

Supplemental Material for Experimental observation of two-dimensional Anderson localization with the atomic kicked rotor

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The following Supplemental Material discusses:

- I. The main improvements in our experimental setup allowing the 2D Anderson localization to be observed.
- II. The mapping of the quasiperiodic kicked rotor onto a 2D Anderson model, indicating the essentials of this mapping and its peculiarities relevant for the present work.
- III. The classical diffusion in the 2-frequency quasiperiodic kicked rotor.
- IV. A schematic derivation of the localization length, indicating the essential steps of the calculation in the case of the quasiperiodic kicked rotor with 2 incommensurate frequencies and the origin of the exponential dependence on the anisotropy.

SETUP IMPROVEMENTS

In our previous experiments, in order to minimize coupling with gravity, the standing wave (SW) was horizontal, see Fig. 1a. This geometry has several drawbacks. For 1000 kicks, the atoms would fall of 3.8 mm, to be compared to the 1.5 mm of the SW waist. Moreover, the SW was built by retro-reflecting the incoming beam, which leads to a ~ 1 m optical path difference between the two beams overlapping on the atomic cloud region, a source of phase noise detrimental to dynamical localization. These restrictions limited the kick number to less than 200 in our previous experiments. In order to overcome this limit, we built a new SW system with several improvements, as shown in Fig. 1b. The SW is now vertical, and, between kicks, the atoms fall freely. The intensity inhomogeneity in the transverse direction was also reduced. The SW is now formed by two independent beams, which has many advantages, as each arm can be independently controlled, both in amplitude and phase through the radio-frequency wave that drives the acousto-optic modulators. This allows us to accurately cancel gravity effects, by imposing a linear frequency chirp to one of the arms with respect to the other, so that

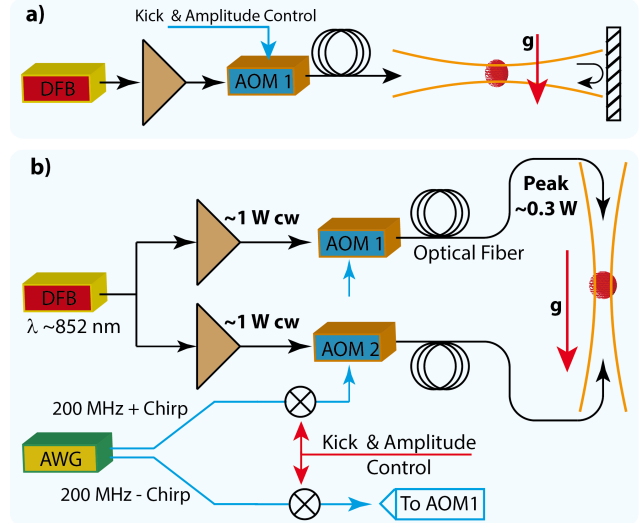


Figure 1. Evolution of the experimental setup. a) Horizontal standing wave setup used in former experiments. The kicks are controlled by a single acousto-optic modulator (AOM), and the standing wave is built by retro-reflecting the incoming laser beam. b) Schematic view of the new vertical setup used in the present work. A Distributed Feedback Laser Diode (DFB) seeds two optical amplifiers. The kick-sequence temporal modulation is produced by an arbitrary wave generator (AWG). In addition to the pulsed sequence, a linear chirp is added so that the standing wave “falls” with acceleration g , simultaneously with the atomic cloud.

the SW itself “falls” with acceleration g . A kicked rotor is thus realized in the free-falling reference frame. Finally, the SW phase noise induced by the laser linewidth is minimized by accurately balancing the optical paths of the arms to better than 1 cm. This is performed by directly minimizing the kinetic energy dispersion at 1000 kicks with a 1D kicked rotor.

MAPPING OF THE QUASIPERIODIC KICKED ROTOR ONTO A 2D ANDERSON MODEL

The mapping of the quasiperiodic kicked rotor (QPKR) onto a 2D Anderson model has been described in [1–3]. Because it is useful to derive an approximate ex-

pression for the localization length, we summarize here the basics of the calculation.

The starting point is the following Hamiltonian:

$$\mathcal{H} = \frac{p_1^2}{2} + \omega_2 p_2 + K \cos x_1 [1 + \varepsilon \cos x_2] \sum_n \delta(t - n) \quad (1)$$

which describes a *periodically* kicked, two-dimensional “rotor”; we use here quotations marks around “rotor” because of the unusual form of the kinetic energy along the direction 2, linear instead of quadratic. It is a matter of algebra [3] to show that the temporal evolution of an initial state initially localized at $x_2 = \phi_2$, but with arbitrary wave-function along x_1 :

$$\Psi(x_1, x_2, t = 0) \equiv \Xi(x_1, t = 0) \delta(x_2 - \phi_2) \quad (2)$$

leads to the state:

$$\Psi(x_1, x_2, t) = \Xi(x_1, t) \delta(x_2 - \phi_2 - \omega_2 t), \quad (3)$$

where the wavefunction $\Xi(x_1, t)$ *exactly* obeys the time-dependent Schrödinger equation of the quasiperiodic kicked rotor:

$$H = \frac{p_1^2}{2} + K \cos x_1 [1 + \varepsilon \cos(\omega_2 t + \phi_2)] \sum_n \delta(t - n). \quad (4)$$

Thus, the evolution of the quasiperiodic kicked rotor can be thought of as the evolution of a 2D periodic “rotor”, with a peculiar initial state perfectly localized in the x_2 direction and thus completely delocalized in the conjugate p_2 direction.

The Hamiltonian Eq. (1) being time-periodic, one can use the Floquet theorem and look at its Floquet eigenstates. Let $|\phi\rangle$ denote an eigenstate of the evolution operator over one period with eigenvalue $\exp(-iE/\hbar)$, and consider the state $|\chi\rangle = (1 + iW)^{-1}|\phi\rangle$ where $W(x_1, x_2) \equiv \tan[K \cos x_1 (1 + \varepsilon \cos x_2) / 2\hbar]$. The Hamiltonian being periodic in x_1 and x_2 (we take periodic boundary conditions), $|\chi\rangle$ can be expanded on the basis of plane waves $|m_1, m_2\rangle$ – eigenstates of the momentum operators in both directions with eigenvalues $m_1\hbar$ and $m_2\hbar$: $|\chi\rangle = \sum_{m_1, m_2} \chi_{m_1, m_2} |m_1, m_2\rangle$. It is again a matter of algebra [2, 4] to show that the coefficients χ_{m_1, m_2} obey the following equation:

$$\epsilon_{m_1, m_2} \chi_{m_1, m_2} + \sum_{r_1, r_2 \neq 0} W_{r_1, r_2} \chi_{m_1 + r_1, m_2 + r_2} = 0, \quad (5)$$

where ϵ_{m_1, m_2} is given by:

$$\epsilon_{m_1, m_2} = \tan \left\{ \frac{1}{2} \left[\left(\hbar \frac{m_1^2}{2} + \omega_2 m_2 \right) - \frac{E}{\hbar} \right] \right\} \quad (6)$$

and the W_{r_1, r_2} are the two-fold Fourier components of the doubly-periodic function W .

Equation (5) can be interpreted as the eigenvalue equation for a Anderson-like model on a 2D lattice with sites labeled (m_1, m_2) , with hopping described by W_{r_1, r_2} and on-site energies ϵ_{m_1, m_2} . There are however four differences with respect to a usual Anderson model:

- The hopping matrix elements W_{r_1, r_2} are not limited to nearest neighbors. They do however decrease fast enough at large (r_1, r_2) so that this difference does not change the qualitative behavior (i.e. localization).
- The on-site energies ϵ_{m_1, m_2} are not random, but rather a deterministic pseudo-random sequence. Provided π, \hbar, ω_2 are incommensurate, the sequence has no periodicity and localization properties similar to those of a truly random model are expected (and observed numerically).
- The hopping is anisotropic, governed by the K/\hbar coefficient along direction 1 and by the $\varepsilon K/\hbar$ coefficient along direction 2. For $\varepsilon = 0$, the 2D system appears as a series of chains along direction 1, which are uncoupled along direction 2, and a 1D Anderson model is recovered.
- Floquet quasi-eigenstates with different quasi-energies E are associated with the same energy 0 in Eq. (5), but with different realizations of the disorder, Eq. (6), all having the same statistical properties. This is why, for given values of parameters \hbar, K, ε , all quasi-eigenstates have the same localization length [5].

In order to understand localization properties of the quasiperiodic kicked rotor, it is thus sufficient to study transport and localization on the equivalent anisotropic Anderson-like 2D model Eq. (1).

CLASSICAL DIFFUSION

It is well known that the classical dynamics of the periodic kicked rotor is described by the Standard Map [6]. A similar map can be constructed for the quasiperiodic kicked rotor. The classical evolution over one period of the Hamiltonian Eq. (4) is given by the following map:

$$\begin{aligned} p_{1n+1} &= p_{1n} + K \sin x_{1n} (1 + \varepsilon \cos x_{2n}), \\ p_{2n+1} &= p_{2n} + K \varepsilon \cos x_{1n} \sin x_{2n}, \\ x_{1n+1} &= x_{1n} + p_{1n+1}, \\ x_{2n+1} &= x_{2n} + \omega_2, \end{aligned}$$

where $y_n = y(t = n + \epsilon)$, $y = x_i, p_i$ ($i = 1, 2$). If K is sufficiently large, the classical dynamics is almost fully chaotic [7]. For the 1D problem ($\varepsilon = 0$), this takes place for $K \gtrsim 6$. For the 2D problem (non vanishingly small ε), we found an almost fully chaotic dynamics for $K \gtrsim 4$, and this is why we did not perform any experiment below this value. In the chaotic regime, the kicks in momentum have a pseudo-random sign (depending on x_1 and x_2), making the dynamics in momentum space appear as a pseudo-random walk, i.e. a chaotic diffusive process at

long time. Note however that the diffusion is anisotropic because kicks along p_2 are typically smaller than kicks along p_1 by a factor ε . In the limit of large K it is easy to evaluate the diffusion tensor, defined as

$$D_{ij} = \lim_{t \rightarrow +\infty} \frac{\langle p_i p_j \rangle}{t} \quad (7)$$

by assuming that positions of consecutive kicks are uniform uncorrelated variables (see [7] for the essentially identical calculation in 3D). It is diagonal in the (1,2) directions with:

$$\begin{aligned} D_{11} &\approx (K^2/4)(1 + \varepsilon^2/2), \\ D_{22} &\approx K^2 \varepsilon^2/8, \\ D_{i \neq j} &\approx 0. \end{aligned} \quad (8)$$

Numerical simulations of the classical dynamics [7] fully confirm that the classical dynamics is an anisotropic diffusion; however, at not too large K , correlations between successive kicks are responsible for oscillatory corrections to the diffusion constant of the order of few ten percents.

LOCALIZATION LENGTH

The calculation of the localization length turns out to be a bit tricky in 2D. It has been discussed in the literature mainly in the context of electrons in disordered potentials [8, 9]. The calculation for the quasiperiodic rotor follows the same lines. We sketch here only the general method, comparing the key steps with the case of electrons in disordered potentials.

The starting point is to take into account the weak localization effect due to closed loops in the system. In the weak scattering regime, where this correction is small, it takes the following form for electrons (or massive particles) in disordered potentials:

$$\frac{D}{\mathcal{D}} = 1 - \frac{1}{\pi\rho} \int \frac{1}{(2\pi)^2} \frac{d\mathbf{q}}{Dq^2} \quad (9)$$

where D is the classical (Boltzmann) diffusion constant and \mathcal{D} the quantum modified one. The term after 1 in the right-hand side is the weak localization correction. It depends on the density of states ρ and involves a two-dimensional integral over the vector \mathbf{q} which is a momentum, conjugate of position. The integrand is nothing but the classical diffusive kernel (at zero frequency, hence infinite time). Formally, the integral diverges both for small and large q , so that appropriate cut-offs must be set. The natural short distance (large q) cut-off is the mean free path ℓ (at shorter scale, the dynamics is not diffusive). The short q cut-off can be taken proportional to $1/L$, where L is the size of the system. One then obtains a size-dependent diffusion constant $\mathcal{D}(L)$. Taking for the

density of states in 2D its disorder-free value $1/2\pi$ (we take the mass of the particle and \hbar as unity), one obtains:

$$\frac{\mathcal{D}(L)}{D} = 1 - \frac{2}{\pi k \ell} \ln \left(\frac{L}{\ell} \right) \quad (10)$$

where k is the wavevector of the particle. In these units, the classical diffusion constant is $D = k\ell/2$. The logarithmic dependence in L is a crucial ingredient, as we will see below. It is intimately related to the low- q divergence of the integral.

Equation (10) shows that the diffusion constant decreases with the system size. Of course, this cannot be correct for arbitrary large size, as the correction is computed assuming it is small. This ceases to be true when the right hand side term in Eq. (10) vanishes. This gives an order of magnitude of the size at which the diffusion constant vanishes, that is the localization length. One then gets

$$\xi = \ell \exp \left(\frac{\pi k \ell}{2} \right). \quad (11)$$

Of course, this is only a very approximate expression. The self-consistent theory of localization [8, 9] is an attempt to be a bit more quantitative. It predicts essentially the same exponential dependence for the localization length.

For the periodic kicked rotor, the weak localization corrections have been computed in [10]. These results have been extended to the quasiperiodic rotor in [11, 12]. There are essentially two modifications:

- The term depending on the density of states (which is meaningless for a time-periodic system where the density of states of the Floquet Hamiltonian is infinite) $1/\pi\rho$ must be replaced by $2k^2$.
- Because the classical dynamics is an anisotropic diffusion, the diffusive kernel is now $(D_{11}q_1^2 + D_{22}q_2^2)^{-1}$. When performing the integral over \mathbf{q} , it is not entirely clear how to choose the cut-offs. The simplest choice is to take the large q_1 and q_2 cut-offs scaling like the (anisotropic) mean free paths, i.e. respectively proportional to $1/\sqrt{D_{11}}$ and $1/\sqrt{D_{22}}$. This is however arbitrary and questionable. It is important to understand that the choice of different cut-offs will affect the prefactors, but not the key point, namely the logarithmic dependence of the integral on the system size.

With this simple choice of cut-offs, the weak localization correction reads:

$$\frac{\mathcal{D}_{11}(L)}{D_{11}} = \frac{\mathcal{D}_{22}(L)}{D_{22}} = 1 - \frac{k^2}{\pi\sqrt{D_{11}D_{22}}} \ln \left(\frac{L}{l} \right) \quad (12)$$

where l is a yet unspecified short scale cut-off.

With the same reasoning as the one for electrons, one obtains the approximate expression for the localization length (in momentum space for the kicked rotor):

$$p_{\text{loc}} = l \exp\left(\frac{\pi\sqrt{D_{11}D_{22}}}{k^2}\right). \quad (13)$$

In the $\varepsilon \rightarrow 0$ limit, one must recover the localization length of the periodic kicked rotor, which fixes $l = K^2/4k$. By inserting the values of the diffusion constants, Eqs. (8) into Eq. (13), we finally obtain:

$$p_{\text{loc}} = \frac{K^2}{4k} \exp\left(\frac{\pi\varepsilon K^2}{\sqrt{32}k^2}\right). \quad (14)$$

Because of the questionable assumptions on the cut-offs, it might well be that the coefficient $\pi/\sqrt{32}$ is not exact and requires some correction. This is a difficult question left for further investigation. We just emphasize that the exponential behavior of the localization length with the scaling parameter $\varepsilon K^2/k^2$ comes from the prefactors in the integral over \mathbf{q} and from the logarithmic singularity of the integral. It is thus a very robust phenomenon, unlikely to be affected by the shortcomings of the self-consistent theory of localization. Ultimately, the *experimentally observed* exponential dependence of the localization length *proves* that the weak localization correction is logarithmic with the system size. This, in turn, implies that dimension $d = 2$ is the lowest critical dimension of the Anderson transition. Indeed, a modification of the dimension would introduce an extra q^{d-2} factor in the integral, i.e. would change the logarithm dependence in L in Eq. (12) to an algebraic one, incompatible with the exponential behavior of the localization length.

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