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► **To cite this version:**

Stanislaw Migórski, Anna Ochal, Mircea Sofonea. Analysis of a piezoelectric contact problem with subdifferential boundary condition. Proceedings of the Royal Society of Edinburgh: Section A, Mathematics, 2014, 144 (05), pp.1007-1025. 10.1017/S0308210513000607 . hal-01142125

HAL Id: hal-01142125

<https://hal.science/hal-01142125>

Submitted on 8 Feb 2023

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Analysis of a piezoelectric contact problem with subdifferential boundary condition

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We consider a mathematical model which describes the frictionless contact between a piezoelectric body and a foundation. The contact process is quasi-static and the foundation is assumed to be insulated. The novelty of the model consists in the fact that the material behaviour is described with an electro-elastic–visco-plastic constitutive law and the contact is modelled with a subdifferential boundary condition. We derive a variational formulation of the problem which is in the form of a system coupling two nonlinear integral equations with a history-dependent hemivariational inequality and a time-dependent linear equation. We prove the existence of a weak solution to the problem and, under additional assumptions, its uniqueness. The proof is based on a recent result on history-dependent hemivariational inequalities obtained by Migórski, Ochal and Sofonea in 2011.

1. Introduction

The piezoelectric effect is characterized by the appearance of electric charges on surfaces of some crystals after deformation. Its reverse effect consists of the generation of stress and strain in crystals under the action of the electric field on the boundary. Materials undergoing piezoelectric effects are called piezoelectric materials. Their study requires techniques and results from electromagnetic theory and continuum mechanics. Foundations of the theory of piezoelectricity were laid down by Voigt [26], who provided the first mathematical model of a linearly elastic material, which takes into account the interaction between mechanical and electrical properties. General models for elastic materials with piezoelectric effect can be found in [15–17, 24, 25], and more recent models in [1, 4, 27, 28].

Currently, there is considerable interest in contact problems involving piezoelectric materials. Indeed, such problems are important from a practical point of view, since most of the measuring equipment used in radioelectronics and electro-acoustics involve piezoelectric sensors in contact with rigid or deformable supports. Static contact problems for electric–elastic materials were considered in [2, 12], where various numerical approaches based on finite-element discretization were presented together with numerical simulations. A slip-dependent frictional contact problem

for electro-elastic materials was studied in [21], and a frictional problem with normal compliance for electro-viscoelastic materials was considered in [22]. In the latter two references, weak formulations of the corresponding problems were derived and existence and uniqueness results for the weak solutions were proved.

The present paper is devoted to the study of a mathematical model that describes the frictionless contact between a piezoelectric body and a foundation, within the mathematical theory of contact mechanics. We model the material behaviour by a constitutive law of the form

$$\boldsymbol{\sigma}' = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}') - \mathcal{P}^T \mathbf{E}(\varphi') + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \mathbf{D}, \mathbf{E}(\varphi)), \quad (1.1)$$

$$\mathbf{D}' = \boldsymbol{\beta} \mathbf{E}(\varphi') + \mathcal{P} \boldsymbol{\varepsilon}(\mathbf{u}') + G(\mathbf{D}, \mathbf{E}(\varphi), \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})), \quad (1.2)$$

in which $\boldsymbol{\sigma}$ denotes the stress tensor, \mathbf{u} is the displacement field, φ is the electric potential and \mathbf{D} represents the electric displacement field. Also, $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain tensor, $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, \mathcal{A} is the elasticity tensor, \mathcal{P} represents the third-order piezoelectric tensor, \mathcal{P}^T is its transpose and $\boldsymbol{\beta}$ denotes the electric permittivity tensor. Moreover, \mathcal{G} and G are given nonlinear constitutive functions and, here and below, the prime denotes the time derivative with respect to the time variable.

Note that when $\mathcal{G} = \mathbf{0}$ and $G = \mathbf{0}$ the constitutive law (1.1), (1.2) leads to the constitutive law of a linearly electro-elastic materials, that is,

$$\begin{aligned} \boldsymbol{\sigma} &= \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{P}^T \mathbf{E}(\varphi), \\ \mathbf{D} &= \boldsymbol{\beta} \mathbf{E}(\varphi) + \mathcal{P} \boldsymbol{\varepsilon}(\mathbf{u}). \end{aligned}$$

Contact problems with such materials have been considered in [2,12,21]. Also, in the purely mechanical case (i.e. $\mathcal{P} = \mathbf{0}$ and \mathcal{G} independent of \mathbf{D} and $\mathbf{E}(\varphi)$), constitutive equations of the form (1.1) were considered in [6,11]. There, various examples and mechanical interpretations concerning the corresponding elastic-visco-plastic constitutive laws were provided. In the purely electric case (i.e. $\mathcal{P} = \mathbf{0}$ and G independent of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\mathbf{u})$), electric relationships of the form (1.2) were considered in [23]. Constitutive equations of the form (1.1), (1.2) in which \mathcal{G} does not depend on \mathbf{D} and G does not depend on $\boldsymbol{\sigma}$ were introduced in [10]. There, the unique solvability of a piezoelectric contact problem was proved, error estimates for fully discrete schemes were obtained and numerical simulations were performed. Finally, a frictionless contact problem for piezoelectric materials with a constitutive law of the form (1.1), (1.2) has been considered in [3]. There, the contact was modelled with normal compliance and unilateral constraint and the weak solvability of the problem was provided by using arguments of variational inequalities and convex analysis.

Besides the use of the general constitutive law (1.1), (1.2), the novelty of this paper arises in the fact that here we assume that the contact is described with a sub-differential boundary condition, associated to a non-convex potential. Considering such conditions allows us to describe the contact with various foundations, including foundations with a linearly elastic, elastic perfectly plastic or rigid perfectly plastic (with or without hardening or softening) behaviour, as shown in [14, ch. 7]. Also, it leads to a new and non-standard mathematical model that we investigate by using an abstract argument developed in [13].

We also mention that, besides existence and uniqueness results, a complete analysis of evolution mathematical models includes the investigation of the regularity and the long-time behaviour of solutions as well as error estimates for effective approximation schemes. Details can be found in [29,30] and the references therein. This analysis is important and, for our piezoelectric contact problem, it will be studied elsewhere.

The rest of the paper is structured as follows. In §2 we present some preliminary material. In §3 we describe the model of the process of frictionless contact between the electro-elastic–visco-plastic body and the foundation. In §4 we list the assumptions on the data and derive a variational formulation of the model. Then we state our existence and uniqueness results (theorem 4.1). The proof of the theorem is provided in §5. It is based on a recent result in the study of history-dependent hemivariational inequalities obtained in [13].

2. Preliminaries

In this section we introduce a preliminary material to be used in the variational analysis of the piezoelectric contact problem that will be introduced in the next section. For further details, we refer the reader to [5,7,8,19] as well as to the recent monograph [14].

Given a normed space $(E, \|\cdot\|_E)$ we denote by E^* its dual space, and $\langle \cdot, \cdot \rangle_{E^* \times E}$ will represent the duality pairing of E and E^* . Let $h: E \rightarrow \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of h at $x \in E$ in the direction $v \in E$, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}$$

and the generalized gradient of h at x , denoted by $\partial h(x)$, is a subset of a dual space E^* given by

$$\partial h(x) = \{\zeta \in E^* \mid h^0(x; v) \geq \langle \zeta, v \rangle_{E^* \times E} \text{ for all } v \in E\}.$$

A locally Lipschitz function h is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $Dh(x; v)$ exists and satisfies $h^0(x; v) = Dh(x; v)$ for all $v \in E$. The symbol w - E is used for the space E endowed with the weak topology. The space of all linear and continuous operators from a normed space E to a normed space F is denoted by $\mathcal{L}(E, F)$.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded subset of \mathbb{R}^d with a Lipschitz continuous boundary $\partial\Omega$ and $\Gamma \subseteq \partial\Omega$. Let Y be a closed subspace of $H^1(\Omega; \mathbb{R}^s)$, $s \geq 1$, $H = L^2(\Omega; \mathbb{R}^s)$ and $Z = H^\rho(\Omega; \mathbb{R}^s)$ with $\rho \in (1/2, 1)$. Denoting by $i: Y \rightarrow Z$ the embedding, by $\gamma: Z \rightarrow L^2(\Gamma; \mathbb{R}^s)$ and $\gamma_0: H^1(\Omega; \mathbb{R}^s) \rightarrow H^{1/2}(\Gamma; \mathbb{R}^s) \subset L^2(\Gamma; \mathbb{R}^s)$ the trace operators, we get $\gamma_0 v = \gamma(iv)$ for all $v \in Y$. For simplicity, in what follows we omit the notation of the embedding i and we write $\gamma_0 v = \gamma v$ for all $v \in Y$.

From the theory of Sobolev spaces, we know that (Y, H, Y^*) and (Z, H, Z^*) form evolution triples of spaces and the embedding $Y \subset Z$ is compact. We denote by c_0 the embedding constant of Y into Z , by $\|\gamma\|$ the norm of the trace in $\mathcal{L}(Z, L^2(\Gamma; \mathbb{R}^s))$ and by $\gamma^*: L^2(\Gamma; \mathbb{R}^s) \rightarrow Z^*$ the adjoint operator of γ . We also introduce the spaces

$$\mathcal{Y} = L^2(0, T; Y), \quad \mathcal{Z} = L^2(0, T; Z) \quad \text{and} \quad \hat{\mathcal{H}} = L^2(0, T; H),$$

where $0 < T < +\infty$. Since the embeddings $Y \subseteq Z \subseteq H \subseteq Z^* \subseteq Y^*$ are continuous, it is known that the embeddings $\mathcal{Y} \subseteq \mathcal{Z} \subseteq \hat{\mathcal{H}} \subseteq \mathcal{Z}^* \subseteq \mathcal{Y}^*$ are also continuous, where $\mathcal{Z}^* = L^2(0, T; Z^*)$ and $\mathcal{Y}^* = L^2(0, T; Y^*)$.

Consider now the operators A and \mathcal{S} , and the functions j and f , which satisfy the following conditions.

(C1) $A: (0, T) \times Y \rightarrow Y^*$ is such that

- (a) $A(\cdot, v)$ is measurable on $(0, T)$ for all $v \in Y$,
- (b) $A(t, \cdot)$ is hemicontinuous and strongly monotone for almost every (a.e.) $t \in (0, T)$, i.e.

$$\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_{Y^* \times Y} \geq m_1 \|v_1 - v_2\|_Y^2$$

for all $v_1, v_2 \in Y$ with $m_1 > 0$,

- (c) $\|A(t, v)\|_{Y^*} \leq a_0(t) + a_1 \|v\|_Y$ for all $v \in Y$, for a.e. $t \in (0, T)$ with $a_0 \in L^2(0, T)$, $a_0 \geq 0$ and $a_1 > 0$,
- (d) $A(t, 0) = 0$ for a.e. $t \in (0, T)$.

(C2) $\mathcal{S}: \mathcal{Y} \rightarrow \mathcal{Y}^*$ is such that

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_{Y^*} \leq L_S \int_0^t \|u_1(s) - u_2(s)\|_Y ds$$

for all $u_1, u_2 \in \mathcal{Y}$, for a.e. $t \in (0, T)$ with $L_S > 0$.

(C3) $j: \Gamma \times (0, T) \times \mathbb{R}^s \rightarrow \mathbb{R}$ is such that

- (a) $j(\cdot, \cdot, \xi)$ is measurable on $\Gamma \times (0, T)$ for all $\xi \in \mathbb{R}^s$ and there exists $e \in L^2(\Gamma; \mathbb{R}^s)$ such that $j(\cdot, \cdot, e(\cdot)) \in L^1(\Gamma \times (0, T))$,
- (b) $j(x, t, \cdot)$ is locally Lipschitz on \mathbb{R}^s for a.e. $(x, t) \in \Gamma \times (0, T)$,
- (c) $\|\partial j(x, t, \xi)\|_{\mathbb{R}^s} \leq \bar{c}_0 + \bar{c}_1 \|\xi\|_{\mathbb{R}^s}$ for a.e. $(x, t) \in \Gamma \times (0, T)$, all $\xi \in \mathbb{R}^s$ with $\bar{c}_0, \bar{c}_1 \geq 0$,
- (d) $(\zeta_1 - \zeta_2) \cdot (\xi_1 - \xi_2) \geq -m_2 \|\xi_1 - \xi_2\|_{\mathbb{R}^s}^2$ for all $\zeta_i, \xi_i \in \mathbb{R}^s$, $\zeta_i \in \partial j(x, t, \xi_i)$, $i = 1, 2$, for a.e. $(x, t) \in \Gamma \times (0, T)$ with $m_2 \geq 0$,
- (e) $j^0(x, t, \xi; -\xi) \leq \bar{d}_0(1 + \|\xi\|_{\mathbb{R}^s})$ for a.e. $(x, t) \in \Gamma \times (0, T)$, all $\xi \in \mathbb{R}^s$ with $\bar{d}_0 \geq 0$.

(C4) $f \in \mathcal{Y}^*$.

With these data we consider the problem of finding an element $y \in \mathcal{Y}$ such that

$$\langle A(t, y(t)), z \rangle_{Y^* \times Y} + \langle \mathcal{S}y(t), z \rangle_{Y^* \times Y} + \int_{\Gamma} j^0(t, \gamma y(t); \gamma z) d\Gamma \geq \langle f(t), z \rangle_{Y^* \times Y} \quad (2.1)$$

for all $z \in Y$, for a.e. $t \in (0, T)$. To avoid any confusion, we note that, in (2.1) and below, the notation $\mathcal{S}y(t)$ stands for $(\mathcal{S}y)(t)$, i.e. $\mathcal{S}y(t) = (\mathcal{S}y)(t)$ for all $y \in \mathcal{Y}$ and for a.e. $t \in (0, T)$. ∂j and j^0 denote the Clarke subdifferential of a locally Lipschitz function $j(x, t, \cdot)$ and its generalized directional derivative, respectively.

Following the terminology introduced in [13] we refer to hemivariational inequalities of the form (2.1) as *history-dependent hemivariational inequalities*. The main feature of such inequalities consists in the fact that they contain the term in \mathcal{S} , which, at any moment $t \in (0, T)$, depends on the *history* of the solution up to the moment t . This feature is different from the time-dependent hemivariational inequalities studied in the literature, in which, usually, the operators involved in are assumed to depend on the *current* value of the solution $y(t)$.

The following existence and uniqueness result for the hemivariational inequality (2.1) was recently proved in [13].

THEOREM 2.1. *Assume that (C1), (C2) and (C4) hold. If one of the following hypotheses is satisfied, then inequality (2.1) has a solution $y \in \mathcal{Y}$:*

- (i) *conditions (a)–(d) of (C3) and $m_1 > \max\{\sqrt{3}\bar{c}_1, m_2\}c_0^2\|\gamma\|^2$;*
- (ii) *(C3) and $m_1 > m_2c_0^2\|\gamma\|^2$.*

If, in addition,

$$\text{either } j(x, t, \cdot) \text{ or } -j(x, t, \cdot) \text{ is regular on } \mathbb{R}^s \text{ for a.e. } (x, t) \in \Gamma \times (0, T), \quad (2.2)$$

then the solution of (2.1) is unique.

3. The contact problem

The physical setting is as follows. An electro-elastic–visco-plastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with a Lipschitz boundary $\partial\Omega$. The body is subject to the action of body forces of density \mathbf{f}_0 and volume electric charges of density q_0 . The boundary of the body is subject to mechanical and electrical constraints. To describe the mechanical constraints we consider a partition of $\partial\Omega$ into three measurable parts, Γ_D , Γ_N and Γ_C , such that $\text{meas}(\Gamma_D) > 0$. We assume that the body is fixed on Γ_D and surface tractions of density \mathbf{f}_N act on Γ_N . On Γ_C , the body is (or can arrive) in contact with an insulated obstacle, the so-called foundation. The contact is frictionless and it is modelled with a subdifferential boundary condition. To describe the electrical constraints we consider a partition of $\Gamma_D \cup \Gamma_N$ into two measurable sets, Γ_a and Γ_b , such that $\text{meas}(\Gamma_a) > 0$. We assume that the electrical potential vanishes on Γ_a and surface electric charges of density q_b are prescribed on Γ_b . Also, since the foundation is insulated, the electrical charges vanish on the potential contact surface. We assume that the problem is quasi-static, and we study the problem in the time interval $(0, T)$ with $T > 0$. We also use the shorthand notation $Q = \Omega \times (0, T)$, $\Sigma_D = \Gamma_D \times (0, T)$, $\Sigma_N = \Gamma_N \times (0, T)$, $\Sigma_C = \Gamma_C \times (0, T)$, $\Sigma_a = \Gamma_a \times (0, T)$ and $\Sigma_b = \Gamma_b \times (0, T)$ and we denote by \mathbb{S}^d the space of second-order symmetric tensors on \mathbb{R}^d or, equivalently, the space of symmetric matrices of order d .

Then, the classical formulation of the contact problem described above is as follows.

(P) Find a displacement field $\mathbf{u}: Q \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}: Q \rightarrow \mathbb{S}^d$, an electric potential $\varphi: Q \rightarrow \mathbb{R}$ and an electric displacement field $\mathbf{D}: Q \rightarrow \mathbb{R}^d$ such that

$$\boldsymbol{\sigma}' = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}') - \mathcal{P}^T \mathbf{E}(\varphi') + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \mathbf{D}, \mathbf{E}(\varphi)) \quad \text{in } Q, \quad (3.1)$$

$$\mathbf{D}' = \beta \mathbf{E}(\varphi') + \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}') + G(\mathbf{D}, \mathbf{E}(\varphi), \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})) \quad \text{in } Q, \quad (3.2)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } Q, \quad (3.3)$$

$$\text{div } \mathbf{D} = q_0 \quad \text{in } Q, \quad (3.4)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Sigma_D, \quad (3.5)$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{f}_N \quad \text{on } \Sigma_N, \quad (3.6)$$

$$\varphi = 0 \quad \text{on } \Sigma_a, \quad (3.7)$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = q_b \quad \text{on } \Sigma_b, \quad (3.8)$$

$$-\sigma_\nu \in \partial j_\nu(u_\nu - g) \quad \text{on } \Sigma_C, \quad (3.9)$$

$$\boldsymbol{\sigma}_\tau = 0 \quad \text{on } \Sigma_C, \quad (3.10)$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Sigma_C, \quad (3.11)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \varphi(0) = \varphi_0, \mathbf{D}(0) = \mathbf{D}_0 \quad \text{in } \Omega. \quad (3.12)$$

We present brief comments on the equations and conditions in problem (P) and we refer the reader to [14] for more details and mechanical description. First, we note that, here and below, in order to simplify the notation we do not indicate explicitly the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \partial\Omega$ and $t \in (0, T)$. Also, we use the notation $\boldsymbol{\nu}$ for the outward unit normal at $\partial\Omega$.

Equations (3.1) and (3.2) represent the electro-elastic-visco-plastic constitutive law of the material introduced in §1 (see (1.1) and (1.2), respectively). Equations (3.3) and (3.4) are the balance equations for the stress and the electric displacement fields, respectively, in which we recall that Div and div denote the divergence operators for tensor and vector-valued functions. Conditions (3.5) and (3.6) are the displacement and traction boundary conditions, respectively, and conditions (3.7) and (3.8) represent the electric boundary conditions.

Condition (3.9) is the contact condition in which u_ν and σ_ν represent the normal displacement and the normal stress, respectively, g is the gap function and j_ν is a given function that may depend explicitly on the time variable. As usual, ∂j_ν denotes the Clarke subdifferential of j_ν with respect to the last variable. Condition (3.10) is the frictionless condition, and it shows that the friction force, denoted by $\boldsymbol{\sigma}_\tau$, vanishes on the contact surface, during the process. Condition (3.11) shows that there are no electric charges on the potential contact surface, and we use it here since the foundation is assumed to be insulated. Nevertheless, we mention that it is possible to treat contact models in which (3.10) and (3.11) are replaced by more general subdifferential boundary conditions (see [13, 14] for details). Finally, (3.12) represents the initial conditions in which \mathbf{u}_0 , $\boldsymbol{\sigma}_0$, φ_0 and \mathbf{D}_0 denote the initial displacement, the initial stress, the initial electric potential and the initial electric displacement, respectively.

Concrete examples of frictional models that lead to subdifferential boundary conditions of the form (3.9) in which the function j_ν satisfies assumption (C10) can be found in [14, ch.7]. Here, we restrict ourselves to remarking that these examples model a contact with normal compliance, in both the single-valued and the

multi-valued cases. Such examples describe a wide variety of foundations, including foundation with a linear elastic, elastic perfectly plastic or rigid perfectly plastic (with or without hardening or softening) behaviour. Our results below are valid for the corresponding quasi-static frictionless contact problems. Moreover, we note that the explicit dependence of the function j_ν on the time variable allows us to model situations when the contact conditions depend on the temperature, which plays the role of a parameter.

4. Variational formulation and the main result

The system (3.1)–(3.12) represents the classical formulation of the piezoelectric frictionless contact problem we are interested in, and by this we mean that the unknowns and the data are smooth functions such that all the derivatives and all the conditions are satisfied in the usual sense, i.e. at each point and at each time instant. However, it is well known that, in general, the classical formulations of contact problems do not have any solution. Therefore, in order to provide results concerning the well posedness of the model, we need to reformulate problem (P) in a weaker sense, i.e. to derive its variational formulation.

To this end we need to introduce further notation. First, recall that we use $\mathbf{x} = (x_i)$ for a typical point in $\Omega \cup \partial\Omega$ and $\boldsymbol{\nu} = (\nu_i)$ for the outward unit normal at $\partial\Omega$. Here and below the indices i and j run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. Also, the index that follows a comma indicates a partial derivative with the corresponding component of the spatial variable \mathbf{x} . Moreover, the canonical inner products and the corresponding norms on \mathbb{R}^d and \mathbb{S}^d are given by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i, \quad \|\mathbf{v}\|_{\mathbb{R}^d} = (\mathbf{v} \cdot \mathbf{v})^{1/2} \quad \text{for all } \mathbf{u} = (u_i), \quad \mathbf{v} = (v_i) \in \mathbb{R}^d, \\ \boldsymbol{\sigma} : \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \quad \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d. \end{aligned}$$

We use standard notation for Lebesgue spaces and Sobolev spaces. For $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$ we still denote by \mathbf{v} the trace of \mathbf{v} on $\partial\Omega$ and we use the notation v_ν and \mathbf{v}_τ for the normal and tangential components of \mathbf{v} on $\partial\Omega$ given by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}.$$

We recall that the normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by

$$\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

For the mechanical variables \mathbf{u} and $\boldsymbol{\sigma}$ we introduce the spaces

$$\begin{aligned} H &= L^2(\Omega; \mathbb{R}^d), \\ \mathcal{H} &= \{\boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega; \mathbb{S}^d), \\ V &= \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}, \\ \mathcal{H}_1 &= \{\boldsymbol{\tau} \in \mathcal{H} \mid \text{Div } \boldsymbol{\tau} \in H\}. \end{aligned}$$

Recall that condition $\mathbf{v} = \mathbf{0}$ on Γ_D in the definition of the space V is understood in the sense of trace, i.e. $\gamma \mathbf{v} = \mathbf{0}$ almost everywhere on Γ_D . It is well known that

the spaces H , \mathcal{H} , V and \mathcal{H}_1 are Hilbert spaces equipped with the inner products

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_H &= \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx, & (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\tau} \, dx, \\ (\mathbf{u}, \mathbf{v})_V &= (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, & (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_H, \end{aligned}$$

where $\boldsymbol{\varepsilon}: H^1(\Omega; \mathbb{R}^d) \rightarrow \mathcal{H}$ and $\operatorname{Div}: \mathcal{H}_1 \rightarrow H$ denote the *deformation* and the *divergence* operators, respectively, given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The associated norms in H , \mathcal{H} , V and \mathcal{H}_1 are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_V$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Note that the completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $\operatorname{meas}(\Gamma_D) > 0$, which allows us to use the Korn inequality. Moreover, if $\boldsymbol{\sigma}$ is sufficiently smooth, the following Green-type formula holds:

$$\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \operatorname{Div} \boldsymbol{\sigma} \cdot \mathbf{v} \, dx = \int_{\partial\Omega} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \mathbf{v} \, d\Gamma \quad \text{for all } \mathbf{v} \in H^1(\Omega; \mathbb{R}^d). \quad (4.1)$$

For the electrical unknowns φ and \mathbf{D} we need the spaces

$$W = \{\mathbf{D} \in H \mid \operatorname{div} \mathbf{D} \in L^2(\Omega)\}, \quad \Phi = \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \Gamma_a\},$$

which are Hilbert spaces equipped with the standard inner products. Recall that, here and below, $\operatorname{div}: H \rightarrow L^2(\Omega)$ represents the *divergence* operator given by

$$\operatorname{div} \mathbf{D} = (D_{i,i}).$$

Moreover, since $\operatorname{meas}(\Gamma_a)$ is positive, it can be shown that Φ is a Hilbert space with the inner product and the corresponding norm given by

$$(\varphi, \psi)_{\Phi} = (\nabla \varphi, \nabla \psi)_H, \quad \|\psi\|_{\Phi} = \|\nabla \psi\|_H.$$

In addition, it is well known that the inclusions $\Phi \subset L^2(\Omega) \subset \Phi^*$ are continuous and compact, where Φ^* denotes the dual space of Φ . And, finally, if \mathbf{D} is a sufficiently regular function, then

$$\int_{\Omega} \mathbf{D} \cdot \nabla \psi \, dx + \int_{\Omega} \operatorname{div} \mathbf{D} \psi \, dx = \int_{\partial\Omega} \mathbf{D} \cdot \boldsymbol{\nu} \psi \, d\Gamma \quad \text{for all } \psi \in H^1(\Omega). \quad (4.2)$$

In the study of problem (P) we assume that the elasticity tensor, the piezoelectric tensor and the electric permittivity tensor satisfy the following conditions.

(C5) $\mathcal{A} = (a_{ijkl}): \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is such that

- (a) $a_{ijkl} = a_{klij} = a_{jikl} \in L^\infty(\Omega)$, $1 \leq i, j, k, l \leq d$,
- (b) there exists $m_{\mathcal{A}} > 0$ such that $\mathcal{A}(\mathbf{x})\boldsymbol{\tau} : \boldsymbol{\tau} \geq m_{\mathcal{A}} \|\boldsymbol{\tau}\|_{\mathbb{S}^d}^2$ for all $\boldsymbol{\tau} \in \mathbb{S}^d$, for a.e. $\mathbf{x} \in \Omega$.

(C6) $\mathcal{P} = (p_{ijk}): \Omega \times \mathbb{S}^d \rightarrow \mathbb{R}^d$ is such that

$$p_{ijk} = p_{ikj} \in L^\infty(\Omega), \quad 1 \leq i, j, k \leq d.$$

(C7) $\beta = (\beta_{ij}): \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is such that

- (a) $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$, $1 \leq i, j \leq d$,
- (b) there exists $m_\beta > 0$ such that $\beta(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq m_\beta \|\boldsymbol{\xi}\|_{\mathbb{R}^d}^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, for a.e. $\mathbf{x} \in \Omega$.

The nonlinear constitutive functions \mathcal{G} and G are assumed to satisfy the following.

(C8) $\mathcal{G}: \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{S}^d$ is such that

- (a) there exists $L_G > 0$ such that

$$\begin{aligned} & \|\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \mathbf{D}_1, \mathbf{E}_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \mathbf{D}_2, \mathbf{E}_2)\|_{\mathbb{S}^d} \\ & \leq L_G (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbb{S}^d} + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} + \|\mathbf{D}_1 - \mathbf{D}_2\|_{\mathbb{R}^d} + \|\mathbf{E}_1 - \mathbf{E}_2\|_{\mathbb{R}^d}) \end{aligned}$$

for all $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, $\mathbf{D}_1, \mathbf{D}_2, \mathbf{E}_1, \mathbf{E}_2 \in \mathbb{R}^d$, for a.e. $\mathbf{x} \in \Omega$,

- (b) the mapping $\mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \mathbf{D}, \mathbf{E})$ is measurable on Ω , for all $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d$ and $\mathbf{D}, \mathbf{E} \in \mathbb{R}^d$,
- (c) the mapping $\mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ belongs to \mathcal{H} .

(C9) $G: \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{S}^d \rightarrow \mathbb{R}^d$ is such that

- (a) there exists $L_G > 0$ such that

$$\begin{aligned} & \|G(\mathbf{x}, \mathbf{D}_1, \mathbf{E}_1, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - G(\mathbf{x}, \mathbf{D}_2, \mathbf{E}_2, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\|_{\mathbb{R}^d} \\ & \leq L_G (\|\mathbf{D}_1 - \mathbf{D}_2\|_{\mathbb{R}^d} + \|\mathbf{E}_1 - \mathbf{E}_2\|_{\mathbb{R}^d} + \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbb{S}^d} + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d}) \end{aligned}$$

for all $\mathbf{D}_1, \mathbf{D}_2, \mathbf{E}_1, \mathbf{E}_2 \in \mathbb{R}^d$, $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, for a.e. $\mathbf{x} \in \Omega$,

- (b) the mapping $\mathbf{x} \mapsto G(\mathbf{x}, \mathbf{D}, \mathbf{E}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})$ is measurable on Ω , for all $\mathbf{D}, \mathbf{E} \in \mathbb{R}^d$ and $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d$,
- (c) the mapping $\mathbf{x} \mapsto G(\mathbf{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ belongs to H .

These assumptions are reasonable from physical point of view (see, for example, the comments in [6,9,11,18,20,23]). In some applications, \mathcal{G} and G are linear functions.

The contact potential j_ν satisfies the following hypothesis.

(C10) $j_\nu: \Sigma_C \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

- (a) $j_\nu(\cdot, \cdot, r)$ is measurable on Σ_C for all $r \in \mathbb{R}$ and there exists $e_1 \in L^2(\Gamma_C)$ such that $j_\nu(\cdot, \cdot, e_1(\cdot)) \in L^1(\Sigma_C)$,
- (b) $j_\nu(\mathbf{x}, t, \cdot)$ is locally Lipschitz on \mathbb{R} for a.e. $(\mathbf{x}, t) \in \Sigma_C$,
- (c) $|\partial j_\nu(\mathbf{x}, t, r)| \leq c_{0\nu} + c_{1\nu}|r|$ for all $r \in \mathbb{R}$, for a.e. $(\mathbf{x}, t) \in \Sigma_C$ with $c_{0\nu}, c_{1\nu} \geq 0$,
- (d) $(\zeta_1 - \zeta_2)(r_1 - r_2) \geq -m_\nu |r_1 - r_2|^2$ for all $\zeta_i \in \partial j_\nu(\mathbf{x}, t, r_i)$, $r_i \in \mathbb{R}$, $i = 1, 2$, for a.e. $(\mathbf{x}, t) \in \Sigma_C$ with $m_\nu \geq 0$,
- (e) $j_\nu^0(\mathbf{x}, t, r; -r) \leq d_\nu(1 + |r|)$ for all $r \in \mathbb{R}$, for a.e. $(\mathbf{x}, t) \in \Sigma_C$ with $d_\nu \geq 0$.

We also assume that the densities of body forces, tractions, volume and surface electric charge have the regularities

$$\mathbf{f}_0 \in L^2(0, T; H), \quad \mathbf{f}_N \in L^2(0, T; L^2(\Gamma_N; \mathbb{R}^d)), \quad (\text{C11})$$

$$q_0 \in L^2(0, T; L^2(\Omega)), \quad q_b \in L^2(0, T; L^2(\Gamma_b)). \quad (\text{C12})$$

The initial data satisfy

$$\mathbf{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in \mathcal{H}, \quad \varphi_0 \in \Phi, \quad \mathbf{D}_0 \in H, \quad (\text{C13})$$

and the gap function is such that

$$g \in L^\infty(\Gamma_C), \quad g \geq 0 \text{ almost everywhere on } \Gamma_C. \quad (\text{C14})$$

Finally, we recall that the link between the tensor \mathcal{P} and its transpose, denoted \mathcal{P}^T , is given by

$$\mathcal{P}\boldsymbol{\sigma} \cdot \boldsymbol{\xi} = \boldsymbol{\sigma} : \mathcal{P}^\text{T}\boldsymbol{\xi} \quad \text{for all } \boldsymbol{\sigma} \in \mathbb{S}^d, \boldsymbol{\xi} \in \mathbb{R}^d, \quad \text{almost everywhere in } \Omega. \quad (4.3)$$

We turn now to the variational formulation of problem (P). Assume in what follows that $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$ are sufficiently regular functions that satisfy (3.1)–(3.12) and let $t \in [0, T]$ be given. We integrate (3.1) and (3.2) over $(0, t)$ with the initial conditions (3.12) to obtain

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \mathcal{P}^\text{T}\mathbf{E}(\varphi(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{D}(s), \mathbf{E}(\varphi(s))) \, ds \\ &\quad + \boldsymbol{\sigma}_0 - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_0) + \mathcal{P}^\text{T}\mathbf{E}(\varphi_0), \end{aligned} \quad (4.4)$$

$$\begin{aligned} \mathbf{D}(t) &= \beta\mathbf{E}(\varphi(t)) + \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t G(\mathbf{D}(s), \mathbf{E}(\varphi(s)), \boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s))) \, ds \\ &\quad + \mathbf{D}_0 - \beta\mathbf{E}(\varphi_0) - \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}_0). \end{aligned} \quad (4.5)$$

Let $\mathbf{v} \in V$. Then, using (3.3), (3.6), (3.10) and (4.1), we have

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} = (\mathbf{f}_0(t), \mathbf{v})_H + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v} \, d\Gamma + \int_{\Gamma_C} \sigma_\nu(t) v_\nu \, d\Gamma \quad \text{for a.e. } t \in (0, T). \quad (4.6)$$

On the other hand, combining the definition of the Clarke subdifferential with (3.9), we have

$$\int_{\Gamma_C} \sigma_\nu(t) v_\nu \, d\Gamma \geq - \int_{\Gamma_C} j_\nu^0(t, u_\nu(t) - g; v_\nu) \, d\Gamma \quad \text{for a.e. } t \in (0, T). \quad (4.7)$$

Consider the function $\mathbf{f}: (0, T) \rightarrow V^*$ given by

$$\langle \mathbf{f}(t), \mathbf{v} \rangle_{V^* \times V} = (\mathbf{f}_0(t), \mathbf{v})_H + (\mathbf{f}_N(t), \mathbf{v})_{L^2(\Gamma_N; \mathbb{R}^d)} \quad (4.8)$$

for all $\mathbf{v} \in V$ and for a.e. $t \in (0, T)$. We combine (4.6)–(4.8) to obtain

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + \int_{\Gamma_C} j_\nu^0(t, u_\nu(t) - g; v_\nu) \, d\Gamma \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{V^* \times V} \quad \text{for a.e. } t \in (0, T). \quad (4.9)$$

Similarly, for all $\psi \in \Phi$, from (3.4), (3.8), (3.11) and (4.2) we deduce that

$$(\mathbf{D}(t), \nabla \psi)_H + \langle q(t), \psi \rangle_{\Phi^* \times \Phi} = 0 \quad \text{for a.e. } t \in (0, T), \quad (4.10)$$

where $q: (0, T) \rightarrow \Phi^*$ is given by

$$\langle q(t), \psi \rangle_{\Phi^* \times \Phi} = (q_0(t), \psi)_{L^2(\Omega)} - (q_b(t), \psi)_{L^2(\Gamma_b)} \quad (4.11)$$

for all $\psi \in \Phi$ and for a.e. $t \in (0, T)$.

We gather the equalities (4.4), (4.5), (4.10) and inequality (4.9) to obtain the following variational formulation of problem (P).

(P_V) Find a displacement field $\mathbf{u}: (0, T) \rightarrow V$, a stress field $\boldsymbol{\sigma}: (0, T) \rightarrow \mathcal{H}_1$, an electric potential $\varphi: (0, T) \rightarrow \Phi$ and an electric displacement field $\mathbf{D}: (0, T) \rightarrow W$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \mathcal{P}^T \mathbf{E}(\varphi(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{D}(s), \mathbf{E}(\varphi(s))) ds \\ &\quad + \boldsymbol{\sigma}_0 - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_0) + \mathcal{P}^T \mathbf{E}(\varphi_0) \quad \text{for a.e. } t \in (0, T), \end{aligned} \quad (4.12)$$

$$\begin{aligned} \mathbf{D}(t) &= \beta \mathbf{E}(\varphi(t)) + \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t G(\mathbf{D}(s), \mathbf{E}(\varphi(s)), \boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}(s))) ds \\ &\quad + \mathbf{D}_0 - \beta \mathbf{E}(\varphi_0) - \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}_0) \quad \text{for a.e. } t \in (0, T) \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + \int_{\Gamma_C} j_\nu^0(t, u_\nu(t) - g; v_\nu) d\Gamma \\ \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{V^* \times V} \quad \text{for all } \mathbf{v} \in V \text{ for a.e. } t \in (0, T), \end{aligned} \quad (4.14)$$

$$(\mathbf{D}(t), \nabla \psi)_H + \langle q(t), \psi \rangle_{\Phi^* \times \Phi} = 0 \quad \text{for all } \psi \in \Phi \text{ for a.e. } t \in (0, T). \quad (4.15)$$

Note that problem (P_V) represents a system involving two nonlinear integral equations, a hemivariational inequality and a time-dependent linear differential equation. One of the main features of this system arises in the coupling between the mechanical unknowns \mathbf{u} , $\boldsymbol{\sigma}$ with the electrical unknowns φ and \mathbf{D} , which appears in the nonlinear equations (4.12) and (4.13).

In order to study the solvability of (P_V) we need the space $\tilde{Z} = H^\delta(\Omega; \mathbb{R}^d)$, where $\delta \in (1/2, 1)$ is fixed. We denote by $c_e > 0$ the embedding constant of V into \tilde{Z} . Moreover, we introduce the trace operator $\tilde{\gamma}: \tilde{Z} \rightarrow L^2(\Gamma_C; \mathbb{R}^d)$ and denote by $\|\tilde{\gamma}\|$ its norm in $\mathcal{L}(\tilde{Z}, L^2(\Gamma_C; \mathbb{R}^d))$.

Our main result in the study of problem (P_V) that we state here and prove in the next section is the following.

THEOREM 4.1. *Assume (C5)–(C9), (C11)–(C14) and let one of the following hypotheses hold:*

- (i) *conditions (a)–(d) of (C10) and $\min\{m_{\mathcal{A}}, m_\beta\} > \max\{\sqrt{3}c_{1\nu}, m_\nu\}c_e^2\|\tilde{\gamma}\|^2$;*
- (ii) *(C10) and $\min\{m_{\mathcal{A}}, m_\beta\} > m_\nu c_e^2\|\tilde{\gamma}\|^2$.*

Then problem (P_V) has at least one solution that satisfies

$$\mathbf{u} \in L^2(0, T; V), \quad (4.16)$$

$$\boldsymbol{\sigma} \in L^2(0, T; \mathcal{H}_1), \quad (4.17)$$

$$\varphi \in L^2(0, T; \Phi), \quad (4.18)$$

$$\mathbf{D} \in L^2(0, T; W). \quad (4.19)$$

If, in addition, we assume that

$$\text{either } j_\nu(\mathbf{x}, t, \cdot) \text{ or } -j_\nu(\mathbf{x}, t, \cdot) \text{ is regular for a.e. } (\mathbf{x}, t) \in \Sigma_C, \quad (4.20)$$

then problem (P_V) has a unique solution.

A quadruple of functions $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$ which satisfies (4.12)–(4.15) is called a *weak solution* of problem (P). Assume that (C5)–(C9) and (C11)–(C14) hold. Then, it follows from theorem 4.1 that, under the assumptions (i) or (ii) above, there exists at least unique weak solution of problem (P), with regularity (4.16)–(4.19). And if, in addition, (4.20) holds, the weak solution is unique.

5. Proof of theorem 4.1

The proof of theorem 4.1 will be carried out in several steps. Below in this section we assume that (C5)–(C9) and (C11)–(C14) hold, without recalling these hypotheses explicitly. Moreover, if X_1 and X_2 are two Hilbert spaces endowed with the inner products $(\cdot, \cdot)_{X_1}$ and $(\cdot, \cdot)_{X_2}$ and the associated norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, we denote by $X_1 \times X_2$ the product space together with the canonical inner product $(\cdot, \cdot)_{X_1 \times X_2}$ and the associated norm $\|\cdot\|_{X_1 \times X_2}$. Everywhere in this section c will denote a positive generic constant whose value may change from place to place. In addition, in order to simplify the writing, in the following lemma we shall write

$$\tilde{\mathcal{G}}(\mathbf{u}, \varphi, \boldsymbol{\sigma}, \mathbf{D})$$

instead of

$$\mathcal{G}(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{P}^\top \mathbf{E}(\varphi) + \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}), \mathbf{E}(\varphi), \boldsymbol{\beta}\mathbf{E}(\varphi) + \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{D})$$

and

$$\tilde{\mathcal{G}}(\mathbf{u}, \varphi, \boldsymbol{\sigma}, \mathbf{D})$$

instead of

$$\mathcal{G}(\boldsymbol{\beta}\mathbf{E}(\varphi) - \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{D}, \mathbf{E}(\varphi), \boldsymbol{\varepsilon}(\mathbf{u}), \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{P}^\top \mathbf{E}(\varphi) + \boldsymbol{\sigma}).$$

The first step is given by the following existence and uniqueness result.

LEMMA 5.1. *For all $\mathbf{u} \in L^2(0, T; V)$ and $\varphi \in L^2(0, T; \Phi)$ there exists a unique pair of functions*

$$(\boldsymbol{\sigma}^I(\mathbf{u}, \varphi), \mathbf{D}^I(\mathbf{u}, \varphi)) \in W^{1,2}(0, T; \mathcal{H} \times H)$$

such that

$$\begin{aligned}\boldsymbol{\sigma}^I(\mathbf{u}, \varphi)(t) &= \int_0^t \tilde{\mathcal{G}}(\mathbf{u}(s), \varphi(s), \boldsymbol{\sigma}^I(\mathbf{u}, \varphi)(s), \mathbf{D}^I(\mathbf{u}, \varphi)(s)) \, ds \\ &\quad + \boldsymbol{\sigma}_0 - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_0) + \mathcal{P}^\top \mathbf{E}(\varphi_0)\end{aligned}$$

and

$$\begin{aligned}\mathbf{D}^I(\mathbf{u}, \varphi)(t) &= \int_0^t \tilde{\mathcal{G}}(\mathbf{u}(s), \varphi(s), \boldsymbol{\sigma}^I(\mathbf{u}, \varphi)(s), \mathbf{D}^I(\mathbf{u}, \varphi)(s)) \, ds \\ &\quad + \mathbf{D}_0 - \beta \mathbf{E}(\varphi_0) - \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}_0),\end{aligned}$$

for all $t \in [0, T]$.

Proof. Let $(\mathbf{u}, \varphi) \in L^2(0, T; V \times \Phi)$ be given. We introduce the operator

$$\Lambda: L^2(0, T; \mathcal{H} \times H) \rightarrow W^{1,2}(0, T; \mathcal{H} \times H) \subset L^2(0, T; \mathcal{H} \times H)$$

defined by

$$\begin{aligned}\Lambda(\boldsymbol{\sigma}, \mathbf{D})(t) &= (\Lambda_1(\boldsymbol{\sigma}, \mathbf{D})(t), \Lambda_2(\boldsymbol{\sigma}, \mathbf{D})(t)) \quad \text{for all } (\boldsymbol{\sigma}, \mathbf{D}) \in L^2(0, T; \mathcal{H} \times H), \\ \Lambda_1(\boldsymbol{\sigma}, \mathbf{D})(t) &= \int_0^t \tilde{\mathcal{G}}(\mathbf{u}(s), \varphi(s), \boldsymbol{\sigma}(s), \mathbf{D}(s)) \, ds + \boldsymbol{\sigma}_0 - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_0) + \mathcal{P}^\top \mathbf{E}(\varphi_0), \\ \Lambda_2(\boldsymbol{\sigma}, \mathbf{D})(t) &= \int_0^t \tilde{\mathcal{G}}(\mathbf{u}(s), \varphi(s), \boldsymbol{\sigma}(s), \mathbf{D}(s)) \, ds + \mathbf{D}_0 - \beta \mathbf{E}(\varphi_0) - \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}_0)\end{aligned}$$

for all $t \in [0, T]$. The operator Λ depends on the pair (\mathbf{u}, φ) but, for simplicity, we do not indicate this dependence explicitly.

Let $(\boldsymbol{\sigma}_1, \mathbf{D}_1), (\boldsymbol{\sigma}_2, \mathbf{D}_2) \in L^2(0, T; \mathcal{H} \times H)$ and let $t \in [0, T]$. Then

$$\begin{aligned}&\|\Lambda(\boldsymbol{\sigma}_1, \mathbf{D}_1)(t) - \Lambda(\boldsymbol{\sigma}_2, \mathbf{D}_2)(t)\|_{\mathcal{H} \times H} \\ &\leq \left\| \int_0^t \tilde{\mathcal{G}}(\mathbf{u}(s), \varphi(s), \boldsymbol{\sigma}_1(s), \mathbf{D}_1(s)) \, ds - \int_0^t \tilde{\mathcal{G}}(\mathbf{u}(s), \varphi(s), \boldsymbol{\sigma}_2(s), \mathbf{D}_2(s)) \, ds \right\|_{\mathcal{H}} \\ &\quad + \left\| \int_0^t \tilde{\mathcal{G}}(\mathbf{u}(s), \varphi(s), \boldsymbol{\sigma}_1(s), \mathbf{D}_1(s)) \, ds - \int_0^t \tilde{\mathcal{G}}(\mathbf{u}(s), \varphi(s), \boldsymbol{\sigma}_2(s), \mathbf{D}_2(s)) \, ds \right\|_H.\end{aligned}$$

We use (C8) and (C9) to deduce that

$$\begin{aligned}\|\Lambda(\boldsymbol{\sigma}_1, \mathbf{D}_1)(t) - \Lambda(\boldsymbol{\sigma}_2, \mathbf{D}_2)(t)\|_{\mathcal{H} \times H} \\ \leq c \int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}} + \|\mathbf{D}_1(s) - \mathbf{D}_2(s)\|_H) \, ds,\end{aligned}$$

and therefore

$$\begin{aligned}\|\Lambda(\boldsymbol{\sigma}_1, \mathbf{D}_1)(t) - \Lambda(\boldsymbol{\sigma}_2, \mathbf{D}_2)(t)\|_{\mathcal{H} \times H} \\ \leq c \int_0^t \|(\boldsymbol{\sigma}_1(s), \mathbf{D}_1(s)) - (\boldsymbol{\sigma}_2(s), \mathbf{D}_2(s))\|_{\mathcal{H} \times H} \, ds,\end{aligned}$$

where $c > 0$. It is clear that the operator Λ is well defined and takes values in $W^{1,2}(0, T; \mathcal{H} \times H)$. We iterate this inequality n times and use a standard argument

to see that, for n large, the operator A^n is a contraction on the space $L^2(0, T; \mathcal{H} \times H)$. Therefore, the Banach fixed argument shows that the operator A has a unique fixed point, denoted $(\boldsymbol{\sigma}^I(\mathbf{u}, \varphi), \mathbf{D}^I(\mathbf{u}, \varphi))$, which concludes the proof. \square

We continue with the following equivalence result.

LEMMA 5.2. *A quadruple of functions $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{D}, \varphi)$ is a solution to problem (P_V) if and only if, for a.e. $t \in (0, T)$, we have*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathcal{P}^T \nabla \varphi(t) + \boldsymbol{\sigma}^I(\mathbf{u}, \varphi)(t), \quad (5.1)$$

$$\mathbf{D}(t) = -\beta \nabla \varphi(t) + \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathbf{D}^I(\mathbf{u}, \varphi)(t), \quad (5.2)$$

$$\begin{aligned} & (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\mathcal{P}^T \nabla \varphi(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\boldsymbol{\sigma}^I(\mathbf{u}, \varphi)(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \\ & \quad + \int_{\Gamma_C} j_{\nu}^0(t, u_{\nu}(t) - g; v_{\nu}) \, d\Gamma \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{V^* \times V} \\ & \quad \text{for all } \mathbf{v} \in V, \end{aligned} \quad (5.3)$$

$$\begin{aligned} & (\beta \nabla \varphi(t), \nabla \psi)_H - (\mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \nabla \psi)_H - (\mathbf{D}^I(\mathbf{u}, \varphi)(t), \nabla \psi)_H = \langle q(t), \psi \rangle_{\Phi^* \times \Phi} \\ & \quad \text{for all } \psi \in \Phi. \end{aligned} \quad (5.4)$$

Proof. Lemma 5.2 is a direct consequence of the notation $\mathbf{E}(\varphi) = -\nabla \varphi$ combined with the definition of the functions $\boldsymbol{\sigma}^I$ and \mathbf{D}^I , introduced in lemma 5.1. \square

To proceed, we denote by X the space $X = V \times \Phi$ and let $\mathcal{X} = L^2(0, T; X)$. We also consider the operators $A: X \rightarrow X^*$, $\mathcal{S}: \mathcal{X} \rightarrow \mathcal{X}^*$ and the function $f \in \mathcal{X}^*$ defined by

$$\begin{aligned} \langle Ax, y \rangle_{X^* \times X} &= (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\mathcal{P}^T \nabla \varphi, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \\ & \quad - (\mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}), \nabla \psi)_H + (\beta \nabla \varphi, \nabla \psi)_H \\ & \quad \text{for all } x = (\mathbf{u}, \varphi), \quad y = (\mathbf{v}, \psi) \in X, \end{aligned} \quad (5.5)$$

$$\begin{aligned} \langle \mathcal{S}x(t), y \rangle_{X^* \times X} &= (\boldsymbol{\sigma}^I(\mathbf{u}, \varphi)(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} - (\mathbf{D}^I(\mathbf{u}, \varphi)(t), \nabla \psi)_H \\ & \quad \text{for all } x = (\mathbf{u}, \varphi) \in \mathcal{X}, \quad y = (\mathbf{v}, \psi) \in X, \quad t \in [0, T], \end{aligned} \quad (5.6)$$

$$f = (\mathbf{f}, q). \quad (5.7)$$

Note that the definition of the operators A and \mathcal{S} follows from Riesz's representation theorem.

The next step is provided by the following result.

LEMMA 5.3. *The functions $\mathbf{u} \in L^2(0, T; V)$ and $\varphi \in L^2(0, T; \Phi)$ satisfy (5.3) and (5.4) if and only if $x = (\mathbf{u}, \varphi) \in \mathcal{X}$ satisfies the hemivariational inequality*

$$\begin{aligned} \langle Ax(t), y \rangle_{X^* \times X} + \langle \mathcal{S}x(t), y \rangle_{X^* \times X} + \int_{\Gamma_C} j_{\nu}^0(t, u_{\nu}(t) - g; v_{\nu}) \, d\Gamma & \geq \langle f(t), y \rangle_{X^* \times X} \\ & \quad \text{for all } y = (\mathbf{v}, \psi) \in X, \quad \text{for a.e. } t \in (0, T). \end{aligned} \quad (5.8)$$

Proof. We add inequality (5.3) and equality (5.4). Then we use the definitions (5.5)–(5.7) to obtain (5.8).

Conversely, assume that $x = (\mathbf{u}, \varphi) \in \mathcal{X}$ represents a solution of (5.8). We test (5.8) with $y = (\mathbf{v}, 0) \in X$, then with $y = (0, \pm\psi) \in X$, where \mathbf{v} is an arbitrary element of V and ψ is an arbitrary element in Φ . As a result we obtain (5.3) and (5.4), respectively, which concludes the proof. \square

We continue with the following existence and uniqueness result.

LEMMA 5.4. *Assume (C5)–(C9), (C11)–(C14) and let one of the hypotheses (i) and (ii) of theorem 4.1 hold. Then there exists at least one solution $x \in \mathcal{X}$ of the hemivariational inequality (5.8). In addition, if (4.20) holds, the solution is unique.*

Proof. We apply theorem 2.1 on the space $Y = X$. To this end, we show that the assumptions (C5) and (C7) on the tensors \mathcal{A} and β imply that the operator $A: X \rightarrow X^*$ defined by (5.5) satisfies (C1). Indeed, it is obvious that (a) of (C1) holds and, since A is linear, (d) of (C1) holds too. Moreover, from the Hölder inequality, we have

$$\begin{aligned} & |\langle Ax, y \rangle_{X^* \times X}| \\ & \leq \int_{\Omega} (\|\mathcal{A}\varepsilon(\mathbf{u})\|_{\mathbb{S}^d} \|\varepsilon(\mathbf{v})\|_{\mathbb{S}^d} + \|\mathcal{P}^T \nabla \varphi\|_{\mathbb{S}^d} \|\varepsilon(\mathbf{v})\|_{\mathbb{S}^d} \\ & \quad + \|\mathcal{P}\varepsilon(\mathbf{u})\|_{\mathbb{R}^d} \|\nabla \psi\|_{\mathbb{R}^d} + \|\beta \nabla \varphi\|_{\mathbb{R}^d} \|\nabla \psi\|_{\mathbb{R}^d}) \, dx \\ & \leq \|\mathcal{A}\| \|\mathbf{u}\|_V \|\mathbf{v}\|_V + \|\mathcal{P}\| (\|\varphi\|_{\Phi} \|\mathbf{v}\|_V + \|\mathbf{u}\|_V \|\psi\|_{\Phi}) + \|\beta\| \|\varphi\|_{\Phi} \|\psi\|_{\Phi} \end{aligned}$$

for all $x = (\mathbf{u}, \varphi)$, $y = (\mathbf{v}, \psi) \in X$, where

$$\begin{aligned} \|\mathcal{A}\| &= \max_{i,j,k,l} \|a_{ijkl}\|_{L^\infty(\Omega)}, \\ \|\mathcal{P}\| &= \|\mathcal{P}^T\| = \max_{i,j,k} \|p_{ijk}\|_{L^\infty(\Omega)}, \\ \|\beta\| &= \max_{i,j} \|\beta_{ij}\|_{L^\infty(\Omega)}. \end{aligned}$$

Therefore, we conclude that

$$\|Ax\|_{X^*} \leq (\|\mathcal{A}\| + 2\|\mathcal{P}\| + \|\beta\|) \|x\|_X \quad \text{for all } x \in X,$$

i.e. (c) of (C1) holds with $a_0 \equiv 0$ and $a_1 > 0$. Thus, A is a bounded operator; hence, it is also continuous.

On the other hand, using (C5)–(C7), we have

$$\begin{aligned} & \langle Ax, x \rangle_{X^* \times X} \\ & = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{u}))_{\mathcal{H}} + (\mathcal{P}^T \nabla \varphi, \varepsilon(\mathbf{u}))_{\mathcal{H}} + (\mathcal{P}\varepsilon(\mathbf{u}), \nabla \varphi)_H + (\beta \nabla \varphi, \nabla \varphi)_H \\ & \geq m_{\mathcal{A}} \|\varepsilon(\mathbf{u})\|_{\mathcal{H}}^2 + m_{\beta} \|\nabla \varphi\|_H^2 \\ & \geq \min\{m_{\mathcal{A}}, m_{\beta}\} \|x\|_X^2 \end{aligned}$$

for all $x = (\mathbf{u}, \varphi) \in X$. Here we exploited the equality

$$(\mathcal{P}^T \nabla \varphi, \varepsilon(\mathbf{u}))_{\mathcal{H}} = (\mathcal{P}\varepsilon(\mathbf{u}), \nabla \varphi)_H \quad \text{for all } \mathbf{u} \in V, \varphi \in \Phi,$$

which is a direct consequence of (4.3). We deduce that the coercivity, and therefore the strong monotonicity condition (b) of (C1), holds with $m_1 = \min\{m_{\mathcal{A}}, m_{\beta}\}$. This implies that the operator A satisfies (C1).

Next, we check that the operator $\mathcal{S}: \mathcal{X} \rightarrow \mathcal{X}^*$ defined by (5.6) satisfies condition (C2). To this end we note that, for any $t \in [0, T]$ and $x_i = (\mathbf{u}_i, \varphi_i) \in \mathcal{X}$, $i = 1, 2$, we have

$$\begin{aligned} & \|\mathcal{S}(\mathbf{u}_1, \varphi_1)(t) - \mathcal{S}(\mathbf{u}_2, \varphi_2)(t)\|_{X^*} \\ & \leq c(\|\boldsymbol{\sigma}^I(\mathbf{u}_1, \varphi_1)(t) - \boldsymbol{\sigma}^I(\mathbf{u}_2, \varphi_2)(t)\|_{\mathcal{H}} + \|\mathbf{D}^I(\mathbf{u}_1, \varphi_1)(t) - \mathbf{D}^I(\mathbf{u}_2, \varphi_2)(t)\|_H). \end{aligned} \quad (5.9)$$

Then, using the definitions of $\boldsymbol{\sigma}^I$ and \mathbf{D}^I introduced in lemma 5.1, by (C8) and (C9) we obtain

$$\begin{aligned} & \|\boldsymbol{\sigma}^I(\mathbf{u}_1, \varphi_1)(t) - \boldsymbol{\sigma}^I(\mathbf{u}_2, \varphi_2)(t)\|_{\mathcal{H}} + \|\mathbf{D}^I(\mathbf{u}_1, \varphi_1)(t) - \mathbf{D}^I(\mathbf{u}_2, \varphi_2)(t)\|_H \\ & \leq c \int_0^t (\|\boldsymbol{\sigma}^I(\mathbf{u}_1, \varphi_1)(s) - \boldsymbol{\sigma}^I(\mathbf{u}_2, \varphi_2)(s)\|_{\mathcal{H}} + \|\mathbf{D}^I(\mathbf{u}_1, \varphi_1)(s) - \mathbf{D}^I(\mathbf{u}_2, \varphi_2)(s)\|_H \\ & \quad + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V + \|\varphi_1(s) - \varphi_2(s)\|_{\Phi}) ds. \end{aligned}$$

Now, using a Gronwall argument, it follows that

$$\begin{aligned} & \|\boldsymbol{\sigma}^I(\mathbf{u}_1, \varphi_1)(t) - \boldsymbol{\sigma}^I(\mathbf{u}_2, \varphi_2)(t)\|_{\mathcal{H}} + \|\mathbf{D}^I(\mathbf{u}_1, \varphi_1)(t) - \mathbf{D}^I(\mathbf{u}_2, \varphi_2)(t)\|_H \\ & \leq c \int_0^t (\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V + \|\varphi_1(s) - \varphi_2(s)\|_{\Phi}) ds. \end{aligned} \quad (5.10)$$

We combine (5.9) and (5.10) to obtain

$$\|\mathcal{S}(\mathbf{u}_1, \varphi_1)(t) - \mathcal{S}(\mathbf{u}_2, \varphi_2)(t)\|_{X^*} \leq c \int_0^t \|(\mathbf{u}_1(s), \varphi_1(s)) - (\mathbf{u}_2(s), \varphi_2(s))\|_X ds,$$

and therefore (C2) holds.

Subsequently, we define the function $j: \Sigma_C \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ by

$$j(\mathbf{x}, t, \boldsymbol{\xi}, r) = j_\nu(\mathbf{x}, t, \xi_\nu - g(\mathbf{x}))$$

for all $(\boldsymbol{\xi}, r) \in \mathbb{R}^d \times \mathbb{R}$ for a.e. $(\mathbf{x}, t) \in \Sigma_C$. We verify that j satisfies condition (C3). It is obvious to see that j satisfies (a) of (C3) with $\mathbf{e} \in L^2(\Gamma_C; \mathbb{R}^{d+1})$ given by $\mathbf{e}(\mathbf{x}) = ((e_1(\mathbf{x}) + g(\mathbf{x}))\boldsymbol{\nu}, 0)$ for a.e. $\mathbf{x} \in \Gamma_C$, and $j(\mathbf{x}, t, \cdot, \cdot)$ is locally Lipschitz for a.e. $(\mathbf{x}, t) \in \Sigma_C$. Moreover, from [14, lemma 3.39 and proposition 3.37], we obtain

$$j^0(\mathbf{x}, t, \boldsymbol{\xi}, r; \boldsymbol{\varrho}, s) \leq j_\nu^0(\mathbf{x}, t, \xi_\nu - g(\mathbf{x}); \varrho_\nu), \quad (5.11)$$

$$\partial j(\mathbf{x}, t, \boldsymbol{\xi}, r) \subseteq \partial j_\nu(\mathbf{x}, t, \xi_\nu - g(\mathbf{x}))\boldsymbol{\nu} \times \{0\} \quad (5.12)$$

for all $(\boldsymbol{\xi}, r), (\boldsymbol{\varrho}, s) \in \mathbb{R}^d \times \mathbb{R}$ and a.e. $(\mathbf{x}, t) \in \Sigma_C$.

From the inequality

$$\begin{aligned} \|\partial j(\mathbf{x}, t, \boldsymbol{\xi}, r)\|_{\mathbb{R}^{d+1}} & \leq |\partial j_\nu(\mathbf{x}, t, \xi_\nu - g(\mathbf{x}))| \\ & \leq c_{0\nu} + c_{1\nu}|\xi_\nu - g(\mathbf{x})| \\ & \leq c_{0\nu} + c_{1\nu}|g(\mathbf{x})| + c_{1\nu}\|(\boldsymbol{\xi}, r)\|_{\mathbb{R}^{d+1}}, \end{aligned}$$

which is valid for all $(\boldsymbol{\xi}, r) \in \mathbb{R}^d \times \mathbb{R}$ for a.e. $(\mathbf{x}, t) \in \Sigma_C$, we deduce that (c) of (C3) holds with $\bar{c}_0 = c_{0\nu} + c_{1\nu}\|g\|_{L^\infty(\Gamma_C)}$ and $\bar{c}_1 = c_{1\nu}$.

Next, let $(\zeta_i, \eta_i) \in \partial j(\mathbf{x}, t, \xi_i, r_i)$, where $(\zeta_i, \eta_i) \in \mathbb{R}^d \times \mathbb{R}$, $(\xi, r_i) \in \mathbb{R}^d \times \mathbb{R}$, $i = 1, 2$. By (5.12), it follows that $\zeta_i = \zeta_i \nu$ with $\zeta_i \in \partial j_\nu(\mathbf{x}, t, \xi_{i\nu} - g(\mathbf{x}))$, $\eta_i = 0$ and

$$\begin{aligned} ((\zeta_1, \eta_1) - (\zeta_2, \eta_2)) \cdot ((\xi_1, r_1) - (\xi_2, r_2)) &= (\zeta_1 - \zeta_2) \nu \cdot (\xi_1 - \xi_2) \\ &= (\zeta_1 - \zeta_2)(\xi_{1\nu} - \xi_{2\nu}) \geq -m_\nu |\xi_{1\nu} - \xi_{2\nu}|^2 \\ &\geq -m_\nu \|(\xi_1, r_1) - (\xi_2, r_2)\|_{\mathbb{R}^{d+1}}^2 \end{aligned}$$

for a.e. $(\mathbf{x}, t) \in \Sigma_C$. This proves that (d) of (C3) holds with $m_2 = m_\nu$.

From the inequality (5.11), the subadditivity of the function $j_\nu^0(\mathbf{x}, t, q; \cdot)$ for all $q \in \mathbb{R}$ and a.e. $(\mathbf{x}, t) \in \Sigma_C$ and hypothesis (d) of (C10), we deduce

$$\begin{aligned} j^0(\mathbf{x}, t, \xi, r; -\xi, -r) &\leq j_\nu^0(\mathbf{x}, t, \xi_\nu - g(\mathbf{x}); -\xi_\nu) \\ &\leq j_\nu^0(\mathbf{x}, t, \xi_\nu - g(\mathbf{x}); -(\xi_\nu - g(\mathbf{x}))) + j_\nu^0(\mathbf{x}, t, \xi_\nu - g(\mathbf{x}); -g(\mathbf{x})) \\ &\leq d_\nu(1 + |\xi_\nu - g(\mathbf{x})|) + \max\{\eta(-g(\mathbf{x})) \mid \eta \in \partial j_\nu(\mathbf{x}, t, \xi_\nu - g(\mathbf{x}))\} \\ &\leq d_\nu(1 + \|\xi\|_{\mathbb{R}^d} + \|g\|_{L^\infty(\Gamma_C)}) + |g(\mathbf{x})|(c_{0\nu} + c_{1\nu}|\xi_\nu - g(\mathbf{x})|) \\ &\leq \tilde{d}(1 + \|(\xi, r)\|_{\mathbb{R}^{d+1}}) \end{aligned}$$

for all $(\xi, r) \in \mathbb{R}^d \times \mathbb{R}$ and a.e. $(\mathbf{x}, t) \in \Sigma_C$ with $\tilde{d} > 0$. We conclude from the above that the function j satisfies (C3).

Next, it follows from (C11), (C12), (4.8), (4.11) and (5.7) that $f \in \mathcal{X}^*$, and therefore condition (C4) is also satisfied. We also observe that, with the notation above, the assumptions (i) and (ii) in theorem 4.1 are equivalent to the corresponding assumptions (i) and (ii) in theorem 2.1. Finally, the regularity hypothesis (4.20) implies the Clarke regularity (2.2).

Lemma 5.4 is now a consequence of theorem 2.1. \square

We now have all the ingredients to provide the proof of our main existence and uniqueness result.

Proof of theorem 4.1. Assume that either (i) or (ii) holds. Then, using lemma 5.4 we obtain the existence (and the uniqueness if (4.20) holds) of a solution $x = (\mathbf{u}, \varphi) \in \mathcal{X}$ for the hemivariational inequality (5.8). Moreover, by lemma 5.3, we know that $x = (\mathbf{u}, \varphi)$ is a solution (the unique solution if (4.20) holds) of the system (5.3), (5.4).

Consider now the functions σ and \mathbf{D} defined by equalities (5.1), (5.2), for a.e. $t \in (0, T)$. It follows that $\sigma \in L^2(0, T; \mathcal{H})$ and $\mathbf{D} \in L^2(0, T; H)$ and, moreover, by lemma 5.2 we deduce that quadruple of functions $(\mathbf{u}, \sigma, \mathbf{D}, \varphi)$ is a solution to problem (P_V). We test (4.14) with $\mathbf{v} \in C_0^\infty(\Omega; \mathbb{R}^d)$. Then we take $\psi \in C_0^\infty(\Omega)$ in (4.15) to obtain that

$$\text{Div } \sigma(t) + \mathbf{f}_0(t) = \mathbf{0}, \quad \text{div } \mathbf{D}(t) = q_0(t)$$

almost everywhere in Ω for a.e. $t \in (0, T)$. Next, we use assumptions (C11) and (C12) to deduce that $\text{Div } \sigma \in L^2(0, T; H)$, $\text{div } \mathbf{D} \in L^2(0, T; L^2(\Omega))$, and therefore $\sigma \in L^2(0, T; \mathcal{H}_1)$ and $\mathbf{D} \in L^2(0, T; W)$. We conclude from the above that the quadruple of functions $(\mathbf{u}, \sigma, \mathbf{D}, \varphi)$ satisfies (4.16)–(4.19), which completes the

existence part of the theorem. The uniqueness part (under the additional assumption (4.20)) is a consequence of lemma 5.2 combined with the unique solvability of the system (5.3), (5.4), proved above. \square

Acknowledgements

This research was supported by the Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme under Grant Agreement no. 295118 and the National Science Centre of Poland under the Maestro Advanced Project no. DEC-2012/06/A/ST1/00262. S.M. and A.O. were also partly supported by the National Science Centre of Poland under Grant no. N N201 604640 and the International Project co-financed by the Ministry of Science and Higher Education of the Republic of Poland under Grant no. W111/7.PR/2012.

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