigma \in K_i, \sigma \partial_i = \sum_{j=0}^i (-1)^j \sigma d_j$

Assume a semi-simplicial set is constructed through an incremental process. Let  $S^i$  be the semi-simplicial set at step  $i,$  and  $(C^i, \partial^i)$  be its associated chain complex. Assume an operation  $op$  is applied to  $S^i,$  producing  $S^{i+1}.$  It is important that the simplices of  $S^i$  correspond to the generators of  $(C^i, \partial^i):$  since basic operations are local ones, computing  $(C^{i+1}, \partial^{i+1})$  by modifying  $(C^i, \partial^i)$  with respect to  $op$  is more efficient than to compute it from  $S^{i+1}$  by

applying definition 13. So, it is assumed in the following that any simplex of a semi-simplicial set corresponds to a generator of the associated chain complex (it is explained in section 3.2.3 that the chain complex may contain other generators, which correspond to simplices which have been removed from the semi-simplicial set).

In order to optimize the homology computation at each step, a homological equivalence  $\Upsilon^i : (C^i, \partial^i) \rightleftarrows (C^{S^i}, \partial^{S^i})$  is associated with  $S^i$ . A true optimization is obtained when the computational cost of both  $\Upsilon^i$  and the homology of  $(C^{S^i}, \partial^{S^i})$  is lower than the computational cost of the homology of  $(C^i, \partial^i)$ .

In the following, when complexity issues are discussed:

- the complexity of a chain group  $C$ , denoted  $|C|$ , is the number of its generators;
- let  $\Phi$  be a chain group morphism, the complexity of  $\Phi(c)$  is the number of generators involved in the resulting chain. The complexity of  $\Phi$ , denoted  $|\Phi|$ , is the sum of the complexities of  $\Phi$  applied to all generators;
- the complexity of a chain complex  $(C, \partial)$ , denoted  $|C, \partial|$ , is the sum of the complexities of  $C$  and  $\partial$ .

### 3.2. Cone operation - adding / removing isolated simplices

#### 3.2.1. Cone

**Theorem 2.** *Let  $S^1$  be a semi-simplicial set and  $(C^1, \partial^1)$  be its associated chain complex. Let  $S^2$  be the cone of  $S^1$  and  $(C^2, \partial^2)$  be its associated chain complex.*

*Let  $\Upsilon^2 : (C^2, \partial^2) \xrightarrow{\rho^2} (C^2, \partial^2) \xrightarrow{\rho^{S^2}} (C^{S^2}, \partial^{S^2})$  be such that (with the notations of definition 11 page 18):*

- $\rho^2 = ((C^2, \partial^2), (C^2, \partial^2), h^2 = 0, f^2 = id, g^2 = id)$
- $\rho^{S^2} = ((C^2, \partial^2), (C^{S^2}, \partial^{S^2}), h^{S^2}, f^{S^2}, g^{S^2})$ , such that:
  - $C^{S^2}$  contains a unique generator  $\sigma^2$ , which is a 0-dimensional one;
  - $\partial^{S^2} = 0$ ;
  - let  $\mu$  be any generator of  $C^2$ ; if  $\mu$  corresponds to an  $i$ -simplex of  $S^1$ , then  $\mu h^{S^2} = (-1)^{i+1} \mu \phi$ , else  $\mu h^{S^2} = 0$ ;

- let  $\nu$  be any generator of  $C^2$ ; if the dimension of  $\nu$  is 0, then  $\nu f^{S^2} = \sigma^2$ , else  $\nu f^{S^2} = 0$ ;
- $\sigma^2 g^{S^2} = \sigma$ .

$\Upsilon^2$  is a homological equivalence which can be computed in linear time according to  $(C^1, \partial^1)$ . The size of  $\Upsilon^2$  is linear according to  $(C^1, \partial^1)$ .

The proof is in Annex 6.6 page 62 in [ADLP15]. Note that  $(C^{S^2}, \partial^{S^2})$  is the homology of  $(C^2, \partial^2)$ , since  $\partial^{S^2}$  is null : it is a well-known result that the homology of a cone is trivial.

### 3.2.2. Adding isolated simplices

As a consequence, defining a homological equivalence  $\Upsilon^\sigma$  associated with a semi-simplicial set  $S_\sigma$ , containing a complete isolated simplex and its boundary, is straightforward. So, given a semi-simplicial set  $S^1$  together with its corresponding homological equivalence  $\Upsilon^1$ , the homological equivalence  $\Upsilon^2$  associated with  $S^2$ , obtained by adding  $S_\sigma$  to  $S^1$ , is the direct sum of  $\Upsilon^\sigma$  and  $\Upsilon^1$ .

### 3.2.3. Removing isolated simplices

Let  $S^1$  be a semi-simplicial set, and  $\Upsilon^1 : (C^1, \partial^1) \xrightarrow{\rho^1} (C^{B^1}, \partial^{B^1}) \xrightarrow{\rho^{S^1}} (C^{S^1}, \partial^{S^1})$  be its associated homological equivalence. Let  $S_\sigma$  be a complete isolated simplex (with its boundary) of  $S^1$ , and let  $S^2$  be the result of removing  $S_\sigma$  from  $S^1$ . Since  $S_\sigma$  is a connected component of  $S^1$ ,  $(C^1, \partial^1)$  is the direct sum of two chain complexes  $(C^\sigma, \partial^\sigma)$  and  $(C^2, \partial^2)$  associated respectively with  $S_\sigma$  and  $S^2$ . Unfortunately,  $(C^{B^1}, \partial^{B^1})$  is not, in the general case, the direct sum of two chain complexes corresponding to  $(C^\sigma, \partial^\sigma)$  and  $(C^2, \partial^2)$ ; so, we don't know how to directly deduce from  $\Upsilon^1$  a homological equivalence associated with  $S^2$ . Nevertheless, it is simple to associate  $\Upsilon^1$  with  $S^2$  and to mark the generators of  $C^1$  which correspond to  $S_\sigma$ : it is thus easy to deduce the homology of  $S^2$  from the homology of  $(C^{S^1}, \partial^{S^1})$  and the marked generators of  $C^1$ , since the homology of  $S_\sigma$  is characterized by a unique 0-dimensional generator. But it is not straightforward to deduce the generators of the homology groups of  $S^2$  from those computed from  $(C^{S^1}, \partial^{S^1})$ .

### 3.3. Identification operation and its inverse

#### 3.3.1. Construction of a short exact sequence

**Lemma 1.** (cf. example 5) *An effective short exact sequence can be associated with any identification, in linear time according to the size of the chain complex associated with the semi-simplicial set.*

**Proof 1.** *We use here the notations of Definition 9 (cf. page 10).*

Let  $S^1 = (K^1, (d_i^1)_{i \in [0, n]})$  be a semi-simplicial set and  $(C^1, \partial^1)$  be its associated chain complex. Assume that identifications of simplices have to be performed on  $S^1$ . These identifications can involve other identifications in the boundary of these simplices. All these identifications can be easily deduced and we assume that all simplices to be identified are structured in the following way:  $I = \{I_1, I_2, \dots, I_k\}$  where each  $I_u$  contains simplices to be identified together and if  $\sigma_1^1$  and  $\sigma_2^1$  belong to  $I_u$  then  $\forall p$ , either  $\sigma_1^1 d_p^1 = \sigma_2^1 d_p^1$  or  $\exists q$ , such that  $\sigma_1^1 d_p^1 \in I_q$  and  $\sigma_2^1 d_p^1 \in I_q$ .

Let  $S^2 = (K^2, (d_i^2)_{i \in [0, n]})$  be the result of all identifications and  $(C^2, \partial^2)$  be its associated chain complex. Note that  $|C^2, \partial^2| \leq |C^1, \partial^1|$ , and that its computation time is linear in  $|C^1, \partial^1|$  (in practice, if  $S^1$  is modified in order to produce  $S^2$ , the modification time is sub-linear).

Below,  $(C^0, \partial^0)$ ,  $i, j, r, s$  are defined, so that:

$$0 \xrightarrow{0} (C^0, \partial^0) \xleftarrow[r]{i} (C^1, \partial^1) \xleftarrow[s]{j} (C^2, \partial^2) \xrightarrow{0} 0$$

is a short exact sequence.

1. Let  $j$  be the surjective mapping between  $S^1$  and  $S^2$ , corresponding to the identification. More precisely, let  $\sigma^1$  be a simplex of  $S^1$ : if  $\sigma^1$  does not belong to  $I$ , then a simplex of  $S^2$  is uniquely associated with  $\sigma^1$ ; else  $u$  exists, such that  $\sigma^1 \in I_u$ , and all simplices of  $I_u$  are uniquely associated with their resulting simplex in  $S^2$ . Thus due to the definition of  $I$ ,  $j$  is obviously a surjective chain-complex morphism. Moreover,  $|j| = |C^1|$ .
2. For each  $I_u$ , a representative simplex  $\sigma_u^1$  is chosen. Let  $s$  be the mapping defined by: let  $\sigma^2$  be a simplex of  $S^2$ ; either  $\sigma^2$  is the image by  $j$  of a unique simplex  $\sigma^1$  and  $\sigma^2 s = \sigma^1$ , or there exists  $u$  such that  $\sigma^2$  is the image by  $j$  of all simplices in  $I_u$  and  $\sigma^2 s = \sigma_u^1$ .  $s$  is obviously a graded module morphism, and  $|s| = |C^2|$ .
3. Let us define  $C^0$ ,  $i$ , and  $\partial^0$  by

- for each  $\sigma^1 \in I$  which is not a representative simplex, there exists in  $C^0$  a generator  $\sigma^0$  of same dimension<sup>9</sup>. So,  $|C^0|$  is (almost) the number of identified simplices, and  $|C^0| \leq |C^1|$ ;
- for each  $\sigma^0 \in C^0$ ,  $\sigma^0 i = \sigma_u^1 - \sigma^1$ , where  $\sigma^0$  corresponds to  $\sigma^1$  and  $\sigma_u^1$  is the representative simplex of  $\sigma^1$ .  $i$  is obviously an injective graded module morphism, and  $|i| = 2|C^0|$ ;
- $\partial^0$  is defined in order to satisfy:  $\sigma^0 \partial^0 i = \sigma^0 i \partial^1$  where  $\sigma^0$  is a generator of  $C^0$ . As  $\sigma^0 i = \sigma_u^1 - \sigma^1$ ,  $\sigma^0 \partial^0 i = \sigma_u^1 \partial^1 - \sigma^1 \partial^1 = \sum_p (-1)^p (\sigma_u^1 d_p^1 - \sigma^1 d_p^1)$ . For a given  $p$ :
  - either  $\sigma_u^1 d_p^1 = \sigma^1 d_p^1$ ,
  - or  $q$  exists such that  $\sigma_u^1 d_p^1$  and  $\sigma^1 d_p^1$  belong to  $I_q$  (see definition of  $I$  above). In this case:
    - \* either  $\sigma_u^1 d_p^1$  or  $\sigma^1 d_p^1$  is the representative element of  $I_q$  and an element of  $C^0$  corresponds to  $\pm(\sigma_u^1 d_p^1 - \sigma^1 d_p^1)$ ,
    - \* or  $\sigma_u^1 d_p^1 - \sigma^1 d_p^1$  is the image by  $i$  of the difference of two elements of  $C^0$ , as  $\sigma_u^1 d_p^1 - \sigma^1 d_p^1 = (\sigma_q^1 - \sigma^1 d_p^1) - (\sigma_q^1 - \sigma_u^1 d_p^1)$  where  $\sigma_q^1$  is the representative element of  $I_q$ .

Since  $i$  is injective, it is thus possible to define  $\sigma^0 \partial^0 \in C^0$  in unique way, and  $|\partial^0| \leq 2|\partial^1|$  (in fact,  $|\partial^0|$  is at most twice the size of  $\partial^1$  restricted to the identified simplices).

- $\partial^0$  is a boundary operator, since  $\forall \sigma^0 \in C^0$ ,  $\sigma^0 \partial^0 \partial^0 i = \sigma^0 i \partial^1 \partial^1 = 0$  and  $i$  is injective.
- Thus  $(C^0, \partial^0)$  is a chain complex and  $i$  is a chain-complex morphism.

4. Let  $\sigma^1$  be a simplex of  $S^1$ . Either  $\sigma^1$  does not belong to  $I$  and  $\sigma^1 r = 0$ . Or  $\sigma^1$  belongs to some  $I_u$ ; let  $\sigma_u^1$  be the representative simplex of  $I_u$ : either  $\sigma^1 = \sigma_u^1$  and  $\sigma^1 r = 0$ , or  $\sigma^1 r = -\sigma^0$  where  $\sigma^0 i = \sigma_u^1 - \sigma^1$ .  $r$  is obviously a graded module morphism, and  $|r| = |C^0|$ ;

So, the computation of  $(C^0, \partial^0)$ ,  $i$ ,  $j$ ,  $r$  and  $s$  is linear in the size of  $(C^1, \partial^1)$ . We check below the properties of  $i$ ,  $j$ ,  $r$ ,  $s$ .

- $\forall \sigma^0 \in C^0$ ,  $\sigma^0 i r = (\sigma_u^1 - \sigma^1) r = -\sigma^1 r = -(-\sigma^0) = \sigma^0$ , thus  $ir = id_{|C^0|}$ ;

---

<sup>9</sup>In other words, a graded module morphism exists between  $C^0$  and the subset of  $I$  which contains the non representative simplices.

- $\forall \sigma^1 \in S^1$ ,
  - either  $\sigma^1 = \sigma_u^1$  is a representative simplex of some  $I_u$  then  $\sigma_u^1 ri + \sigma_u^1 js = 0 + \sigma_u^2 s = \sigma_u^1$ ,
  - or  $\sigma^1$  belongs to some  $I_u$ , such that  $\sigma_u^1 \neq \sigma^1$  is the representative simplex of  $I_u$ .  $\sigma^1 ri + \sigma^1 js = -(\sigma_u^1 - \sigma^1) + \sigma_u^1 = \sigma^1$ ,
  - or  $\sigma^1$  does not belong to  $I$ , then  $\sigma^1 ri + \sigma^1 js = 0 + \sigma^1$ .

Thus  $ri + js = id|_{C^1}$ ;

- $\forall \sigma^2 \in S^2$ ,  $\sigma^2 sj = \sigma^2$  and thus  $sj = id|_{C^2}$ .

**Example 5.** Let  $S^1$  be the semi-simplicial set depicted on Fig. 8(a). Assume edges  $j^1$  and  $n^1$  (resp.  $p^1$  and  $q^1$ ,  $m^1$  and  $r^1$ ) are identified into  $j^2$  (resp.  $p^2$ ,  $m^2$ ). This involves the identification of vertices  $a^1$  and  $f^1$  (resp.  $c^1$ ,  $e^1$  and  $h^1$ ) into  $a^2$  (resp.  $c^2$ ), producing the semi-simplicial set  $S^2$  depicted on Fig. 8(b). Let:

- $I = \{\{a^1, f^1\}, \{j^1, n^1\}, \{c^1, e^1, h^1\}, \{p^1, q^1\}, \{m^1, r^1\}\}$
- $j$  is defined by:  $a^1 j = f^1 j = a^2$ ;  $j^1 j = n^1 j = j^2$ ;  $c^1 j = e^1 j = h^1 j = c^2$ ;  $p^1 j = q^1 j = p^2$ ;  $m^1 j = r^1 j = m^2$ ;  $b^1 j = b^2$ ;  $d^1 j = d^2$ ;  $g^1 j = g^2$ ;  $i^1 j = i^2$ ;  $k^1 j = k^2$ ;  $l^1 j = l^2$ ;  $o^1 j = o^2$ ;  $s^1 j = s^2$ ;  $t^1 j = t^2$ ;  $u^1 j = u^2$ ;  $v^1 j = v^2$ ;  $w^1 j = w^2$ ;
- $a^1, j^1, c^1, p^1, m^1$  are chosen as representative simplices. Thus  $a^2 s = a^1$ ;  $j^2 s = j^1$ ;  $c^2 s = c^1$ ;  $p^2 s = p^1$ ;  $m^2 s = m^1$ ;  $b^2 s = b^1$ ;  $d^2 s = d^1$ ;  $g^2 s = g^1$ ;  $i^2 s = i^1$ ;  $k^2 s = k^1$ ;  $l^2 s = l^1$ ;  $o^2 s = o^1$ ;  $s^2 s = s^1$ ;  $t^2 s = t^1$ ;  $u^2 s = u^1$ ;  $v^2 s = v^1$ ;  $w^2 s = w^1$ ;
- Let  $C^0 = \{f^0, n^0, e^0, h^0, q^0, r^0\}$ ; thus  $f^0 i = a^1 - f^1$ ;  $n^0 i = j^1 - n^1$ ;  $e^0 i = c^1 - e^1$ ;  $h^0 i = c^1 - h^1$ ;  $q^0 i = p^1 - q^1$ ;  $r^0 i = m^1 - r^1$ .  $f^0 \partial^0 = 0$ ;  $n^0 \partial^0 = f^0 - e^0$ ;  $e^0 \partial^0 = 0$ ;  $h^0 \partial^0 = 0$ ;  $q^0 \partial^0 = h^0 - e^0$ ;  $r^0 \partial^0 = h^0$ . More precisely:
  - $n^0 i \partial^1 = (j^1 - n^1) \partial^1 = a^1 - c^1 - f^1 + e^1 = (a^1 - f^1) - (c^1 - e^1) = (f^0 - e^0) i$ ;
  - $q^0 i \partial^1 = (p^1 - q^1) \partial^1 = e^1 - g^1 - h^1 + g^1 = e^1 - h^1 = (c^1 - h^1) - (c^1 - e^1) = (h^0 - e^0) i$ ;



$$-r^0 i \partial^1 = (m^1 - r^1) \partial^1 = c^1 - d^1 - h^1 + d^1 = c^1 - h^1 = h^0 i;$$

- $f^1 r = -f^0$ ;  $n^1 r = -n^0$ ;  $e^1 r = -e^0$ ;  $h^1 r = -h^0$ ;  $q^1 r = -q^0$ ;  $r^1 r = -r^0$ ;  
for any other simplex  $\sigma$ ,  $\sigma r = 0$ .

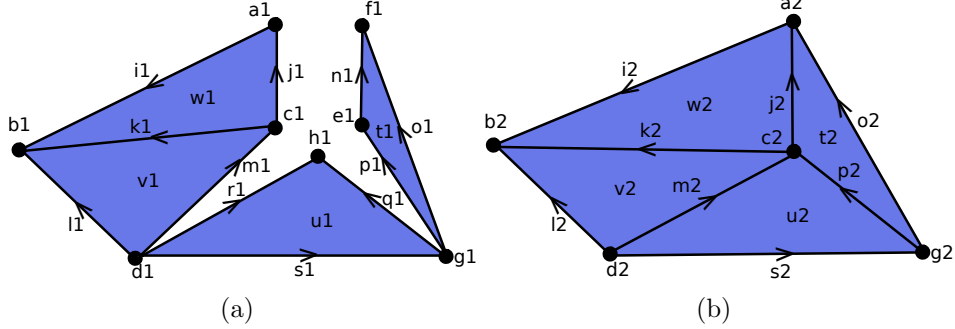


Figure 8: Identifications of several simplices of  $S^1$  (left), producing  $S^2$  (right)

### 3.3.2. Identification

**Theorem 3.** Let  $S^1$  be a semi-simplicial set,  $(C^1, \partial^1)$  be its associated chain complex and  $\Upsilon^1 : (C^1, \partial^1) \xleftarrow{\rho^1} (C^{B1}, \partial^{B1}) \xrightarrow{\rho^{S1}} (C^{S1}, \partial^{S1})$  be a homological equivalence. Let  $S^2$  be the result of any identification applied to  $S^1$ , and  $(C^2, \partial^2)$  be its associated chain complex. A homological equivalence  $\Upsilon^2 : (C^2, \partial^2) \xleftarrow{\rho^2} (C^{B2}, \partial^{B2}) \xrightarrow{\rho^{S2}} (C^{S2}, \partial^{S2})$  can be deduced from  $\Upsilon^1$  and the identification, by one application of the SES theorem (cf. theorem 1 Part 1, page 12).

**Proof 2.** Let:

- $(C^0, \partial^0)$ ,  $i, j, r, s$  be defined as in the proof of lemma 1;
- $\rho^0 = ((C^0, \partial^0), (C^0, \partial^0), h^0 = 0, f^0 = id, g^0 = id)$ ;
- $\rho^{s0} = ((C^0, \partial^0), (C^{S0}, \partial^{S0}), h^{s0}, f^{s0}, g^{s0})$  be any reduction defined on  $(C^0, \partial^0)$  (at least a reduction exists :  $((C^0, \partial^0), (C^0, \partial^0), h^{s0} = 0, f^{s0} = id, g^{s0} = id)$ );
- $\Upsilon^0 : (C^0, \partial^0) \xleftarrow{\rho^0} (C^0, \partial^0) \xrightarrow{\rho^{s0}} (C^{S0}, \partial^{S0})$ ;

By applying the SES theorem, we get (using the notations of this theorem):

- $i^B : (C^0, \partial^0) \rightarrow (C^{B1}, \partial^{B1}),$
- $i^S : (C^{S0}, \partial^{S0}) \rightarrow (C^{S1}, \partial^{S1}),$
- and  $\Upsilon^2 : (C^2, \partial^2) \xleftarrow{\rho^2} \text{Cone}(i^B) = (C^{B2}, \partial^{B2}) \xrightarrow{\rho^{S2}} \text{Cone}(i^S) = (C^{S2}, \partial^{S2}).$

**Example 6.** Fig. 9 shows two construction steps. The initial semi-simplicial set  $S$  contains two complete 2-dimensional simplices and their boundaries. Two edges incident to the upper triangle are identified at the first construction step. At the second step, the resulting edge is identified with an edge incident to the other 2-dimensional simplex.

More precisely, figure 9 shows the semi-simplicial sets together with (representations of) homological equivalences, and illustrates the application of theorem 3. The "big" complex of the homological equivalence associated with  $S$  is a chain complex which is isomorphic the chain complex associated with  $S$ ; the "small" complex contains only two 0-dimensional generators, i.e. it is the homology of  $S$ . The first identification is described on the left of figure 9. The short exact sequence on the top is constructed as described in the proof of lemma 1; we have associated a homological equivalence with the chain complex top left. Then  $i^B$ ,  $i^S$  and the homological equivalence associated with the semi-simplicial set resulting from the identification are computed as described in the proof of theorem 3. The second identification is described on the right of figure 9.

### 3.3.3. Complexity

The complexity of the computation of the resulting homological equivalence  $\Upsilon^2$  is related to the complexity of  $\Upsilon^1$ , and thus depends on the operations which have been applied to construct  $S^1$ : cf. corollary 1 in Annex 6.2. It is thus not possible to estimate the complexity in the general case, since other operations have been defined for constructing semi-simplicial sets, and thus can be applied during a construction process.

Nevertheless, if only identification and cone operations are applied, then the following two results are obtained.

First, if only identifications are applied to a semi-simplicial set, at most all  $i$ -dimensional simplices are identified into one  $i$ -simplex, for any  $i$  (and any further identification is impossible); thus the number of generators of

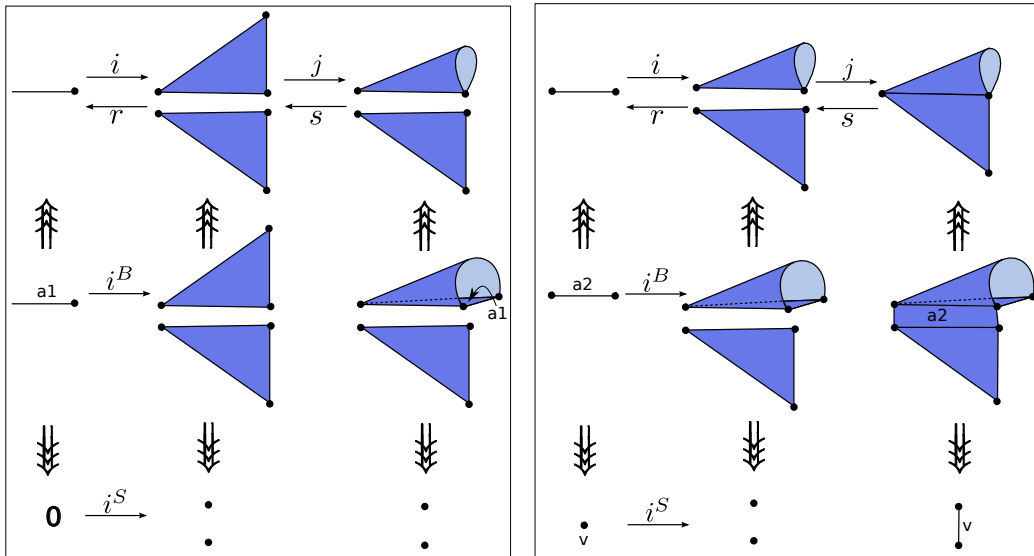


Figure 9: Incremental computation of homological equivalences

the resulting "big" complex is at most twice the number of initial simplices, since any identification of two simplices "adds" a corresponding generator in the "big" complex. So, the number of generators in  $\Upsilon^2$  is at most twice the number of generators of  $\Upsilon^1$  (if  $k$  simplices of  $S^1$  are identified into a unique simplex of  $S^2$ , then the number of generators of  $C^2$  (resp.  $C^{B2}$ ,  $C^{S2}$ ) is almost the number of generators of  $C^1$  minus  $k$  (resp. plus  $k$ , plus  $k$ )).

Second, assume a semi-simplicial set  $S$  is constructed by cones and identifications: cf. figure 10. More precisely, let  $S'$  be defined in the following way: a complete simplex (and its boundary) is associated in  $S'$  with any main simplex of  $S$ . A homological equivalence can thus be associated with any complete simplex, in which the "big" complex corresponds to the complete simplex, and the "small" complex is made of one 0-dimensional generator. The homological equivalence associated with  $S'$  is the sum of all these homological equivalences. All the necessary identifications are applied in order to get  $S$ , by increasing dimensions. For any identification, apply the construction described in theorem 3, where  $\rho^{s_0} = ((C^0, \partial^0), (C^0, \partial^0), h^{s_0} = 0, f^{s_0} = id, g^{s_0} = id)$ . It can be shown that the size and the computation time of the resulting homological equivalence  $\Upsilon$  is linear in the size of  $S'$  (cf. Annex 6.2), and thus does not depend on the number of the construction steps.

**Remark 1.** *These two results show in some way that the incremental method induces a reasonable additional cost, since (under the previous assumptions):*

- *the cost (in size) for maintaining a homological equivalence is at most four times the cost for maintaining a single chain complex;*
- *the cost (in time) related to the modification of the homological equivalence is linear in the size of the modification.*

*If compared with the cost of a Smith Normal Form for instance, it seems thus efficient to maintain at each construction step a homological equivalence, in order to compute the homology of the object from its associated small chain complex.*

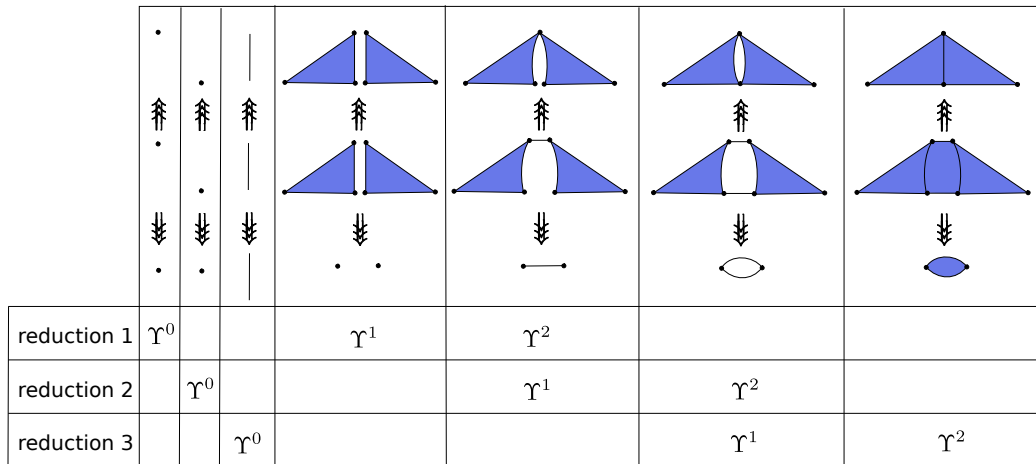


Figure 10: Construction of a semi-simplicial set by cones and identifications. A homological equivalence is computed at each step. The four columns top right describe homological equivalences corresponding to the initial semi-simplicial set and to the three semi-simplicial sets constructed by applying identifications. The columns top left describe, for the three identifications, the homological equivalences corresponding to the identified simplices. The three rows down denote the homological equivalences corresponding to the three identifications, with the notations of theorem 3.

### 3.3.4. Optimizations related to small complexes

In practice, when an identification is performed, "simplifications" can be applied at two moments (cf. Fig. 11), e.g. applying elementary reductions

(cf. annex 6.1) or computing a homological equivalence corresponding to a Smith Normal Form [BMLH10]:

- when computing homological equivalence  $\Upsilon^0$  (using the notations of theorem 3), and more precisely  $\rho^{s_0}$  (i.e. when  $(C^{s_0}, \partial^{s_0})$  is deduced from  $(C^0, \partial^0)$ ); so it is interesting here to simultaneously identify simplices and simplices of their boundaries (such simplifications cannot be applied when identifying simplices by increasing dimensions, for instance);
- after  $\Upsilon^2$  has been computed, it could be possible, and interesting, to simplify its "small" complex, since  $\Upsilon^2$  at one construction step is  $\Upsilon^1$  at the next construction step.

Of course, such simplifications induce more computing time, and it can thus be interesting to elaborate a strategy, with respect to the constructions which are performed.

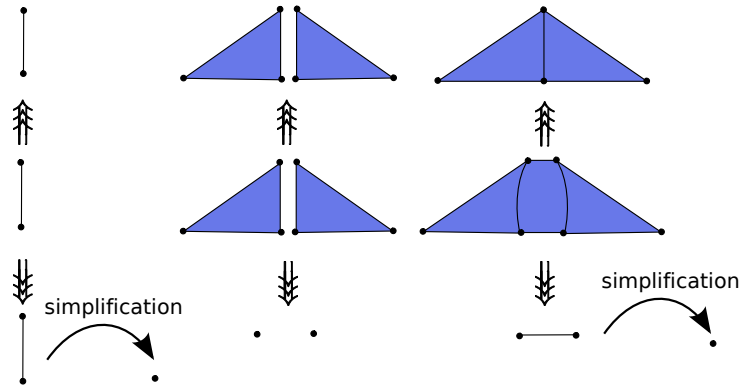


Figure 11: Two kinds of simplifications allowing to reduce the small complex of the resulting homological equivalence. In this example,  $2 * 2$  vertices and 2 edges are identified at the same step. The small complexes of  $\Upsilon^0$  and  $\Upsilon^2$  can be simplified.

Note that for low dimensions, look-up tables can be precomputed in order to store homological equivalences corresponding to identification configurations:

- for main simplices and their boundaries for obtaining the initial  $\Upsilon^1$ : cf. Fig. 12(a)
- for identified simplices for obtaining  $\Upsilon^0$  at each step: cf. Fig. 12(b).

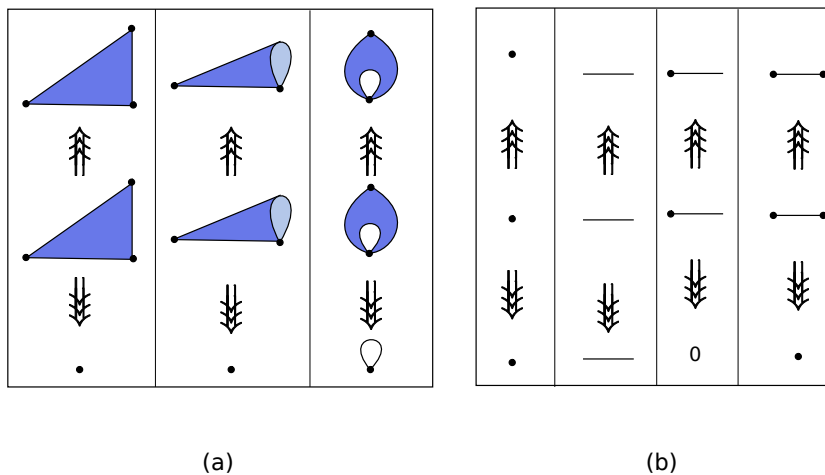


Figure 12: (a) a part of the lookup table regarding main triangles and their boundaries. (b) lookup table for identified 0- and 1-simplices.

### 3.3.5. Inverse of identification

The formal definition of this operation is not recalled here. Let  $S$  be a semi-simplicial set to which this operation is applied, and let  $S'$  be the resulting semi-simplicial set. So,  $S$  could be obtained from  $S'$  by applying the identification operation; by applying the construction described in the proof of lemma 1, a short exact sequence

$$0 \xrightarrow{0} (C^0, \partial^0) \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{i} \end{array} (C^1, \partial^1) \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{j} \end{array} (C^2, \partial^2) \xrightarrow{0} 0$$

can be defined, such that  $(C^1, \partial^1)$  (resp.  $(C^2, \partial^2)$ ) is the chain complex associated with  $S'$  (resp.  $S$ ). It is possible to construct a homological equivalence associated with  $(C^0, \partial^0)$ , in a way similar to that described in the proof of theorem 3. If a homological equivalence is associated with  $S$ , then a homological equivalence associated with  $S'$  can be computed by applying the SES theorem (cf. theorem 1 Part 2, page 12).

The computation complexities for the identification operation and its inverse operation are thus similar.

## 3.4. Complements

### 3.4.1. Application to simplicial, cubical and simplicial structures

All results presented above can be directly applied to:

- abstract simplicial complexes [Ago76]; in this case, the identification operation has to be restricted in order to avoid multi-incidence, i.e. in order to avoid the degeneracy of simplices;
- simplicial sets [May67]; in this case, the boundary operator is restricted to the non degenerate simplices (i.e. the boundary of a simplex is defined according to definition 13, except that the coefficients of degenerate simplices are null).

Cubical sets [Ser51, BH81] and simploidal sets [Gug57, PFL09, LP16] are combinatorial structures close to simplicial sets, such that cells are cubes or simploids, i.e. products of simplices. For these structures, adding and identifying cubes or simploids are also basic operations for which similar results can be obtained:

- it is possible to define a homological equivalence for any complete isolated cube or simploid (and its boundary) such that the small chain complex is reduced to a single 0-dimensional generator<sup>10</sup>;
- the short exact sequence construction described in the proof of lemma 1 can easily be adapted for cubical and simploidal structures, so a theorem equivalent to theorem 3 can be obtained.

### 3.4.2. Comparison with Mayer-Vietoris algorithm

At last, note that tools and results of Effective Homology have also been applied by Boltcheva *et al* [BMLH10] for the conception of an algorithm called *Mayer-Vietoris (MV) algorithm*, which computes the homological information of abstract simplicial complexes from the homological information of their sub-complexes and their intersections. This algorithm has been applied for the *Manifold-Connected (MC) decomposition* of abstract simplicial complexes [BCMA<sup>+</sup>11]. More precisely, a MC decomposition of an abstract simplicial complex  $A$  is a decomposition of  $A$  into sub-complexes which satisfy combinatorial properties close to that of quasi-manifolds (cf. Section 4.1.1). Given two sub-complexes  $B$  and  $C$ , the MV algorithm computes the homological information of  $B \cup C$  from the homological informations of  $B$  and  $C$ , and the homological information of  $B \cap C$ .

---

<sup>10</sup>Moreover, given two homological equivalences associated with the operands of the cartesian product operation, it is possible to compute a homological equivalence associated with its result: cf. [RS06] p.56.

In fact, the MV algorithm corresponds to a particular case of theorem 3, even if it is restricted to abstract simplicial complexes.

Let  $A$  be an abstract simplicial complex,  $B$  and  $C$  be two subcomplexes such that  $A = B \cup C$ . The MV algorithm is based on the existence of a short exact sequence  $\xi$ <sup>11</sup>

$$0 \xrightarrow{0} B \cap C \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{i} \end{array} B \oplus C \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{j} \end{array} B \cup C = A \xrightarrow{0} 0$$

This construction corresponds to the identification, in  $B \oplus C$ , of the parts corresponding to  $B \cap C$ , and the short exact sequence associated with the identification is also  $\xi$ .

Conversely, it is not always possible to apply MV algorithm for any identification applied to a semi-simplicial set. For instance, let  $O$  be an object composed by an edge and its two incident vertices; if these two vertices have to be identified, then it is not possible to decompose  $O$  into a direct sum and to apply MV algorithm. Moreover, even if it is possible to apply MV algorithm, it can be inefficient: if the identification is applied to simplices belonging to the same connected component  $O$ , it would be necessary to compute a partition of  $O$  into two connected components, and the corresponding homological equivalences.

In this way, theorem 3 generalizes the approach of Boltcheva *et al*, since the identification operation is a basic construction operation.

## 4. Application to "cellular" combinatorial structures

### 4.1. "Cellular" combinatorial structures

More precisions about the notions presented in this section can be found in [Bri93, Lie94, Ber99, DL14].

#### 4.1.1. Numbered semi-simplicial sets

**Definition 14.** (cf. Fig. 13) A numbered semi-simplicial set  $S = (K, (d_j), \nu)$  is a semi-simplicial set together with a mapping  $\nu : K^0 \rightarrow \mathbb{N}$  such that the numbering  $(i_0, \dots, i_n)$  of each  $n$ -simplex  $\sigma$ , i.e. the sequence of integers associated with its 0-faces by  $\nu$ , satisfies  $i_k \neq i_l$  for  $k \neq l$ .

---

<sup>11</sup>By abuse of notation,  $X$  denotes either an abstract simplicial complex and its associated chain complex.



Note that any simplex is a complete one. Moreover, for each  $n$ -simplex  $\sigma$  numbered  $(i_0, \dots, i_j, \dots, i_n)$ , with  $i_0 < \dots < i_j < \dots < i_n$ , it is always possible to define the face operators such that  $\sigma d_j$  is numbered  $(i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_n)$ ,  $0 \leq j \leq n$ . Note that numbered semi-simplicial sets as defined here are slightly more general than that defined in [DL14, ADLP15], which satisfy the constraint 1 below.

**Definition 15.** (cf. Fig. 13) An  $i$ -cell is a 0-simplex numbered  $i$  together with the subset of its star made of the simplices whose numberings have  $i$  as highest integer.

The dimension of any simplex of a  $i$ -cell is lower than or equal to  $i$ . Moreover, the set of cells is a partition of  $K$ . Note also that, given any  $k$ -simplex  $\sigma$  belonging to a given  $i$ -cell  $c$ ,  $\sigma d_p$  belongs to  $c$  for any  $p \in [0, k-1]$ . The *boundary of a cell* is its simplicial boundary minus the simplices which belong to the cell itself: it is thus composed by cells of lower dimensions (cf. Fig. 13(b)).

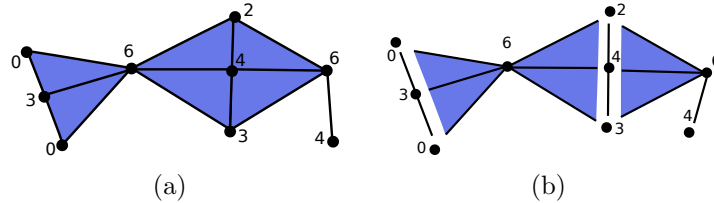


Figure 13: (a) A numbered semi-simplicial set, represented in a simplified way, since the boundary operators can be retrieved from the numbering. (b) Its cells: two 0-cells, one 2-cell, one 3-cell, two 4-cells, and two 6-cells.

Any numbered semi-simplicial set can be constructed by adding isolated numbered simplices (and their boundaries), and by identifying simplices having same numbering.

Often, cellular combinatorial structures correspond to subclasses of numbered semi-simplicial sets, which can be defined by adding constraints:

1. to the definition. For instance, the dimension of a cell is that of its main simplices if each *main*  $n$ -simplex  $\sigma \in K$  satisfies (cf. Fig. 14):
  - if  $n = 0$  then  $\sigma\nu = 0$ ;

- if  $n > 0$  then its numbering is  $(0, \dots, n)$ ;
2. and/or to the construction operations. For instance, cells can be constructed by applying *cellular cone* operations, and they can be identified by applying *cellular identifications*. Let  $S = (K, (d_j), \nu)$  be a numbered semi-simplicial set:
- assume  $\sigma\nu < n$  for any 0-simplex  $\sigma \in K$ ; the  $n$ -dimensional cellular cone of  $S$  is the cone of  $(K, (d_j))$ , where the new vertex is numbered  $n$ , and the numberings of all other vertices is defined by  $\nu$ ; note that this cellular cone contains only one  $n$ -cell;
  - the cellular identification of two  $i$ -cells  $c_1$  and  $c_2$  having the same boundary is defined if an isomorphism  $\phi$  exists between  $c_1$  and  $c_2$ . In this case, the cellular identification consists in identifying the simplices of  $c_1$  and  $c_2$  with respect to  $\phi$ . This operation can easily be generalized in order to identify cells and cells of their boundaries (if any two cells which are identified together are isomorphic).

For instance, the numbered semi-simplicial set represented on figure 13 can be constructed by cellular cones and cellular identifications.

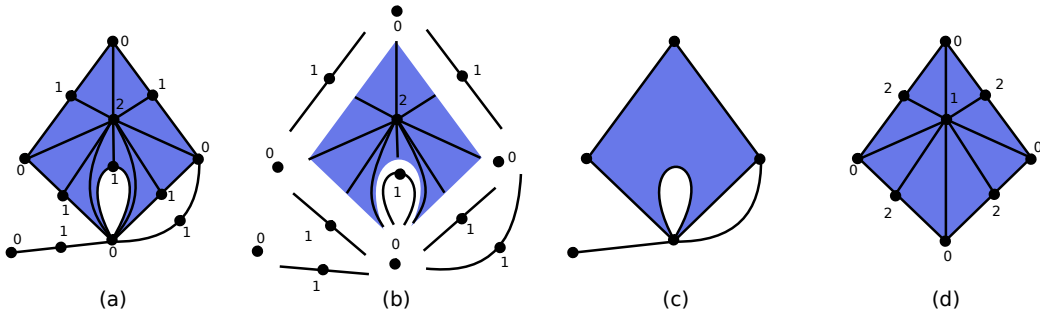


Figure 14: (a) A numbered semi-simplicial set. (b) Its cells. (c) The corresponding “cellular object”. (d) A numbered simplicial quasi-manifold which is not a cellular quasi-manifold, since the 1-cell is incident to four vertices.

Among these subclasses, *cellular quasi-manifolds* are of particular interest; they can be defined in the following way:

**Definition 16.** A numbered  $n$ -dimensional simplicial quasi-manifold is a numbered semi-simplicial set, which can be constructed by (cf. Fig. 14(d)):

- adding  $n$ -simplices numbered  $(0, \dots, n)$  and their boundaries.
- identifying  $(n - 1)$ -simplices (and their boundaries), in such a way that at most two  $n$ -simplices belong to the star of any  $(n - 1)$ -simplex.

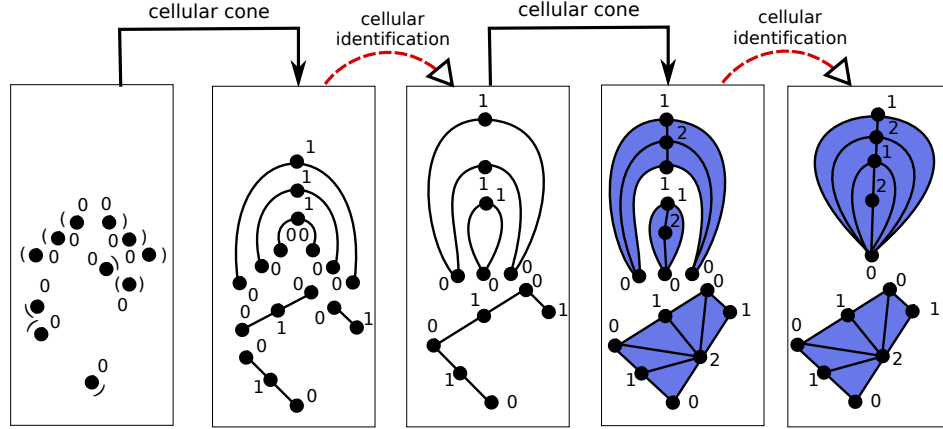


Figure 15: Construction of a cellular quasi-manifold, applying cellular cones and cellular identifications.

**Definition 17.** An  $n$ -dimensional cellular quasi-manifold is (cf. Fig. 16(a), (b) and Fig. 15 ):

1. for  $n = 0$ , a set of 0-simplices numbered  $(0)$ , partitioned into connected components made of one or two 0-simplices<sup>12</sup>;
2. for  $n \geq 1$ :
  - either the  $n$ -dimensional cellular cone of a connected  $(n - 1)$ -dimensional cellular quasi-manifold;
  - or the result of the cellular identification of two  $(n - 1)$ -cells  $c_1$  and  $c_2$  of an  $n$ -dimensional cellular quasi-manifold, s.t.:
    - each  $(n - 1)$ -simplex of  $c_1$  and  $c_2$  is incident to one  $n$ -simplex,
    - an isomorphism  $\phi$  exists between  $c_1$  and  $c_2$ . If  $c_1 = c_2$ , then  $\phi$  is an automorphism different from the identity s.t.  $\phi = \phi^{-1}$ .

<sup>12</sup>A connected component containing two 0-simplices describes the topology of a 0-sphere.

Note that contrary to numbered simplicial quasi-manifolds, any cell of a cellular quasi-manifold is the inside of a cellular quasi-manifold: cf. Fig. 16(a); cf. Fig. 14(d) for a counter-example; note also that even if the numbered semi-simplicial set represented on Fig. 14(a) is not a cellular quasi-manifold, its cells are cells of cellular quasi-manifolds.

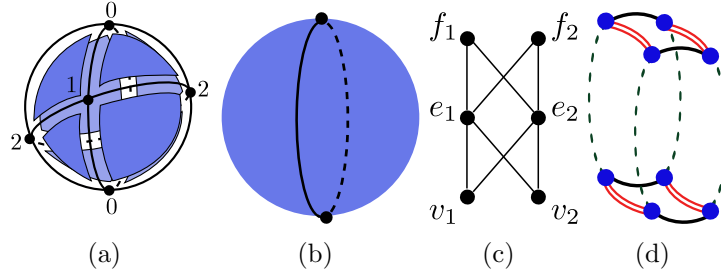


Figure 16: (a) A cellular quasi-manifold. (b), (c) and (d) The corresponding "cellular object", incidence graph and gmap. The graphical conventions for representing incidence graphs and gmaps are described on Fig. 17 and Fig. 18.

#### 4.1.2. Incidence graphs - combinatorial maps

"Cellular" combinatorial structures can be thought of as optimizations for the representation of subclasses of numbered semi-simplicial sets.

For instance, the cell is the basic element for incidence graphs. Note that many classes of incidence graphs have been defined (cf. [Ede87, RO89, Bri93, Ber99]<sup>13</sup>). The "basic" notion can be defined in the following way. An  $n$ -dimensional incidence graph  $G$  is a graph  $(V, E)$  such that (cf. Fig. 17(a), (d), (g)):

- $V = \bigcup_{i \in [0, n]} V_i$ , such that  $V_i \neq \emptyset$  for any  $i$ ;  $c_i \in V_i$  is an  $i$ -cell;
- $E \subset \bigcup_{i \in [1, n]} V_i \times V_{i-1}$ ; any  $i$ -cell is the origin of at least one edge, if  $i > 0$ .

A *face path*  $(c_i, c_{i-1}, \dots, c_j)$ , with  $i \geq j$ , is a sequence of cells such that  $(c_k, c_{k-1}) \in E$  for  $k \in [j + 1, i]$ . Let  $c$  be an  $i$ -cell;  $c'$  is a  $j$ -face of  $c$  if a face path  $(c_i, c_{i-1}, \dots, c_j)$  exists, such that  $c = c_i$  and  $c' = c_j$ . The boundary of  $c$  is the set of (or the subgraph restricted to) its faces.

<sup>13</sup>They are sometimes referred to as *orders*, *Hasse diagrams*, etc.

Any incidence graph can be constructed by the *extension* and *cell identification* operations:

- the extension of an  $(n - 1)$ -dimensional incidence graph is an incidence graph of dimension  $n$ , obtained by adding a unique  $n$ -cell  $c$ , and by adding an edge  $(c, c')$  for each  $(n - 1)$ -cell  $c'$  (cf. Fig. 17(d), (g));
- the identification of two cells  $c$  and  $c'$  consists in replacing  $c$  and  $c'$  by a unique cell  $c''$ , and in modifying the edges accordingly (cf. Fig. 17(a), (d)). This operation is often constrained in order to avoid modifications of the cell structures (cf. below).

The *simplicial analog* of an incidence graph is an  $n$ -dimensional numbered semi-simplicial set, such that (cf. Fig. 17(a), (b), (d), (e)):

- a 0-simplex numbered  $i$  is associated with any  $i$ -cell;
- a  $k$ -simplex  $\sigma$  is associated with any subsequence  $(c_{i_k}, \dots, c_{i_0})$  of a face path;  $\sigma d_j$  is the  $(k - 1)$  simplex associated with  $(c_{i_k}, \dots, c_{i_{j+1}}, c_{i_{j-1}}, \dots, c_{i_0})$ .

The extension of an incidence graph corresponds to the cellular cone of its simplicial analog.

Let  $c_1$  and  $c_2$  be two cells satisfying: the boundaries of  $c_1$  and  $c_2$  are equal, and no cell contains both  $c_1$  and  $c_2$  in its boundary; in this case, applying the cell identification operation to the incidence graph corresponds to the application of the cellular identification operation to its simplicial analog. Note that  $c_1$  and  $c_2$  are isomorphic (i.e. a one-to-one mapping exists between their associated sets of simplices, which preserves the numbering); moreover, the structure of any cell (i.e. its associated set of simplices) is not modified by the operation. The cell identification operation can be generalized in order to identify two cells having isomorphic boundaries.

Subclasses of incidence graphs have been defined, for instance  $n$ -surfaces, whose simplicial analogs are cellular quasi-manifolds [DCB05, ADLL08]: cf. Fig. 16(c).

At last, note that *multi-incidence* is not taken into account by incidence graphs; for instance, it is not possible to represent an edge loop, i.e. an edge incident twice to a vertex. In order to take multi-incidence into account, the

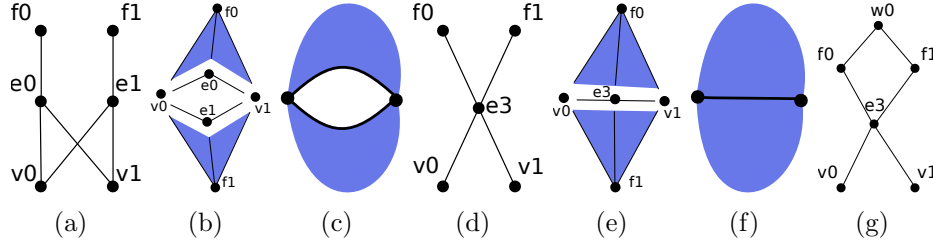


Figure 17: (a) An incidence graph. (b), (c) The corresponding numbered semi-simplicial set and "cellular object". (d), (e) and (f) are obtained from (a), (b) and (c) by identifying edges  $e_0$  and  $e_1$ . (g) is the result of the extension of (d). The graphical conventions for (b) and (e) are the following: a vertex labelled  $v_i$  (resp.  $e_i, f_i$ ) is numbered (0) (resp. (1), (2)).

basic element of a combinatorial map is the *dart* [Vin83, Lie94], and not the cell; a dart corresponds to a main simplex of a cell in the simplicial analog.

Different subclasses of combinatorial maps have been defined [DL14]. For instance, given  $n \geq 0$ , an  $n$ -gmap is defined by an  $(n + 2)$ -tuple  $G = (D, \alpha_0, \dots, \alpha_n)$  such that:

- $D$  is a finite set of darts;
- $\forall i, 0 \leq i \leq n, \alpha_i : D \rightarrow D$  is an involution;
- $\forall i, 0 \leq i \leq n - 2, \forall j, i + 2 \leq j \leq n, \alpha_i \alpha_j$  is an involution.

By convention,  $G = (D)$  is a  $(-1)$ -gmap, where  $D$  is a set of darts.

Let  $G = (D, \alpha_0, \dots, \alpha_n)$  be an  $n$ -gmap. A connected component of  $G$  is an *orbit*<sup>14</sup> relatively to  $\langle \alpha_0, \dots, \alpha_n \rangle$ . An  $i$ -dimensional cell of  $G$  is an orbit relatively to  $\langle \alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n \rangle$ .

For example, the gmap in Fig. 18 contains three connected components. On the top, the connected component contains two faces: a square face  $\langle \alpha_0, \alpha_1 \rangle (d26) = \{d_{20}, \dots, d_{27}\}$ , and a pentagon  $\langle \alpha_0, \alpha_1 \rangle (d28) = \{d_{12}, \dots, d_{19}, d_{28}, d_{29}\}$ . These two faces share the edge  $\langle \alpha_0, \alpha_2 \rangle (d19) = \{d_{19}, d_{20}, d_{27}, d_{28}\}$ . Each face is incident only once to this edge since each face

<sup>14</sup>Let  $\Phi = \{\pi_0, \dots, \pi_n\}$  be a set of permutations defined on a set  $D$ . The permutation group of  $D$  generated by  $\Phi$  is denoted  $\langle \Phi \rangle = \langle \pi_0, \dots, \pi_n \rangle$ . The orbit of an element  $d \in D$  relatively to  $\langle \Phi \rangle$ , denoted  $\langle \Phi \rangle (d)$ , is the set  $\{d\phi \mid \phi \in \langle \Phi \rangle\}$ . It denotes also the structure  $(D^d = \langle \Phi \rangle (d), \pi_0|_{D^d}, \dots, \pi_n|_{D^d})$ , where  $\pi_i|_{D^d}$  denotes the restriction of  $\pi_i$  to  $D^d$ .

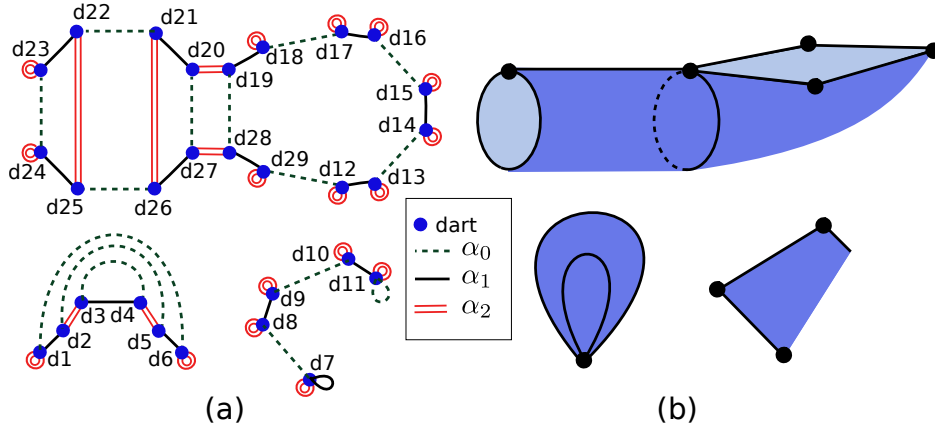


Figure 18: Graphical representation of a 2-gmap: the conventions for representing darts and involutions are described between (a) and (b). (a) a 2-gmap made of 3 connected components. (b) The corresponding “cellular objects”.

and the edge share a single orbit relatively to  $\alpha_0$ . Similarly the square face makes a cylinder since it is incident twice to the edge  $\langle d_{21} \rangle = \{d_{21}, d_{22}, d_{25}, d_{26}\}$  (the face and the edge share two orbits relatively to  $\alpha_0$  i.e.  $\langle \alpha_0 \rangle (d_{21})$  and  $\langle \alpha_0 \rangle (d_{26})$ ).

Any  $n$ -gmap  $G = (D, \alpha_0, \dots, \alpha_n)$  can be obtained from the  $(-1)$ -gmap  $G = (D)$  by applying *extension*, and *sewing* operations<sup>15</sup>, defined below (cf. figure 19 and figure 20). The extension of  $G = (D, \alpha_0, \dots, \alpha_n)$  is the  $(n + 1)$ -gmap  $G' = (D, \alpha_0, \dots, \alpha_n, \alpha_{n+1} = id)$ . The sewing of two darts  $d_1$  and  $d_2$  by  $\alpha_n$  in  $G = (D, \alpha_0, \dots, \alpha_n)$  can be defined if:

- $d_1 \neq d_2$ ,
- $d_1 \alpha_n = d_1$  and  $d_2 \alpha_n = d_2$ ,
- $O_{d_1} = \langle \alpha_0, \dots, \alpha_{n-2} \rangle (d_1)$  and  $O_{d_2} = \langle \alpha_0, \dots, \alpha_{n-2} \rangle (d_2)$  are isomorphic; let  $\phi$  be the corresponding isomorphism,
- if  $O_{d_1} = O_{d_2}$ , then  $\phi = \phi^{-1}$ .

<sup>15</sup>Gmaps can be handled through other sets of operations [Lie94, BSP<sup>+</sup>05, DL03, DD09, BPA<sup>+</sup>10].

It is the gmap  $G' = (D, \alpha_0, \dots, \alpha_{n-1}, \alpha'_n)$ , such that<sup>16</sup>:

- $\forall d \in D - (O_{d_1} \cup O_{d_2}), d\alpha'_n = d\alpha_n$
- $\forall d \in O_{d_1}, d\alpha'_n = d\phi$
- if  $O_{d_1} \neq O_{d_2}$ , then  $\forall d \in O_{d_2}, d\alpha'_n = d\phi^{-1}$ .

Gmaps *without self-bending* [CMP06, ADLP15] are constructed by exclusively sewing distinct orbits (i.e.  $O_{d_1} \neq O_{d_2}$ ).

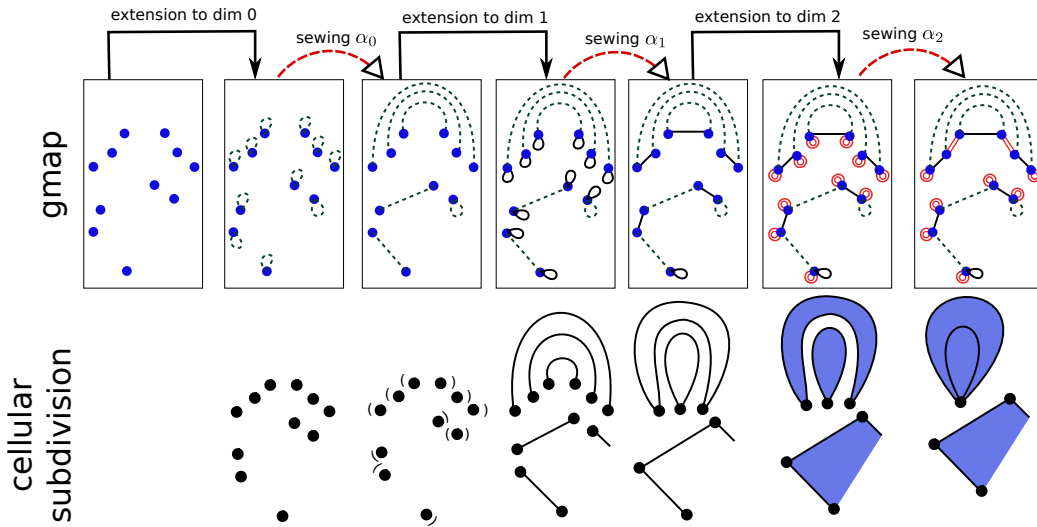


Figure 19: The 2-gmap  $G' = (D, \alpha_0, \alpha_1, \alpha_2)$  on the right is obtained from the  $(-1)$ -gmap  $G = (D)$  on the left by applying a sequence of extension and sewing operations. The corresponding “cellular object” is represented under each gmap. Note that identifying two edges consists in sewing two isomorphic orbits relatively to  $\alpha_0$ . This construction corresponds to the one represented in Fig. 15.

Generalized maps are equivalent to cellular quasi-manifolds. The conversion processes between an  $n$ -gmap  $G$  and its corresponding cellular quasi-manifold  $M$  are (cf. Figs. 19 and 15):

<sup>16</sup>Note that a general sewing operation is defined for any  $n$ -gmap, and any dimension  $i, 0 \leq i \leq n$  [Lie94].



- $M \rightarrow G$  : a dart is associated with each  $n$ -simplex ; two darts corresponding to two  $n$ -simplices sharing a  $(n - 1)$ -simplex numbered  $(0, \dots, i - 1, i + 1, \dots, n)$  are linked by  $\alpha_i$ ; in fact, all  $\alpha_i$  correspond to adjacency relations between  $n$ -simplices of  $M$ . If  $n = 0$ ,  $\alpha_0$  links darts corresponding to 0-simplices belonging to the same connected component.
- $G \rightarrow M$  : Let  $N = [0, n]$ . For any non-empty  $I \subseteq N$ , an  $(|I| - 1)$ -simplex<sup>17</sup> is associated with any orbit  $\langle \rangle_{N-I}$ ; for any  $j$ , for any  $\sigma$  corresponding to an orbit  $\langle \rangle_{N-I}(b)$ ,  $\sigma d_j$  is the simplex corresponding to  $\langle \rangle_{N-I'}(b)$ , where  $I'$  is  $I$  without its  $j^{\text{th}}$  element. When  $n = 0$ , each connected component of  $M$  corresponds to a connected component of  $G$ .

Note that the sewing (resp. extension) operation for gmaps is equivalent to the cellular identification (resp. cellular cone) operation for cellular quasi-manifolds (resp. connected cellular quasi-manifolds)<sup>18</sup>.

#### 4.2. Simplicial chain complex and cellular operations

Let  $CO$  be the subset of numbered semi-simplicial sets, which can be constructed by adding / removing cells (and their boundaries), such that each cell (and its boundary) is the result of a cellular cone, and by applying cellular identifications. In particular,  $CO$  contains the simplicial analogs of incidence graphs and combinatorial maps without self bending, as defined in the previous section. So, the incremental construction of numbered semi-simplicial sets of  $CO$  is first studied (cf. sections 4.2.1 and 4.2.2); then the results are applied for the construction of incidence graphs or combinatorial maps (cf. section 4.2.3).

Let  $M$  be a numbered semi-simplicial set of  $CO$ . In order to compute efficiently the homology at each step of a construction process, a homological equivalence is associated with  $M$ , and *also with any cell* of  $M$ : in particular when cells are identified, their associated homological equivalences can be used in order to efficiently compute the homological equivalences of the result.

---

<sup>17</sup> $|I|$  denotes the number of elements of  $I$ .

<sup>18</sup>More generally, all important notions can be directly translated: for instance, the notions of cells in cellular quasi-manifolds can be directly retrieved through the notion of compacted cells of gmaps: cf. property 5, [ADLP15].

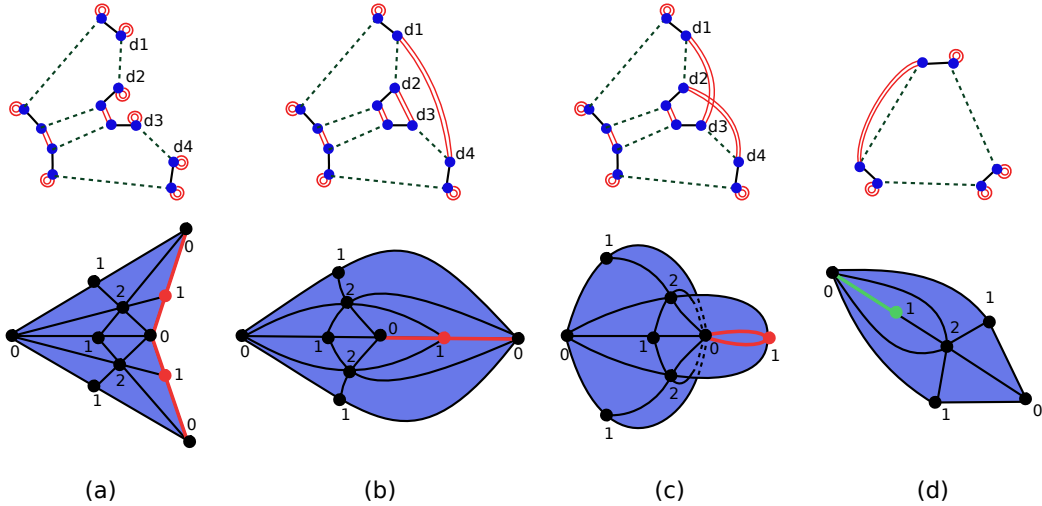


Figure 20: (a) a gmap and its corresponding cellular quasi manifold. (b) On the top, sewing of two isomorphic orbits of dimension 1 (i.e. two edges), s.t.  $d1\phi = d4$  and  $d2\phi = d3$ . On the bottom, the corresponding identifications in the associated cellular quasi-manifold. (c) On the top, sewing of two isomorphic edges s.t.  $d1\phi = d3$  and  $d2\phi = d4$ . On the bottom, the corresponding identifications in the associated cellular quasi-manifold. Note that in this case, three 0-cells are identified. (d) a gmap with self-bending: an edge is identified with itself. In the associated cellular quasi-manifold, the corresponding 1-cell is not isomorphic to other edges.

Note that homological equivalences can be associated with isolated 0-cells. The simplicial analog of a 0-cell is a 0-simplex numbered (0). A homological equivalence can be easily built: the 3 chain complexes are isomorphic, homotopy operator  $h$  is null,  $f$  and  $g$  mappings correspond to the isomorphisms. Figure 21 illustrates the homological equivalences for a 0-gmap made of three darts in two connected components.

The incremental computations of the homological equivalences (for the resulting cellular structure and its cells) related to a cellular cone operation or to a cellular identification operation are studied in sections 4.2.1 and 4.2.2. It is then possible to deduce how to take into account operations for adding and removing cells, or the inverse operation of the cellular identification: this is not described in this paper, since the methods are similar to those presented for the simplicial case (cf. section 3).

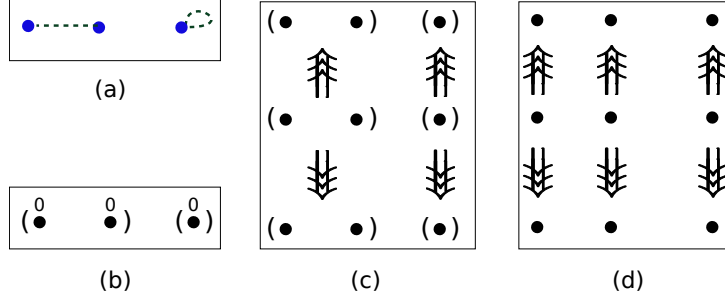


Figure 21: (a) a 0-gmap, (b) its corresponding cellular quasi-manifold, (c) homological equivalences related to the connected components, (d) homological equivalences related to the cells.

#### 4.2.1. Cellular cone

**Theorem 4.** Let  $M^1$  be a numbered semi-simplicial set of  $CO$ , and  $U^1$  be its homological equivalences. Let  $M^2$  be the cellular cone of  $M^1$ .

$M^2$  and associated homological equivalences  $U^2$  can be computed in linear time in the sizes of  $M^1$  and  $U^1$  respectively.

**Proof 3.** The numbered semi-simplicial set  $M^2$  is obtained by applying a cellular cone operation on  $M^1$ : the size of  $M^2$  is thus linear according to the size of  $M^1$ , and its computation time is also linear in the size of  $M^1$  (cf. definition 11 page 18). In a first step, it is explained how to obtain a homological equivalence for  $M^2$ . In a second step, it is explained how to obtain homological equivalences for the cells of  $M^2$ .

#### Homological equivalence for the numbered semi-simplicial set

Since a cellular cone is applied to  $M^1$ , theorem 2 can be applied (cf. page 20). More precisely, the associated homological equivalence  $\Upsilon^2$  is the following:  $(C^{B^2}, \partial^{B^2})$  is isomorphic to  $(C^2, \partial^2)$ , and  $(C^{S^2}, \partial^{S^2})$  is made of a single vertex with a null boundary operator since it is a well known result that the homology of a cone is trivial. The computation of  $(C^2, \partial^2)$  is linear in the size of  $M^1$  since it is a cone. So, the same result holds for  $(C^{B^2}, \partial^{B^2})$ , and the computation time of  $(C^{S^2}, \partial^{S^2})$  is constant. The two reductions  $\rho^2 : C^{B^2} \rightarrow C^2$  and  $\rho^{S^2} : C^{B^2} \rightarrow C^{S^2}$  are defined as follows:

$\rho^2$  : the homotopy operator  $h^2$  is null, the mappings  $f^2$  and  $g^2$  are isomorphisms. Thus, the time complexity for computing  $\rho^2$  is linear in the size of  $M^1$ .

$\rho^{S^2}$  : the homotopy operator  $h^{S^2}$  associates with each  $k$ -simplex  $\sigma \in M^1$ , the corresponding  $(k+1)$ -simplex  $\tau$  in the cone, s.t.  $\sigma h^{S^2} = (-1)^{k+1} \tau$ ,  $h^{S^2}$  is null everywhere else. The mapping  $f^{S^2}$  is null everywhere except for the vertex of the cone and the initial vertices of  $M^1$ , which are associated with the single vertex of  $C^{S^2}$ . The mapping  $g^{S^2}$  associates the vertex of  $C^{S^2}$  with the vertex of the cone in  $C^{B^2}$ . Thus, the time complexity for computing  $\rho^{S^2}$  is linear in the size of  $M^1$ .

*Homological equivalences for cells*

Let  $n$  be the number associated with the vertex  $v$  of the cone. For each dimension lower than  $n$ , neither cells nor their homological equivalences are modified. The computation time is thus linear in the size of  $U^1$  restricted to these cells (in practice, nothing is changed: so, the modification time is null).

Let  $c$  be the  $n$ -cell of  $M^2$ . Let

$$\Upsilon^1 : (C^1, \partial^1) \xleftarrow{\rho^1} (C^{B^1}, \partial^{B^1}) \xrightarrow{\rho^{S^1}} (C^{S^1}, \partial^{S^1})$$

be the homological equivalence associated with  $M^1$ . The homological equivalence

$$\Upsilon^2 : (C^2, \partial^2) \xleftarrow{\rho^2} (C^{B^2}, \partial^{B^2}) \xrightarrow{\rho^{S^2}} (C^{S^2}, \partial^{S^2})$$

associated with  $c$  is deduced from  $\Upsilon^1$  in the following way:

- The new complex  $(C^2, \partial^2)$  is defined by  $C^2 = C_0^2 \oplus \dots \oplus C_n^2$ , where:
  - $C_0^2$  has only one generator corresponding to the vertex  $v$  of the cone;
  - for  $i \geq 1$ ,  $C_i^2$  is isomorphic to  $C_{i-1}^1$ . Let  $\phi$  be the sum of all these isomorphisms.

The boundary operator  $\partial^2$  is defined as  $\partial'^2 + \delta^2$  where:

- for any  $i \geq 1$ ,  $\partial'_i{}^2 = \phi^{-1} \partial_{i-1}^1 \phi$ ,
- $\delta^2$  is null everywhere, except  $\delta_1^2$  which associates  $v$  with any generator of  $C_1^2$ .

Note that  $\partial'^2$  is a boundary operator. Moreover  $\partial^2$  is a boundary operator since:

- $\forall i \neq 1, \partial_i^2 = \partial_i'^2$ ;

- remember that any 1-simplex of a numbered semi-simplicial set is incident to two distinct 0-simplices ; so, and also due to the definition of a cone, for any generator  $\sigma \in C_2^2$ ,  $\sigma\partial^2 = e - e'$  where  $e \neq e'$  and  $e\partial_1^2 = e'\partial_1^2 = v$ . So,  $\delta^2$  is a perturbation of  $\partial^2$ .

The computation time of  $(C^2, \partial^2)$  is linear in the size of  $M^1$ . In practice, the main changes are related to the boundaries of 1-simplices of the  $n$ -cell  $c$ , so the modification time is linear in the number of 0-simplices of  $M^1$ .

- The new complex  $(C^{B2}, \partial^{B2})$  is defined by  $C^{B2} = C_0^{B2} \oplus \dots \oplus C_n^{B2}$ , where:
  - $C_0^{B2}$  has only one generator  $v^B$ ,
  - for any  $i \geq 1$ ,  $C_i^{B2}$  is isomorphic to  $C_{i-1}^{B1}$ . Let  $\phi^B$  be the sum of all these isomorphisms.

Let  $\partial'^{B2}$  be defined by: for any  $i$ ,  $\partial_i'^{B2} = (\phi^B)^{-1}\partial_{i-1}^{B1}\phi^B$ . Since  $\partial^{B1}$  is a boundary operator,  $\partial'^{B2}$  is also a boundary operator.

Let  $\rho = ((C^{B2}, \partial'^{B2}), (C^2, \partial^2), h, f, g)$  be such that:

- $h = (\phi^B)^{-1}h^1\phi^B$ ,
- $f_{|(C^{B2}-\{v^B\})} = (\phi^B)^{-1}f^1\phi$ ,  $v^B f = v$ ,
- $g_{|(C^2-\{v\})} = \phi^{-1}g^1\phi^B$ ,  $vg = v^B$ ,

where  $h^1, f^1, g^1$  are the homotopy operator and morphisms of  $\rho^1$ . Since  $\rho^1$  is a reduction,  $\rho$  is also a reduction.

Since  $\delta^2$  is a perturbation of  $\partial^2$ , the Easy Perturbation Lemma applies (cf. property 3 page 9). So:

- $\delta^{B2} = f\delta^2g$  is a perturbation of  $\partial'^{B2}$ , and thus  $\partial^{B2} = \partial'^{B2} + \delta^{B2}$  is a boundary operator,
- $\rho^{B2} = ((C^{B2}, \partial^{B2}), (C^2, \partial^2), h, f, g)$  is a reduction.

The computation time of  $C^{B2}$ ,  $\partial^{B2}$  and the application of the Easy Perturbation Lemma are linear in the size of  $M^1$ . In practice, the main changes are related to the boundary of generators of  $C_1^{B2}$ . Thus the modification time is linear in the number of generators of  $C_0^{B1}$ .

- The new complex  $(C^{S^2}, \partial^{S^2})$  is defined by  $C^{S^2} = C_0^{S^2} \oplus \dots \oplus C_n^{S^2}$ , where:
  - $C_0^{S^2}$  has only one generator  $v^S$ ,
  - for any  $i \geq 1$ ,  $C_i^{S^2}$  is isomorphic to  $C_{i-1}^{S^1}$ . Let  $\phi^S$  be the sum of all these isomorphisms.

Let  $\partial^{S^2}$  be defined by: for any  $i$ ,  $\partial_i^{S^2} = (\phi^S)^{-1} \partial_{i-1}^{S^1} \phi^S$ . Since  $\partial^{S^1}$  is a boundary operator,  $\partial^{S^2}$  is also a boundary operator.

Let  $\rho^S = ((C^{B^2}, \partial^{B^2}), (C^{S^2}, \partial^{S^2}), h^S, f^S, g^S)$  be such that:

- $h^S = (\phi^B)^{-1} h^{S^1} \phi^B$ ,
- $f_{|(C^{B^2} - \{v^B\})}^S = (\phi^B)^{-1} f^{S^1} \phi^B$ ,  $v^B f = v^S$ ,
- $g_{|(C^{S^2} - \{v^S\})}^S = (\phi^S)^{-1} g^1 \phi^B$ ,  $v^S g = v^B$ ,

where  $h^{S^1}, f^{S^1}, g^{S^1}$  are the homotopy operator and morphisms of  $\rho^{S^1}$ . Since  $\rho^{S^1}$  is a reduction,  $\rho^S$  is a reduction.

Moreover,  $\delta^{B^2}$  is a perturbation of  $\partial^{B^2}$  s.t.  $\delta^{B^2} h^S = 0$ , and thus  $\delta^{B^2}$  satisfies the nilpotency hypothesis: indeed,  $\delta^2$  is null everywhere except on  $C_1^2$ ; so  $\delta^{B^2}$  is null everywhere except on  $C_1^{B^2}$ , since  $\delta^{B^2} = f \delta^2 g$ . Let  $\gamma$  be a chain of  $C_1^{B^2}$ .  $\gamma \delta_1^{B^2} h^S = \alpha v g h^S = \alpha v^B h^S = 0$ , since  $\delta^2$  associates  $v$  with any generator of  $C_1^2$ .

So,  $\partial^{S^2}$  and the reduction  $\rho^{S^2}$  can be deduced from  $\rho^S$  by applying the Basic Perturbation Lemma (cf. property 4 page 9), where the mappings  $\Phi$  and  $\Psi$  are equal to the identity.

The computation time of  $C^{S^2}$ ,  $\partial^{S^2}$  and the application of the Basic Perturbation Lemma are linear in the number of the generators of  $C^{B^1}$  (this is due to the fact that  $\delta^{B^2} h^S = 0$ ). In practice, the main changes are related to the boundary of generators of  $C_1^{S^2}$ . Thus the modification time is linear in the number of generators of  $C_0^{S^1}$ .

### Complexity

Each homological equivalence restricted to  $M^2$  or to one of its cells can be computed in linear time and space, according to the size of the initial corresponding homological equivalence, so the whole complexity of  $U^2$  is linear in the size of  $U^1$ .

In practice, it is only necessary to compute homological equivalences for:

- $M^2$ : this is linear in the size of  $M^1$ ,
- the  $n$ -cells: this is linear in the size of the corresponding homological equivalences of  $U^1$ .

#### 4.2.2. Cellular identification

**Theorem 5.** Let  $M^1$  be a numbered semi-simplicial set of CO, and  $U^1$  be its homological equivalences. Let  $M^2$  be the result of the cellular identification of isomorphic cells. Homological equivalences  $U^2$  associated with  $M^2$  can be directly deduced from  $U^1$ , by one application of the SES theorem (cf. theorem 1 Part 1 page 12).

#### Proof 4.

Homological equivalences for cells

Let  $c$  be a cell of  $M^2$ :

- either  $c$  corresponds to a cell  $c'$  of  $M^1$ , i.e.  $c'$  is not affected by the identification; so the homological equivalence associated with  $c$  is a copy of the one of  $c'$ ;
- or  $c$  results from the identification of several cells of  $M^1$ ; since all these cells are isomorphic, a copy of the homological equivalence associated with one of the initial cells can be associated with  $c$ . In practice, it should be interesting to choose the most “efficient” homological equivalence, for instance according to the size of its small chain complex.

So the computation time is linear in the size of  $U^1$  restricted to the cells. In practice nothing is modified, so the modification time is null.

Homological equivalence for  $M^2$

Preliminary remarks and notations

Let

$$\Upsilon^1 : (C^1, \partial^1) \xleftarrow{\rho^1} (C^{B1}, \partial^{B1}) \xrightarrow{\rho^{S1}} (C^{S1}, \partial^{S1})$$

be the homological equivalence associated with  $M^1 = (K^1, (d_i^1)_{i=0,\dots,n})$ .

Let  $(C^2, \partial^2)$  be the chain complex associated with  $M^2$ .

Remember that the cellular identification of cells of  $M^1$  implies the identification of cells of their boundaries if necessary; the condition for the identification is: “to be identified, two cells have to be isomorphic”. In the following, “cell” denotes either the corresponding set of simplices, or the set of simplices

together with the internal face relations, or the corresponding generators of a chain complex, or the corresponding sub chain complex.

The cells which have to be identified can be structured in the following way:  $I = \{I_1, I_2, \dots, I_k\}$ , such that each  $I_u$  contains all cells which have to be identified together. Moreover, for each  $I_u$ , a representative cell  $c_u$  is chosen; let  $\phi_u$  be the surjective morphism which maps any non representative cell of  $I_u$  (i.e. different from  $c_u$ ) onto  $c_u$ .

So, if  $c_1$  and  $c_2$  belong to  $I_u$ ,  $c_1$  and  $c_2$  are isomorphic ; let  $\phi_{1,2}$  be the corresponding isomorphism. Let  $\sigma_1 \in c_1$  and  $\sigma_2 \in c_2$  be two  $p$ -simplices, such that  $\sigma_1 \phi_{1,2} = \sigma_2$  and  $p > 0$ . So, either  $\sigma_1 d_p = \sigma_2 d_p$ , or  $\sigma_1 d_p$  and  $\sigma_2 d_p$  belong to two isomorphic cells of the same  $I_q$ , such that  $\sigma_1 d_p$  and  $\sigma_2 d_p$  are associated by the corresponding isomorphism. At last, remember that for any  $j, 0 \leq j \leq p - 1$ ,  $\sigma_1 d_j$  (resp.  $\sigma_2 d_j$ ) belongs to  $c_1$  (resp.  $c_2$ ), and thus  $\sigma_1 d_j \phi_{1,2} = \sigma_2 d_j$ .

Construction of a homological equivalence  $\Upsilon^0$

$$\Upsilon^0 : (C^0, \partial^0) \xleftarrow{\rho^0} (C^{B0}, \partial^{B0}) \xrightarrow{\rho^{S0}} (C^{S0}, \partial^{S0})$$

Let

$$\Upsilon'^0 : (C^0, \partial^0) \xleftarrow{\rho'^0} (C^{B'0}, \partial^{B'0}) \xrightarrow{\rho'^{S0}} (C^{S'0}, \partial^{S'0})$$

be the sum of all homological equivalences associated with<sup>19</sup> the non representative cells of  $I$ . Let  $\phi_{0,1}$  be the bijection between  $C^0$  and  $I$  minus all its representative cells. For each cell of  $C^0$ ,  $\phi_{0,1}$  restricted to this cell is an isomorphism with the corresponding cell of  $C^1$ .

Let  $\delta^0$  be defined in the following way (cf. Fig. 22). For any  $p$ -dimensional generator  $\sigma$  of  $C^0$ , such that  $p > 0$ , let  $\sigma^1 = \sigma \phi_{0,1}$  and  $\sigma_u^1 = \sigma^1 \phi_u$ :

- if  $\sigma_u^1 d_p = \sigma^1 d_p$ , then  $\sigma \delta^0 = 0$ ;
- else  $v$  exists, such that  $\sigma_u^1 d_p$  and  $\sigma^1 d_p$  belong to  $I_v$ . In this case:

---

<sup>19</sup>or equivalent homological equivalences, since the homological equivalences associated with isomorphic cells are equivalent; in other words, for each  $u$  between 1 and  $k$ , we choose one "best" homological equivalence for each  $I_u$ , and  $\Upsilon'^0$  contains  $q_u$  copies of this homological equivalence,  $q_u$  being the number of non representative cells of  $I_u$ .



- if  $\sigma_u^1 d_p$  (resp.  $\sigma^1 d_p$ ) belongs to the representative cell of  $I_v$ ,  $\sigma^1 d_p$  (resp.  $\sigma_u^1 d_p$ ) does not belong to this representative cell, and a generator  $\sigma_s$  (resp.  $\sigma_r$ ) exists in  $C^0$ , such that  $\sigma_s \phi_{0,1} = \sigma^1 d_p$  (resp.  $\sigma_r \phi_{0,1} = \sigma_u^1 d_p$ ); in this case,  $\sigma \delta^0 = (-1)^p \sigma_s$  (resp.  $\sigma \delta^0 = -(-1)^p \sigma_r$ );
- at last, if neither  $\sigma_u^1 d_p$  nor  $\sigma^1 d_p$  belong to the representative cell of  $I_v$ , let  $\sigma_r$  (resp.  $\sigma_s$ ) be the generator of  $C^0$  associated with  $\sigma_u^1 d_p$  (resp.  $\sigma^1 d_p$ ); in this case,  $\sigma \delta^0 = (-1)^p (\sigma_s - \sigma_r)$ .

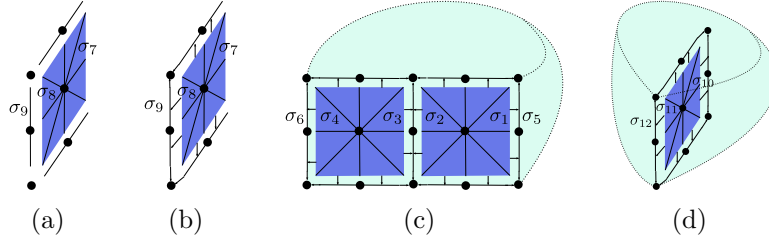


Figure 22: The identification of the two 2-cells (and their boundaries) of (c) produces the object (d). (a) and (b) Representations of the chain complexes  $(C^0, \partial^0)$  and  $(C^0, \delta^0)$ . Since  $\sigma_3 d_2 = \sigma_2 d_2$ ,  $\sigma_7 \delta^0 = 0$ . Since  $\sigma_4 d_2 = \sigma_6 \neq \sigma_5 = \sigma_1 d_2$ ,  $\sigma_8 \delta^0 = \pm \sigma_9$ , depending on the choice of the representative cells.

Note that, if  $\sigma$  belongs to a homological equivalence corresponding to an identified  $p$ -dimensional cell,  $\sigma \delta^0$  contains generators belonging to homological equivalences corresponding to cells of dimension lower than  $p$ . Moreover,  $\delta^0$  is a perturbation of  $\partial^0$ . Let  $i : C^0 \rightarrow C^1$  be defined by: for any generator  $\sigma$  of  $C^0$ ,  $\sigma i = \sigma_u^1 - \sigma^1$ .  $i$  is thus an injective graded module morphism. Let  $\sigma$  be a  $p$ -dimensional generator of  $C^0$ .  $\sigma(\partial^0 + \delta^0)i = \sigma \partial^0 i + \sigma \delta^0 i$ . Since  $\phi_{0,1}$  and  $\phi_u$ , restricted to cells, are isomorphisms,  $\sigma \partial^0 i = \sum_{j=0}^{p-1} (\sigma i) d_j = \sum_{j=0}^{p-1} (\sigma_u^1 - \sigma^1) d_j$ . It is easy to deduce from the four cases of the definition of  $\delta^0$  that  $\sigma(\partial^0 + \delta^0)i = \sigma i \partial^1$ . Since  $i$  is injective,  $\partial^0 = \partial^0 + \delta^0$  is a boundary operator, and  $\delta^0$  is a perturbation of  $\partial^0$ .

Reduction  $\rho^0 : (C^0, \partial^0) \xrightarrow{\rho^0} (C^{B0}, \partial^{B0})$  can be deduced from reduction  $\rho^0 : ((C^{B0}, \partial^{B0}), (C^0, \partial^0), h^0, f^0, g^0)$  and  $\delta^0$  by applying the Easy Perturbation Lemma, where  $\partial^{B0} = \partial^0 + \delta^0$  and  $\delta^{B0} = f^0 \delta^0 g^0$ .

Reduction  $\rho^{S0} : (C^{B0}, \partial^{B0}) \xrightarrow{\rho^{S0}} (C^{S0}, \partial^{S0})$  can be deduced from reduction  $\rho^{S0} : ((C^{B0}, \partial^{B0}), (C^{S0}, \partial^{S0}), h^{S0}, f^{S0}, g^{S0})$  and  $\delta^{B0}$  by applying the Basic

*Perturbation Lemma.* Indeed, it is easy to show that  $\delta^{B^0}$  satisfies the nilpotency hypothesis. Let  $c$  be a generator of  $C^{B^0}$ :  $c$  belongs to some homological equivalence corresponding to an identified  $p$ -dimensional cell, and  $cf^0$  belongs to the same homological equivalence. If  $cf^0$  is not null, let  $c'$  be a generator of  $cf^0$ . Then  $c'\delta^0$  is either null, or a sum of generators of  $C^0$ : in the latter case, each of these generators belongs to a homological equivalence corresponding to a cell of lower dimension than  $p$  (cf. definition of  $\delta^0$  above), so do the generators of  $c'\delta^0 g^0$  and of  $c'\delta^0 g^0 h^{S^0}$ . So,  $m$  exists, such that  $c(\delta^{B^0} h^{S^0})^m = 0$ ; moreover,  $m \leq n$ , the dimension of  $M^1$ .

Construction of an effective short exact sequence

1. A surjective chain complex morphism  $j : (C^1, \partial^1) \rightarrow (C^2, \partial^2)$  can be deduced from the cellular identification; in particular, given  $u$  between 1 and  $k$ , all cells of  $I_u$  are mapped onto a unique resulting cell.
2. Let  $s$  be the mapping defined in the following way. Let  $c^2$  be a cell of  $C^2$ :
  - either  $c^2$  is the image by  $j$  of a unique cell  $c^1$ , and  $c^2 s = c^1$ ;
  - or  $u$  exists, such that  $c^2$  is the image by  $j$  of all cells of  $I_u$ , and  $c^2 s$  is the representative cell of  $I_u$ .

The mapping  $s$  is a graded module morphism (in fact, restricted to each cell, it is a sub chain complex morphism).

3.  $i : C^0 \rightarrow C^1$  has been defined in the previous paragraph, when proving that  $\delta^0$  is a perturbation of  $\partial^0$ ; it has also been proved that  $i$  is an injective chain complex morphism;
4.  $r : C^1 \rightarrow C^0$  associates any cell with 0, except the non representative cells of  $I$ : let  $c$  be such a cell, then  $cr = -c\phi_{0,1}^{-1}$ .

As for the simplicial case, it is easy to prove that  $(C^0, \partial^0), (C^1, \partial^1), (C^2, \partial^2), j, s, i, r$  define an effective short exact sequence.

*Conclusion and complexity*

It is clear that the SES theorem can be applied in order to deduce a homological equivalence

$$\Upsilon^2 : (C^2, \partial^2) \xleftarrow{\rho^2} (C^{B^2}, \partial^{B^2}) \xrightarrow{\rho^{S^2}} (C^{S^2}, \partial^{S^2})$$

associated with  $M^2$ , from  $\Upsilon^0, \Upsilon^1, j, s, i, r$ .

**Remark 2.** *Note that the constructions of  $\Upsilon^0, i, j, r, s$  are linear in the size of their arguments, except the application of the Basic Perturbation Lemma (for the construction of  $\Upsilon^0$ ), which depends also upon the dimension of the identified cells, i.e. upon  $n$ , the dimension of  $M^1$ . The computational complexity involved by the application of the SES theorem has already been studied: cf. section 3.3.3.*

*Other remarks*

1. Similarly to the simplicial case studied in section 3, simplification techniques as elementary reductions or reductions related to Smith Form [BMLH10] can be applied at any moment in order to reduce the size of the small complex of any homological equivalence (associated with a connected component or a cell). More precisely:
  - For the cellular cone operation. It is not possible to simplify the homological equivalences associated with  $M^2$ ; the homological equivalences associated with cells of dimensions lower than  $n$  are not modified; so, it is not useful to try to simplify them, if it has been done at a construction step before. But it can be useful to try to simplify the homological equivalence associated with the  $n$ -cell: remember that (the homological equivalence  $\Upsilon_c$  associated with)  $n$ -cell  $c$  is deduced from (the homological equivalence  $\Upsilon_{M^1}$  associated with) a preexisting  $(n-1)$ -dimensional numbered semi-simplicial set  $M^1$ . Even if it is not possible to simplify  $\Upsilon_{M^1}$ , it could be possible to simplify  $\Upsilon_c$ : cf. Fig. 23;
  - For the cellular identification operation. The homological equivalences associated with cells after the application of the operation are equal to homological equivalences associated with cells before: so, it is not useful to try to simplify them, if it has been done at a construction step before. For  $M^2$ , it can be interesting (as usual) to try to simplify the homological equivalence  $\Upsilon^2$  resulting from the application of the SES theorem. Moreover, it can also be interesting to try to simplify  $\Upsilon^0$  before the application of the SES theorem, since it results from the "linking" of all homological equivalences associated with the identified cells with respect to the boundary relations of these cells. This is similar to the simplicial case, when simplices and simplices of their boundaries are simultaneously identified: simplifications are possible, which are

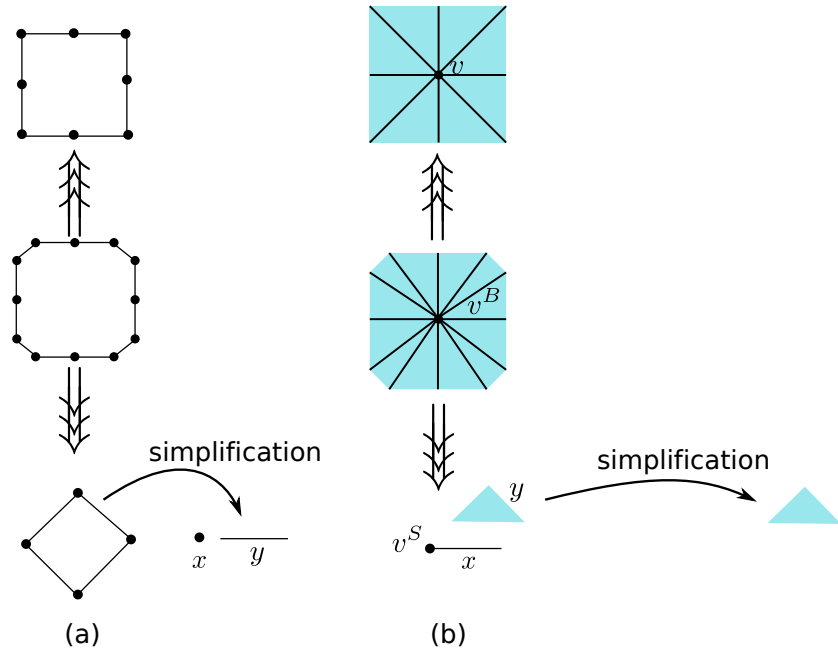


Figure 23: (a) A homological equivalence corresponding to a numbered semi-simplicial set. (b) The homological equivalence of the 2-cell deduced from (a) by the extension operation. The small complexes computed for both homological equivalences can be simplified, as shown on (a) and (b).

not possible when simplices of different dimensions are identified independently (cf. Fig. 9 and Fig. 10).

2. The construction of  $\Upsilon^2$  described in the proof of theorem 5 is based on the construction of  $\Upsilon^0$  and one application of the SES theorem. One application of the Basic Perturbation Lemma is necessary in order to construct  $\Upsilon^0$ , involving a complexity which is linear in the size of the homological equivalences associated with the identified cells times their dimensions<sup>20</sup>. Note that it could be possible to construct a homological equivalence associated with  $M^2$  in  $n$  steps, by identifying cells from dimension 0 until dimension  $n - 1$ : SES theorem would be thus applied  $n$  times, but the complexity would be worse, and the possibility of

<sup>20</sup>This corresponds to the application of the Basic Perturbation Lemma: cf. nilpotency hypothesis.

simplifying  $\Upsilon^0$  before applying the SES theorem would be lost.

3. As for the "simplicial case", look-up tables can be used:
  - if the number of "types" of cells is low; lookup tables can be used in order to memorize simplified homological equivalences which can then be used later for the identification of cells of the same types. In practice, when a homological equivalence has been computed, it is memorized in order to be used for the other isomorphic cells.
  - For low dimensions; lookup tables can also be used in order to memorize homological equivalences corresponding to identification configurations (i.e. corresponding to  $\Upsilon^0$ ). For instance in dimension 2 for cellular quasi-manifolds, there exist one configuration for the identification of 2 vertices, and a few number of configurations for the identification of two edges<sup>21</sup>. For dimension 3, the number of configurations for the identification of 2-cells can be low, depending on a particular application.

To take full benefit of lookup tables if cellular combinatorial structures are handled, efficient methods should be used to compute isomorphisms of parts of such structures (cf. [SDdJ15] for instance).

#### *4.2.3. Application to incidence graphs and combinatorial maps*

As for the "simplicial case", it is important to maintain a strong correspondence between the combinatorial structure and its associated chain complex, in order to efficiently "propagate" the operation applied to the structure on the associated chain complex (more generally, to its associated homological equivalence and to the homological equivalences associated with its cells). The correspondence is straightforward for the "simplicial case", since any simplex of a semi-simplicial set is associated with a generator of the associated chain complex: so, a "link" can be maintained between a simplex and its associated generator, or a single representation can be used for both. If a simplex is concerned when an operation is applied, it is thus easy to propagate any modification applied to the simplex on its associated

---

<sup>21</sup>For instance, assume the two edges contain two 1-simplices each: so two possible isomorphisms exist, i.e. two ways for sewing the two edges. The different sewing configurations are distinguished by taking also into account the different configurations for the incident vertices: is an edge incident to one or two vertices ? are vertices incident to both edges ?

generator.

If cellular operations are applied to a "cellular" combinatorial structure, a strong correspondence has to be maintained between the cellular structure, its simplicial analog and the simplicial chain complexes associated with it and also with its cells. Remember that cells make a partition of the simplicial analog, and each simplex corresponds to a generator. It is thus possible to maintain a single representation for any  $p$ -simplex  $\sigma$  of the simplicial analog and its two associated generators  $g_c$  and  $g_s$  of the chain complexes associated with its cell and with the whole cellular structure. Moreover, the boundaries of  $g_c$  and  $g_s$  are equal, except that  $\sigma d_p$  does not appear in the boundary of  $g_c$ . Remember also that it is important, when an identification operation is applied, to know  $\sigma d_p$  for constructing a homological equivalence characterizing the identification. It is thus easy to conceive a unique representation for both the simplicial analog and the chain complexes corresponding to the cells and to the structure itself, such that the size of the representation is the size of the simplicial analog, and such that the representation provides all necessary informations in order to efficiently compute all results when an operation is applied.

The correspondence between the cellular combinatorial structure and its simplicial analog (or more generally the representation mentioned above) depends upon the nature of the cellular combinatorial structure. Assume it is a combinatorial map; then any main simplex of any cell can be represented by a dart. So, a "link" can be maintained between each main simplex of a cell and its representative dart; it is easy to maintain such links when extension or identification operations are applied. Moreover, if an identification operation is applied, it is easy to propagate the isomorphisms between the cells of the combinatorial map (which map their darts) onto their associated simplices and the simplices of their boundaries (i.e. the simplices which are not main simplices of cells), due to the correspondence between "adjacency operators" on darts and face operators on simplices (cf. Section 4.1.2).

Assume the cellular combinatorial structure is an incidence graph. Given a  $p$ -cell  $c$  of the graph, a main simplex of the associated cell in its simplicial analog corresponds to a longest face path issued from  $c$ . A correspondence similar to that described for combinatorial maps can thus be defined for incidence graphs by linking such face paths with main simplices of cells. Note

that these links can be implicit, if a *list* of incident  $(p - 1)$ -cells is associated with any  $p$ -cell. Indeed, face paths can be traversed with respect to some order, and the corresponding simplices can be stored with respect to this order; the drawback of this space optimization is that it may be necessary to modify the order of some simplices when cells are identified, increasing the time complexity.

*Extension for gmap with self-bending*

When two darts belonging to the same  $(n - 1)$ -cell are sewn together, the resulting  $(n - 1)$ -cell is bent onto itself (cf. Fig. 24 and Section 4.1.2). Moreover, the bending of a cell on itself can induce the bending on themselves of cells of its boundary. So, it is necessary to compute:

- a homological equivalence for each cell  $c$  bent onto itself; it can be computed by applying the SES theorem to  $c$  (since almost the "half" of  $c$  is identified with the other "half"), but it is necessary to compute a homological equivalence  $\Upsilon_c^0$  corresponding to the identified part. We don't know yet how to take into account the homological equivalence associated with  $c$  in order to compute  $\Upsilon_c^0$ , but  $\Upsilon_c^0$  can be computed as described in Subsection 3.3 (cf. proof of lemma 1 page 22 and theorem 3 page 25);
- a homological equivalence for the resulting connected components. The principle is the same as that described in the proof of theorem 5; but for each cell  $c$  bent on itself, it is necessary to use the homological equivalence  $\Upsilon_c^0$  computed previously instead of the homological equivalence associated with  $c$ .

*4.3. Optimizations for cellular chain complexes*

Let  $CC$  be the subset of "cellular" combinatorial structures which satisfy properties described in [Bas10, ADLP15, ALP15], mainly:

- any cell is a cellular quasi-manifold (without its boundary);
- no cell is bent on itself;

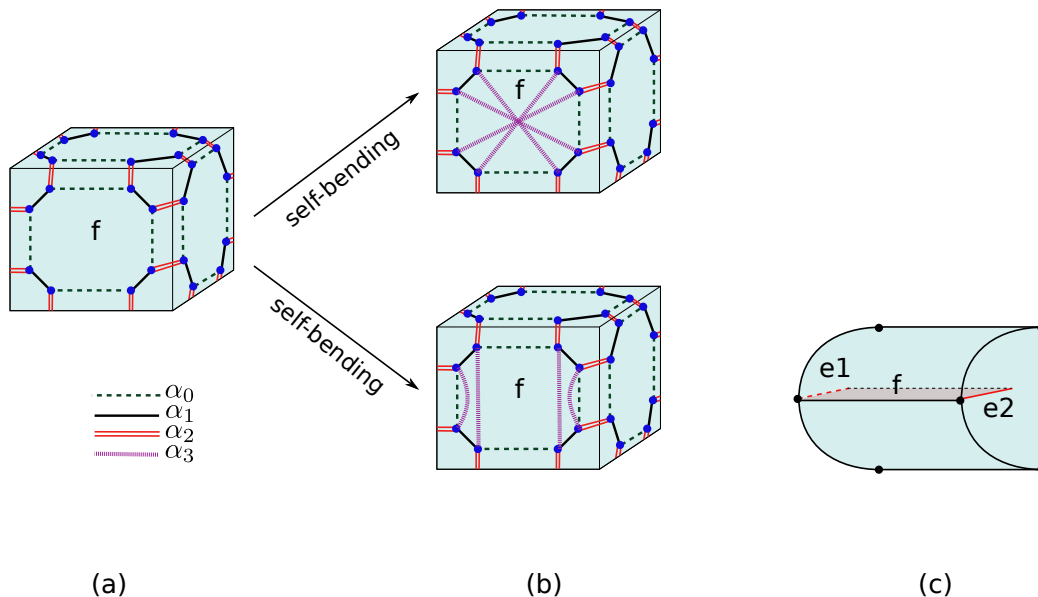


Figure 24: (a) a 3-gmap encoding a cube ( $\alpha_3$  is not represented, since  $\alpha_3 = \text{identity}$ ). (b) two possible identifications of face  $f$  with itself ( $\alpha_3$  is represented only for the darts of  $f$ ). In the bottom case, the self-bending of  $f$  induces the self-bending of edges of its boundary. (c) a geometric representation of the self-bending of  $f$  (bottom case): the two edges  $e1$  and  $e2$  of the boundary of  $f$  are also bent on themselves.

- any  $p$ -cell is orientable, i.e. its set of  $p$ -simplices<sup>22</sup> can be partitioned into two subsets, such that two  $p$ -simplices sharing a  $(p - 1)$ -simplex belong to distinct subsets; each subset is an *orientation* of the cell<sup>23</sup>;
- the *canonical* boundary of any  $p$ -cell is homologically equivalent to a  $(p - 1)$ -sphere<sup>24</sup>.

<sup>22</sup>Note that even if the simplicial analog is mentioned, it is never explicitly handled nor represented.

<sup>23</sup>Orienting a cell consists in choosing one of its orientation; this can be done explicitly, for instance by choosing one dart for combinatorial maps (cf. [ADLP15] p. 60), or implicitly by associating a sign with any edge of an incidence graph (cf. [ALP15] p. 22).

<sup>24</sup>The canonical boundary of a cell  $c$  is the cellular quasi-manifold  $M$  such that  $c$  is the interior of the cellular cone of  $M$ ; for instance, when no multi-incidence occurs, the canonical boundary *is* the boundary.



It has been proved that it is possible to associate a cellular chain complex with any cellular structure of  $CC$ , such that it is homologically equivalent to the corresponding simplicial chain complex. More precisely, let  $M$  be a structure of  $CC$ , and  $(C, \partial)$  be its cellular chain complex. Each cell of  $M$  is oriented (cf. Fig. 25), and a generator of  $C$  is associated with any oriented cell of  $M$ . Let  $c$  be a  $p$ -cell of  $M$ ;  $c\partial = \sum_{c' \in C_{p-1}} (c : c')c'$ , where  $(c : c') \in \mathbb{Z}$  is the *incidence number* of the pair  $(c, c')$ . This incidence number can be easily computed on the "cellular" combinatorial structure: cf. Definition 9 in [ADLP15] p. 60 and Definition 15 in [ALP15] p. 22. In fact, chain  $c\partial$  can be easily computed during a traversal of the  $(p - 1)$ -cells incident to  $c$  (obviously, if  $c'$  is not incident to  $c$ , then  $(c : c') = 0$ ).

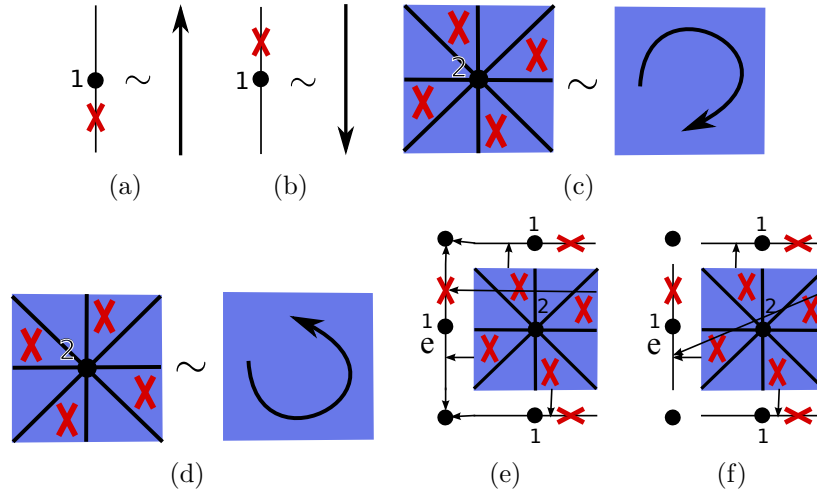


Figure 25: (a) and (b) (resp. (c) and (d)) the two orientations of an edge (resp. a face). (e) edge  $e$  appears  $+1$  and  $-1$  times in the boundary of the face, the corresponding object is a cylinder. (f) edge  $e$  appears  $-2$  times in the boundary of the face, the corresponding cellular object is a Möbius strip.

### Extension

Let  $M^1$  be an  $(n - 1)$ -dimensional "cellular" combinatorial structure of  $CC$ , such that its associated cellular chain complex  $(C^1, \partial^1)$  is homologically equivalent to an  $(n-1)$ -sphere, and let  $M^2$  be the cellular cone of  $M^1$ . Assume a homological equivalence  $\Upsilon^1 : (C^1, \partial^1) \xleftarrow{\rho^1} (C^{B1}, \partial^{B1}) \xrightarrow{\rho^{S1}} (C^{S1}, \partial^{S1})$  is associated with  $M^1$ . It is easy to deduce from  $\Upsilon^1$  a homological equivalence

$\Upsilon^2 : (C^2, \partial^2) \xleftarrow{\rho^2} (C^{B2}, \partial^{B2}) \xrightarrow{\rho^{S2}} (C^{S2}, \partial^{S2})$  associated with  $M^2$ .  $C^2$  (resp.  $C^{B2}, C^{S2}$ ) is deduced from  $C^1$  (resp.  $C^{B1}, C^{S1}$ ) by adding an  $n$ -dimensional generator. A perturbation  $\delta^2$  is then defined on  $C^2$ , in order to associate its boundary with the  $n$ -dimensional generator: it is the sum of all  $(n - 1)$ -generators corresponding to the  $(n - 1)$ -cells of the boundary of the  $n$ -cell (with a coefficient equal to  $\pm 1$  with respect to the relative orientations of the cells), and  $\partial^2 = \partial^1 + \delta^2$ . Then, the Easy Perturbation Lemma is applied, defining  $\delta^{B2}, \partial^{B2}$  and  $\rho^2$ . Perturbation  $\delta^{B2}$  satisfies the nilpotency hypothesis, since the homotopy operator of  $\rho^{S1}$  is null on  $C_{n-1}^{B1}$ ; thus, the Basic Perturbation Lemma can be applied, defining  $\partial^{S2}$  and  $\rho^{S2}$ .  $(C^{S2}, \partial^{S2})$  can then be simplified<sup>25</sup>. This proves the following theorem.

**Theorem 6.** *Let  $M^1$  be a "cellular" combinatorial structure of  $CC$ , and  $\Upsilon^1$  be its associated homological equivalence. Let  $M^2$  be the extension of  $M^1$ .  $M^2$  and an associated homological equivalence can be computed in linear time in the sizes of  $M^1$  and  $\Upsilon^1$  respectively.*

Of course, a homological equivalence  $\Upsilon'^2 : (C^2, \partial^2) \xleftarrow{\rho^S} (C'^{S2}, \partial'^{S2})$  could be directly computed, such that  $C'^{S2}$  contains one generator of dimension 0, but it would be necessary to compute  $\rho^S$  from scratch.

### Identification

Let  $M^1$  be an  $n$ -dimensional "cellular" combinatorial structure of  $CC$ , and let  $\Upsilon^1 : (C^1, \partial^1) \xleftarrow{\rho^1} (C^{B1}, \partial^{B1}) \xrightarrow{\rho^{S1}} (C^{S1}, \partial^{S1})$  be its associated homological equivalence. Assume cells of  $M^1$  are identified, producing  $M^2$  and its associated cellular chain complex  $(C^2, \partial^2)$ . As before, two cells which are identified together are isomorphic, as they are isomorphic to the resulting cell (the isomorphism between identified cells can be extended by a morphism on their boundaries, such that this morphism restricted to two cells of the boundary which are identified together is an isomorphism). More precisely, let  $c$  be a cell of  $M^1$ , and let  $\phi_c$  be the isomorphism which associates  $c$  with its corresponding cell in  $M^2$ ; let  $\phi$  be the sum of all such isomorphisms.

Assume cells  $c_1^1$  and  $c_2^1$  are identified into  $c^2$ , i.e.  $c_1^1\phi = c_2^1\phi = c^2$ ; let  $\phi_{1,2}$  denotes the isomorphism associating  $c_1^1$  and  $c_2^1$ . Let  $o_{c_1^1:c_2^1}$  (resp.  $o_{c_1^1:c^2}$ )

---

<sup>25</sup>If "small" chain complexes are simplified after an operation is applied,  $C^{S1}$  contains one 0-dimensional generator and one  $(n - 1)$ -dimensional generator; and the "simplified"  $C^{S2}$  contains one 0-dimensional generator.

denotes the relative orientations of  $c_1^1$  and  $c_2^1$  (resp.  $c_1^1$  and  $c^2$ ). More precisely,  $o_{c_1^1:c_2^1} = 1$  if their chosen orientations are associated by  $\phi_{1,2}$ , otherwise  $o_{c_1^1:c_2^1} = -1$ ; the same holds for  $o_{c_1^1:c^2}$  with respect to  $\phi$ .

Let  $c_1^1 \partial^1 = \sum_y (c_1^1 : y)y$ . It can be easily proved that

$$c^2 \partial^2 = c_1^1 \phi \partial^2 = o_{c_1^1:c^2} \sum_y o_{y:\phi}(c_1^1 : y)y\phi$$

Now, assume all cells of  $C^1$  are partitioned into  $I = \{I_1, \dots, I_k\}$ , such that  $I_u$  contains all cells which are identified together (if a cell is not identified, it is the single element of its corresponding subset); for each  $I_u$ , a representative cell  $c_u^1$  is chosen; in particular, if a cell is not identified, it is its own representative cell. A homological equivalence  $\Upsilon^0 : (C^0, \partial^0) \xleftarrow{\rho^0} (C^{B0}, \partial^{B0}) \xrightarrow{\rho^{S0}} (C^{S0}, \partial^{S0})$ , and a short exact sequence  $((C^0, \partial^0), (C^1, \partial^1), (C^2, \partial^2), i, j, r, s)$  can be defined in the following way:

- let  $j : C^1 \rightarrow C^2$  be defined by:  $c^1 j = o_{c^1:c^1} \phi c^1$ , for any  $c^1 \in C^1$ .  $j$  is a chain complex morphism;
- let  $s : C^2 \rightarrow C^1$  be defined by:  $c^2 s = o_{c_u^1:c^2} c_u^1$ , where  $c_u^1$  is the representative cell such that  $c_u^1 \phi = c^2$ , for any  $c^2 \in C^2$ .  $s$  is a graded morphism such that  $sj = id_{C^2}$ ;
- $C^0$  is such that, for each non representative cell  $c^1$  of  $C^1$ , a generator  $c^0$  of same dimension exists in  $C^0$ ; let  $\phi^0$  be the corresponding one-to-one mapping;
- let  $r : C^1 \rightarrow C^0$  be defined by:  $c_u^1 r = 0$  for any representative cell  $c_u^1$ ;  $c^1 r = -o_{c^1:c_u^1} c^0$  for any non representative cell  $c^1$ , where  $c_u^1$  is its corresponding representative cell, and  $c^0 \phi^0 = c^1$ ;  $r$  is a graded module morphism;
- let  $i : C^0 \rightarrow C^1$  be defined by:  $c^0 i = c_u^1 - o_{c^1:c_u^1} c^1$  for any  $c^0 \in C^0$ , where  $c^1 = c^0 \phi^0$  and  $c_u^1$  is the representative cell associated with  $c^1$ ;  $i$  is a graded module morphism;
- $\partial^0$  is defined so that  $\partial^0 i = i \partial^1$ ; this is similar to the previous "simplicial" and "cellular" cases (cf. Section 3.3 and 4.2.2); so,  $i$  is a chain complex morphism, such that  $ir = id_{C^0}$  and  $ri + js = id_{C^1}$ ;

- as for the "simplicial case",  $(C^{B0}, \partial^{B0}) = (C^0, \partial^0)$ ,  $\rho^0 = ((C^0, \partial^0), (C^0, \partial^0), f^0 = id_{C^0}, g^0 = id_{C^0}, h^0 = 0)$ , and  $\rho^{S0}$  is a reduction which can be computed by "simplifying"  $(C^0, \partial^0)$ .

**Theorem 7.** *Let  $M^1$  be a "cellular" combinatorial structure of  $CC$ , and let  $\Upsilon^1$  be its associated homological equivalence. Let  $M^2$  be the result of the identification of isomorphic cells. Then a homological equivalence  $\Upsilon^2 : (C^2, \partial^2) \xrightarrow{\rho^2} (C^{B2}, \partial^{B2}) \xrightarrow{\rho^{S2}} (C^{S2}, \partial^{S2})$  associated with  $M^2$  can be computed by one application of the SES theorem.*

**Remark 3.** *Note that the construction of  $\Upsilon^0$  and  $i, j, r, s$  is linear in the size of their arguments. The computational complexity involved by the application of the SES theorem has already been studied: cf. section 3.3.3 page 26 and remark 1 page 27.*

*Implementation remark*

The correspondence between a cellular combinatorial structure  $S$  and its associated cellular chain complex  $C$  can easily be maintained. Assume the cellular combinatorial structure is an incidence graph. Then a generator of  $C$  is associated with each cell of  $S$ . The relative orientation of an  $i$ -cell and an  $(i - 1)$ -cell of its boundary can be explicitly represented in  $S$  by associating a sign  $+$  or  $-$  with the corresponding edge of the graph: cf. [ALP15] page 22 (it is not efficient to compute such orientations at each time it is needed); the corresponding incidence number is then  $+1$  or  $-1$  respectively.

Assume the cellular combinatorial structure is a combinatorial map. Any cell can be represented by choosing one of its darts; then a generator of  $C$  is associated with each chosen dart of  $S$ . Moreover, a chosen dart represents one of the two possible orientations of the corresponding cell: it is thus easy to check the relative orientation of two cells, and to compute their incidence number (cf. [ADLP15] pages 60-61).

## 5. Perspectives

Future works will deal with the study of:

- other construction operations for simplicial and "cellular" combinatorial structures: variants of the cone and the identification (and its inverse), variants of the extrusion and more generally of the cartesian product, contraction or removal of cells (and the inverse operations) [DL03], chamfering, split, etc;

- simplification operations for chain complexes: elementary reduction (cf. Annex 6.1), computation of vector fields [RS10], computation of SNF reduction [BMLH10], etc;
- the complexities related to both kinds of operations, i.e the complexities of the operations themselves, but also the complexities related to the computed homological equivalences.

At last, the results will be experimented for "big" objects constructed by applying numerous local operations.

### Acknowledgments:

The authors gratefully acknowledge Francis Sergeraert for many helpful discussions and encouragements. Many thanks also to Pol Vanhaecke et Claude Quitté.

## 6. Annex

### 6.1. Elementary reduction

An *elementary reduction* is defined as follows.

**Definition 18.** *Let  $(C^0, \partial^0)$  be a chain complex. Let  $x$  and  $y$  be two elements of  $C^0$  such that  $x$  appears in the boundary of  $y$  with a coefficient  $\epsilon = \pm 1$ . The homotopy operator  $h^{01}$  is defined by:*

- $xh^{01} = \epsilon y$
- $\forall z \neq x, z h^{01} = 0$

*Then:*

- $C^1$  is a graded module isomorphic to  $C^0 - \{x, y\}$  by a one-to-one mapping  $\phi$ .
- $f^{01} : C^0 \longrightarrow C^1$  is such that:
  - $\forall z \neq x, y, z f^{01} = z\phi$
  - $y f^{01} = 0$
  - $x f^{01} = (-\epsilon y \partial^0 + x)\phi$

- $g^{01} : C^1 \longrightarrow C^0$  is such that:  
 $\forall z, zg^{01} = z\phi^{-1} - z\phi^{-1}\partial^0 h^{01}$

- $\partial^1$  is such that:  
 $\forall z, z\partial^1 = z\phi^{-1}\partial^0 f^{01}$

This elementary reduction is denoted  $x \rightarrow \epsilon y$ .

**Property 5.** An elementary reduction is a reduction.

The proof is recalled in [AFLP15] (Annex 6.2 page 50)

**Example 7.** Sequence of elementary reductions

A sequence of elementary reductions:

- $\rho^{01} = ((C^0, \partial^0), (C^1, \partial^1), h^{01}, f^{01}, g^{01}),$
- $\rho^{12} = ((C^1, \partial^1), (C^2, \partial^2), h^{12}, f^{12}, g^{12}),$
- $\rho^{23} = ((C^2, \partial^2), (C^3, \partial^3), h^{23}, f^{23}, g^{23}),$
- $\rho^{34} = ((C^3, \partial^3), (C^4, \partial^4), h^{34}, f^{34}, g^{34})$

is represented on Fig. 26 (graphical representation) and Fig. 27 (formal definition).

The properties of any reduction  $(C, C', f, g, h)$  can be explained on an elementary reduction example, for instance, on  $\rho^{01}$  (see Figure 26(b)).

The homotopy operator,  $h$ , associates the elements to be suppressed through the reduction process (for instance here,  $h^{01}$  links  $b^0$  and  $-g^0$ ). Roughly speaking, it generalizes the notion of collapse [KMM04].

$f$  and  $g$  are inverse to each other on the remaining elements. The element which is "shrunk" has a null image by  $f$  (for instance,  $g^0 f^{01} = 0$ ). The element associated with it in the reduction is sent on the remaining part of the boundary of the "shrunk" element (for instance,  $b^0 f^{01} = a^1$ ). This explains why  $gf = id$  and  $id - fg = h\partial + \partial h$  since  $id - fg$  is the difference in the initial object between an element and the elements corresponding to the elements on which it is sent. For instance:

- for  $h^0$ :  
 $- h^0(id - f^{01}g^{01}) = h^0 - (h^0 - g^0) = g^0,$

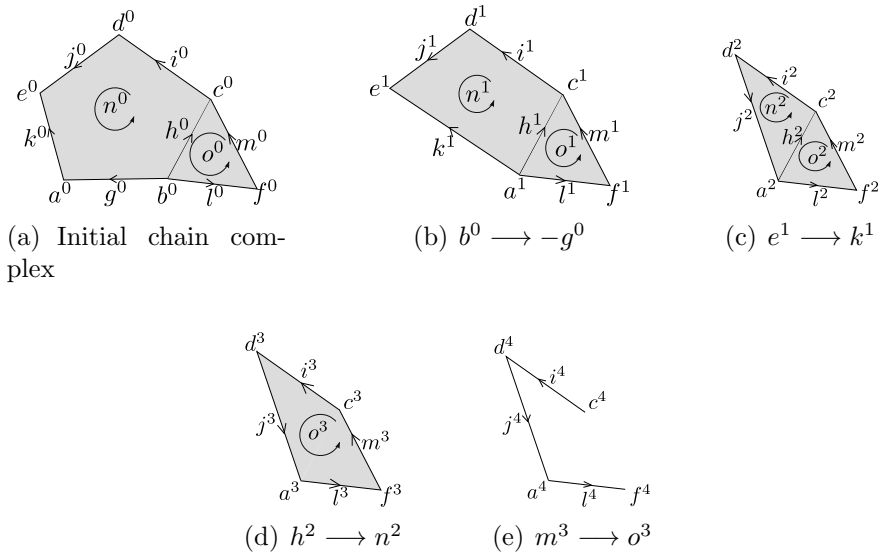


Figure 26: Sequence of elementary reductions (remember that  $x \rightarrow \epsilon y$  denotes the elementary reduction characterized by  $xh = \epsilon y$ ). The *generators* of the different chain groups are graphically represented. A  $i$ -dimensional generator is represented by a  $i$ -dimensional cell. A 0-generator appears in the boundary of a 1-generator with 1 as coefficient (resp.  $-1$ ) if the corresponding vertex is the extremity (resp. origin) of the corresponding edge. Each face is assumed to be oriented counterclockwise: a 1-generator appears in the boundary of a 2-generator with 1 as coefficient (resp.  $-1$ ) if the orientation of the associated edge corresponds to (resp. is inverse to) the orientation of the associated face. The exponent associated with each cell allows to distinguish the different chain complexes.

$C^0$	$a^0$	$\mathbf{b}^0$	$c^0$	$d^0$	$e^0$	$f^0$	$g^0$	$h^0$	$i^0$	$j^0$	$k^0$	$l^0$	$m^0$	$n^0$	$o^0$
$\partial^0$	0	0	0	0	0	0	$a^0 b^0$	$c^0 b^0$	$d^0 c^0$	$e^0 d^0$	$e^0 a^0$	$f^0 b^0$	$c^0 f^0$	$-g^0 + h^0 + i^0 + j^0 - k^0$	$l^0 + m^0 - h^0$
$\mathbf{h}^{01}$	0	$-\mathbf{g}^0$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^{01}$	$a^1$	$a^1$	$c^1$	$d^1$	$e^1$	$f^1$	0	$h^1$	$i^1$	$j^1$	$k^1$	$l^1$	$m^1$	$n^1$	$o^1$
$C^1$	$a^1$		$c^1$	$d^1$	$\mathbf{e}^1$	$f^1$		$h^1$	$i^1$	$j^1$	$k^1$	$l^1$	$m^1$	$n^1$	$o^1$
$\partial^1$	0		0	0	0	0		$c^1 a^1$	$d^1 c^1$	$e^1 d^1$	$e^1 a^1$	$f^1 a^1$	$c^1 f^1$	$h^1 + i^1 + j^1 - k^1$	$l^1 + m^1 - h^1$
$g^{01}$	$a^0$		$c^0$	$d^0$	$e^0$	$f^0$		$h^0 - g^0$	$i^0$	$j^0$	$k^0$	$l^0 - g^0$	$m^0$	$n^0$	$o^0$
$\mathbf{h}^{12}$	0		0	0	$\mathbf{k}^1$	0		0	0	0	0	0	0	0	0
$f^{12}$	$a^2$		$c^2$	$d^2$	$a^2$	$f^2$		$h^2$	$i^2$	$j^2$	0	$l^2$	$m^2$	$n^2$	$o^2$
$C^2$	$a^2$		$c^2$	$d^2$		$f^2$		$\mathbf{h}^2$	$i^2$	$j^2$		$l^2$	$m^2$	$n^2$	$o^2$
$\partial^2$	0		0	0		0		$c^2 a^2$	$d^2 c^2$	$a^2 d^2$		$f^2 a^2$	$c^2 f^2$	$h^2 + i^2 + j^2$	$l^2 + m^2 - h^2$
$g^{12}$	$a^1$		$c^1$	$d^1$		$f^1$		$h^1$	$i^1$	$j^1 - k^1$		$l^1$	$m^1$	$n^1$	$o^1$
$\mathbf{h}^{23}$	0		0	0		0		$\mathbf{n}^2$	0	0		0	0	0	0
$f^{23}$	$a^3$		$c^3$	$d^3$		$f^3$		$-i^3 - j^3$	$i^3$	$j^3$		$l^3$	$m^3$	0	$o^3$
$C^3$	$a^3$		$c^3$	$d^3$		$f^3$			$i^3$	$j^3$		$l^3$	$\mathbf{m}^3$		$o^3$
$\partial^3$	0		0	0		0			$d^3 c^3$	$a^3 d^3$		$f^3 a^3$	$c^3 f^3$		$i^3 + j^3 + l^3 + m^3$
$g^{23}$	$a^2$		$c^2$	$d^2$		$f^2$			$i^2$	$j^2$		$l^2$	$m^2$		$n^2 + o^2$
$\mathbf{h}^{34}$	0		0	0		0			0	0		0	$\mathbf{o}^3$		0
$f^{34}$	$a^4$		$c^4$	$d^4$		$f^4$			$i^4$	$j^4$		$l^4$	$-i^4 - j^4 - l^4$		0
$C^4$	$a^4$		$c^4$	$d^4$		$f^4$			$i^4$	$j^4$		$l^4$			
$\partial^4$	0		0	0		0			$d^4 c^4$	$a^4 d^4$		$f^4 a^4$			
$g^{34}$	$a^3$		$c^3$	$d^3$		$f^3$			$i^3$	$j^3$		$l^3$			

Figure 27: Formal definition of the sequence of elementary reductions depicted on Fig. 26. For instance, the row starting with  $C^0$  contains its generators, from dimension 0 to dimension 2. The following rows, until the row starting with  $C^1$ , contains the definitions of the mappings defined on  $C^0$ , e.g.  $b^0 h^{01} = -g^0$ , since  $-g^0$  is at the intersection of the row corresponding to  $h^{01}$  and of the column corresponding to  $b^0$  in the row associated with  $C^0$ .



$$\begin{aligned}
& - h^0 h^{01} \partial^0 = 0 \\
& - h^0 \partial^0 h^{01} = (c^0 - b^0) h^{01} = 0 - (-g^0);
\end{aligned}$$

• for  $b^0$ :

$$\begin{aligned}
& - b^0 (id - f^{01} g^{01}) = b^0 - a^0, \\
& - b^0 h^{01} \partial^0 = -g^0 \partial^0 = -(a^0 - b^0), \\
& - b^0 \partial^0 h^{01} = 0.
\end{aligned}$$

More generally, we can see in example 8 that:

- $h^0 (id - f^{03} g^{03}) = h^0 - (-i^3 - j^3) g^{03} = h^0 - (-i^0 - (j^0 - k^0)) = h^0 + i^0 + j^0 - k^0,$
- $h^0 h^{03} \partial^0 = n^0 \partial^0 = -g^0 + h^0 + i^0 + j^0 - k^0,$
- $h^0 \partial^0 h^{03} = (c^0 - b^0) h^{03} = 0 - (-g^0).$

The three last properties can be explained in the following way:

- $hf = 0$ , since for each element  $x$ , either  $xh = 0$ , or  $xh \neq 0$  and this "shrunked" element has no corresponding element in the resulting chain complex.
- $gh = 0$ , since  $g$  associates with a remaining element an element of the initial chain complex, which is either an element which remains (on which  $h = 0$ ) or a "shrunked" one (on which  $h = 0$  also). For instance,  $j^2 g^{12} = j^1 - k^1$ , intuitively  $j^1$  has been extended on  $-k^1$  through  $e^1$  (note also that  $e^1$  has no preimage in  $C^2$ ).
- $hh = 0$ , since for each element  $x$ , either  $xh = 0$ , or  $xh \neq 0$  is a "shrunked" element which cannot be used in order to "shrink" another element.

**Example 8.** *Composition of elementary reductions*

Let  $\rho^{02} = ((C^0, \partial^0), (C^2, \partial^2), h^{02}, f^{02}, g^{02})$  (resp.  $\rho^{03} = ((C^0, \partial^0), (C^3, \partial^3), h^{03}, f^{03}, g^{03})$ ) be the composition of  $\rho^{01}$  and  $\rho^{12}$  (resp.  $\rho^{02}$  and  $\rho^{23}$ ) of example 7.  $\rho^{02}$  and  $\rho^{03}$  are described in Fig. 28.

$C^0$	$a^0$	$b^0$	$c^0$	$d^0$	$e^0$	$f^0$	$g^0$	$h^0$	$i^0$	$j^0$	$k^0$	$l^0$	$m^0$	$n^0$	$o^0$
$\partial^0$	0	0	0	0	0	0	$a^0-b^0$	$c^0-b^0$	$d^0-c^0$	$e^0-d^0$	$e^0-a^0$	$f^0-b^0$	$c^0-f^0$	$-g^0+h^0+$ $i^0+j^0-k^0$	$l^0+$ $m^0-h^0$
$h^{02}$	0	$-g^0$	0	0	$k^0$	0	0	0	0	0	0	0	0	0	0
$f^{02}$	$a^2$	$a^2$	$c^2$	$d^2$	$a^2$	$f^2$	0	$h^2$	$i^2$	$j^2$	0	$l^2$	$m^2$	$n^2$	$o^2$
$C^2$	$a^2$		$c^2$	$d^2$		$f^2$		$h^2$	$i^2$	$j^2$		$l^2$	$m^2$	$n^2$	$o^2$
$\partial^2$	0		0	0		0		$c^2-a^2$	$d^2-c^2$	$a^2-d^2$		$f^2-a^2$	$c^2-f^2$	$h^2+i^2+j^2$	$l^2+$ $m^2-h^2$
$g^{02}$	$a^0$		$c^0$	$d^0$		$f^0$		$h^0-g^0$	$i^0$	$j^0-k^0$		$l^0-g^0$	$m^0$	$n^0$	$o^0$
$h^{03}$	0	$-g^0$	0	0	$k^0$	0	0	$n^0$	0	0	0	0	0	0	0
$f^{03}$	$a^3$	$a^3$	$c^3$	$d^3$	$a^3$	$f^3$	0	$-i^3-j^3$	$i^3$	$j^3$	0	$l^3$	$m^3$	0	$o^3$
$C^3$	$a^3$		$c^3$	$d^3$		$f^3$			$i^3$	$j^3$		$l^3$	$m^3$		$o^3$
$\partial^3$	0		0	0		0			$d^3-c^3$	$a^3-d^3$		$f^3-a^3$	$c^3-f^3$		$i^3+j^3+$ $l^3+m^3$
$g^{03}$	$a^0$		$c^0$	$d^0$		$f^0$			$i^0$	$j^0-k^0$		$l^0-g^0$	$m^0$		$n^0+o^0$

Figure 28: Composition of elementary reductions depicted in Fig. 26.

**Remark 4.** *Not any reduction can be decomposed into elementary reductions. For instance, let us consider the chain complex:*

$$\cdots 0 \longrightarrow C_i = \mathbb{Z} \times \mathbb{Z} \xrightarrow{\partial^i} C_{i-1} = \mathbb{Z} \times \mathbb{Z} \longrightarrow 0 \cdots$$

with  $\forall j > i$  or  $j < i - 1$ ,  $C_j = 0$ , and

$$\partial^i = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

Note that  $C_i$  and  $C_{i-1}$  have the same generators  $(1, 0)$  and  $(0, 1)$ .

Let the null complex be

$$\cdots 0 \longrightarrow D_i = 0 \xrightarrow{0} D_{i-1} = 0 \longrightarrow 0 \cdots$$

with  $\forall j > i$  or  $j < i - 1$ ,  $D_j = 0$ .

We can easily check that a reduction of  $(C, \partial)$  onto  $(D, 0)$  is defined by:

- The homotopy operator  $h$  such that

$$h_{i-1} = \begin{pmatrix} 3/2 & -2 \\ -1 & 3/2 \end{pmatrix} : C_{i-1} \longrightarrow C_i$$

and  $\forall j \neq i - 1, h_j$  is null.

- $f : C \rightarrow D$  and  $g : D \rightarrow C$ , both null.

The generators  $(0, 1)$  and  $(1, 0)$  appear with a coefficient different from  $\pm 1$  in the boundary of each generator. Hence there exists no elementary reduction.

## 6.2. Simplicial identification

**Corollary 1.** Using the notations of lemma 1 and theorem 3, let:

- $\rho^1 = ((C^{B1}, \partial^{B1}), (C^1, \partial^1), h^1, f^1, g^1);$
- $\rho^{S1} = ((C^{B1}, \partial^{B1}), (C^{S1}, \partial^{S1}), h^{S1}, f^{S1}, g^{S1});$
- $\rho^2 = ((C^{B2}, \partial^{B2}), (C^2, \partial^2), h^2, f^2, g^2),$
- $\rho^{S2} = ((C^{B2}, \partial^{B2}), (C^{S2}, \partial^{S2}), h^{S2}, f^{S2}, g^{S2}).$

Then:

- $|i^B| \leq 2|g^1|;$
- $|i^S| \simeq 2|g^{S0}g^1f^{S1}|$  (by abuse of notations, by assimilating the elements of  $C^0$  with the corresponding elements of  $C^1$ );
- $|(C^{B2}, \partial^{B2})| = |(C^0, \partial^0) + |C^{B1}, \partial^{B1}| + |i^B| \leq 2|C^{B1}, \partial^{B1}| + |i^B|;$
- $|(C^{S2}, \partial^{S2})| = |(C^{S0}, \partial^{S0}) + |C^{S1}, \partial^{S1}| + |i^S|;$
- $|h^2| = |f^1| + |h^1|;$
- $|f^2| = |f^1|;$
- $|g^2| \leq |\partial^1| + |g^1|;$
- $|h^{S2}| \simeq |h^{S0}| + 2|h^{S0}g^1h^{S1}| + |h^{S1}|;$
- $|f^{S2}| \simeq |f^{S0}| + 2|h^{S0}g^1f^{S1}| + |f^{S1}|;$
- $|g^{S2}| \simeq |g^{S0}| + 2|g^{S0}g^1h^{S1}| + |g^{S1}|.$

**Proof 5.** *The proof is left to the reader: it is mainly the application of the formulas which appear in the sketch of the proof of theorem 1.*

*Identifications by increasing dimensions*

At the initial step, the semi-simplicial set  $S^1$  contains complete simplices, each one being a connected component. The associated homological equivalence  $\Upsilon^1 : (C^1, \partial^1) \xleftarrow{\rho^1} (C^1, \partial^1) \xrightarrow{\rho^{S^1}} (C^{S^1}, \partial^{S^1})$  is defined by:

- $(C^1, \partial^1)$  is the chain complex associated with  $S^1$ ;
- $\rho^1 = ((C^1, \partial^1), (C^1, \partial^1), h^1 = 0, f^1 = id, g^1 = id)$ ;
- $(C^{S^1}, \partial^{S^1})$  is such that:
  - $C^{S^1}$  contains only 0-dimensional generators, each one corresponding to a complete simplex of  $S^1$ ;
  - $\partial^{S^1} = 0$ ;
- $\rho^{S^1} = ((C^1, \partial^1), (C^{S^1}, 0), h^{S^1}, f^{S^1}, g^{S^1})$  corresponds to the classical reduction of a set of cones.

Step  $k$  corresponds to the identifications of all concerned  $(k-1)$ -simplices:

- $S^k$  is the current semi-simplicial set. Its associated homological equivalence is  $\Upsilon^k : (C^k, \partial^k) \xleftarrow{\rho^k} (C^{Bk}, \partial^{Bk}) \xrightarrow{\rho^{S^k}} (C^{S^k}, \partial^{S^k})$ , with:
  - $\rho^k = ((C^{Bk}, \partial^{Bk}), (C^k, \partial^k), h^k, f^k, g^k)$ ;
  - $\rho^{S^k} = ((C^{Bk}, \partial^{Bk}), (C^{S^k}, \partial^{S^k}), h^{S^k}, f^{S^k}, g^{S^k})$ ;
- $S^{k+1}$  is the resulting simplicial set and  $\Upsilon^{k+1}$  is its associated homological equivalence;
- $\Upsilon_{id}^k : (C_{id}^k, 0) \xleftarrow{\rho_{id}^k} (C_{id}^k, 0) \xrightarrow{\rho_{id}^{S^k}} (C_{id}^k, 0)$ , with  $h_{id}^k = 0, f_{id}^k = id, g_{id}^k = id$ , is the homological equivalence corresponding to the identified part;
- Let  $i, j, r, s, i^B, i^S$  be defined as in the proof of theorem 3;
- $|C^{k+1}| = |C^k| - |C_{id}^k| = |C^1| - \sum_{i=1}^k |C_{id}^i|$ ;
- $|\partial^{k+1}|$  is the size of  $\partial^1$  restricted to the generators of  $C^{k+1}$ , so  $|\partial^{k+1}| = |\partial^1| - \sum_{i=2}^k |C_{id}^i|$ ;

- $|i^B|$ : each  $(k-1)$ -generator of  $C_{id}^k$  is associated by  $i$  with two  $(k-1)$ -generators  $\gamma_1$  and  $\gamma_2$  of  $C^k$ , corresponding to two simplices  $\sigma_1$  and  $\sigma_2$  of  $S^k$ . The image of  $\gamma_1$  (resp.  $\gamma_2$ ) by  $g^k$  in  $C^{Bk}$  is a chain generated by itself and at most  $k$   $(k-1)$ -generators corresponding to the identified  $(k-2)$ -simplices of the boundary of  $\sigma_1$  (resp.  $\sigma_2$ ): cf. the size of  $g^k$  below.

$$\text{So } |i^B| \leq 2(1+k) |C_{id}^k|;$$

- $|i^S|$ : the explanation is similar to that of  $|i^B|$ , followed by the application of  $f^{Sk}$ .  $f^{Sk}$  is not null only for the generators which correspond to identified simplices (cf.  $|f^{Sk}|$  below). For instance,  $\gamma_1 g^k f^{Sk}$  is a chain generated by at most  $k$   $(k-1)$ -generators corresponding to the identified  $(k-2)$ -simplices of the boundary of  $\sigma_1$ .

$$\text{So } |i^S| \leq 2k |C_{id}^k|;$$

- $|C^{Bk+1}| = |C^{Bk}| + |C_{id}^k| = |C^1| + \sum_{i=1}^k |C_{id}^i|$ ;
- $|\partial^{Bk+1}| = |i^B| + |\partial^{Bk}| \leq |\partial^1| + 2(|C_{id}^1| + \sum_{i=2}^k (i+1) |C_{id}^i|)$ ;
- $|C^{Sk+1}| = |C^{Sk}| + |C_{id}^k| = |C^{S1}| + \sum_{i=1}^k |C_{id}^i|$ ;
- $|\partial^{Sk+1}| = |i^S| + |\partial^{Sk}| \leq 2(|C_{id}^1| + \sum_{i=2}^k i |C_{id}^i|)$ ;
- $h^{k+1} = \begin{pmatrix} 0 & 0 \\ f^k_r & h^k \end{pmatrix}$ ;  $r$  is defined only for generators corresponding to identified simplices. So  $|h^{k+1}| = \sum_{i=1}^k |C_{id}^i|$ ;
- $f^{k+1} = \begin{pmatrix} 0 \\ f^k_j \end{pmatrix}$ , so each generator of  $C^{Bk+1}$  corresponding to a simplex of  $S^1$  is associated with itself. The images of other generators are null. So  $|f^{k+1}| = |C^1|$ ;
- $g^{k+1} = (-s\partial^k_r \quad sg^k)$ . Let  $\sigma$  be a simplex of  $S^{k+1}$  and  $\gamma$  be its associated generator in  $C^{k+1}$ . If the dimension of  $\sigma$  is different from  $k$ , then the image of  $\gamma$  by  $s\partial^k_r$  is null, else it is a chain which contains at most  $(k+1)(k-1)$ -generators corresponding to the identified simplices of the boundary of  $\sigma$ .

Regarding  $sg^k$  the explanation is similar for the generators of  $C^{k+1}$  of dimension lower than  $k$  (this corresponds to the previous identifications); moreover each generator corresponding to a simplex of  $C^{k+1}$  is associated with itself in  $C^{Bk+1}$ .

$$\begin{aligned} \text{So } |g^{k+1}| &\leq |C^{k+1}| + \sum_{i=1}^k (i+1) |C_i^{k+1}| \\ &= |C^1| - (|C_{id}^1| + \sum_{i=2}^k (i+1) |C_{id}^i|) + \sum_{i=1}^k (i+1) |C_i^1|; \end{aligned}$$

- $h^{Sk+1} = \begin{pmatrix} 0 & 0 \\ 0 & h^{Sk} \end{pmatrix}$ ; thus  $|h^{Sk+1}| = |h^{S1}| \simeq |C^1| / 2$ ;
- $f^{Sk+1} = \begin{pmatrix} id & 0 \\ 0 & f^{Sk} \end{pmatrix}$ ; compared to  $f^{Sk}$ , the correspondence between the generators corresponding to the identified simplices is added. Thus  $|f^{Sk+1}| = |C_0^1| + \sum_{i=1}^k |C_{id}^i|$ ;
- $g^{Sk+1} = \begin{pmatrix} id & -i^B h^{Sk} \\ 0 & g^{Sk} \end{pmatrix}$ ; compared to  $g^{Sk}$ ,
  - the correspondence between the generators corresponding to the identified simplices,
  - the correspondence between the generators of  $C_{id}^k$  and chains which contain at most two generators of  $C^{Bk+1}$ , due to the definition of  $i^B$  and  $h^{Sk}$ ,

are added.

So, each generator of  $C_{id}^k$  is associated with a chain containing at most three generators of  $C^{Bk+1}$ , and  $|g^{Sk+1}| = |C_0^{S1}| + 3 \sum_{i=1}^k |C_{id}^i|$ , where  $|C_0^{S1}| = |C^{S1}|$  is the number of main simplices of  $S^1$ .

Let  $n$  be the dimension of  $S^1$ . Let  $A^1 + B^1$  (resp.  $A^n + B^n$ ) be the size of  $\Upsilon^1$  (resp.  $\Upsilon^n$ ) where  $A^1$  (resp.  $A^n$ ) denotes the size of the chains complexes of  $\Upsilon^1$  (resp.  $\Upsilon^n$ ).

- $A^1 = 2(|C^1| + |\partial^1|) + |C^{S1}| = |C^{S1}| + 2 \sum_{i=0}^n |C_i^1| + \sum_{i=1}^n (i+1) |C_i^1|$ ;
- $B^1 \simeq 2 |C^1| + |C^1| / 2 + |C^{S1}| + |C_0^1|$ ;

- $A^n = |C^n| + |\partial^n| + |C^{Bn}| + |\partial^{Bn}| + |C^{Sn}| + |\partial^{Sn}|$   
 $\leq A^1 + 5 |C_{id}^1| + 3 \sum_{i=2}^n (i+1) |C_{id}^i|$ ;
- $B^n \leq B^1 + 4 \sum_{i=1}^n |C_{id}^i| + \sum_{i=1}^n (i+1) |C_i^1| - \sum_{i=2}^n (i+1) |C_{id}^i|$ .

So,  $A^n + B^n \leq A^1 + B^1 + 5 |C_{id}^1| + 4 \sum_{i=1}^n |C_{id}^i| + 2 \sum_{i=2}^n (i+1) |C_{id}^i| + |\partial^1|$ . Since  $|C_{id}^i| \leq |C_{i-1}^1|$ ,  $\sum_{i=1}^n |C_{id}^i| \leq |C^1|$  and  $\sum_{i=2}^n (i+1) |C_{id}^i| \leq |\partial^1|$ . The size of  $\Upsilon^n$  is thus linear according to the size of  $\Upsilon^1$ .

We can easily deduce that the time complexity is linear according to  $n$  times the size of  $\Upsilon^1$ . More precisely, when only the modifications are computed at each step, the computing time is linear according to the size of  $S^1$ .

[ADLL08] S. Alayrangues, X. Daragon, J.-O. Lachaud, and P. Lienhardt. Equivalence between closed connected n-g-maps without multi-incidence and n-surfaces. *J. Math. Imaging Vis.*, 32(1):1–22, 2008.

[ADLP15] S. Alayrangues, G. Damiand, P. Lienhardt, and S. Peltier. Homology of cellular structures allowing multi-incidence. *Discrete and Computational Geometry*, 54:42–77, 2015.

[AFLP15] S Alayrangues, L Fuchs, P Lienhardt, and S Peltier. Incremental Computation of the Homology of Generalized Maps: An Application of Effective Homology Results. <https://hal.archives-ouvertes.fr/hal-01142760>, April 2015.

[Ago76] M. K. Agoston. *Algebraic Topology, a first course*. Pure and applied mathematics. Marcel Dekker Ed., 1976.

[ALP15] S. Alayrangues, P. Lienhardt, and S. Peltier. Conversion between chains of maps and chains of surfaces; application to the computation of incidence graphs homology. Research report, Université de Poitiers, 2015. <https://hal.archives-ouvertes.fr/hal-01130543/file/RR-2014-11.pdf>.

[Bas10] T. Basak. Combinatorial cell complexes and poincaré duality. *Geometriae Dedicata*, 147:357–387, 2010.

[BCMA<sup>+</sup>11] D. Boltcheva, D. Canino, S. Merino Aceituno, J.-C. Léon, L. De Floriani, and F. Hétroy. An iterative algorithm for homology

- computation on simplicial shapes. *Computer-Aided Design*, 43(11):1457–1467, 2011.
- [Ber99] G. Bertrand. New notions for discrete topology. In *Proc. of 8th Disc. Geom. for Computer Imagery, Marne-la-Vallée*, volume 1568 of *LNCS*, pages 218–228. Springer-Verlag, 1999.
- [BH81] Ronald Brown and Philip J. Higgins. On the algebra of cubes. *Journal of Pure and Applied Algebra*, 21(3):233 – 260, 1981.
- [BMLH10] D. Boltcheva, S. Merino, J.-C. Léon, and F. Hétroy. Constructive Mayer-Vietoris Algorithm: Computing the Homology of Unions of Simplicial Complexes. Research Report RR-7471, INRIA, December 2010. <http://hal.inria.fr/inria-00542717/PDF/RR-7471.pdf>.
- [Bou89a] N. Bourbaki. *Algebra 5: Chapter 10*. Elements of mathematics. Springer, 1989.
- [Bou89b] N. Bourbaki. *Algebra I: Chapters 1-3*. Elements of mathematics. Springer, 1989.
- [BPA<sup>+</sup>10] T. Bellet, M. Poudret, A. Arnould, P. Le Gall, and L. Fuchs. Designing a Topological Modeler Kernel: A Rule-Based Approach. In *Shape Modeling International (SMI'10)*, Aix-en-Provence, France, 2010.
- [Bri93] E. Brisson. Representing geometric structures in d dimensions: Topology and order. *Discrete & Computational Geometry*, 9:387–426, 1993.
- [BSP<sup>+</sup>05] S. Brandel, S. Schneider, M. Perrin, N. Guiard, J.-F. Rainaud, P. Lienhardt, and Y. Bertrand. Automatic building of structured geological models. *Journal of Computing and Information Science in Engineering, Volume 5, numéro 2*, 2005.
- [CL06] A.V. Chernavsky and V.P. Leksine. Unrecognizability of manifolds. *Annals of Pure and Applied Logic*, 141(3):325 – 335, 2006. Papers presented at the Second St. Petersburg Days of Logic and Computability Conference on the occasion of the centennial of Andrey Andreevich Markov, Jr. Papers presented at the Second St. Petersburg Days of Logic and Computability Conference on the occasion of the centennial of Andrey Andreevich Markov, Jr.



- [CMP06] David Cardoze, Gary Miller, and Todd Phillips. Representing topological structures using cell-chains. In Myung-Soo Kim and Kenji Shimada, editors, *Geometric Modeling and Processing - GMP 2006*, volume 4077 of *Lecture Notes in Computer Science*, pages 248–266. Springer Berlin / Heidelberg, 2006. 10.1007/11802914\_18.
- [DCB05] X. Daragon, M. Couprie, and G. Bertrand. Discrete surfaces and frontier orders. *Journal of Mathematical Imaging and Vision*, 23:379–399, 2005.
- [DD09] A. Dupas and G. Damiand. Region merging with topological control. *Discrete Applied Mathematics*, 157(16):3435–3446, August 2009.
- [DHSW03] J.-G. Dumas, F. Heckenbach, B. D. Saunders, and V. Welker. Computing simplicial homology based on efficient smith normal form algorithms. In *Algebra, Geometry, and Software Systems*, pages 177–206, 2003.
- [DKMW11] P. Dlotko, T. Kaczynski, M. Mrozek, and T. Wanner. Coreduction homology algorithm for regular cw-complexes. *Discrete & Computational Geometry*, 46(2):361–388, 2011.
- [DL03] G. Damiand and P. Lienhardt. Removal and contraction for n-dimensional generalized maps. In *Proc. of 11th International Conference on Discrete Geometry for Computer Imagery (DGCI)*, volume 2886 of *Lecture Notes in Computer Science*, pages 408–419, Naples, Italy, November 2003. Springer Berlin/Heidelberg.
- [DL14] G. Damiand and P. Lienhardt. *Combinatorial Maps: Efficient Data Structures for Computer Graphics and Image Processing*. A K Peters/CRC Press, September 2014.
- [DSV01] J. G. Dumas, B.D. Saunders, and G. Villard. On efficient sparse integer matrix smith normal form computations. *J. of Symbolic Computation*, 2001.
- [Ede87] H. Edelsbrunner. *Algorithms in Computational Geometry*. Springer, 1987.
- [EH10] Herbert Edelsbrunner and John Harer. *Computational Topology - an Introduction*. American Mathematical Society, 2010.

- [EZ50] S. Eilenberg and J. A. Zilber. Semi-simplicial complexes and singular homology. *Annals of Mathematics , Second Series*, 51(3):499–513, 1950.
- [GDJMR09] R. González-Díaz, M. J. Jiménez, B. Medrano, and P. Real. Chain homotopies for object topological representations. *Discrete Applied Mathematics*, 157(3):490–499, 2009.
- [Gie96] M. Giesbrecht. Probabilistic computation of the smith normal form of a sparse integer matrix. In *Proceedings of the Second Int. Symp. on Algorithmic Number Theory, LNCS*, volume 1122, pages 173–186. Springer-Verlag, 1996.
- [Gug57] V. K. A. M. Gugenheim. On supercomplexes. *Transactions of the American Mathematical Society*, 85(1):35–51, May 1957.
- [KMM04] T. Kaczynski, K. Mischaikow, and M. Mrozek. *Computational Homology*. Applied Mathematical Sciences. Springer, 2004.
- [Lie94] P. Lienhardt. N-dimensional generalized combinatorial maps and cellular quasi-manifolds. *Int. J. on Comput. Geom. & App.*, 4(3):275–324, 1994.
- [LP16] P. Lienhardt and S. Peltier. Simplicoidal Sets. Technical report, Université de Poitiers, February 2016. working paper.
- [May67] J. P. May. *Simplicial Objects in Algebraic Topology*. Van Nostrand, 1967.
- [MB09] M. Mrozek and B. Batko. Coreduction homology algorithm. *Discrete and Computational Geometry*, 41(1):96–118, 2009.
- [PAFL06] S. Peltier, S. Alayrangués, L. Fuchs, and J.-O. Lachaud. Computation of homology groups and generators. *Comput. & Graph.*, 30:62–69, feb. 2006.
- [PFL09] S. Peltier, L. Fuchs, and P. Lienhardt. Simplicoidal sets: Definitions, operations and comparison with simplicial sets. *Discrete App. Math.*, 157:542–557, feb. 2009.
- [RO89] J. Rossignac and M. O’Connor. A dimension-independant model for pointsets with internal structures and incomplete boundaries. *Geometric modeling for Product Engineering*, pages 145–180, 1989.

- [RS06] J. Rubio and F. Sergeraert. Constructive homological algebra and applications, August, 28 - September, 02 2006. <http://arxiv.org/abs/1208.3816>.
- [RS10] A. Romero and F. Sergeraert. Discrete vector fields and fundamental algebraic topology. *CoRR*, abs/1005.5685, 2010. <http://arxiv.org/abs/1005.5685>.
- [SDdJ15] C. Solnon, G. Damiand, C. de la Higuera, and J.-C. Janodet. On the complexity of submap isomorphism and maximum common submap problems. *Pattern Recognition (PR)*, 48(2):302–316, February 2015.
- [Ser] F. Sergeraert. The kenzo program. <https://www-fourier.ujf-grenoble.fr/~sergerar/kenzo/>.
- [Ser51] J.-P. Serre. Homologie singulière des espaces fibrés. *The Annals of Mathematics*, 54(3):425–505, November 1951.
- [Sto96] A. Storjohann. Near optimal algorithms for computing smith normal forms of integer matrices. In Y. N. Lakshman, editor, *Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation*, pages 267–274. ACM, 1996.
- [Vin83] A. Vince. Combinatorial maps. *Journal of Combinatorial Theory Series B*, 34:1–21, 1983.