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ANALYSIS OF A CONTACT PROBLEM WITH NORMAL COMPLIANCE, FINITE PENETRATION AND NONMONOTONE SLIP DEPENDENT FRICTION

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We consider a mathematical model which describes the frictional contact between a linearly elastic body and an obstacle, the so-called foundation. The process is static and the contact is modeled with normal compliance condition of such a type that the penetration is restricted with unilateral constraint. The friction is modeled with a nonmonotone law in which the friction bound depends both on the tangential displacement and on the value of the penetration. In order to approximate the contact conditions, we consider a regularized problem wherein the contact is modeled by a standard normal compliance condition without finite penetrations. For each problem, we derive a variational formulation and an existence result of the weak solutions of regularized problems is obtained. Next, we prove the convergence of the weak solutions of regularized problems to the weak solution of the initial nonregularized problem. Finally, we provide a numerical validation of this convergence result. To this end we introduce a discrete scheme for the numerical approximation of the frictional contact problems. The solution of the resulting nonsmooth and nonconvex frictional contact problems is found, basing on approximation by a sequence of nonsmooth convex programming problems. Some numerical simulation results are presented in the study of an academic two-dimensional example.

Keywords: Static; linearly elastic material; unilateral contact; normal compliance; regularization; asperities; nonmonotone friction law; weak solution; hemivariational equality; finite element method; quasi-augmented Lagrangian method; Newton method; numerical experiments.

Mathematics Subject Classification 2010: 35J87, 47J22, 49J40, 65N30

1. Introduction

Phenomena of contact and friction between deformable bodies or between deformable and rigid bodies abound in industry and everyday life. For this reason, a considerable effort has been developed in their modeling, mathematical analysis and numerical simulations. Owing to their inherent complexity, frictional contact phenomena lead to nonlinear, nonsmooth and nonconvex mathematical problems.

The contact with a deformable foundation is modeled in the literature by the so-called normal compliance contact condition. It assigns a reactive normal pressure that depends on the interpenetration of the asperities on the body's surface and those on the foundation. The normal compliance condition was first introduced in [19, 26] in the study of dynamic contact problems with elastic and viscoelastic materials. Nevertheless, its principal criticism is the fact that, in principle, this condition allows a unlimited penetration of the foundation, which is not realistic from physical point of view. For this reason, various versions of the normal compliance condition have been introduced, recently, in the literature [14, 31, 32, 18].

Furthermore, the friction is generally modeled in the literature by the so-called Coulomb friction law which is an adequate representation of friction for the analysis of many physical systems. But the classical monotone Coulomb law which states that during the slipping, the friction coefficient remains constant, has shown its limits for friction-induced phenomena such as stick-slip motion. Then many authors in the literature propose nonmonotone versions of the friction law [28, 12, 10, 11, 30, 20]. Moreover, in some type of contact such as that in which the interface asperities play a significant role then the relationship between contact force and frictional force is not exactly simple and so the frictional force is not entirely independent of the contact area of the surfaces. Then it is interesting to propose a nonmonotone friction law in relation to the normal compliance contact with finite penetration.

The aim of this paper is to study a frictional contact problems in which the contact is modeled with normal compliance of such a type that the penetration, characterized by the size of the asperities, is restricted with unilateral constraint. Furthermore, the friction is modeled with a nonmonotone law in which the friction bound depends on the tangential displacement and the size of the asperities. The behavior of the material is modeled with a linear elastic constitutive law. This problem can be viewed as a preliminary work for the study of incremental formulations related to evolutionary problems.

A rather similar frictional contact model was studied recently in the article of Eck *et al.* (see [9]). In that smoother approach, however, the maximal possible indentation that leads to a complete flattening of the asperities is never achieved for a finite normal stress, and only the framework of monotone laws was taken into account.

The present paper represents a continuation of [4, 3]. In [4], a quasi-static frictionless contact problem for viscoplastic materials was considered; the process was assumed to be quasi-static and the contact was modeled by using the normal

compliance condition with finite penetration. In [3], a bilateral contact problem with nonmonotone friction was studied. In contrast, in the present paper we consider two frictional contact problems with normal compliance, finite penetration and nonmonotone friction law. The first problem is characterized by normal compliance in which the penetration is restricted by unilateral constraint and the second problem represents a regularization of the first problem by considering unlimited penetration. For the regularized problem, we prove the existence of the weak solution. A trait of novelty of this paper arises from the fact that here we state and prove the convergence of the solution of the nonmonotone friction problem with normal compliance and unlimited penetration to the solution of the nonmonotone friction problem with normal compliance and finite penetration. This convergence result leads to the solvability of the initial nonregularized problem. And, finally, we provide numerical simulations which illustrate the mechanical behavior of the contact model and provide a numerical validation of the convergence result.

The rest of the paper is structured as follows. In Sec. 2 we introduce the notation we shall use as well as some preliminary material. In Sec. 3 we present the classical formulation of the contact problems, we list the assumptions on the data and derive the variational formulation of the problems. Then, in Sec. 4 we state and prove the existence of the weak solution of the approximate problems. Section 5 is devoted to state and prove the convergence result, Theorem 5.1, which represents the main result of this paper. In Sec. 6 the numerical solution of the frictional contact problems is presented. And, finally, in Sec. 7 we present some numerical simulations on an academic two-dimensional example including a numerical validation of the convergence result.

2. Notation and Preliminaries

In this section we present the notation we shall use and some preliminary material. For further details we refer the reader to [8, 13, 27].

We denote by \mathbb{S}^d the space of second-order symmetric tensors on \mathbb{R}^d ($d \leq 3$ in applications), while “ \cdot ” and $|\cdot|$ will represent the inner product and the Euclidean norm on \mathbb{S}^d and \mathbb{R}^d , respectively, i.e.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i, & |\mathbf{v}| &= (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, & |\boldsymbol{\tau}| &= (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d. \end{aligned}$$

Here and below the indices i and j run between 1 and d , and the summation convention over repeated indices is adopted.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ and $\boldsymbol{\tau}$ the associated tangent vector. We shall use the notation

$$\begin{aligned} H &= L^2(\Omega)^d = \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega)\}, & Q &= \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \\ H_1 &= \{\mathbf{u} = (u_i) \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in Q\}, & Q_1 &= \{\boldsymbol{\sigma} \in Q \mid \text{Div } \boldsymbol{\sigma} \in H\}. \end{aligned}$$

Here $\varepsilon : H_1 \rightarrow Q$ and $\text{Div} : Q_1 \rightarrow H$ are the *deformation* and *divergence* operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j}),$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H , Q , H_1 and Q_1 are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_H &= \int_{\Omega} u_i v_i \, dx, & (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q &= \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx, \\ (\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_Q, & (\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\text{Div } \boldsymbol{\sigma}, \text{Div } \boldsymbol{\tau})_H, \end{aligned}$$

for $\mathbf{u}, \mathbf{v} \in H_1$ and $\boldsymbol{\sigma}, \boldsymbol{\tau} \in Q_1$. The associated norms on these spaces are denoted by $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{Q_1}$, respectively.

Let $H_{\Gamma} = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \rightarrow H_{\Gamma}$ be the trace map. For every element $\mathbf{v} \in H_1$ we still write \mathbf{v} to denote the trace $\gamma\mathbf{v}$ of \mathbf{v} on Γ and we denote by v_{ν} and v_{τ} the *normal* and *tangential* components of \mathbf{v} on the boundary Γ given by

$$v_{\nu} = \mathbf{v} \cdot \boldsymbol{\nu}, \quad v_{\tau} = \mathbf{v} - v_{\nu} \boldsymbol{\nu}.$$

Let H_{Γ}^* be the dual of H_{Γ} and let (\cdot, \cdot) denote the duality pairing between H_{Γ}^* and H_{Γ} . For every $\boldsymbol{\sigma} \in Q_1$, there exists an element $\boldsymbol{\sigma}\boldsymbol{\nu} \in H_{\Gamma}^*$ such that

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_Q + (\text{Div } \boldsymbol{\sigma}, \mathbf{v})_H = (\boldsymbol{\sigma}\boldsymbol{\nu}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in H_1.$$

Moreover, if $\boldsymbol{\sigma}$ is a smooth (say C^1) function, then

$$(\boldsymbol{\sigma}\boldsymbol{\nu}, \mathbf{v}) = \int_{\Gamma} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \mathbf{v} \, d\Gamma \quad \text{for all } \mathbf{v} \in H_1.$$

We also denote by σ_{ν} and $\boldsymbol{\sigma}_{\tau}$ the *normal* and *tangential* traces of $\boldsymbol{\sigma}$ and we recall that, when $\boldsymbol{\sigma}$ is smooth enough and Γ has sufficient regularity (for example C^2), then

$$\sigma_{\nu} = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_{\tau} = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_{\nu} \boldsymbol{\nu}.$$

Note that the additional smoothness of boundary Γ is required for the strong formulations of the problems analyzed in the sequel to make sense, however weak formulations can be considered with Γ Lipschitz.

Finally, we recall definitions of the generalized derivative and gradient (see [6]). Let X be a Banach space and X^* be its dual. The Clarke generalized directional derivative of a locally Lipschitz function $h : X \rightarrow \mathbb{R}$ at the point $x \in X$ in the direction $v \in X$, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

The Clarke subdifferential of h at $x \in X$, denoted by $\partial h(x)$, is a subset of X^* given by $\partial h(x) = \{\zeta \in X^* : h^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X\}$. We recall that Clarke

directional derivative has the representation $h^0(x; v) = \sup_{\xi \in \partial h(x)} \{ \langle \xi, v \rangle_{X^* \times X} \}$ (see [6]).

We define pseudomonotone and generalized pseudomonotone multivalued operators and formulate the theorem on their surjectivity (see [21]).

Definition 2.1. Let X be a reflexive Banach space and $A: X \rightarrow 2^{X^*}$ be a multivalued operator. We say that A is

(a) *pseudomonotone* if

- (1) A has values which are nonempty, bounded, closed and convex,
- (2) A is upper semicontinuous from every finite-dimensional subspace of X to X^* endowed with the weak topology,
- (3) if $\{u_n\} \subset X$ with $u_n \rightarrow u$ weakly in X and $u_n^* \in Au_n$ is such that

$$\limsup \langle u_n^*, u_n - u \rangle_{X^* \times X} \leq 0,$$

then for every $y \in X$ there exists $u^*(y) \in Au$ such that

$$\langle u^*(y), u - y \rangle_{X^* \times X} \leq \liminf \langle u_n^*, u_n - y \rangle_{X^* \times X}.$$

(b) *generalized pseudomonotone* if for any sequence $\{u_n\} \subset X$, $\{u_n^*\} \subset X^*$ with $u_n^* \in Au_n$, $u_n \rightarrow u$ weakly in X , $u_n^* \rightarrow u^*$ weakly in X^* and

$$\limsup \langle u_n^*, u_n - u \rangle_{X^* \times X} \leq 0,$$

we have $u^* \in Au$ and $\langle u_n^*, u_n \rangle_{X^* \times X} \rightarrow \langle u^*, u \rangle_{X^* \times X}$.

The next result shows that every pseudomonotone operator is generalized pseudomonotone, while the converse holds under an additional boundedness condition.

Proposition 2.1. Let X be a reflexive Banach space and $A: X \rightarrow 2^{X^*}$ be an multivalued operator.

- (i) If A is pseudomonotone operator, then A is generalized pseudomonotone.
- (ii) If A is generalized pseudomonotone operator which is bounded (i.e. maps bounded sets into bounded ones) and for each $u \in X$ Au is nonempty, closed and convex subset of X^* , then A is pseudomonotone.

Definition 2.2. Let X be a reflexive Banach space and $A: X \rightarrow 2^{X^*}$ be a multivalued operator. We say that A is *coercive* if

$$\lim_{\|x\| \rightarrow \infty} \frac{\inf \{ \langle u^*, u \rangle_{X^* \times X} \mid u^* \in Au \}}{\|x\|} = +\infty.$$

The following is the main surjectivity result for pseudomonotone and coercive operators.

Theorem 2.1. Let X be a reflexive Banach space and $A: X \rightarrow 2^{X^*}$ be pseudomonotone and coercive. Then A is surjective, i.e. $R(A) = X^*$.

3. Mechanical Problems and Variational Formulations

3.1. Classical formulation

In this section we describe the model for the nonmonotone frictional contact with normal compliance and finite penetrations as well as a family of auxiliary models used for its approximation.

The physical setting is as follows (Fig. 1). A linearly elastic body occupies an open bounded connected set $\Omega \subset \mathbb{R}^d$ ($d \leq 3$ in applications) with a Lipschitz boundary Γ that is partitioned into three disjoint parts $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$ with Γ_1 , Γ_2 and Γ_3 being relatively open, and $\text{meas}(\Gamma_1) > 0$. The body is clamped on Γ_1 and thus the displacement field vanishes there. A volume force of density \mathbf{f}_0 acts in Ω and a surface traction of density \mathbf{f}_2 acts on Γ_2 . The body is in frictional contact with an obstacle on Γ_3 . The contact is modeled by a normal compliance contact with finite penetration in which the penetration corresponds to the flattening of the asperities. We model the friction by a nonmonotone friction law wherein the friction bound depends on the slip tangential displacement and the size of asperities g . The material is linearly elastic and the process is assumed to be static.

In the study of the frictional contact problems presented below we need the following assumptions on their data.

- $H(\mathcal{E})$: the elasticity operator $\mathcal{E} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is a bounded symmetric positive definite fourth-order tensor, i.e.

$$\begin{cases} \text{(a) } \mathcal{E}_{ijkl} \in L^\infty(\Omega), & 1 \leq i, j, k, l \leq d; \\ \text{(b) } \mathcal{E}\sigma \cdot \tau = \sigma \cdot \mathcal{E}\tau, & \forall \sigma, \tau \in \mathbb{S}^d, \text{ a.e. in } \Omega; \\ \text{(c) } \mathcal{E}\tau \cdot \tau \geq m|\tau|^2 & \forall \tau \in \mathbb{S}^d, \text{ a.e. in } \Omega \text{ with } m > 0. \end{cases}$$

- $H(f)$: the force and the traction densities satisfy

$$\mathbf{f}_0 \in L^2(\Omega)^d, \quad \mathbf{f}_2 \in L^2(\Gamma_2)^d. \quad (3.1)$$

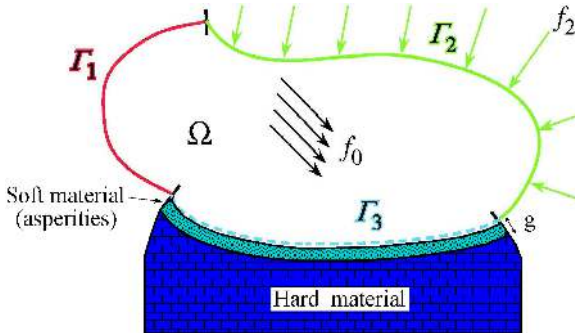


Fig. 1. Physical setting of the contact problem.

- $H(\mu)$: the friction coefficient μ satisfies

$$\begin{cases} \text{(a) } \mu : [0, \infty) \rightarrow \mathbb{R} \text{ is measurable;} \\ \text{(b) } \mu(0) = \lim_{x \rightarrow 0^+} \mu(x) > 0; \\ \text{(c) } |\mu(t)| \leq c(1+t) \quad \forall t \geq 0 \text{ with } c > 0; \\ \text{(d) } \mu(t) \geq -d \left(1 + \frac{1}{t}\right) \quad \forall t > 0 \text{ with } d > 0. \end{cases}$$

- $H(g)$: the gap function $g : \Gamma_3 \rightarrow [0, \infty)$ satisfies

$$\begin{cases} \text{(a) } g \text{ is continuous;} \\ \text{(b) } g(x) \leq G \quad \forall x \in \Gamma_3 \text{ with } G > 0. \end{cases}$$

- $H(p)$: the compliance function $p : (-\infty, G] \rightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) } |p(s)| \leq C_p, \quad \forall s \in \mathbb{R} \text{ with } C_p > 0; \\ \text{(b) } p \text{ is continuous.} \end{cases}$$

- $H(N)$: the friction bound $N : \Gamma_3 \times \mathbb{R} \rightarrow [0, \infty)$ satisfies

$$\begin{cases} \text{(a) } N(\cdot, s) : \Gamma_3 \rightarrow [0, \infty) \text{ is measurable } \quad \forall s \in \mathbb{R}; \\ \text{(b) } N(x, \cdot) : \mathbb{R} \rightarrow [0, \infty) \text{ is continuous a.e. on } \Gamma_3; \\ \text{(c) } N(x, s) \leq K, \quad \forall s \in \mathbb{R} \text{ a.e. on } \Gamma_3 \text{ with } K > 0. \end{cases}$$

We also define auxiliary function $\bar{p} : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\bar{p}(x, s) = \begin{cases} p(s) & \text{for } s \leq g(x), \\ p(g(x)) & \text{for } s > g(x). \end{cases}$$

In the sequel the dependence on $x \in \Gamma_3$ of the functions g , \bar{p} and N is sometimes omitted for convenience of notation.

Remark 3.1. In particular case, since μ corresponds to the physical resistance force, it is nonnegative, so the condition $H(\mu)(d)$ holds obviously.

The classical formulation of the mechanical problem is the following.

Problem \mathcal{P}_M . Find a displacement $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$ such that,

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \quad (3.2)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = 0 \quad \text{in } \Omega, \quad (3.3)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad (3.4)$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2, \quad (3.5)$$

$$\sigma_\nu + p(u_\nu) \leq 0, \quad u_\nu - g \leq 0, \quad (\sigma_\nu + p(u_\nu))(u_\nu - g) = 0 \quad \text{on } \Gamma_3, \quad (3.6)$$

$$\begin{aligned} |\boldsymbol{\sigma}_\tau| &\leq N(u_\nu)\mu(|\mathbf{u}_\tau|) \quad \text{if } \mathbf{u}_\tau = \mathbf{0}, \\ -\boldsymbol{\sigma}_\tau &= N(u_\nu)\mu(|\mathbf{u}_\tau|)\frac{\mathbf{u}_\tau}{|\mathbf{u}_\tau|} \quad \text{if } \mathbf{u}_\tau \neq \mathbf{0}, \end{aligned} \quad \text{on } \Gamma_3. \quad (3.7)$$

Condition (3.6) was first introduced in [14] and represents a unilateral contact with finite penetration in which the compliance function p can take the following form:

$$p(r) = c_1 r_+, \quad (3.8)$$

where $p(\cdot)$ is a nonnegative prescribed function which vanishes for negative argument and c_1 is a positive constant related to the stiffness of the foundation. This condition shows that when there is separation between the body and the obstacle (i.e. when $u_\nu < 0$), then the reaction of the foundation vanishes (since $\sigma_\nu = 0$); moreover, the penetration is limited (since $u_\nu \leq g$) and g represents its maximum value. When $0 \leq u_\nu < g$ then the reaction of the foundation is uniquely determined by the normal displacement (since $-\sigma_\nu = p(u_\nu)$) and, when $u_\nu = g$, the normal stress is not uniquely determined but is submitted to the restriction $-\sigma_\nu \geq p(g)$. Such a condition shows that the contact follows a normal compliance condition but up to the limit g and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap g . For this reason we refer to the contact condition (3.6) as a normal compliance contact condition in which the penetration is restricted by unilateral constraint or, for simplicity, a normal compliance condition with finite penetration. We conclude from above that this case models a surface contact foundation characterized by a thin layer of asperities of thickness g . Also, note that when $g > 0$ and $p = 0$, condition (3.6) becomes the Signorini contact condition in a form with a gap function and a perfectly rigid foundation,

$$u_\nu \leq g, \quad \sigma_\nu \leq 0, \quad \sigma_\nu(u_\nu - g) = 0 \quad \text{on } \Gamma_3 \times (0, \infty).$$

It is also reasonable to consider $p(u_\nu) \neq 0$ for $u_\nu < 0$. In such case p can represent reaction between the body and the foundation which acts if there is no contact, for example, the adhesive force (Fig. 2(a)).

In (3.7) $N(u_\nu)\mu(|\mathbf{u}_\tau|)$ represents the magnitude of the limiting friction traction at which slip begins. The friction bound and more precisely, the friction coefficient μ depends on the tangential displacement $|\mathbf{u}_\tau|$. The strict inequality in (3.7) holds in the stick zone and the equality holds in the slip zone. This physical model of slip-dependent friction was introduced in [28] for geophysical context of earthquakes' modeling and it also was studied in [10–12, 30, 20]. The purpose of the term $N(u_\nu)$ is to propose a friction bound which depends proportionally on the value of the

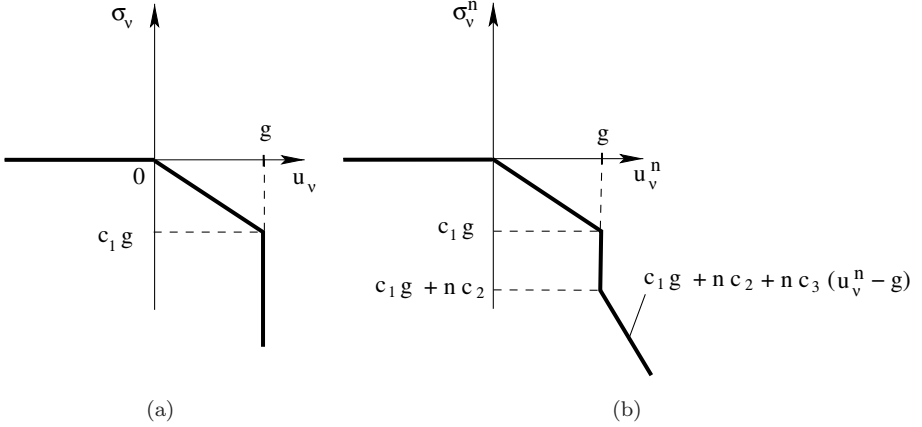


Fig. 2. Contact models: (a) the contact condition (3.6); (b) the contact conditions (3.10).

penetration u_ν and on the size of the asperities g used in the normal compliance condition with finite penetration (3.6). An example of function N would be

$$N(x, \eta) = \begin{cases} 0 & \text{for } \eta \leq 0, \\ S \frac{\eta}{g(x)} & \text{for } \eta \in (0, g(x)), \\ S & \text{for } \eta \geq g(x). \end{cases} \quad (3.9)$$

In the above formula the value $S \geq 0$ is a given value. If the normal displacement belongs to the interval $(0, g)$ then due to the presence of the asperities in the contact surface the friction occurs. If in turn the normal displacement is less than 0, then the body loses contact with the surface and in consequence the tangent stress is equal to zero. The friction model proposed in (3.7) can be viewed as an extension of a nonmonotone friction law which takes into account the evolution of the flattening of the asperities on the contact interface. This aspect of the evolution of the real contact area via the flattening of the asperities was also proposed in friction laws by Shaw [29] and Burdekin *et al.* [5].

Next we define the approximate problems corresponding to Problem \mathcal{P}_M . Let $n \in \mathbb{N}$.

Problem \mathcal{P}_M^n . Find a displacement $\mathbf{u}^n : \Omega \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}^n : \Omega \rightarrow \mathbb{S}^d$ such that, (3.2)–(3.5) and (3.7) hold for $\mathbf{u} = \mathbf{u}^n$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^n$, and

$$-\sigma_\nu^n \in \begin{cases} \{p(u_\nu^n)\} & \text{if } u_\nu^n < g, \\ [p(g), p(g) + nc_2] & \text{if } u_\nu^n = g, \\ \{p(g) + nc_2 + nc_3(u_\nu^n - g)\} & \text{if } u_\nu^n > g, \end{cases} \quad \text{on } \Gamma_3. \quad (3.10)$$

In (3.10) c_2 and c_3 are arbitrary nonnegative constants such that $c_2 + c_3 > 0$. These coefficients are related to the stiffness of the foundation after the limit g is reached. This contact condition can be interpreted physically as follows. First,

when $0 \leq u_\nu^n < g$, the contact process corresponds to the flattening of the asperities which leads in a second step ($u_\nu^n = g$) to the stiffening of the thin layer of asperities of the foundation. Finally, after this complete flattening of the asperities and this stiffening ($u_\nu^n > g$), the foundation recovers a smoothness rigidity with additional penetration (Fig. 2(b)).

In the sequel Problem \mathcal{P}_M will be approximated by the sequence of Problems \mathcal{P}_M^n , as $n \rightarrow \infty$.

Due to the basic properties of the Clarke subdifferential (cf. [6]), the right-hand side of the friction conditions (3.7) can be written as a subdifferential of a locally Lipschitz, possibly nonconvex superpotential j_τ which depends on the tangential displacement \mathbf{u}_τ . In fact, if the function $j_\tau : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by

$$j_\tau(\boldsymbol{\xi}) = \int_0^{|\boldsymbol{\xi}|} \mu(t) dt, \quad (3.11)$$

then, we can prove that under assumptions $H(\mu)$ (a)–(c) the conditions (3.7) are equivalent to the following subdifferential inclusion:

$$-\boldsymbol{\sigma}_\tau \in N(u_\nu) \partial j_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_3,$$

where $\partial j_\tau(\boldsymbol{\xi})$ denotes the Clarke subdifferential of j_τ (Fig. 3). By means of basic calculations one can easily prove the following lemma on the properties of the function j_τ .

Lemma 3.1. *If the assumptions $H(\mu)$ (a)–(d) hold, then the function j_τ defined by (3.11) has the following properties:*

- $H(j_\tau)$

$$\begin{cases} \text{(a) } j_\tau \text{ is locally Lipschitz;} \\ \text{(b) } |\boldsymbol{\eta}| \leq c(1 + |\boldsymbol{\xi}|) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d, \quad \boldsymbol{\eta} \in \partial j_\tau(\boldsymbol{\xi}); \\ \text{(c) } j_\tau^0(\boldsymbol{\xi}; -\boldsymbol{\xi}) \leq d(1 + |\boldsymbol{\xi}|) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d. \end{cases}$$

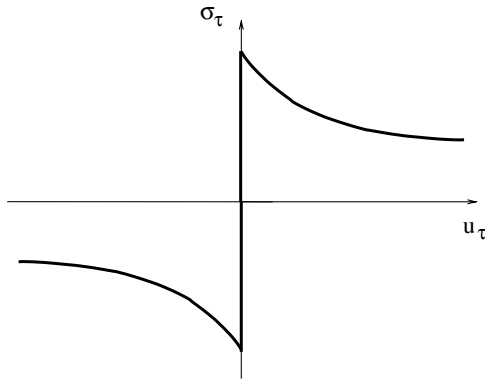


Fig. 3. Nonmonotone friction law (3.7).

If we define the functionals $j_\nu : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$, $j_\nu^n : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$j_\nu(x, \eta) = \begin{cases} 0 & \text{if } \eta \leq g(x), \\ \infty & \text{if } \eta > g(x), \end{cases}$$

$$j_\nu^n(x, \eta) = \begin{cases} 0 & \text{if } \eta \leq g(x), \\ nc_2\eta + \frac{nc_3}{2}(\eta - g(x))^2 & \text{if } \eta > g(x), \end{cases}$$

then the conditions (3.6) and (3.10) are respectively equivalent to

$$-\sigma_\nu \in p(u_\nu) + \partial_{\text{conv}} j_\nu(x, u_\nu), \quad -\sigma_\nu^n \in \bar{p}(x, u_\nu^n) + \partial j_\nu^n(x, u_\nu^n) \quad \text{on } \Gamma_3,$$

where both subdifferentials are taken with respect to the second variable. Since j_ν is not locally Lipschitz, the subdifferential in the first inclusion cannot be understood in the sense of Clarke, only in the sense of convex analysis, while in the second one the subdifferential is understood in both senses and they correspond to each other.

3.2. Variational formulations

In this section we formulate the above problems in a weak sense. To this end, we introduce the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}.$$

We also define the convex set

$$K = \{v \in V, v_\nu \leq g(x) \text{ on } \Gamma_3\}.$$

Since $\text{meas}(\Gamma_1) > 0$, the Korn inequality holds, thus there exists $C_K > 0$ which depends only on Ω and Γ_1 such that

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \geq C_K \|\mathbf{v}\|_{H_1} \quad \text{for all } \mathbf{v} \in V. \quad (3.12)$$

A proof of the Korn inequality may be found in [25, p. 79]. On V , we consider the inner product given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q \quad \text{for all } \mathbf{u}, \mathbf{v} \in V,$$

and let $\|\cdot\|$ be the associated norm, i.e.

$$\|\mathbf{v}\| = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \quad \text{for all } \mathbf{v} \in V. \quad (3.13)$$

It follows from (3.12) and (3.13) that $\|\cdot\|_{H_1}$ and $\|\cdot\|$ are equivalent norms on V and therefore $(V, \|\cdot\|)$ is a real Hilbert space. The duality pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$. Moreover, by the Sobolev trace theorem and (3.12) we have

a constant C_0 depending only on the domain Ω , Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq C_0 \|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in V. \quad (3.14)$$

Next we define the operator $B: V \rightarrow V^*$ and the bilinear form $a: V \times V \rightarrow \mathbb{R}$ by

$$\langle B\mathbf{u}, \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}) = (\mathcal{E}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_Q \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

We also denote by \mathbf{f} the element of V^* given by

$$\langle \mathbf{f}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} \, d\Gamma \quad (3.15)$$

for all $\mathbf{v} \in V$. From (3.1), it follows that the integrals in (3.15) are well defined. Proceeding in a standard way, we obtain the following variational formulations of Problems \mathcal{P}_M and \mathcal{P}_M^n .

Problem \mathcal{P}_V . Find the displacement field $\mathbf{u} \in K$ and the friction density $\boldsymbol{\sigma}_\tau \in L^2(\Gamma_3)^d$ such that for all $\mathbf{v} \in K$ we have

$$\begin{aligned} & \langle B\mathbf{u} - \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle + \int_{\Gamma_3} p(u_\nu)(v_\nu - u_\nu) \, d\Gamma \\ & \geq \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau) \, d\Gamma, \quad \text{with } -\boldsymbol{\sigma}_\tau \in N(u_\nu) \partial j_\tau(\mathbf{u}_\tau) \text{ a.e. on } \Gamma_3. \end{aligned} \quad (3.16)$$

Problem \mathcal{P}_V^n . Find the displacement field $\mathbf{u}^n \in V$, friction density $\boldsymbol{\sigma}_\tau^n \in L^2(\Gamma_3)^d$ and normal stress $\sigma_\nu^n \in L^2(\Gamma_3)$ such that for all $\mathbf{v} \in V$ we have

$$\begin{aligned} \langle B\mathbf{u}^n - \mathbf{f}, \mathbf{v} \rangle &= \int_{\Gamma_3} \boldsymbol{\sigma}_\tau^n \cdot \mathbf{v}_\tau + \sigma_\nu^n v_\nu \, d\Gamma, \\ & \text{with } -\boldsymbol{\sigma}_\tau^n \in N(u_\nu^n) \partial j_\tau(\mathbf{u}_\tau^n) \text{ a.e. on } \Gamma_3, \\ & \text{and } -\sigma_\nu^n \in \bar{p}(u_\nu^n) + \partial j_\nu^n(u_\nu^n) \text{ a.e. on } \Gamma_3. \end{aligned} \quad (3.17)$$

Problem \mathcal{P}_V is called a boundary variational-hemivariational inequality, while Problem \mathcal{P}_V^n is called a boundary hemivariational inequality.

Remark 3.2. Due to the growth condition $H(j_\tau)$ (b), conditions $H(N)$ (c), $H(p)$ (a) and (3.10) we have that the stress densities $\boldsymbol{\sigma}_\tau, \boldsymbol{\sigma}_\tau^n \in L^2(\Gamma_3)^d$ as well as $\sigma_\nu^n \in L^2(\Gamma_3)$ and the terms in (3.16) and (3.17) with these quantities, which in general would have the form of dualities on the spaces of boundary functions, have a form of integrals (see [15, 16]). The normal stress in Problem \mathcal{P}_V belongs to the space X_ν^* , where $X_\nu = \{v_\nu : v \in V\}$ and can be represented as an element from $H^{-\frac{1}{2}}(\Gamma_3)$ provided the boundary of Ω is smooth enough (cf. [17, Theorem 1.27, p. 57]).

Remark 3.3. Theorems 4.1 and 5.1 proved in Secs. 4 and 5 establish the existence of solutions to Problems \mathcal{P}_V^n and \mathcal{P}_V . Note that nothing is known on the uniqueness of solutions, generally we expect that there may be more than one solution unless the functions μ and p satisfy one-sided Lipschitz condition with a bound on the constants dependent on the coercivity constant of B (see [3] for detailed analysis of bilateral contact with friction).

4. Existence Result for the Approximate Problem

The goal of this section is to provide a theorem on existence of weak solution of the Problem \mathcal{P}_V^n .

Theorem 4.1. *If the assumptions $H(\mathcal{E})(a)$ –(c), $H(\mu)(a)$ –(d) and $H(f)$ hold, then Problem \mathcal{P}_V^n has a solution for every $n \in \mathbb{N}$.*

First we formulate a simple lemma, which can be proved by direct calculations.

Lemma 4.1. *If the assumptions $H(\mathcal{E})(a)$ –(c) hold then the operator $B \in \mathcal{L}(V, V^*)$ is symmetric (i.e. for all $\mathbf{u}, \mathbf{v} \in V$, $\langle B\mathbf{u}, \mathbf{v} \rangle = \langle B\mathbf{v}, \mathbf{u} \rangle$) and coercive (i.e. for all $\mathbf{u} \in V$, $\langle B\mathbf{u}, \mathbf{u} \rangle \geq m\|\mathbf{u}\|^2$ with $m > 0$).*

We will also need the following auxiliary lemma.

Lemma 4.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $j: \mathbb{R}^d \rightarrow \mathbb{R}$ be locally Lipschitz such that its Clarke subdifferential $\xi \rightarrow \partial j(\xi)$ is bounded on bounded sets. We assume that the sequences $u^n \rightarrow u$ in $L^p(\Omega; \mathbb{R}^d)$ and $\xi^n \rightarrow \xi$ weakly in $L^q(\Omega; \mathbb{R}^d)$, where $1 \leq p < \infty$ and $1 < q < \infty$. If*

$$\xi^n(x) \in \partial j(u^n(x)) \quad \text{for a.e. } x \in \Omega,$$

then

$$\xi(x) \in \partial j(u(x)) \quad \text{for a.e. } x \in \Omega.$$

Proof. It follows the lines of Step 2 of the proof of [24, Theorem 4.1]. Since $u^n \rightarrow u$ strongly in $L^p(\Omega; \mathbb{R}^d)$ then for a subsequence of $\{u^n\}$ (again denoted by the same symbol) one gets $u^n(x) \rightarrow u(x)$ for a.e. $x \in \Omega$. Thus, by Egoroff's theorem, for any $\varepsilon > 0$ there exists a subset $\omega \subset \Omega$, such that $\text{meas}(\omega) < \varepsilon$ and

$$u^n \rightarrow u \text{ strongly in } L^\infty(\Omega \setminus \omega; \mathbb{R}^d).$$

For any $v \in L^\infty(\Omega \setminus \omega; \mathbb{R}^d)$, by the definition of Clarke subdifferential, there holds

$$\int_{\Omega \setminus \omega} \xi^n(x) \cdot v(x) d\Omega \leq \int_{\Omega \setminus \omega} j^0(u^n(x); v(x)) d\Omega. \quad (4.1)$$

Passing to the limit in (4.1) and using the weak convergence of ξ^n , we have

$$\int_{\Omega \setminus \omega} \xi(x) \cdot v(x) d\Omega = \lim_{n \rightarrow \infty} \int_{\Omega \setminus \omega} \xi^n(x) \cdot v(x) d\Omega \leq \limsup_{n \rightarrow \infty} \int_{\Omega \setminus \omega} j^0(u^n(x); v(x)) d\Omega. \quad (4.2)$$

Moreover, we have for $x \in \Omega \setminus \omega$

$$j^0(u^n(x); v(x)) = \sup_{\eta \in \partial j(u^n(x))} \eta \cdot v(x) \leq \|v\|_{L^\infty(\Omega \setminus \omega; \mathbb{R}^d)} \sup_{\eta \in \partial j(u^n(x))} |\eta|.$$

So from the uniform convergence of $\{u^n\}$ and boundedness of the multifunction $\xi \rightarrow \partial j(\xi)$, it follows that $j^0(u^n(x); v(x))$ is bounded from above and thus we are

in position to invoke the Fatou lemma in (4.2) which gives

$$\int_{\Omega \setminus \omega} \xi(x) \cdot v(x) d\Omega \leq \int_{\Omega \setminus \omega} \limsup_{n \rightarrow \infty} j^0(u^n(x); v(x)) d\Omega. \quad (4.3)$$

From (4.3) and the upper semicontinuity of the Clarke directional derivative, we obtain

$$\int_{\Omega \setminus \omega} \xi(x) \cdot v(x) d\Omega \leq \int_{\Omega \setminus \omega} j^0(u(x); v(x)) d\Omega. \quad (4.4)$$

Since in (4.4) v is arbitrary, it easily follows that

$$\xi(x) \in \partial j(u(x)) \quad \text{a.e. in } \Omega \setminus \omega.$$

Finally, since $\varepsilon > 0$ is arbitrary, we get the assertion. \square

Proof of Theorem 4.1. We define an auxiliary multivalued operator $M^n : V \rightarrow 2^{V^*}$ as follows:

$$\xi \in M^n(\mathbf{u}) \Leftrightarrow \langle \xi, \mathbf{v} \rangle = \int_{\Gamma_3} ((\bar{p}(u_\nu) + \eta_\nu)v_\nu + N(u_\nu)\boldsymbol{\eta}_\tau \mathbf{v}_\tau) d\Gamma \quad \text{for all } \mathbf{v} \in V,$$

where $\eta_\nu(x) \in \partial j_\nu^n(u_\nu(x))$ and $\boldsymbol{\eta}_\tau(x) \in \partial j_\tau(\mathbf{u}_\tau(x))$ for a.e. $x \in \Gamma_3$. Now the problem under consideration reads as follows: find $\mathbf{u}^n \in V$ such that

$$B\mathbf{u}^n + M^n(\mathbf{u}^n) \ni f.$$

The basic properties of the Clarke subdifferential (see [6, Theorem 2.7.2]) guarantee that for each solution \mathbf{u}^n of this problem, there exist $\boldsymbol{\sigma}_\tau^n$ and σ_ν^n such that $(\mathbf{u}^n, \boldsymbol{\sigma}_\tau^n, \sigma_\nu^n)$ solves Problem \mathcal{P}_V^n . We need to verify that the operator $B + M^n$ is coercive and pseudomonotone and exploit the surjectivity Theorem 2.1.

$B + M^n$ is pseudomonotone. Linearity, continuity and coercivity of B imply its pseudomonotonicity. Thus we only need the pseudomonotonicity of M^n and we exploit the fact that the sum of two pseudomonotone mappings is pseudomonotone (see [21]). We show generalized pseudomonotonicity of M^n and use Proposition 2.1. Growth conditions on ∂j_ν^n , ∂j_τ and boundedness of N and \bar{p} as well as the fact that ∂j_ν^n , ∂j_τ are nonempty imply that also $M^n(\mathbf{u})$ is nonempty for $\mathbf{u} \in V$. The same argument implies that M^n maps bounded sets into bounded sets. Convexity of the set $M^n(\mathbf{u})$ in V^* follows directly from the convexity of the Clarke subdifferentials ∂j_τ and ∂j_ν^n . Now we show that $M^n(\mathbf{u})$ is closed. To this end let us take $\{\eta_k\}_{k=1}^\infty \subset M^n(\mathbf{u})$. It means that for $\mathbf{v} \in V$, we have

$$\langle \xi^k, \mathbf{v} \rangle = \int_{\Gamma_3} (\bar{p}(u_\nu) + \eta_\nu^k)v_\nu + N(u_\nu)\boldsymbol{\eta}_\tau^k \mathbf{v}_\tau d\Gamma, \quad (4.5)$$

where $\eta_\nu^k(x) \in \partial j_\nu^n(u_\nu(x))$ and $\boldsymbol{\eta}_\tau^k(x) \in \partial j_\tau(\mathbf{u}_\tau(x))$ for a.e. $x \in \Gamma_3$. The growth condition $H(j_\tau)$ (b) implies that $\|\boldsymbol{\eta}_\tau^k\|_{L^2(\Gamma_3; \mathbb{R}^d)}$ is bounded and the definition of j_ν^n implies that $\|\eta_\nu^k\|_{L^2(\Gamma_3; \mathbb{R})}$ is also bounded. Thus we can extract the subsequences, denoted again by index k , such that $\boldsymbol{\eta}_\tau^k \rightharpoonup \boldsymbol{\eta}_\tau$ weakly in $L^2(\Gamma_3; \mathbb{R}^d)$ and $\eta_\nu^k \rightharpoonup \eta_\nu$

weakly in $L^2(\Gamma_3)$. From Lemma 4.2, it follows that $\boldsymbol{\eta}_\tau(x) \in \partial j_\tau(\mathbf{u}_\tau(x))$ and $\eta_\nu(x) \in \partial j_\nu^n(u_\nu(x))$ for a.e. $x \in \Gamma_3$. Thus, we can pass to the limit in (4.5) and obtain

$$\langle \xi, \mathbf{v} \rangle = \int_{\Gamma_3} (\bar{p}u_\nu) + \eta_\nu v_\nu + N(u_\nu)\boldsymbol{\eta}_\tau \mathbf{v}_\tau d\Gamma \quad \text{for } \mathbf{v} \in V,$$

which means that $\xi \in M^n(\mathbf{u})$. To conclude the proof of pseudomonotonicity, let $\mathbf{u}^k \rightarrow \mathbf{u}$ weakly in V and $\xi^k \rightarrow \xi$ weakly in V^* with $\xi^k \in M^n(\mathbf{u}^k)$. Therefore for $\mathbf{v} \in V$, we have

$$\langle \xi^k, \mathbf{v} \rangle = \int_{\Gamma_3} (\bar{p}(u_\nu^k) + \eta_\nu^k)v_\nu + N(u_\nu^k)\boldsymbol{\eta}_\tau^k \mathbf{v}_\tau d\Gamma, \quad (4.6)$$

where $\eta_\nu^k(x) \in \partial j_\nu^n(u_\nu^k(x))$ and $\boldsymbol{\eta}_\tau^k(x) \in \partial j_\tau(\mathbf{u}_\tau^k(x))$ for a.e. $x \in \Gamma_3$. Using the same argument as in the proof of closedness of $M^n(\mathbf{u})$ and the fact, that the weakly convergent sequence $\{\mathbf{u}^k\}$ is bounded in V , we can extract the subsequences, denoted again by index k , such that $\boldsymbol{\eta}_\tau^k \rightarrow \boldsymbol{\eta}_\tau$ weakly in $L^2(\Gamma_3; \mathbb{R}^d)$ and $\eta_\nu^k \rightarrow \eta_\nu$ weakly in $L^2(\Gamma_3)$. From Lemma 4.2, it follows that $\boldsymbol{\eta}_\tau(x) \in \partial j_\tau(\mathbf{u}_\tau(x))$ and $\eta_\nu(x) \in \partial j_\nu^n(u_\nu(x))$ a.e. on Γ_3 . By the compactness of the trace operator and the continuity of the operators of taking the normal and tangent components, we have that $\mathbf{u}_\tau^k \rightarrow \mathbf{u}_\tau$ strongly in $L^2(\Gamma_3; \mathbb{R}^d)$ and $u_\nu^k \rightarrow u_\nu$ strongly in $L^2(\Gamma_3)$. In order to pass to the limit in Eq. (4.6), we will calculate its terms separately. First we extract a subsequence of $\{u_\nu^k\}$ (denoted again by the same index) which converges to u_ν pointwise almost everywhere on Γ_3 . Since p is continuous, we have $\bar{p}(u_\nu^k(x)) \rightarrow \bar{p}(u_\nu(x))$ for a.e. $x \in \Gamma_3$. Moreover $|\bar{p}(u_\nu^k(x))v_\nu(x)| \leq C_p$, so using the Lebesgue dominated convergence theorem, we have

$$\int_{\Gamma_3} \bar{p}(u_\nu^k)v_\nu d\Gamma \rightarrow \int_{\Gamma_3} \bar{p}(u_\nu)v_\nu d\Gamma \quad \text{for all } \mathbf{v} \in V.$$

Since $\eta_\nu^k \rightarrow \eta_\nu$ weakly in $L^2(\Gamma_3)$, we immediately get

$$\int_{\Gamma_3} \eta_\nu^k v_\nu d\Gamma \rightarrow \int_{\Gamma_3} \eta_\nu v_\nu d\Gamma.$$

For the last term in (4.6), we have

$$\int_{\Gamma_3} N(u_\nu^k)\boldsymbol{\eta}_\tau^k \mathbf{v}_\tau d\Gamma = \int_{\Gamma_3} N(u_\nu)\boldsymbol{\eta}_\tau^k \mathbf{v}_\tau d\Gamma + \int_{\Gamma_3} (N(u_\nu^k) - N(u_\nu))\boldsymbol{\eta}_\tau^k \mathbf{v}_\tau d\Gamma.$$

From $H(N)(c)$, the first integral tends to $\int_{\Gamma_3} N(u_\nu)\boldsymbol{\eta}_\tau \mathbf{v}_\tau d\Gamma$. We need to show that the second integral also tends to zero. From [7, Proposition 2.2.41], we extract from the sequences u_ν^k and \mathbf{u}_τ^k subsequences, denoted again by the same index, which converge, respectively, to u_ν and \mathbf{u}_τ pointwise almost everywhere on Γ_3 and $|u_\nu^k(x)| \leq h_\nu(x)$, $|\mathbf{u}_\tau^k(x)| \leq h_\tau(x)$ for a.e. $x \in \Gamma_3$ with $h_\nu, h_\tau \in L^2(\Gamma_3)$. By the continuity of N and the growth condition on ∂j_τ the integrand tends to zero almost everywhere. Moreover, $|(N(u_\nu^k(x)) - N(u_\nu(x)))\boldsymbol{\eta}_\tau^k(x)\mathbf{v}_\tau(x)| \leq 2Kc(1 + |\mathbf{u}_\tau^k(x)|)|\mathbf{v}_\tau(x)| \leq 2Kc(1 + h_\tau(x))|\mathbf{v}_\tau(x)|$, so we obtain the conclusion by applying the dominated Lebesgue convergence theorem.

$B+M^n$ is coercive. Coercivity of $B+M^n$ follows from condition (c) of Lemma 3.1, coercivity of B and assumptions $H(p)$ and $H(N)$. Indeed if $\xi \in B\mathbf{u} + M^n(\mathbf{u})$, for $\mathbf{u} \in V$ then

$$\langle \xi, \mathbf{u} \rangle \geq m\|\mathbf{u}\|^2 - C_p \int_{\Gamma_3} |u_\nu| d\Gamma + \int_{\Gamma_3} \eta_\nu u_\nu d\Gamma - K \int_{\Gamma_3} d(1 + |\mathbf{u}_\tau|) d\Gamma,$$

where $\eta_\nu \in \partial j_\nu^n(u_\nu)$. Now we observe that $0 \in \partial j_\nu^n(0)$. Hence from the fact that ∂j_ν^n is monotone it follows that $\eta_\nu u_\nu \geq 0$ for a.e. $x \in \Gamma_3$. Moreover $\|u_\nu\|_{L^1(\Gamma_3)} \leq C_1 \|\mathbf{u}\|$ and $\|\mathbf{u}_\tau\|_{L^1(\Gamma_3; \mathbb{R}^d)} \leq C_2 \|\mathbf{u}\|$ with $C_1, C_2 > 0$. It follows that

$$\frac{\langle \xi, \mathbf{u} \rangle}{\|\mathbf{u}\|} \geq m\|\mathbf{u}\| - C_p C_1 - Kd \left(\frac{\text{meas}(\Gamma_3)}{\|\mathbf{u}\|} - C_2 \right),$$

and the coercivity is shown. \square

5. Convergence Result

In this section, we formulate the main theorem of this work which corresponds to the convergence of the weak solution the approximate problems \mathcal{P}_V^n to the weak solution of the initial nonregularized Problem \mathcal{P}_V when the penalty parameter n tends to infinity.

Theorem 5.1. *If the assumptions $H(\mathcal{E})(a)-(c)$, $H(\mu)(a)-(d)$, $H(g)(a)-(b)$, $H(p)(a)-(b)$, $H(N)(a)-(c)$, $H(f)$ hold and $(\mathbf{u}^n, \boldsymbol{\sigma}_\tau^n, \sigma_\nu^n)$ is a solution of Problem \mathcal{P}_V^n , then, for a subsequence, we have $\mathbf{u}^n \rightharpoonup \mathbf{u}$ weakly in V , $\boldsymbol{\sigma}_\tau^n \rightharpoonup \boldsymbol{\sigma}_\tau$ weakly in $L^2(\Gamma_3; \mathbb{R}^d)$, where $(\mathbf{u}, \boldsymbol{\sigma}_\tau)$ is a solution of Problem \mathcal{P}_V .*

Remark 5.1. In the proof of Theorem 5.1, it is actually shown that, for every cluster point \mathbf{u} of the sequence $\{\mathbf{u}^n\}$ in the weak topology of V , one can find the stress $\boldsymbol{\sigma}_\tau$ such that $(\mathbf{u}, \boldsymbol{\sigma}_\tau)$ solves Problem \mathcal{P}_V . It is moreover shown that such cluster point exists and thus the solvability of the initial nonregularized Problem \mathcal{P}_V is provided.

Proof of Theorem 5.1. We start with an *a priori* estimate. Let us take $\mathbf{v} = \mathbf{u}^n$ in Eq. (3.17). By the coercivity of B and the continuity of \mathbf{f} , we have

$$m\|\mathbf{u}^n\|^2 \leq \int_{\Gamma_3} \sigma_\nu^n u_\nu^n d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau^n \cdot \mathbf{u}_\tau^n d\Gamma + \|\mathbf{f}\|_{V^*} \|\mathbf{u}^n\|. \quad (5.1)$$

By the formula (3.10), the first integral in (5.1) can be estimated as follows:

$$\int_{\Gamma_3} \sigma_\nu^n u_\nu^n d\Gamma = \int_{\Gamma_3} (-\bar{p}(u_\nu^n) - \eta_\nu^n) u_\nu^n d\Gamma \leq C_p \int_{\Gamma_3} |u_\nu^n| d\Gamma,$$

where $\eta_\nu^n(x) \in \partial j_\nu^n(x, u_\nu^n(x))$, for a.e. $x \in \Gamma_3$ and by the definition of j_ν^n the term with η_ν^n is nonpositive. In order to estimate the second integral, we use the definition of the Clarke subdifferential, the sign-condition $H(j_\tau)(c)$ and $H(N)$. Thus, we

obtain

$$m\|\mathbf{u}^n\|^2 \leq C_p \int_{\Gamma_3} |u_\nu^n| d\Gamma + Kd \int_{\Gamma_3} (1 + |\mathbf{u}^n|) d\Gamma + \|\mathbf{f}\|_{V^*} \|\mathbf{u}^n\|$$

and after a simple calculation, one gets

$$\|\mathbf{u}^n\| \leq M_1, \quad (5.2)$$

where $M_1 > 0$ depends only on $\|\mathbf{f}\|_{V^*}$, m , K , d , C_p , Γ_3 and Ω . By the growth condition $H(j_\tau)(b)$ and $H(N)(c)$, it follows that

$$\|\boldsymbol{\sigma}_\tau^n\|_{L^2(\Gamma_3; \mathbb{R}^d)} \leq M_2, \quad (5.3)$$

where $M_2 > 0$ depends on the problem data and it is independent of n . It follows that, for a subsequence, we have

$$\mathbf{u}^n \rightharpoonup \mathbf{u} \text{ weakly in } V, \quad (5.4)$$

$$\boldsymbol{\sigma}_\tau^n \rightharpoonup \boldsymbol{\sigma}_\tau \text{ weakly in } L^2(\Gamma_3; \mathbb{R}^d). \quad (5.5)$$

By the compactness of the trace operator, we also have

$$\mathbf{u}_\tau^n \rightarrow \mathbf{u}_\tau \text{ strongly in } L^2(\Gamma_3; \mathbb{R}^d), \quad (5.6)$$

$$u_\nu^n \rightarrow u_\nu \text{ strongly in } L^2(\Gamma_3). \quad (5.7)$$

It remains to show that $(\mathbf{u}, \boldsymbol{\sigma}_\tau)$ solves Problem \mathcal{P}_V .

First, we prove that $\mathbf{u} \in K$. To this end, we take $\mathbf{v} := \mathbf{u}^n$ in (3.17) and use (3.10). Thus

$$\begin{aligned} \langle B\mathbf{u}^n, \mathbf{u}^n \rangle - \langle \mathbf{f}, \mathbf{u}^n \rangle &= \int_{\Gamma_3} \boldsymbol{\sigma}_\tau^n \mathbf{u}_\tau^n d\Gamma - \int_{\Gamma_3} \bar{p}(u_\nu^n) u_\nu^n - \int_{\Gamma_3^-} \eta_\nu^n g d\Gamma \\ &\quad - \int_{\Gamma_3^>} (nc_2 + nc_3(u_\nu^n - g)) u_\nu^n d\Gamma, \end{aligned}$$

where $\eta_\nu^n \in [0, nc_2]$, $\Gamma_3^> = \Gamma_3 \cap \{u_\nu^n > g\}$ and $\Gamma_3^- = \Gamma_3 \cap \{u_\nu^n = g\}$. Since B is positively defined (see Lemma 4.1), we get

$$\begin{aligned} &c_2 \|(u_\nu^n - g)^+\|_{L^1(\Gamma_3)} + c_3 \|(u_\nu^n - g)^+\|_{L^2(\Gamma_3)}^2 \\ &\leq \frac{1}{n} (\|\mathbf{f}\|_{V^*} \|\mathbf{u}^n\| + \|\boldsymbol{\sigma}_\tau^n\|_{L^2(\Gamma_3; \mathbb{R}^d)} \|\mathbf{u}_\tau^n\|_{L^2(\Gamma_3; \mathbb{R}^d)} \\ &\quad + \|\bar{p}(u_\nu^n)\|_{L^2(\Gamma_3)} \|u_\nu^n\|_{L^2(\Gamma_3)}) \leq \frac{\tilde{C}}{n}, \end{aligned}$$

with some $\tilde{C} > 0$, where $(u_\nu^n - g)^+$ denotes the positive part of the function $u_\nu^n - g$. The last inequality comes from (5.2), (5.3), $H(p)$ and the boundedness of the trace

operator. Hence

$$(u_\nu^n - g)^+ \rightarrow 0 \text{ strongly in } L^1(\Gamma_3).$$

From (5.7) we also have

$$(u_\nu^n - g)^+ \rightarrow (u_\nu - g)^+ \text{ strongly in } L^1(\Gamma_3),$$

which means that $(u_\nu - g)^+ = 0$ for a.e. $x \in \Gamma_3$, so $\mathbf{u} \in K$.

Next, we show that $(\mathbf{u}, \boldsymbol{\sigma}_\tau)$ satisfies (3.16). We take an arbitrary $\mathbf{v} \in V$ and from the monotonicity of ∂j_ν^n , we obtain

$$\begin{aligned} \int_{\Gamma_3} \eta_\nu^n (v_\nu - u_\nu^n) d\Gamma &= \int_{\Gamma_3} (\eta_\nu^n - \xi^n) (v_\nu - u_\nu^n) d\Gamma + \int_{\Gamma_3} \xi^n (v_\nu - u_\nu^n) d\Gamma \\ &\leq \int_{\Gamma_3} \xi^n (v_\nu - u_\nu^n) d\Gamma, \end{aligned} \quad (5.8)$$

where $\eta_\nu^n(x) \in \partial j_\nu^n(u_\nu^n(x))$ for a.e. $x \in \Gamma_3$ and $-\sigma_\nu^n = \bar{p}(u_\nu^n) + \eta_\nu^n$, which holds for any $\xi^n \in L^2(\Gamma_3; \mathbb{R}^d)$ such that

$$\xi^n(x) \in \partial j_\nu^n(v_\nu(x)) \text{ a.e. on } \Gamma_3. \quad (5.9)$$

Now let $\mathbf{v} \in K$ and we put $\mathbf{v} - \mathbf{u}^n$ instead of \mathbf{v} in (3.17). After comparison with (5.8) and using (5.9) we obtain

$$\begin{aligned} \langle B\mathbf{u}^n - f, \mathbf{v} - \mathbf{u}^n \rangle &- \int_{\Gamma_3} \boldsymbol{\sigma}_\tau^n (\mathbf{v}_\tau - \mathbf{u}_\tau^n) d\Gamma \\ &\geq - \int_{\Gamma_3} \bar{p}(u_\nu^n) (v_\nu - u_\nu^n) d\Gamma - \int_{\Gamma_3} \xi^n (v_\nu - u_\nu^n) d\Gamma \\ &= - \int_{\Gamma_3} \bar{p}(u_\nu^n) (v_\nu - u_\nu^n) d\Gamma. \end{aligned} \quad (5.10)$$

The last equality holds, in particular, for $\xi^n(x) = 0$ for a.e. $x \in \Gamma_3$ since it satisfies (5.9) for $\mathbf{v} \in K$. Using the monotonicity of B , we have

$$\begin{aligned} \langle B\mathbf{u}^n - f, \mathbf{v} - \mathbf{u}^n \rangle &= \langle B\mathbf{u}^n - B\mathbf{v} + B\mathbf{v} - f, \mathbf{v} - \mathbf{u}^n \rangle \\ &= -\langle B\mathbf{u}^n - B\mathbf{v}, \mathbf{u}^n - \mathbf{v} \rangle + \langle B\mathbf{v} - f, \mathbf{v} - \mathbf{u}^n \rangle \\ &\leq \langle B\mathbf{v} - f, \mathbf{v} - \mathbf{u}^n \rangle. \end{aligned} \quad (5.11)$$

From (5.10) and (5.11), we obtain

$$\begin{aligned} \langle B\mathbf{v} - f, \mathbf{v} - \mathbf{u}^n \rangle + \int_{\Gamma_3} \bar{p}(u_\nu^n) (v_\nu - u_\nu^n) d\Gamma - \int_{\Gamma_3} \boldsymbol{\sigma}_\tau^n (\mathbf{v}_\tau - \mathbf{u}_\tau^n) d\Gamma &\geq 0 \\ &\text{for all } \mathbf{v} \in K. \end{aligned} \quad (5.12)$$

We need to pass to the limit in the second term. From (5.7) for a subsequence we have $u_\nu^n(x) \rightarrow u(x)$ for a.e. $x \in \Gamma_3$ and $|u_\nu^n(x)| \leq h(x)$ with $h \in L^2(\Gamma_3)$. By the continuity of p , we have $\bar{p}(x, u_\nu^n(x))(v_\nu(x) - u_\nu^n(x)) \rightarrow \bar{p}(x, u_\nu(x))(v_\nu(x) - u_\nu(x))$

for a.e. $x \in \Gamma_3$. Moreover, from $H(p)$, there holds $|\bar{p}(x, u_\nu^n(x))(v_\nu(x) - u_\nu^n(x))| \leq C_p(|v_\nu(x)| + h(x))$ for a.e. $x \in \Gamma_3$ and therefore from Lebesgue dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\Gamma_3} \bar{p}(u_\nu^n)(v_\nu - u_\nu^n) d\Gamma = \int_{\Gamma_3} \bar{p}(u_\nu)(v_\nu - u_\nu) d\Gamma.$$

Using the last relation as well as the convergences (5.4)–(5.6), we can pass to the limit in (5.12) and obtain

$$\langle B\mathbf{v} - f, \mathbf{v} - \mathbf{u} \rangle + \int_{\Gamma_3} \bar{p}(u_\nu)(v_\nu - u_\nu) d\Gamma - \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \mathbf{u}_\tau) d\Gamma \geq 0$$

for all $\mathbf{v} \in K$. (5.13)

Since K is convex, we can replace \mathbf{v} in (5.13) by $\mathbf{u} + \varepsilon(\mathbf{v} - \mathbf{u})$, where $\varepsilon \in [0, 1]$. If we assume that $\varepsilon \rightarrow 0$, we obtain (3.16).

Now we prove that $-\boldsymbol{\sigma}_\tau \in N(u_\nu)\partial j_\tau(\mathbf{u}_\tau)$ a.e. on Γ_3 . From (3.17), we have

$$-\boldsymbol{\sigma}_\tau^n(x) = N(u_\nu^n(x)) \cdot \boldsymbol{\eta}^n(x) \quad \text{a.e. on } \Gamma_3,$$

where

$$\boldsymbol{\eta}^n(x) \in \partial j_\tau(\mathbf{u}_\tau^n(x)) \quad \text{a.e. on } \Gamma_3.$$

From $H(j_\tau)$ (b) the sequence $\boldsymbol{\eta}^n$ is bounded in $L^2(\Gamma_3; \mathbb{R}^d)$ and, for a subsequence, we have $\boldsymbol{\eta}^n \rightharpoonup \boldsymbol{\eta}$ weakly in $L^2(\Gamma_3; \mathbb{R}^d)$. So by (5.6) and Lemma 4.2, we have

$$\boldsymbol{\eta}(x) \in \partial j_\tau(\mathbf{u}_\tau(x)) \quad \text{a.e. on } \Gamma_3.$$

Proceeding analogously as in the proof of Theorem 4.1, we have

$$\lim_{n \rightarrow \infty} \int_{\Gamma_3} N(u_\nu^n) \boldsymbol{\eta}^n \mathbf{v} d\Gamma = \int_{\Gamma_3} N(u_\nu) \boldsymbol{\eta} \mathbf{v} d\Gamma \quad \text{for all } \mathbf{v} \in L^2(\Gamma_3; \mathbb{R}^d).$$

Hence, it follows that $\boldsymbol{\sigma}_\tau \in N(u_\nu)\partial j_\tau(\mathbf{u}_\tau)$, which completes the proof of the theorem. \square

6. Numerical Algorithm

The numerical strategy presented in this section is based on a sequence of convex programming problems; more details can be found in [22, 23]. This approach to solve Problem \mathcal{P}_V is realized by using an iterative procedure in which for each iteration the friction bound $N(u_\nu) \times \mu(\mathbf{u}_\tau)$ is fixed to a given function depending on the normal displacement u_ν and on the tangential displacement solution \mathbf{u}_τ found in the previous iteration. The frictional contact conditions are treated by using a numerical approach based on the combination of the penalized method and the augmented Lagrangian method. Then, the nonsmooth convex problems arising during the iterative process can be solved by classical numerical methods.

First of all, let us consider some materials for the discretization step. Assume that Ω is a polyhedral domain. Moreover, let $\{\mathcal{T}^h\}$ be a regular family of triangular

finite element partitions of $\overline{\Omega}$ that are compatible with the boundary decomposition $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$, i.e. if one side of an element $T \in \mathcal{T}^h$ has more than one point on Γ , then the side lies entirely on $\overline{\Gamma_1}$, $\overline{\Gamma_2}$ or $\overline{\Gamma_3}$. The space V is approximated by the finite-dimensional space $V^h \subset V$ of continuous and piecewise affine functions, that is,

$$V^h = \{\mathbf{v}^h \in [C(\overline{\Omega})]^d : \mathbf{v}^h|_T \in [P_1(T)]^d \forall T \in \mathcal{T}^h, \\ \mathbf{v}^h = \mathbf{0} \text{ at the nodes on } \Gamma_1\},$$

where $P_1(T)$ represents the space of polynomials of degree less than or equal to one in T .

For the discretization of the normal contact terms, we consider the space $X_\nu^h = \{v_\nu^h|_{\Gamma_3} : \mathbf{v}^h \in V^h\}$ equipped with its usual norm. Let us consider the discrete space of piecewise constants $Y_\nu^h \subset L^2(\Gamma_3)$ related to the discretization of the normal stress σ_ν . We also introduce the function $\varphi : X_\nu^h \rightarrow (-\infty, +\infty]$ and the operator $L : X_\nu^h \rightarrow Y_\nu^h$ defined by

$$\varphi(u_\nu^h) = \int_{\Gamma_3} I_{\mathbb{R}_-}(u_\nu^h - g) d\Gamma, \quad \forall u_\nu^h \in X_\nu^h, \\ L : X_\nu^h \rightarrow Y_\nu^h \quad \langle Lu_\nu^h, v_\nu^h \rangle_{Y_\nu^h, X_\nu^h} = \int_{\Gamma_3} p(u_\nu^h) v_\nu^h d\Gamma, \quad \forall u_\nu^h, v_\nu^h \in X_\nu^h,$$

where $I_{\mathbb{R}_-}$ represents the indicator function of the set $\mathbb{R}_- = (-\infty, 0]$. Let us also consider the boundary interpolation operator $\Pi_\nu^h : X_\nu^h \rightarrow Y_\nu^h$. See [15, 16] for more details about the discretization spaces. Then, we note that the normal compliance condition with finite penetration (3.6) leads to the following discrete subdifferential inclusion

$$-\sigma_\nu^h \in \partial\varphi(\Pi_\nu^h u_\nu^h) + Lu_\nu^h \quad \text{in } Y_\nu^h.$$

For the discretization of the tangential friction terms, let us consider the space $X_\tau^h = \{v_\tau^h|_{\Gamma_3} : \mathbf{v}^h \in V^h\}$ equipped with its usual norm. We also consider the discrete space of piecewise constants $Y_\tau^h \subset L^2(\Gamma_3)^d$ related to the discretization of the friction density σ_τ . In the same way that for the contact part, we note that the friction condition (3.7) leads to the following discrete subdifferential inclusion

$$-\sigma_\tau^h \in N(|\Pi_\nu^h u_\nu^h|) \mu (|\Pi_\tau^h \mathbf{u}_\tau^h|) \partial|\Pi_\tau^h \mathbf{u}_\tau^h| \quad \text{in } Y_\tau^h,$$

where $\Pi_\tau^h : X_\tau^h \rightarrow Y_\tau^h$ represents the boundary interpolation operator from X_τ^h to Y_τ^h (see [15, 16]).

Using the previous discretization notations, the numerical solution of the non-smooth nonconvex variational problem \mathcal{P}_V is based on the following iterative algorithm:

Let $\epsilon > 0$ and $\mathbf{u}^{(0)}$ be given.

Then, for $k = 0, 1, \dots$,

Problem $\mathcal{P}_{V_C^h}$. Find a displacement field $\mathbf{u}^{h,(k+1)} \in V^h$,

a contact stress $\sigma_\nu^{h,(k+1)} \in Y_\nu^h$ and a friction stress field $\sigma_\tau^{h,(k+1)} \in Y_\tau^h$
such that, for $\forall \mathbf{v}^h \in V^h$

$$\langle B\mathbf{u}^{h,(k+1)} - \mathbf{f}, \mathbf{v}^h \rangle = \int_{\Gamma_3} \sigma_\nu^{h,(k+1)} v_\nu^h d\Gamma + \int_{\Gamma_3} \sigma_\tau^{h,(k+1)} \cdot \mathbf{v}_\tau^h d\Gamma \quad (6.1)$$

$$\text{with } -\sigma_\nu^{h,(k+1)} \in \partial\varphi(\Pi_\nu^h u_\nu^{h,(k+1)}) + Lu_\nu^{h,(k+1)} \quad \text{on } \Gamma_3 \quad (6.2)$$

$$\text{and } -\sigma_\tau^{h,(k+1)} \in N(|\Pi_\nu^h u_\nu^{h,(k)}|)\mu(|\Pi_\tau^h \mathbf{u}_\tau^{h,(k)}|)\partial|\Pi_\tau^h \mathbf{u}_\tau^{h,(k+1)}| \quad \text{on } \Gamma_3 \quad (6.3)$$

until $\|\mathbf{u}^{h,(k+1)} - \mathbf{u}^{h,(k)}\| \leq \epsilon \|\mathbf{u}^{h,(k)}\|$

and $\|\sigma^{h,(k+1)} - \sigma^{h,(k)}\|_{L^2(\Gamma_3)^d} \leq \epsilon \|\sigma^{h,(k)}\|_{L^2(\Gamma_3)^d}$.

Here, k represents the index of the iterative procedure. In Problem $\mathcal{P}_{V_C^h}$ the discrete stress σ^h on the contact boundary Γ_3 can be viewed as a Lagrange stress multiplier. This numerical strategy leads to the solution of a nonsmooth convex problem $\mathcal{P}_{V_C^h}$ at each iteration k .

In the rest of this section, to simplify the notation and the readability, we skip the dependence of various variables with respect to the discretization parameters n , k and h , i.e. for example, we write \mathbf{u} instead of \mathbf{u}_n^{hk} .

For the numerical treatment of the nonsmooth convex Problem $\mathcal{P}_{V_C^h}$, we use the penalized method for the normal compliance contact term combined with the augmented Lagrangian approach for the unilateral condition and Coulomb friction law. To this end, we consider additional fictitious nodes for the Lagrange multiplier in the initial mesh. The construction of these nodes depends on the contact element used for the geometrical discretization of the interface Γ_3 . In the case of the numerical example presented below, the discretization is based on “node-to-rigid” contact element, which is composed by one node of Γ_3 and one Lagrange multiplier node. This contact interface discretization is characterized by a finite-dimensional subspace $H_{\Gamma_3}^h \subset Y^h$, where $Y^h = Y_\nu^h \cup Y_\tau^h$. Let N_{tot}^h be the total number of nodes and denote by α^i the basis functions used to define the space V^h for $i = 1, \dots, N_{\text{tot}}^h$. Moreover, let $N_{\Gamma_3}^h$ represent the number of nodes on the interface Γ_3 and let μ^i be the shape functions of the finite element space $H_{\Gamma_3}^h$, for $i = 1, \dots, N_{\Gamma_3}^h$, i.e.

$$H_{\Gamma_3}^h = \left\{ \gamma^h \in Y^h : \gamma^h = \sum_{i=1}^{N_{\Gamma_3}^h} \gamma^i \mu^i \right\}.$$

Usually, if a P_1 finite element method is used for the displacement, then a P_0 finite element method is considered for the multipliers. The expression of functions $\mathbf{v}^h \in V^h$ and $\gamma^h \in H_{\Gamma_3}^h$ is given by

$$\mathbf{v}^h = \sum_{i=1}^{N_{\text{tot}}^h} \mathbf{v}^i \alpha^i \quad \forall \mathbf{v}^h \in V^h \quad \text{and} \quad \gamma^h = \sum_{i=1}^{N_{\Gamma_3}^h} \gamma^i \mu^i \quad \forall \gamma^h \in H_{\Gamma_3}^h, \quad (6.4)$$

where \mathbf{v}^i represent the values of the corresponding functions \mathbf{v}^h at the i th node of \mathcal{T}^h . Also, γ^i denotes the values of the function γ^h at the i th node of the contact element discretization of the contact interface. More details about this discretization step can be found in [2, 15, 16, 32].

In the following, we denote by λ_ν and λ_τ the normal and tangential components of the Lagrange multiplier $\boldsymbol{\lambda}$ on Γ_3 given by $\lambda_\nu = \boldsymbol{\lambda} \cdot \boldsymbol{\nu}$, $\lambda_\tau = \boldsymbol{\lambda} - \lambda_\nu \boldsymbol{\nu}$. Then, it can be shown that the augmented Lagrangian approach on Problem \mathcal{P}_{V^h} is governed at each incremental step k by a system of nonlinear equations of the following form.

Problem \mathcal{P}_L . Find a displacement field $\mathbf{u} \in \mathbb{R}^{d \cdot N_{\text{tot}}^h}$ and a stress multiplier field $\boldsymbol{\lambda} \in \mathbb{R}^{d \cdot N_{\Gamma_3}^h}$ such that

$$\mathbf{R}(\mathbf{u}, \boldsymbol{\lambda}) = \tilde{\mathbf{G}}(\mathbf{u}) + \mathcal{F}(\mathbf{u}, \boldsymbol{\lambda}) = \mathbf{0}, \quad (6.5)$$

where $\tilde{\mathbf{G}}$ and \mathcal{F} are defined below.

First, the vectors \mathbf{u} and $\boldsymbol{\lambda}$ represent the displacement and the Lagrange multiplier generalized vectors, respectively, defined by

$$\mathbf{u} = \{\mathbf{u}^i\}_{i=1}^{N_{\text{tot}}^h}, \quad \boldsymbol{\lambda} = \{\lambda^i\}_{i=1}^{N_{\Gamma_3}^h}, \quad (6.6)$$

where \mathbf{u}^i represents the value of the corresponding function \mathbf{u}^h at the i th node of \mathcal{T}^h . Also, λ^i denotes the value of the corresponding function λ^h at the i th node of the contact element of the discretized contact interface. The generalized elastic term $\tilde{\mathbf{G}}(\mathbf{u}) \in \mathbb{R}^{d \cdot N_{\text{tot}}^h} \times \mathbb{R}^{d \cdot N_{\Gamma_3}^h}$ is defined by $\tilde{\mathbf{G}}(\mathbf{u}) = (\mathbf{G}(\mathbf{u}), \mathbf{0}_{d \cdot N_{\Gamma_3}^h})$, where $\mathbf{0}_{d \cdot N_{\Gamma_3}^h}$ is the zero element of $\mathbb{R}^{d \cdot N_{\Gamma_3}^h}$, $\mathbf{G}(\mathbf{u}) \in \mathbb{R}^{d \cdot N_{\text{tot}}^h}$ denotes the term given by

$$(\mathbf{G}(\mathbf{u}) \cdot \mathbf{v})_{\mathbb{R}^{d \times N_{\text{tot}}^h}} = \langle B\mathbf{u}^h - \mathbf{f}, \mathbf{v}^h \rangle \quad \forall \mathbf{v} = \{\mathbf{v}^i\}_{i=1}^{N_{\text{tot}}^h},$$

\mathbf{v}^h is defined by (6.4). The frictional contact operator $\mathcal{F}(\mathbf{u}, \boldsymbol{\lambda})$, which allows to deal with the frictional contact conditions (6.2) and (6.3), is defined by

$$\begin{aligned} & (\mathcal{F}(\mathbf{u}, \boldsymbol{\lambda}) \cdot (\mathbf{v}, \boldsymbol{\gamma}))_{\mathbb{R}^{d \cdot N_{\text{tot}}^h} \times \mathbb{R}^{d \cdot N_{\Gamma_3}^h}} \\ &= \int_{\Gamma_3} \nabla \mathbf{u} \mathcal{P}_c([u_\nu]_g) \cdot \mathbf{v}^h d\Gamma + \int_{\Gamma_3} \nabla \mathbf{u} [l_\nu(\mathbf{u}^h, \lambda^h) + l_\tau(\mathbf{u}^h, \lambda^h)] \cdot \mathbf{v}^h d\Gamma \\ &+ \int_{\Gamma_3} \nabla \lambda [l_\nu(\mathbf{u}^h, \lambda^h) + l_\tau(\mathbf{u}^h, \lambda^h)] \cdot \boldsymbol{\gamma}^h d\Gamma \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^{d \cdot N_{\text{tot}}^h}, \\ & \quad \forall \boldsymbol{\lambda}, \boldsymbol{\gamma} \in \mathbb{R}^{d \cdot N_{\Gamma_3}^h}, \quad \mathbf{u}^h, \mathbf{v}^h \in V^h, \quad \forall \lambda^h, \boldsymbol{\gamma}^h \in H_{\Gamma_3}^h. \end{aligned}$$

Here $\mathcal{P}_c : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that $\mathcal{P}'_c = p$ on $(-\infty, g]$, $[\cdot]_g : \mathbb{R} \rightarrow \mathbb{R}$ is the function defined by

$$[u]_g = \begin{cases} u & \text{if } u \leq g, \\ 0 & \text{if } u > g, \end{cases}$$

and $\nabla_{\mathbf{x}}$ represents the gradient operator with respect the variable \mathbf{x} ; also, l'_ν and l'_τ denote respectively the augmented Lagrangian functionals given by

$$l_\nu(\mathbf{u}^h, \lambda_\nu^h) = -\frac{1}{2r_\nu} (\lambda_\nu^h)^2 + \frac{1}{2r_\nu} [(\lambda_\nu^h + r_\nu(u_\nu^h - g))]^2, \quad (6.7)$$

$$l_\tau(\mathbf{u}^h, \boldsymbol{\lambda}_\tau^h) = \mathbf{u}_\tau^h \cdot \boldsymbol{\lambda}_\tau^h + \frac{r_\tau}{2} |\mathbf{u}_\tau^h|^2 - \frac{1}{2r_\tau} (\text{dist}_{C[-N\mu]} \{\boldsymbol{\lambda}_\tau^h + r_\tau \mathbf{u}_\tau^h\})^2, \quad (6.8)$$

where r_ν and r_τ are positive penalty coefficients and the Coulomb convex set $C[-N\mu]$ denotes the convex disk of constant radius $-N\mu$ during the incremental step $k+1$. For more details about the ‘‘quasi’’ Lagrangian method, we refer the reader to [2, 32].

The solution of the nonlinear system (6.5) is based on a linear iterative method similar to that used in the Newton method. This Newton method permits to treat both variables $(\mathbf{u}, \boldsymbol{\lambda})$ simultaneously; then, we consider the pair $\tilde{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\lambda})$ and the iterative scheme can be summarized as follows:

Let $\epsilon > 0$ and $\tilde{\mathbf{u}}_0$ be given.

For $i = 0, \dots$ compute

$$\tilde{\mathbf{u}}_{i+1} = (\mathbf{K}_i + \mathbf{T}_i)^{-1} \cdot \mathbf{R}(\tilde{\mathbf{u}}_i). \quad (6.9)$$

until $\|\tilde{\mathbf{u}}_{i+1} - \tilde{\mathbf{u}}_i\|_{\mathbb{R}^{d \cdot N_{\text{tot}}^h + d \cdot N_{\Gamma_3}^h}} \leq \epsilon \|\tilde{\mathbf{u}}_i\|_{\mathbb{R}^{d \cdot N_{\text{tot}}^h + d \cdot N_{\Gamma_3}^h}}$

and $\|\mathbf{R}(\tilde{\mathbf{u}}_{i+1}) - \mathbf{R}(\tilde{\mathbf{u}}_i)\|_{\mathbb{R}^{d \cdot N_{\text{tot}}^h + d \cdot N_{\Gamma_3}^h}} \leq \epsilon \|\mathbf{R}(\tilde{\mathbf{u}}_i)\|_{\mathbb{R}^{d \cdot N_{\text{tot}}^h + d \cdot N_{\Gamma_3}^h}}$.

Here i represents the Newton iteration index, $\mathbf{K}_i = D_{\tilde{\mathbf{u}}} \mathbf{G}(\mathbf{u})$ denotes the usual elastic stiffness matrix and $\mathbf{T}_i \in \partial_{\tilde{\mathbf{u}}} \mathcal{F}(\tilde{\mathbf{u}}_i)$ is the frictional contact tangent matrix. $D_{\tilde{\mathbf{u}}} \mathbf{G}$ represents the differential of the functions \mathbf{G} with respect to the variable $\tilde{\mathbf{u}}$. $\partial_{\tilde{\mathbf{u}}} \mathcal{F}(\tilde{\mathbf{u}}_i)$ represents the generalized Jacobian of \mathcal{F} at $\tilde{\mathbf{u}}_i$. Usually $\tilde{\mathbf{u}}_i$ is in a region of linearity since the point set of nondifferentiability of the function \mathcal{F} is null measure, then \mathbf{T}_i is reduced to a single classical Jacobian matrix. Each region of differentiability of the operator \mathcal{F} corresponds to a frictional contact state of each contact node of the discretization.

It is easy to see that (6.9) is equivalent to an algebraic linear nonsymmetric system which can be solved by Conjugate Gradient Squared Methods with efficient preconditioners, used here to overcome the poor conditioning of the matrix due to the frictional contact terms, see, for instance, [1]. Details on Computational Contact Mechanics, including algorithms similar to that described above can be found in the monographs [18, 32].

7. Numerical Examples

We consider the physical setting depicted in Fig. 4. There, $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ with $L_1, L_2 > 0$ and

$$\Gamma_1 = \{0\} \times [0, L_2], \quad \Gamma_2 = (\{L_1\} \times [0, L_2]) \cup ([0, L_1] \times \{L_2\}), \quad \Gamma_3 = [0, L_1] \times \{0\}.$$

The domain Ω represents the cross-section of a three-dimensional deformable body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed. On the part $\Gamma_1 = \{0\} \times [0, L_2]$ the body is clamped and, therefore, the displacement field vanishes there. Vertical tractions act on the part $[0, L_1] \times \{L_2\}$ of the boundary and the part $\{L_1\} \times [0, L_2]$ is traction free. No body forces are assumed to act on the body during the process. The body is in frictional contact with an obstacle on the part $\Gamma_3 = [0, L_1] \times \{0\}$ of the boundary.

We model the material's behavior with a constitutive law of the form (3.2) in which elasticity tensor \mathcal{E} satisfies

$$(\mathcal{E}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2,$$

where E is the Young modulus, κ the Poisson ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. The friction is modeled by a nonmonotone law (3.7) in which the friction bound $N(u_\nu)\mu(|\mathbf{u}_\tau|)$ depends on the depth of the penetration u_ν and on the tangential displacement $|\mathbf{u}_\tau|$. For the simulations, the function $N: \mathbb{R} \rightarrow \mathbb{R}^+$ given in (3.9) is taken. Let us also consider, the following friction coefficient $\mu: \mathbb{R}^+ \rightarrow \mathbb{R}$:

$$\mu(|\mathbf{u}_\tau|) = (a - b) \cdot e^{-\alpha|\mathbf{u}_\tau|} + b, \quad (7.1)$$

with $a, b, \alpha > 0$, $a \geq b$.

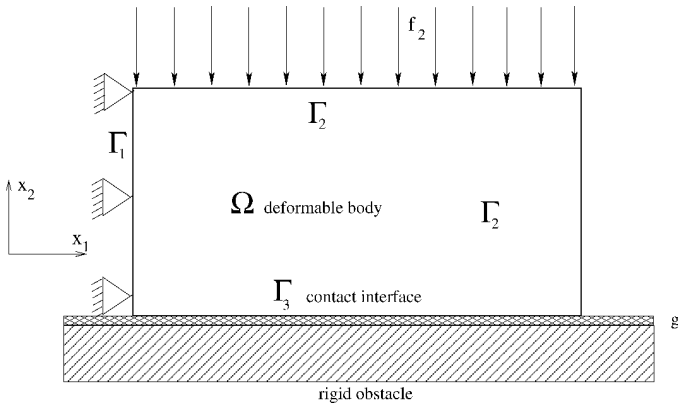


Fig. 4. Initial configuration of the two-dimensional example.

For the computation below we use the following data:

$$L_1 = 2 \text{ m}, \quad L_2 = 1 \text{ m},$$

$$E = 1000 \text{ N/m}^2, \quad \kappa = 0.3,$$

$$\mathbf{f}_0 = (0, 0) \text{ N/m}^2, \quad \mathbf{f}_2 = \begin{cases} (0, 0) \text{ N/m} & \text{on } \{2\} \times [0, 1], \\ (0, -300) \text{ N/m} & \text{on } [0, 2] \times \{1\}, \end{cases}$$

$$a = 1.5, \quad b = 0.5, \quad \alpha = 100, \quad S = 1 \text{ N},$$

$$\mathcal{P}_c(u) = \frac{1}{2} c_1 (u_+)^2, \quad c_1 = 100 \quad \text{and} \quad r_\nu = 0.1, \quad r_\tau = 0.1.$$

stopping criterion: $\epsilon = 10^{-6}$.

Our results are presented in Figs. 5–7 and are described in what follows.

First, in Fig. 5, the deformed configuration as well as the frictional contact forces are plotted both in the case $g = 0 \text{ m}$ and $g = 0.04 \text{ m}$. We note that the case $g = 0 \text{ m}$ corresponds to the classical Signorini contact condition. Moreover, in the case $g = 0.04 \text{ m}$ the contact follows a normal compliance condition as far as the penetration is less than the limit $g = 0.04 \text{ m}$ and, when this limit is reached, it follows a unilateral condition.

In Fig. 6 we present the convergence of solution of problem $\mathcal{P}_{V_h}^n$ to the solution of problem \mathcal{P}_{V_h} . More precisely, we plot four deformed meshes and the associated frictional contact forces at four steps of convergence, for $n = 10, 100, 10^3, 10^4$. One can see that for $n = 10$ all the contact nodes are in strong penetration contact, whereas at $n = 10^4$ the contact nodes are into an admissible finite penetration, since the complete flattening of the asperities of size $g = 0.04 \text{ m}$ was reached.

Numerical convergence. Now, we turn to the numerical validation of the convergence result presented in Theorem 5.1. To this end, besides the numerical approximation of the problem with finite penetration presented above, we consider the solution of the problem with unlimited penetration, \mathcal{P}_V^n , in which $n > 0$ is a penalized parameter. The solution of this problem can be found by solving the nonlinear system (6.5) in which the Lagrange multiplier related to the normal contact stress

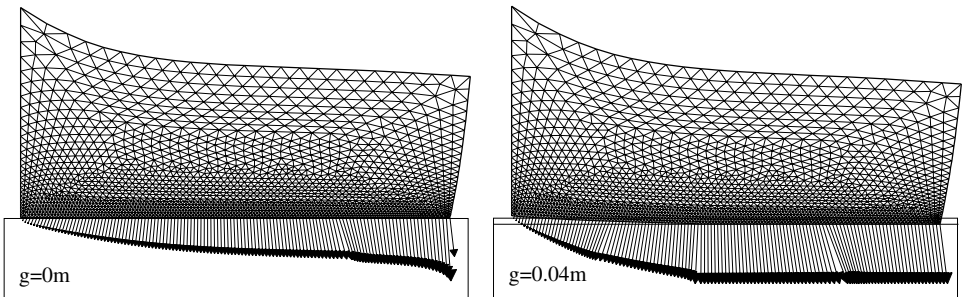


Fig. 5. Deformed meshes and frictional contact forces for $g = 0 \text{ m}$ and $g = 0.04 \text{ m}$.

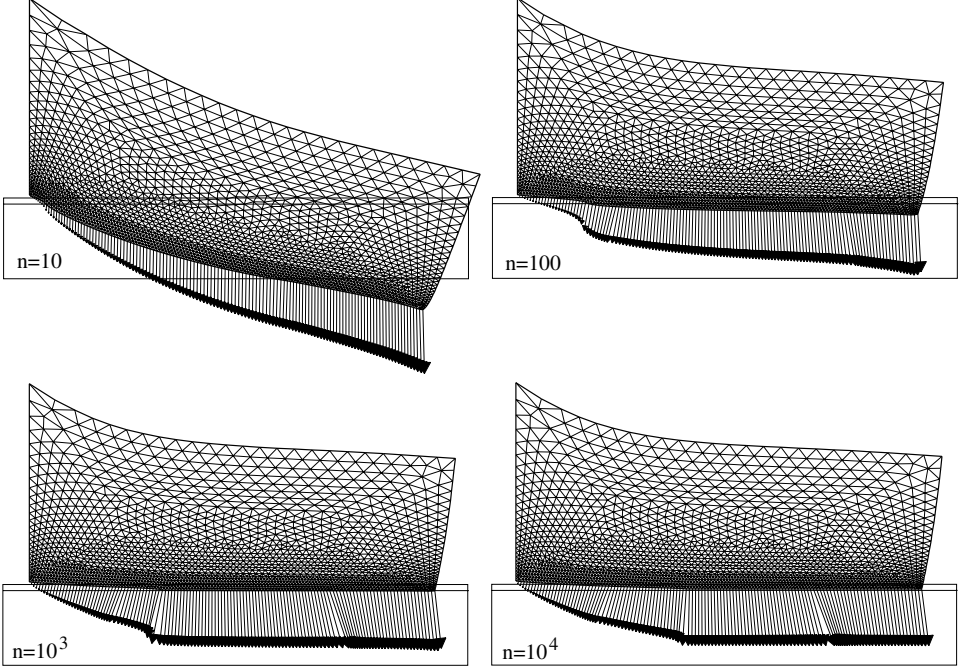


Fig. 6. Deformed meshes and frictional contact forces for $n = 10$, $n = 100$, $n = 10^3$ and $n = 10^4$.

λ_ν vanishes and the parameter r_ν is fixed to the value $n \times c_3$ where c_3 is fixed to 1. To simplify the calculus and the computation, we take c_2 equal to zero. Then, the frictional contact operator $\mathcal{F}(\mathbf{u}, \boldsymbol{\lambda}_\tau)$ is defined by the following form:

$$\begin{aligned}
 & (\mathcal{F}(\mathbf{u}, \boldsymbol{\lambda}_\tau) \cdot (\mathbf{v}, \boldsymbol{\gamma}_\tau))_{\mathbb{R}^{d \cdot N_{\text{tot}}^h} \times \mathbb{R}^{d \cdot N_{\Gamma_3}^h}} \\
 &= \int_{\Gamma_3} r_\nu (u_\nu^h - g)_+ (\nabla \mathbf{u} u_\nu^h) \cdot \mathbf{v}^h d\Gamma + \int_{\Gamma_3} c_1 ([u_\nu^h]_g)_+ (\nabla \mathbf{u} u_\nu^h) \cdot \mathbf{v}^h d\Gamma \\
 &+ \int_{\Gamma_3} \frac{-1}{r_\tau} (\lambda_\tau^h - \text{proj}_{C[-N\mu]}(\lambda_\tau^h + r_\tau \mathbf{u}_\tau^h)) \cdot \boldsymbol{\gamma}_\tau^h d\Gamma, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^{d \cdot N_{\text{tot}}^h}, \\
 &\quad \forall \boldsymbol{\lambda}_\tau, \boldsymbol{\gamma}_\tau \in \mathbb{R}^{d \cdot N_{\Gamma_3}^h}, \quad \forall \mathbf{u}^h, \mathbf{v}^h \in V^h, \quad \forall \lambda_\tau^h, \boldsymbol{\gamma}_\tau^h \in H_{\Gamma_3}^h.
 \end{aligned}$$

Denote by $(\mathbf{u}_n^h, \boldsymbol{\sigma}_n^h)$ and $(\mathbf{u}^h, \boldsymbol{\sigma}^h)$ the discrete solution of the contact problems $\mathcal{P}_{V^h}^n$ and \mathcal{P}_{V^h} , respectively. The numerical estimations of the difference

$$\|\mathbf{u}_n^h - \mathbf{u}^h\|_V + \|\boldsymbol{\sigma}_n^h - \boldsymbol{\sigma}^h\|_Q,$$

for various values of the parameter n , is presented in Fig. 7. It results from here that this difference converges to zero when n tends toward infinity, which represents a numerical validation of the theoretical convergence result obtained in Theorem 5.1.

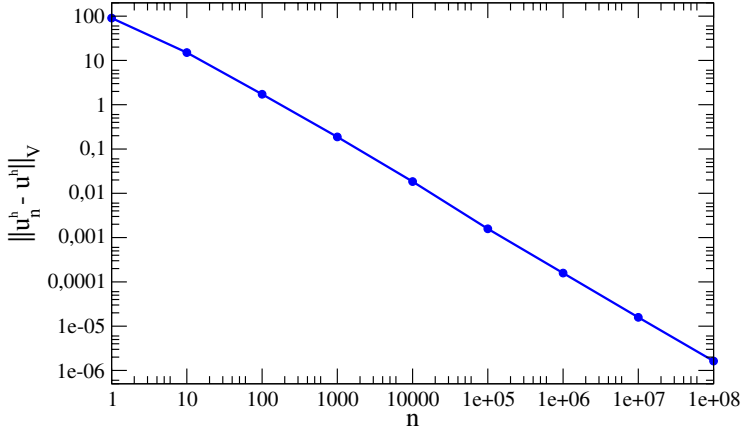


Fig. 7. Numerical validation of the convergence result in Theorem 5.1.

8. Conclusion

The paper presents a contribution on the model of normal compliance contact with nonmonotone slip-dependent friction between an linear elastic body and a foundation. The novelties arise in the combination of contact models with finite and nonrestricted penetrations with a nonmonotone friction law. We consider two problems: one with the nonregularized contact condition and the second with a regularized contact condition. The existence of the weak solution of the regularized problem is obtained by using classical arguments on boundary hemivariational inequalities. The main result concerns a convergence result of the weak solution of the regularized problem to the weak solution of the nonregularized problem which leads to the solvability of this problem. Furthermore, a numerical algorithm based on a sequence of nonsmooth convex programming was implemented and numerical simulations for an academic two-dimensional example were provided in order to illustrate the mechanical behavior of the contact model and to validate the theoretical error estimates convergence. Finally, it may be of interest to continue the work by considering quasi-static or dynamic processes, and study the related evolutionary frictional contact problem.

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