Reflected BSDEs when the obstacle is not right-continuous and optimal stopping

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Abstract

In the first part of the paper, we study reflected backward stochastic differential equations (RBSDEs) with lower obstacle which is assumed to be right upper-semicontinuous but not necessarily right-continuous. We prove existence and uniqueness of the solutions to such RBSDEs in appropriate Banach spaces. The result is established by using some tools from the general theory of processes such as Mertens decomposition of optional strong (but not necessarily right-continuous) supermartingales, some tools from optimal stopping theory, as well as an appropriate generalization of Itô's formula due to Gal'chouk and Lenglart. In the second part of the paper, we provide some links between the RBSDE studied in the first part and an optimal stopping problem in which the risk of a financial position ξ is assessed by an f-conditional expectation $\mathcal{E}^f(\cdot)$ (where f is a Lipschitz driver). We characterize the "value function" of the problem in terms of the solution to our RBSDE. Under an additional assumption of left upper-semicontinuity on ξ, we show the existence of an optimal stopping time. We also provide a generalization of Mertens decomposition to the case of strong $\mathcal{E}^f$-supermartingales.

Keywords : backward stochastic differential equation, reflected backward stochastic differential equation, optimal stopping, f-expectation, strong optional supermartingale, Mertens decomposition, dynamic risk measure

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1 Introduction

Backward stochastic differential equations (BSDEs) have been introduced by Bismut ([2], [3]) in the case of a linear driver. The general theory of existence and uniqueness of solutions to BSDEs has been developed by Pardoux and Peng [23]. Through a result of Feynman-Kac-type, these authors have linked the theory of BSDEs to that of quasilinear parabolic partial differential equations (cf.[24]). BSDEs have found number of applications in finance, among which pricing and hedging of European options and recursive utilities (cf., for instance, [11]). Also, a useful family of operators, the so-called f-conditional expectations, has been defined through the notion of BSDEs and used in the literature on dynamic risk measures (cf., for instance, [25], [28], [14], [1], among others). We recall that the f-conditional expectation at time $t \in [0,T]$ (where $T > 0$ is a fixed final horizon) is the operator which maps a given terminal condition ξ_T to the position at time t of (the first component of) the solution to the BSDE with parameters (f, ξ_T). The operator is denoted $\mathcal{E}^f_{t,T}(\cdot)$.
Reflected backward stochastic differential equations (RBSDEs) can be seen as a variant of BSDEs in which the (first component of the) solution is constrained to remain greater than or equal to a given process called the obstacle. Compared to the case of (non-reflected) BSDEs, there is an additional nondecreasing predictable process which keeps the (first component of the) solution above the obstacle. RBSDEs have been introduced by El Karoui et al. [10] and have proved useful, for instance, in the study of American options. The work by El Karoui et al. [10] considers the case of a Brownian filtration and a continuous obstacle. There have been several extensions of this work to the case of a discontinuous obstacle (cf. [16], [4], [17], [13], [18], [27]). In all these extensions an assumption of right-continuity on the obstacle is made.

In the first part of the present paper we consider a further extension of the theory of RBSDEs to the case where the obstacle is not necessarily right-continuous. Compared to the right-continuous case, the additional nondecreasing process, which "pushes" the (first component of the) solution to stay above the obstacle, is no longer right-continuous. For the sake of simplicity, we place ourselves in the framework of a Brownian filtration, but we note that our results can be generalized to the case of a larger stochastic basis (cf. Section 6). We establish existence and uniqueness of the solution in appropriate Banach spaces. To prove our results we use tools from the general theory of processes such as Mertens decomposition of strong optional (but not necessarily right-continuous) supermartingales (generalizing Doob-Meyer decomposition), some tools from optimal stopping theory, as well as a generalization of Itô's formula to the case of strong optional (but not necessarily right-continuous) semimartingales due to Gal'chouk and Lenglart.
In the second part of the paper, we make some links between the RBSDEs studied in the first part and optimal stopping with f-conditional expectations. More precisely, we are interested in the following optimization problem: we are given a process ξ modelling a dynamic financial position. The risk of ξ is assessed by a dynamic risk measure which (up to a minus sign) is given by an f-conditional expectation. The process ξ is assumed to be right upper-semicontinuous, but not necessarily right-continuous. We aim at stopping the process ξ in such a way that the risk be minimal. We characterize the value of the problem in terms of the unique solution to the RBSDE associated with obstacle ξ and driver f studied in the first part. We show the existence of an optimal stopping time for the problem under an additional assumption of left upper-semicontinuity of ξ, and the existence of an

ε-optimal stopping time in the more general case where this assumption is not made. We provide an optimality criterion characterizing the optimal stopping times for the problem in terms of properties of the "value process". We thus extend some results of [27] to the case where the optimized process ξ is not cadlag. We also establish a comparison principle for the RBSDEs studied in the first part of our paper, as well as a generalization of Mertens decomposition to the case of $\mathcal{E}^f$-strong supermartingales.

The remainder of the paper is organized as follows:

In Section 2 we give some preliminary definitions and properties. In Section 3 we define our RBSDE and we prove existence and uniqueness of the solution. Section 4 is dedicated to our optimal stopping problem with f-conditional expectations. In Subsection 4.1 we formulate and motivate the problem. In Subsection 4.2 we characterize the value function of the problem in terms of the solution of the RBSDE studied in Section 3; we also give an optimality criterion and address the question of existence of ε-optimal and optimal stopping times. In Section 5 we derive some useful additional results: comparison principle for our RBSDEs (Subsection 5.2) and "generalized" Mertens decomposition for $\mathcal{E}^f$-strong supermartingales (Subsection 5.1). In Section 6 we briefly present some further extensions of our work. In the Appendix we make some recalls ("classical" Mertens decomposition, Galchouk-Lenglart's change of variables formula) and we give the proofs of two of the results (Prop. A.5 and Prop. 2.1) used in the main part of the paper.

2 Preliminaries

Let $T > 0$ be a fixed positive real number. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a one-dimensional Brownian motion W. Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the completed natural filtration associated with W. For $t \in [0, T]$, we denote by $\mathcal{T}_{t,T}$ the set of stopping times τ such that $P(t \leq \tau \leq T) = 1$. More generally, for a given stopping time $\nu \in \mathcal{T}_{0,T}$, we denote by $\mathcal{T}_{\nu,T}$ the set of stopping times τ such that $P(\nu \leq \tau \leq T) = 1$.

We denote by $\mathcal{P}$ be the predictable σ-algebra on $\Omega \times [0, T]$.

We use the following notation:

- $L^2(\mathcal{F}_T)$ is the set of random variables which are $\mathcal{F}_T$-measurable and square-integrable.
- $\mathbb{H}^{2,T}$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{\mathbb{H}^{2,T}} := E\left[(\int_0^T |\phi_t|^2 dt)\right] < \infty$.
- $\mathcal{S}^{2,T}$ is the set of real-valued optional processes ϕ such that $|||\phi|||^2_{\mathcal{S}^{2,T}} := E(\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} |\phi_\tau|^2) < \infty$.

When there is no possibility of confusion, we denote $\mathbb{H}^2$ instead of $\mathbb{H}^{2,T}$, $\mathcal{S}^2$ instead of $\mathcal{S}^{2,T}$, as well as $\|\cdot\|^2_{\mathbb{H}^2}$ instead of $\|\cdot\|^2_{\mathbb{H}^{2,T}}$ and $|||\cdot|||^2_{\mathcal{S}^2}$ instead of $|||\cdot|||^2_{\mathcal{S}^{2,T}}$.

Remark 2.1 *Let us emphasize that in our framework of completed brownian filtration every martingale is continuous, up to a modification. We note as well that due to the optional sampling theorem*

every martingale in our framework is strong.
We also recall that in our framework the optional and predictable σ-algebras on $\Omega \times [0,T]$ coincide. In the present paper the terms predictable and optional are thus interchangeable.

Definition 2.1 (Driver, Lipschitz driver) *A function f is said to be a* driver *if*

- $f : \Omega \times [0,T] \times \mathbf{R}^2 \to \mathbf{R}$
  $(\omega, t, y, z) \mapsto f(\omega, t, y, z)$ *is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2)-$ measurable,*
- $E\left[\left(\int_0^T |f(t,0,0)|^2 dt\right)\right] < \infty$.

A driver f is called a Lipschitz driver *if moreover there exists a constant $K \geq 0$ such that $dP \otimes dt$-a.e., for each (y_1, z_1), (y_2, z_2),*

$$|f(\omega, t, y_1, z_1) - f(\omega, t, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|).$$

For a ladlag process ϕ, we denote by ϕ_{t+} and ϕ_{t-} the right-hand and left-hand limit of ϕ at t. We denote by $\Delta_+ \phi_t := \phi_{t_+} - \phi_t$ the size of the right jump of ϕ at t, and by $\Delta \phi_t := \phi_t - \phi_{t_-}$ the size of the left jump of ϕ at t.

We give a useful property of the space $\mathcal{S}^2$.

Proposition 2.1 *The space $\mathcal{S}^2$ endowed with the norm $|||\cdot|||_{\mathcal{S}^2}$ is a Banach space.*

Proof: The proof is given in the Appendix. □

We will also use the following notation:
Let $\beta > 0$. For $\phi \in I\!H^2$, $\|\phi\|_\beta^2 := E[\int_0^T \mathrm{e}^{\beta s} \phi_s^2 ds]$. We note that on the space $I\!H^2$ the norms $\|\cdot\|_\beta$ and $\|\cdot\|_{I\!H^2}$ are equivalent. For $\phi \in \mathcal{S}^2$, we deifine $|||\phi|||_\beta^2 := E[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau} \phi_\tau^2]$. We note that $|||\cdot|||_\beta$ is a norm on $\mathcal{S}^2$ (the proof is similar to that of Prop. 2.1) equivalent to the norm $|||\cdot|||_{\mathcal{S}^2}$.

3 Reflected BSDEs whose obstacles are not cadlag

Let $T > 0$ be a fixed terminal time. Let f be a driver. Let $\xi = (\xi_t)_{t \in [0,T]}$ be a left-limited process in $\mathcal{S}^2$. We suppose moreover that the process ξ is right upper-semicontinuous (r.u.s.c. for short). A process ξ satisfying the previous properties will be called a *barrier*, or an *obstacle*.

Remark 3.2 *Let us note that in the following definitions and results we can relax the assumption of existence of left limits for the obstacle ξ. All the results still hold true provided we replace the process $(\xi_{t-})_{t \in]0,T]}$ by the process $(\underline{\xi}_t)_{t \in]0,T]}$ defined by $\underline{\xi}_t := \limsup_{s \uparrow t, s<t} \xi_s$, for all $t \in]0,T]$. We recall that $\underline{\xi}$ is a predictable process (cf. [6, Thm. 90, page 225]). We call the process $\underline{\xi}$ the left upper-semicontinuous envelope of ξ.*

Definition 3.1 *A process (Y, Z, A, C) is said to be a solution to the reflected BSDE with parameters (f, ξ), where f is a driver and ξ is an obstacle, if*

$(Y, Z, A, C) \in \mathcal{S}^2 \times I\!H^2 \times \mathcal{S}^2 \times \mathcal{S}^2$

$$Y_\tau = \xi_T + \int_\tau^T f(t, Y_t, Z_t)dt - \int_\tau^T Z_t dW_t + A_T - A_\tau + C_{T-} - C_{\tau-} \text{ a.s. for all } \tau \in \mathcal{T}_{0,T}, \tag{3.1}$$

$$Y_t \geq \xi_t \text{ for all } t \in [0, T] \text{ a.s.,} \tag{3.2}$$

A is a nondecreasing right-continuous predictable process with $A_0 = 0, E(A_T) < \infty$ and such that

$$\int_0^T \mathbf{1}_{\{Y_t > \xi_t\}} dA_t^c = 0 \text{ a.s. and } (Y_{\tau-} - \xi_{\tau-})(A_\tau^d - A_{\tau-}^d) = 0 \text{ a.s. for all (predictable) } \tau \in \mathcal{T}_{0,T}, \tag{3.3}$$

C is a nondecreasing right-continuous adapted purely discontinuous process with $C_{0-} = 0, E(C_T) < \infty$ and such that $(Y_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0$ a.s. for all $\tau \in \mathcal{T}_{0,T}$. (3.4)

Here A^c denotes the continuous part of the nondecreasing process A and A^d its discontinuous part.

Remark 3.3 *We note that a process $(Y, Z, A, C) \in \mathcal{S}^2 \times I\!H^2 \times \mathcal{S}^2 \times \mathcal{S}^2$ satisfies equation* (3.1) *in the above definition if and only if*

$$Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + A_T - A_t + C_{T-} - C_{t-} \text{ for all } t \in [0, T] \text{ a.s.}$$

Remark 3.4 *We also note that if (Y, Z, A, C) is a solution to the RBSDE defined above, then $\Delta C_t(\omega) = Y_t(\omega) - Y_{t+}(\omega)$ for all $(\omega, t) \in \Omega \times [0, T]$ outside an evanescent set. This observation is a consequence of equation* (3.1).

Remark 3.5 *If $(Y, Z, A, C) \in \mathcal{S}^2 \times I\!H^2 \times \mathcal{S}^2 \times \mathcal{S}^2$ satisfies the above definition, then the process Y has left and right limits. Moreover, the process $(Y_t + \int_0^t f(s, Y_s, Z_s)ds)_{t\in[0,T]}$ is a strong supermartingale (cf. Definition A.1).*

We first investigate the question of existence and uniqueness of the solution to the RBSDE defined above in the case where the driver f does not depend on y and z. To this purpose, we first prove a lemma which will be used in the sequel.

Lemma 3.2 (A priori estimates) *Let $(Y^1, Z^1, A^1, C^1) \in \mathcal{S}^2 \times I\!H^2 \times \mathcal{S}^2 \times \mathcal{S}^2$ (resp. $(Y^2, Z^2, A^2, C^2) \in \mathcal{S}^2 \times I\!H^2 \times \mathcal{S}^2 \times \mathcal{S}^2$) be a solution to the RBSDE associated with driver $f^1(\omega, t)$ (resp. $f^2(\omega, t)$) and with obstacle ξ. There exists $c > 0$ such that for all $\varepsilon > 0$, for all $\beta \geq \frac{1}{\varepsilon^2}$ we have*

$$\begin{aligned} \|Z^1 - Z^2\|_\beta^2 &\leq \varepsilon^2 \|f^1 - f^2\|_\beta^2 \\ |||Y^1 - Y^2|||_\beta^2 &\leq 2\varepsilon^2(1 + 2c^2)\|f^1 - f^2\|_\beta^2. \end{aligned} \tag{3.5}$$

Proof: Let $\beta > 0$ and $\varepsilon > 0$ be such that $\beta \geq \frac{1}{\varepsilon^2}$. We set $\tilde{Y} := Y^1 - Y^2$, $\tilde{Z} := Z^1 - Z^2$, $\tilde{A} := A^1 - A^2$, $\tilde{C} := C^1 - C^2$, and $\tilde{f}(\omega, t) := f^1(\omega, t) - f^2(\omega, t)$. We note that $\tilde{Y}_T = \xi_T - \xi_T = 0$; moreover,

$$\tilde{Y}_\tau = \int_\tau^T \tilde{f}(t)dt - \int_\tau^T \tilde{Z}_t dW_t + \tilde{A}_T - \tilde{A}_\tau + \tilde{C}_{T-} - \tilde{C}_{\tau-} \text{ a.s. for all } \tau \in \mathcal{T}_{0,T}.$$

Thus we see that $\tilde{Y}$ is an optional (strong) semimartingale (in the vocabulary of [15]) with decomposition $\tilde{Y} = \tilde{Y}_0 + M + A + B$, where $M_t := \int_0^t \tilde{Z}_s dW_s$, $A_t := -\int_0^t \tilde{f}(s)ds - \tilde{A}_t$ and $B_t := -\tilde{C}_{t-}$ (the notation is that of Theorem A.3 and Corollary A.2 from the Appendix). Applying Corollary A.2 to $\tilde{Y}$ gives: almost surely, for all $t \in [0,T]$,

$$\begin{aligned}
\mathrm{e}^{\beta T}\,\tilde{Y}_T^2 = \mathrm{e}^{\beta t}\,\tilde{Y}_t^2 &+ \int_{]t,T]} \beta\,\mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds \\
&- 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{f}(s)ds - 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}d\tilde{A}_s \\
&+ 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s + \int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds \\
&+ \sum_{t<s\le T} \mathrm{e}^{\beta s}(\tilde{Y}_s - \tilde{Y}_{s-})^2 - \int_{[t,T[} 2\,\mathrm{e}^{\beta s}\,\tilde{Y}_s d(\tilde{C})_{s+} + \sum_{t\le s<T} \mathrm{e}^{\beta s}(\tilde{Y}_{s+} - \tilde{Y}_s)^2.
\end{aligned}$$

Thus, we get (recall that $\tilde{Y}_T = 0$): almost surely, for all $t \in [0,T]$,

$$\begin{aligned}
\mathrm{e}^{\beta t}\,\tilde{Y}_t^2 + \int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds = &- \int_{]t,T]} \beta\,\mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds + 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_s\tilde{f}(s)ds + 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}d\tilde{A}_s \\
&- 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s - \sum_{t<s\le T} \mathrm{e}^{\beta s}(\tilde{Y}_s - \tilde{Y}_{s-})^2 \\
&+ 2\int_{[t,T[} \mathrm{e}^{\beta s}\,\tilde{Y}_s d(\tilde{C})_{s+} - \sum_{t\le s<T} \mathrm{e}^{\beta s}(\tilde{Y}_{s+} - \tilde{Y}_s)^2.
\end{aligned}$$

Hence, a.s. for all $t \in [0,T]$,

$$\begin{aligned}
\mathrm{e}^{\beta t}\,\tilde{Y}_t^2 + \int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds \le &- \int_{]t,T]} \beta\,\mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds + 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_s\tilde{f}(s)ds + 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}d\tilde{A}_s \\
&- 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s + 2\int_{[t,T[} \mathrm{e}^{\beta s}\,\tilde{Y}_s d(\tilde{C})_{s+}.
\end{aligned} \tag{3.6}$$

Let us first consider the sum of the first and the second term on the r.h.s. of the above inequality (3.6). By applying the inequality $2ab \le (\frac{a}{\varepsilon})^2 + \varepsilon^2 b^2$, valid for all $a \in \mathbf{R}$, for all $b \in \mathbf{R}$, we get: a.e. for all $t \in [0,T]$,

$$\begin{aligned}
-\int_{]t,T]} \beta\,\mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds + 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_s\tilde{f}(s)ds &\le -\int_{]t,T]} \beta\,\mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds + \frac{1}{\varepsilon^2}\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_s^2 ds + \varepsilon^2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{f}^2(s)ds \\
&= (\frac{1}{\varepsilon^2} - \beta)\int_{]t,T]} \mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds + \varepsilon^2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{f}^2(s)ds.
\end{aligned}$$

As $\beta \ge \frac{1}{\varepsilon^2}$, we have $(\frac{1}{\varepsilon^2} - \beta)\int_{]t,T]} \mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds \le 0$, for all $t \in [0,T]$ a.s.

Next, we show that the last term on the right-hand side of inequality (3.6) is non-positive. More precisely, a.s. for all $t \in [0,T]$, $\int_{[t,T[} \mathrm{e}^{\beta s}\,\tilde{Y}_s d(\tilde{C})_{s+} \le 0$. Indeed, a.s. for all $t \in [0,T]$, $\int_{[t,T[} \mathrm{e}^{\beta s}\,\tilde{Y}_s d(\tilde{C})_{s+} = \int_{[t,T[} \mathrm{e}^{\beta s}\,\tilde{Y}_s d\tilde{C}_s = \sum_{t\le s<T} \mathrm{e}^{\beta s}\,\tilde{Y}_s\Delta\tilde{C}_s$. Now, a.s. for all $s \in [0,T]$,

$$\tilde{Y}_s\Delta\tilde{C}_s = (Y_s^1 - Y_s^2)\Delta C_s^1 - (Y_s^1 - Y_s^2)\Delta C_s^2. \tag{3.7}$$

We use property (3.4), the non-decreasingness of (almost all trajectories of) C^1, and the fact that $Y^2 \geq \xi$ to obtain: a.s. for all $s \in [0,T]$,

$$(Y_s^1 - Y_s^2)\Delta C_s^1 = (Y_s^1 - \xi_s)\Delta C_s^1 - (Y_s^2 - \xi_s)\Delta C_s^1 = 0 - (Y_s^2 - \xi_s)\Delta C_s^1 \leq 0.$$

Similarly, we obtain: a.s. for all $s \in [0,T]$,

$$(Y_s^1 - Y_s^2)\Delta C_s^2 = (Y_s^1 - \xi_s)\Delta C_s^2 + (\xi_s - Y_s^2)\Delta C_s^2 = (Y_s^1 - \xi_s)\Delta C_s^2 + 0 \geq 0.$$

We conclude that the last term on the r.h.s. of equation (3.6) is non-positive (in the above sense). It can be also shown that the third term on the r.h.s. of (3.6) is also non-positive (in the above sense); the proof uses property (3.3) of the definition of the RBSDE and the property $Y^i \geq \xi$, for $i = 1, 2$; the details are similar to those in the case of a cadlag obstacle and are left to the reader (cf., for instance, [27, proof of Prop. A.1]).
The above observations, together with equation (3.6), lead to

$$\mathrm{e}^{\beta t}\,\tilde{Y}_t^2 + \int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds \leq \varepsilon^2 \int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{f}^2(s) ds - 2\int_{]t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s, \text{ for all } t \in [0,T], \text{ a.s.} \tag{3.8}$$

We now show that the last term on the r.h.s. of the previous inequality (3.8) has zero expectation. To this purpose, we show that $E\left[\sqrt{\int_0^T \mathrm{e}^{2\beta s}\,\tilde{Y}_{s-}^2\tilde{Z}_s^2 ds}\right] < \infty$. By using the left-continuity of a.e. trajectory of the process $(\tilde{Y}_{s-})$, we have

$$(\tilde{Y}_{s-})^2(\omega) \leq \sup_{t\in\mathbb{Q}} (\tilde{Y}_{t-})^2(\omega), \text{ for all } s \in (0,T], \text{ for a.e. } \omega \in \Omega. \tag{3.9}$$

On the other hand, for all $t \in (0,T]$, $(\tilde{Y}_{t-})^2 \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} (\tilde{Y}_\tau)^2$ a.s.; hence,

$$\sup_{t\in\mathbb{Q}} (\tilde{Y}_{t-})^2 \leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} (\tilde{Y}_\tau)^2 \text{ a.s.} \tag{3.10}$$

From equations (3.9) and (3.10) we obtain

$$\int_0^T \mathrm{e}^{2\beta s}\,\tilde{Y}_{s-}^2\tilde{Z}_s^2 ds \leq \int_0^T \mathrm{e}^{2\beta s} \sup_{t\in\mathbb{Q}} (\tilde{Y}_{t-})^2 \tilde{Z}_s^2 ds \leq \int_0^T \mathrm{e}^{2\beta s} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} (\tilde{Y}_\tau)^2 \tilde{Z}_s^2 ds \text{ a.s.} \tag{3.11}$$

Using this, together with Cauchy-Schwarz inequality, gives

$$E\left[\sqrt{\int_0^T \mathrm{e}^{2\beta s}\,\tilde{Y}_{s-}^2\tilde{Z}_s^2 ds}\right] \leq E\left[\sqrt{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} (\tilde{Y}_\tau)^2}\sqrt{\int_0^T \mathrm{e}^{2\beta s}\,\tilde{Z}_s^2 ds}\right] \leq |||\tilde{Y}|||_{\mathcal{S}^2} \|\tilde{Z}\|_{2\beta},$$

We have that $|||\tilde{Y}|||_{\mathcal{S}^2} < \infty$ due to the inequality $|||\tilde{Y}|||_{\mathcal{S}^2} \leq |||Y^1|||_{\mathcal{S}^2} + |||Y^2|||_{\mathcal{S}^2}$ and to Y^1 and Y^2 being in $\mathcal{S}^2$. We also have that $\|\tilde{Z}\|_{2\beta} < \infty$, due to the fact that $Z^1, Z^2 \in I\!H^2$ and to the equivalence of the norms $\|\cdot\|_{2\beta}$ and $\|\cdot\|_{I\!H^2}$. We conclude that $E\left[\sqrt{\int_0^T \mathrm{e}^{2\beta s}\,\tilde{Y}_{s-}^2\tilde{Z}_s^2 ds}\right] < \infty$; whence, by standard arguments, we get $E\left[\int_0^T \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s\right] = 0$.

By applying (3.8) with $t = 0$, and by taking expectations on both sides of the resulting inequality, we obtain $\tilde{Y}_0^2 + \|\tilde{Z}\|_\beta^2 \leq \varepsilon^2 \|\tilde{f}\|_\beta^2$. Hence,

$$\|\tilde{Z}\|_\beta^2 \leq \varepsilon^2 \|\tilde{f}\|_\beta^2. \tag{3.12}$$

From (3.8) we also get, for all $\tau \in \mathcal{T}_{0,T}$,

$$\mathrm{e}^{\beta\tau}\,\tilde{Y}_\tau^2 \leq \varepsilon^2 \int_{]0,T]} \mathrm{e}^{\beta s}\,\tilde{f}^2(s)ds - 2\int_{]0,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s + 2\int_{]0,\tau]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s \text{ a.s.}$$

By taking first the essential supremum over $\tau \in \mathcal{T}_{0,T}$ and then the expectation on both sides of the above inequality, we obtain

$$E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau}\,\tilde{Y}_\tau^2] \leq \varepsilon^2\|\tilde{f}\|_\beta^2 + 2E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} |\int_0^\tau \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s|]. \tag{3.13}$$

By using the continuity of a.e. trajectory of the process $(\int_0^t \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s)_{t\in[0,T]}$ (cf. Prop. A.3) and Burkholder-Davis-Gundy inequalities (applied with $p = 1$), we get

$$2E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} |\int_0^\tau \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s|] = 2E[\sup_{t\in[0,T]} |\int_0^t \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s|] \leq 2cE\left[\sqrt{\int_0^T \mathrm{e}^{2\beta s}\,\tilde{Y}_{s-}^2\tilde{Z}_s^2 ds}\right], \tag{3.14}$$

where c is a positive "universal" constant (which does not depend on the other parameters). The same reasoning as that used to obtain equation (3.11) leads to

$$\sqrt{\int_0^T \mathrm{e}^{2\beta s}\,\tilde{Y}_{s-}^2\tilde{Z}_s^2 ds} \leq \sqrt{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau}(\tilde{Y}_\tau)^2 \int_0^T \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds} \text{ a.s.}$$

Combining the two previous inequalities with the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ gives

$$2E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \int_0^\tau \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s] \leq \frac{1}{2}E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau}(\tilde{Y}_\tau)^2] + \frac{1}{2}4c^2E[\int_0^T \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds].$$

From this, together with (3.13), we get

$$\frac{1}{2}|||\tilde{Y}|||_\beta^2 \leq \varepsilon^2\|\tilde{f}\|_\beta^2 + 2c^2\|\tilde{Z}\|_\beta^2.$$

This inequality, combined with the estimate (3.12) on $\|\tilde{Z}\|_\beta^2$, gives

$$|||\tilde{Y}|||_\beta^2 \leq 2\varepsilon^2(1 + 2c^2)\|\tilde{f}\|_\beta^2.$$

□

In the following lemma, we prove existence and uniqueness of the solution to the RBSDE from Definition 3.1 (in the case where the driver f does not depend on y and z) and we characterize the first component of the solution as the "value process" of an optimal stopping problem.

Lemma 3.3 *Suppose that f does not depend on y, z, that is $f(\omega, t, y, z) = f(\omega, t)$, where f is a process in $I\!H^2$. Let (ξ_t) be an obstacle. Then, the RBSDE from Definition 3.1 admits a unique solution $(Y, Z, A, C) \in \mathcal{S}^2 \times I\!H^2 \times \mathcal{S}^2 \times \mathcal{S}^2$, and for each $S \in \mathcal{T}_{0,T}$, we have*

$$Y_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[\xi_\tau + \int_S^\tau f(t)dt \mid \mathcal{F}_S] \quad \text{a.s.} \tag{3.15}$$

Remark 3.6 *If the obstacle (ξ_t) is left upper-semicontinuous (l.u.s.c. for short) along stopping times, then (A_t) is continuous (cf., for instance, the last statement in Thm. 20 of [7, page 429], or [19]).*

The proof of the lemma is divided in several steps. First, we exhibit a "natural candidate" $\overline{Y}$ to be the first component of the solution to the RBSDE with parameters (f, ξ); we prove that $\overline{Y}$ belongs to the space $\mathcal{S}^2$ and we give an estimate of $\left|\left|\left|\overline{Y}\right|\right|\right|^2_{\mathcal{S}^2}$ in terms of $|||\xi|||^2_{\mathcal{S}^2}$ and $\|f\|^2_{I\!H^2}$. In the second step, we exhibit processes Z, A and C such that $(\overline{Y}, Z, A, C)$ is a solution to the RBSDE with parameters (f, ξ). In the third step, we prove that the processes A and C belong to $\mathcal{S}^2$ and we give an estimate of $|||A|||^2_{\mathcal{S}^2}$ and $|||C|||^2_{\mathcal{S}^2}$. In the fourth step, we show that $Z \in I\!H^2$. In the final step, we show the uniqueness of the solution.

Proof:

For $S \in \mathcal{T}_{0,T}$, we define $\overline{Y}(S)$ by

$$\overline{Y}(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[\xi_\tau + \int_S^\tau f(u)du \mid \mathcal{F}_S], \tag{3.16}$$

and $\overline{\overline{Y}}(S)$ by

$$\overline{\overline{Y}}(S) := \overline{Y}(S) + \int_0^S f(u)du = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[\xi_\tau + \int_0^\tau f(u)du \mid \mathcal{F}_S].$$

We note that the process $(\xi_t + \int_0^t f(u)du)_{t\in[0,T]}$ is progressive. Therefore, the family $(\overline{\overline{Y}}(S))_{S\in\mathcal{T}_{0,T}}$ is a supermartingale family (cf. [19, Remark 1.2 with Prop. 1.5]). This observation, combined with [7, Remark (b), page 435], gives the existence of a strong optional supermartingale (which we denote again by $\overline{\overline{Y}}$) such that $\overline{\overline{Y}}_S = \overline{\overline{Y}}(S)$ a.s. for all $S \in \mathcal{T}_{0,T}$. Thus, we have $\overline{Y}(S) = \overline{\overline{Y}}(S) - \int_0^S f(u)du = \overline{\overline{Y}}_S - \int_0^S f(u)du$ a.s. for all $S \in \mathcal{T}_{0,T}$. On the other hand, we know that almost all trajectories of the strong optional supermartingale $\overline{\overline{Y}}$ are ladlag (cf. [7]). Thus, we get that the ladlag optional process $(\overline{Y}_t)_{t\in[0,T]} = (\overline{\overline{Y}}_t - \int_0^t f(u)du)_{t\in[0,T]}$ aggregates the family $(\overline{Y}(S))_{S\in\mathcal{T}_{0,T}}$.

Step 1. By using the definition of $\overline{Y}$ (cf. (3.16)), Jensen's inequality and the triangular inequality, we get

$$|\overline{Y}_S| \le \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[|\xi_\tau| + |\int_S^\tau f(u)du| \mid \mathcal{F}_S] \le E[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} |\xi_\tau| + \int_0^T |f(u)|du \mid \mathcal{F}_S].$$

Thus, we obtain

$$|\overline{Y}_S| \le E[X|\mathcal{F}_S], \tag{3.17}$$

where we have set

$$X := \int_0^T |f(u)|du + \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} |\xi_\tau|. \tag{3.18}$$

Applying Cauchy-Schwarz inequality gives

$$E[X^2] \leq cT\|f\|^2_{I\!H^2} + c|||\xi|||^2_{\mathcal{S}^2} < +\infty, \tag{3.19}$$

where c is a positive constant. Now, inequality (3.17) leads to $|\overline{Y}_S|^2 \leq |E[X|\mathcal{F}_S]|^2$. By taking the essential supremum over $S \in \mathcal{T}_{0,T}$, we get $\operatorname{ess\,sup}_{S\in\mathcal{T}_{0,T}} |\overline{Y}_S|^2 \leq \operatorname{ess\,sup}_{S\in\mathcal{T}_{0,T}} |E[X|\mathcal{F}_S]|^2$. By using Proposition A.3 of the Appendix and Remark 2.1, we get $\operatorname{ess\,sup}_{S\in\mathcal{T}_{0,T}} |\overline{Y}_S|^2 \leq \sup_{t\in[0,T]} |E[X|\mathcal{F}_t]|^2$. By using this inequality and Doob's martingale inequalities in L^2, we obtain

$$E[\operatorname*{ess\,sup}_{S\in\mathcal{T}_{0,T}} |\overline{Y}_S|^2] \leq E[\sup_{t\in[0,T]} |E[X|\mathcal{F}_t]|^2] \leq cE[X^2], \tag{3.20}$$

where $c > 0$ is a constant different from the above one. Finally, combining inequalities (3.20) and (3.19) gives

$$E[\operatorname*{ess\,sup}_{S\in\mathcal{T}_{0,T}} |\overline{Y}_S|^2] \leq cT\|f\|^2_{I\!H^p} + c|||\xi|||^2_{\mathcal{S}^2}, \tag{3.21}$$

where we have again allowed the positive constant c to differ from the above ones.

Step 2. Due to the previous step and to the assumption $f \in I\!H^2$, the strong optional supermartingale $\overline{\overline{Y}}$ is of class (D). Applying Mertens decomposition (cf. Theorem A.1) and a result from optimal stopping theory (cf. [8, Prop. 2.34] or [19]) gives the following

$$\overline{Y}_\tau = -\int_0^\tau f(u)du + M_\tau - A_\tau - C_{\tau-} \text{ a.s. for all } \tau \in \mathcal{T}_{0,T}, \tag{3.22}$$

where M is a (cadlag) uniformly integrable martingale, A is a nondecreasing right-continuous predictable process such that $A_0 = 0$, $E(A_T) < \infty$ and satisfying (3.3), and C is a nondecreasing right-continuous adapted purely discontinuous process such that $C_{0-} = 0$, $E(C_T) < \infty$ and satisfying (3.4). By the martingale representation theorem there exists a unique predictable process Z such that $dM_t = Z_t dW_t$. Moreover, we have $\overline{Y}_T = \xi_T$ a.s. by definition of $\overline{Y}$ (cf. (3.16)). Combining this with equation (3.22) gives equation (3.1). Also by definition of $\overline{Y}$, we have $\overline{Y}_S \geq \xi_S$ a.s. for all $S \in \mathcal{T}_{0,T}$, which, along with Proposition A.4, shows that $\overline{Y}$ satisfies inequality (3.2). In order to conclude that the process $(\overline{Y}, Z, A, C)$ is a solution to the RBSDE with parameters (f, ξ), it remains to show that (Z, A, C) belongs to the space $I\!H^2 \times \mathcal{S}^2 \times \mathcal{S}^2$, which we do in the following steps.

Step 3. Let us define the process $\overline{\overline{A}}$ as the sum of the two non-decreasing processes of the Mertens decomposition of $\overline{\overline{Y}}$. More precisely, we set $\overline{\overline{A}}_t := A_t + C_{t-}$, where the processes (A_t) and (C_{t-}) are given by (3.22). By arguments similar to those used in the proof of inequality (3.17), we see that $|\overline{\overline{Y}}_S| \leq E[X|\mathcal{F}_S]$, where X is the random variable defined in (3.18). This observation, together with Corollary A.1, gives $E\left[\left(\overline{\overline{A}}_T\right)^2\right] \leq cE[X^2]$, where $c > 0$. By combining this inequality with inequality (3.19) , we obtain

$$E\left[\left(\overline{\overline{A}}_T\right)^2\right] \leq cT\|f\|^2_{I\!H^2} + c|||\xi|||^2_{\mathcal{S}^2}, \tag{3.23}$$

where we have again allowed the positive constant c to vary from line to line. We conlude that $\overline{\overline{A}}_T \in L^2$, which, due to the nondecreasingness of $\overline{\overline{A}}$, is equivalent to $\overline{\overline{A}} \in \mathcal{S}^2$.

Step 4. In this step we show that the process Z is in $I\!H^2$. We have

$$\int_0^T Z_s dW_s = \overline{Y}_T + \int_0^T f(s)ds + \overline{\overline{A}}_T - \overline{Y}_0,$$

where $\overline{\overline{A}}$ is the process from Step 3. Since $\overline{\overline{A}}_T \in L^2$, $\overline{Y}_T \in L^2$, $\overline{Y}_0 \in L^2$ and $f \in I\!H^2$, it follows that $\int_0^T Z_s dW_s \in L^2$. Hence, $Z \in I\!H^2$.

Step 5. Let us now prove the uniqueness of the solution. Let (Y, Z, A, C) be a solution of the RBSDE with driver f and obstacle ξ. Then, by the previous Lemma 3.2 (applied with $f^1 = f^2 = f$) we obtain $Y = \overline{Y}$ in $\mathcal{S}^2$, where $\overline{Y}$ is given by (3.16). The uniqueness of Z, A, C follows from the uniqueness of Mertens decomposition of strong optional supermartingales and from the uniqueness of the martingale representation. (We note that the uniqueness of Z can be obtained also by applying the previous Lemma 3.2.) □

In the following theorem we prove existence and uniqueness of the solution to the RBSDE from Definition 3.1 in the case of a general Lipschitz driver f by using a fixed-point theorem. The following remark will be used in the proof.

Remark 3.7 *Let $\beta > 0$. For $\phi \in \mathcal{S}^2$, we have $E[\int_0^T \mathrm{e}^{\beta t} |\phi_t|^2 dt] \leq TE[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau} |\phi_\tau|^2]$. Indeed, by applying Fubini's theorem, we get*

$$E[\int_0^T \mathrm{e}^{\beta t} |\phi_t|^2 dt] = \int_0^T E[\mathrm{e}^{\beta t} |\phi_t|^2] dt \leq \int_0^T E[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau} |\phi_\tau|^2] dt = TE[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau} |\phi_\tau|^2]. \quad (3.24)$$

Theorem 3.4 *Let ξ be a left-limited and r.u.s.c. process in $\mathcal{S}^2$ and let f be a Lipschitz driver. The RBSDE with parameters (f, ξ) from Definition 3.1 admits a unique solution $(Y, Z, A, C) \in \mathcal{S}^2 \times I\!H^2 \times \mathcal{S}^2 \times \mathcal{S}^2$.*
Moreover, if (ξ_t) is assumed l.u.s.c. along stopping times, then (A_t) is continuous.

Proof:

We denote by $\mathcal{B}^2_\beta$ the space $\mathcal{S}^2 \times I\!H^2$ which we equip with the norm $\|(\cdot,\cdot)\|_{\mathcal{B}^2_\beta}$ defined by $\|(Y,Z)\|^2_{\mathcal{B}^2_\beta} := |||Y|||^2_\beta + \|Z\|^2_\beta$, for all $(Y, Z) \in \mathcal{S}^2 \times I\!H^2$. We define a mapping Φ from $\mathcal{B}^2_\beta$ into itself as follows: for a given $(y, z) \in \mathcal{B}^2_\beta$, we let $(Y, Z) = \Phi(y, z)$ be the first two components of the solution to the RBSDE associated with driver $f(s) := f(s, y_s, z_s)$ and with obstacle ξ. Let (A, C) be the associated Mertens process, constructed as in Lemma 3.3. The mapping Φ is well-defined by Lemma 3.3.

Let (y, z) and (y', z') be two elements of $\mathcal{B}^2_\beta$. We set $(Y, Z) = \Phi(y, z)$ and $(Y', Z') = \Phi(y', z')$. We also set $\tilde{Y} := Y - Y'$, $\tilde{Z} := Z - Z'$, $\tilde{y} := y - y'$ and $\tilde{z} := z - z'$.

Let us prove that for a suitable choice of the parameter $\beta > 0$ the mapping Φ is a contraction from the Banach space $\mathcal{B}^2_\beta$ into itself.

By applying Lemma 3.2, we get

$$|||\tilde{Y}|||_\beta^2 + \|\tilde{Z}\|_\beta^2 \leq \varepsilon^2(3+4c^2)\|f(y,z) - f(y',z')\|_\beta^2,$$

for all $\varepsilon > 0$, for all $\beta \geq \frac{1}{\varepsilon^2}$. By using the Lipschitz property of f and the fact that $(a+b)^2 \leq 2a^2+2b^2$, for all $a \in \mathbf{R}$, for all $b \in \mathbf{R}$, we obtain $\|f(y,z)-f(y',z')\|_\beta^2 \leq C_K(\|\tilde{y}\|_\beta^2 + \|\tilde{z}\|_\beta^2)$, where C_K is a positive constant depending on the Lipschitz constant K only. Thus, for all $\varepsilon > 0$, for all $\beta \geq \frac{1}{\varepsilon^2}$, we have

$$|||\tilde{Y}|||_\beta^2 + \|\tilde{Z}\|_\beta^2 \leq \varepsilon^2 C_K(3+4c^2)(\|\tilde{y}\|_\beta^2 + \|\tilde{z}\|_\beta^2).$$

The previous inequality, combined with Remark 3.7, gives

$$|||\tilde{Y}|||_\beta^2 + \|\tilde{Z}\|_\beta^2 \leq \varepsilon^2 C_K(3+4c^2)(T+1)(|||\tilde{y}|||_\beta^2 + \|\tilde{z}\|_\beta^2).$$

Thus, for $\varepsilon > 0$ such that $\varepsilon^2 C_K(3+4c^2)(T+1) < 1$ and $\beta > 0$ such that $\beta \geq \frac{1}{\varepsilon^2}$ the mapping Φ is a contraction. By the Banach fixed-point theorem, we get that Φ has a unique fixed point in $\mathcal{B}_\beta^2$. We thus have the existence and uniqueness of the solution to the RBSDE. □

4 Optimal stopping with f-conditional expectations

4.1 Formulation of the problem

Let $T > 0$ be the terminal time and f be a Lipschitz driver (as before). Let $(\xi_t, 0 \leq t \leq T)$ be a left-limited r.u.s.c. process in $\mathcal{S}^2$ modelling a dynamic financial position. The risk of ξ is assessed by a dynamic risk measure equal, up to a minus sign, to the f-conditional expectation of ξ. More precisely: let $T' \in [0,T]$ be a fixed (for the present) instant before the terminal time T; the gain of the position at T' is equal to $\xi_{T'}$ and the risk at time t, where t runs through the interval $[0,T']$, is assessed by $-\mathcal{E}^f_{t,T'}(\xi_{T'})$. The modelling is similar when $T' \in [0,T]$ is replaced by a more general stopping time $\tau \in \mathcal{T}_{0,T}$. We are interested in stopping the process ξ in such a way that the risk be minimal. We are thus led to formulating the following optimal stopping problem (at the initial time 0):

$$v(0) = -\text{ess}\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^f_{0,\tau}(\xi_\tau). \tag{4.25}$$

As is usual in optimal control, we embed the above problem in a larger class of problems. We thus consider

$$v(S) = -\text{ess}\sup_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau), \tag{4.26}$$

where $S \in \mathcal{T}_{0,T}$. The random variable $v(S)$ corresponds to the minimal risk measure at time S. Our aim is to characterize $v(S)$ for each $S \in \mathcal{T}_{0,T}$, and to study the existence of an S-optimal stopping time $\tau^* \in \mathcal{T}_{S,T}$, i.e. a stopping time $\tau^* \in \mathcal{T}_{S,T}$ such that $v(S) = -\mathcal{E}^f_{S,\tau^*}(\xi_{\tau^*})$ a.s.

4.2 Characterization of the value function as the solution of an RBSDE

In this section, we show that the minimal risk measure v defined by (4.26) coincides with $-Y$, where Y is (the first component of) the solution to the reflected BSDE associated with driver f and obstacle ξ. We also investigate the question of the existence of an ε-optimal stopping time, and that of the existence of an optimal stopping time (under suitable assumptions on the process ξ).

The following terminology will be used in the sequel.

Let Y be a process in $\mathcal{S}^2$. Let f be a Lipschitz driver.

- The process (Y_t) is said to be a strong $\mathcal{E}^f$-supermartingale (resp $\mathcal{E}^f$-submartingale), if $\mathcal{E}^f_{S,\tau}(Y_\tau) \leq Y_S$ (resp. $\mathcal{E}^f_{S,\tau}(Y_\tau) \geq Y_S$) a.s. on $S \leq \tau$, for all $S, \tau \in \mathcal{T}_{0,T}$.
  The process (Y_t) is said to be a strong $\mathcal{E}^f$-martingale if it is both a strong $\mathcal{E}^f$-super and submartingale.

- Let $S, \tau \in \mathcal{T}_{0,T}$ be such that $S \leq \tau$ a.s. The process Y is said to be a strong $\mathcal{E}^f$-supermartingale (resp. a strong $\mathcal{E}^f$-submartingale) on $[S, \tau]$ if for all $\sigma, \mu \in \mathcal{T}_{0,T}$ such that $S \leq \sigma \leq \mu \leq \tau$ a.s., we have $Y_\sigma \geq \mathcal{E}^f_{\sigma,\mu}(Y_\mu)$ a.s. (resp. $Y_\sigma \leq \mathcal{E}^f_{\sigma,\mu}(Y_\mu)$ a.s.) We say that Y is a strong $\mathcal{E}^f$-martingale on $[S, \tau]$ if it is both a strong $\mathcal{E}^f$-super and submartingale on $[S, \tau]$.

Remark 4.8 *We note that a process $Y \in \mathcal{S}^2$ is a strong $\mathcal{E}^f$-martingale on $[S, \tau]$ (where S, $\tau \in \mathcal{T}_{0,T}$ are such that $S \leq \tau$ a.s.) if and only if Y is the solution on $[S, \tau]$ to the BSDE associated with driver f, terminal time τ and terminal condition Y_τ* [1].
It follows that for a process $Y \in \mathcal{S}^2$ to be a strong $\mathcal{E}^f$-martingale on $[S, \tau]$, it is sufficient to have: $Y_\sigma = \mathcal{E}^f_{\sigma,\tau}(Y_\tau)$ a.s., for all $\sigma \in \mathcal{T}_{0,T}$ such that $S \leq \sigma \leq \tau$ a.s.

Property 4.1 *Let f be a Lipschitz driver. Let $S, \tau \in \mathcal{T}_{0,T}$ with $S \leq \tau$ a.s. Let Y be a strong $\mathcal{E}^f$-supermartingale on $[S, \tau]$. The following two assertions are equivalent:*

(i) *The process Y is a strong $\mathcal{E}^f$-martingale on $[S, \tau]$.*

(ii) *$Y_S = \mathcal{E}^f_{S,\tau}(Y_\tau)$ a.s.*

Proof: The implication $(i) \Rightarrow (ii)$ is due to the definition. Let us show the converse implication. Let $\sigma \in \mathcal{T}_{0,T}$ be such that $S \leq \sigma \leq \tau$ a.s. By using (ii) and the consistency property of f-expectations, we obtain

$$Y_S = \mathcal{E}^f_{S,\sigma}\big(\mathcal{E}^f_{\sigma,\tau}(Y_\tau)\big) \text{ a.s.}$$

By using the strong $\mathcal{E}^f$-supermartingale property of Y and the monotonicity of f-expectations, we obtain

$$\mathcal{E}^f_{S,\sigma}\big(\mathcal{E}^f_{\sigma,\tau}\big(Y_\tau\big)\big) \leq \mathcal{E}^f_{S,\sigma}\big(Y_\sigma\big) \leq Y_S \text{ a.s.}$$

[1] We say that the process Y is the solution on $[S, \tau]$ to the BSDE associated with driver f, terminal time τ and terminal condition ζ (where ζ is an $\mathcal{F}_\tau$-measurable square-integrable random variable) if for almost all $\omega \in \Omega$, for all $t \in [0, T]$ such that $S(\omega) \leq t \leq \tau(\omega)$, $Y_t(\omega) = \bar{Y}_t(\omega)$, where $\bar{Y}$ denotes the solution to the BSDE associated with driver f, terminal time τ and terminal condition ζ.

From the previous two equations we get

$$Y_S = \mathcal{E}^f_{S,\sigma}\big(\mathcal{E}^f_{\sigma,\tau}\big(Y_\tau\big)\big) = \mathcal{E}^f_{S,\sigma}\big(Y_\sigma\big) \text{ a.s.}$$

In particular, $\mathcal{E}^f_{S,\sigma}\big(Y_\sigma\big) = \mathcal{E}^f_{S,\sigma}\big(\mathcal{E}^f_{\sigma,\tau}\big(Y_\tau\big)\big)$ a.s. We combine this equality with the inequality $Y_\sigma \geq \mathcal{E}^f_{\sigma,\tau}\big(Y_\tau\big)$ a.s., and we apply the strict comparison theorem for BSDEs to get $Y_\sigma = \mathcal{E}^f_{\sigma,\tau}\big(Y_\tau\big)$ a.s. Hence, the process Y is a strong $\mathcal{E}^f$-martingale on $[S,\tau]$. □

We next show a lemma which will be used in the proof of the main result of this section.

Lemma 4.1 *Let f be a Lipschitz driver and ξ be a left-limited r.u.s.c. process in $\mathcal{S}^2$. Let (Y,Z,A,C) be the solution to the reflected BSDE with parameters (f,ξ) as in Definition 3.1. Let $\varepsilon > 0$ and $S \in \mathcal{T}_{0,T}$. Let τ^ε_S be defined by*

$$\tau^\varepsilon_S := \inf\{t \geq S \colon Y_t \leq \xi_t + \varepsilon\}. \tag{4.27}$$

The following two statements hold:

(i) $Y_{\tau^\varepsilon_S} \leq \xi_{\tau^\varepsilon_S} + \varepsilon$ a.s.

(ii) *The process Y is a strong $\mathcal{E}^f$-martingale on $[S,\tau^\varepsilon_S]$.*

We note that τ^ε_S defined in (4.27) is a stopping time as the *début* after S of a progressive set.

Proof: We first prove statement (i). By way of contradiction, we suppose $P(Y_{\tau^\varepsilon_S} > \xi_{\tau^\varepsilon_S} + \varepsilon) > 0$. We have $\Delta C_{\tau^\varepsilon_S} = C_{\tau^\varepsilon_S} - C_{(\tau^\varepsilon_S)-} = 0$ on the set $\{Y_{\tau^\varepsilon_S} > \xi_{\tau^\varepsilon_S} + \varepsilon\}$. On the other hand, due to Remark 3.4, $\Delta C_{\tau^\varepsilon_S} = Y_{\tau^\varepsilon_S} - Y_{(\tau^\varepsilon_S)+}$. Thus, $Y_{\tau^\varepsilon_S} = Y_{(\tau^\varepsilon_S)+}$ on the set $\{Y_{\tau^\varepsilon_S} > \xi_{\tau^\varepsilon_S} + \varepsilon\}$. Hence,

$$Y_{(\tau^\varepsilon_S)+} > \xi_{\tau^\varepsilon_S} + \varepsilon \text{ on the set } \{Y_{\tau^\varepsilon_S} > \xi_{\tau^\varepsilon_S} + \varepsilon\}. \tag{4.28}$$

We will obtain a contradiction with this statement. Let us fix $\omega \in \Omega$. By definition of $\tau^\varepsilon_S(\omega)$, there exists a non-increasing sequence $(t_n) = (t_n(\omega)) \downarrow \tau^\varepsilon_S(\omega)$ such that $Y_{t_n}(\omega) \leq \xi_{t_n}(\omega) + \varepsilon$, for all $n \in I\!N$. Hence, $\limsup_{n\to\infty} Y_{t_n}(\omega) \leq \limsup_{n\to\infty} \xi_{t_n}(\omega) + \varepsilon$. As the process ξ is r.u.s.c. by assumption, we have $\limsup_{n\to\infty} \xi_{t_n}(\omega) \leq \xi_{\tau^\varepsilon_S}(\omega)$. On the other hand, as $(t_n(\omega)) \downarrow \tau^\varepsilon_S(\omega)$, we have $\limsup_{n\to\infty} Y_{t_n}(\omega) = Y_{(\tau^\varepsilon_S)+}(\omega)$. Thus, $Y_{(\tau^\varepsilon_S)+}(\omega) \leq \xi_{\tau^\varepsilon_S}(\omega) + \varepsilon$, which is in contradiction with (4.28). We conclude that $Y_{\tau^\varepsilon_S} \leq \xi_{\tau^\varepsilon_S} + \varepsilon$ a.s.
Let us now prove statement (ii). By definition of τ^ε_S, we have: for a.e. $\omega \in \Omega$, for all $t \in [S(\omega), \tau^\varepsilon_S(\omega)[$, $Y_t(\omega) > \xi_t(\omega) + \varepsilon$. Hence, for a.e. $\omega \in \Omega$, the function $t \mapsto A^c_t(\omega)$ is constant on $[S(\omega), \tau^\varepsilon_S(\omega)[$; by continuity of almost every trajectory of the process A^c, $A^c_\cdot(\omega)$ is constant on the closed interval $[S(\omega), \tau^\varepsilon_S(\omega)]$, for a.e. ω. Furthermore, for a.e. $\omega \in \Omega$, the function $t \mapsto A^d_t(\omega)$ is constant on $[S(\omega), \tau^\varepsilon_S(\omega)[$. Moreover, $Y_{(\tau^\varepsilon_S)^-} \geq \xi_{(\tau^\varepsilon_S)^-} + \varepsilon$ a.s., which implies that $\Delta A^d_{\tau^\varepsilon_S} = 0$ a.s. Finally, for a.e. $\omega \in \Omega$, for all $t \in [S(\omega), \tau^\varepsilon_S(\omega)[$, $\Delta C_t(\omega) = C_t(\omega) - C_{t-}(\omega) = 0$; therefore, for a.e. $\omega \in \Omega$, for all $t \in [S(\omega), \tau^\varepsilon_S(\omega)[$, $\Delta_+ C_{t-}(\omega) = C_t(\omega) - C_{t-}(\omega) = 0$, which implies that, for a.e. $\omega \in \Omega$, the function $t \mapsto C_{t-}(\omega)$ is constant on $[S(\omega), \tau^\varepsilon_S(\omega)[$. By left-continuity of almost every trajectory of the process (C_{t-}), we get that for a.e. $\omega \in \Omega$, the function $t \mapsto C_{t-}(\omega)$ is constant on the closed interval $[S(\omega), \tau^\varepsilon_S(\omega)]$. Thus, for a.e. $\omega \in \Omega$, the map $t \mapsto A_t(\omega) + C_{t^-}(\omega)$ is constant on $[S(\omega), \tau^\varepsilon_S(\omega)]$. Hence,

Y is the solution on $[S, \tau_S^\varepsilon]$ of the BSDE associated with driver f, terminal time τ_S^ε and terminal condition $Y_{\tau_S^\varepsilon}$. We conclude by using Remark 4.8. □

We now state the main result of this section.

Theorem 4.2 (Characterization theorem) *Let $T > 0$ be the terminal time. Let $(\xi_t, 0 \leq t \leq T)$ be a left-limited r.u.s.c. process in $\mathcal{S}^2$ and let f be a Lipschitz driver. Let (Y, Z, A, C) be the solution to the reflected BSDE with parameters (f, ξ) as in Definition 3.1.*

(i) *For each stopping time $S \in \mathcal{T}_0$, we have*

$$Y_S = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) \quad \text{a.s.} \tag{4.29}$$

(ii) *For each $S \in \mathcal{T}_{0,T}$ and each $\varepsilon > 0$, the stopping time τ_S^ε defined by (4.27) is ($L\varepsilon$)-optimal for problem (4.29), that is*

$$Y_S \;\leq\; \mathcal{E}^f_{S,\tau_S^\varepsilon}(\xi_{\tau_S^\varepsilon}) + L\varepsilon \quad \text{a.s.}\,, \tag{4.30}$$

where L is a constant which only depends on T and the Lipschitz constant K of f.

Remark 4.9 *This result still holds when the assumption of existence of left limits for the process ξ is relaxed (cf. also Remark 3.2).*

Proof:

Let $\varepsilon > 0$ and let $\tau \in \mathcal{T}_{S,T}$. By Proposition A.5 in the Appendix, the process (Y_t) is a strong $\mathcal{E}^f$-supermartingale. Hence, for each $\tau \in \mathcal{T}_{S,T}$, we have

$$Y_S \geq \mathcal{E}^f_{S,\tau}(Y_\tau) \geq \mathcal{E}^f_{S,\tau}(\xi_\tau) \quad a.s.\,,$$

where the second inequality follows from the inequality $Y \geq \xi$ and the monotonicity property of $\mathcal{E}^f(\cdot)$ (with respect to terminal condition). By taking the supremum over $\tau \in \mathcal{T}_{S,T}$, we get

$$Y_S \geq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) \quad a.s. \tag{4.31}$$

It remains to show the converse inequality.

Due to part (ii) of the previous Lemma 4.1 we have $Y_S = \mathcal{E}^f_{S,\tau_S^\varepsilon}(Y_{\tau_S^\varepsilon})$ a.s. From this equality, together with part (i) of Lemma 4.1 and the monotonicity property of $\mathcal{E}^f(\cdot)$, we derive

$$Y_S = \mathcal{E}^f_{S,\tau_S^\varepsilon}(Y_{\tau_S^\varepsilon}) \leq \mathcal{E}^f_{S,\tau_S^\varepsilon}(\xi_{\tau_S^\varepsilon} + \varepsilon) \leq \mathcal{E}^f_{S,\tau_S^\varepsilon}(\xi_{\tau_S^\varepsilon}) + L\varepsilon \quad \text{a.s.}, \tag{4.32}$$

where the last inequality follows from the estimates on BSDEs (see Proposition A.4 [26]). Inequality (4.30) thus holds. From (4.32) we also deduce

$$Y_S \;\leq\; \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) + L\varepsilon \quad \text{a.s.}$$

As ε is an arbitrary positive number, we get $Y_S \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s. By (4.31) this inequality is an equality.

□

We now consider the question of the existence of optimal stopping times for the optimal stopping problem (4.29). We first provide an optimality criterion for the problem (4.29).

Proposition 4.3 (Optimality criterion) *Let $(\xi_t, 0 \leq t \leq T)$ be a left-limited r.u.s.c. process in $\mathcal{S}^2$ and let f be a Lipschitz driver. Let $S \in \mathcal{T}_{0,T}$ and $\hat{\tau} \in \mathcal{T}_{S,T}$. The stopping time $\hat{\tau}$ is S-optimal, i.e. $Y_S = \mathcal{E}^f_{S,\hat{\tau}}(\xi_{\hat{\tau}})$ a.s., if and only if Y is a strong $\mathcal{E}^f$-martingale on $[S, \hat{\tau}]$ with $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ a.s.*

Proof: Let us prove the "only if" part. Suppose that $\hat{\tau}$ is S-optimal, i.e. $Y_S = \mathcal{E}^f_{S,\hat{\tau}}(\xi_{\hat{\tau}})$ a.s. Since by Theorem 4.2 and by Proposition A.5 in the Appendix, Y is a strong $\mathcal{E}^f$-supermartingale, we have

$$Y_S \geq \mathcal{E}^f_{S,\hat{\tau}}(Y_{\hat{\tau}}) \geq \mathcal{E}^f_{S,\hat{\tau}}(\xi_{\hat{\tau}}) \quad a.s.\,,$$

where the last inequality holds because $Y \geq \xi$. It follows that $Y_S = \mathcal{E}^f_{S,\hat{\tau}}(Y_{\hat{\tau}})$ a.s. By Property 4.1, Y is a strong $\mathcal{E}^f$-martingale on $[S, \hat{\tau}]$. Moreover, since $\mathcal{E}^f_{S,\hat{\tau}}(Y_{\hat{\tau}}) = \mathcal{E}^f_{S,\hat{\tau}}(\xi_{\hat{\tau}})$ a.s. with $Y_{\hat{\tau}} \geq \xi_{\hat{\tau}}$ a.s., the strict comparison theorem for BSDEs implies that $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ a.s.
The "if part" is immediate. □

Under an additional assumption of left upper-semicontinuity on the process ξ, we prove that the first time when the value process Y "hits" ξ is optimal. To this purpose, we first give a lemma which is to be compared with Lemma 4.1.

Lemma 4.2 *Let f be a Lipschitz driver. Let $(\xi_t, 0 \leq t \leq T)$ be a left-limited r.u.s.c. and l.u.s.c. process in $\mathcal{S}^2$. Let (Y, Z, A, C) be the solution to the reflected BSDE with parameters (f, ξ). We define τ^*_S by*

$$\tau^*_S := \inf\{u \geq S\colon Y_u = \xi_u\}. \tag{4.33}$$

The following assertions hold:

(i) *$Y_{\tau^*_S} = \xi_{\tau^*_S}$ a.s.*

(ii) *The process Y is a strong $\mathcal{E}^f$-martingale on $[S, \tau^*_S]$.*

Proof: To prove the first statement we note that $Y_{\tau^*_S} \geq \xi_{\tau^*_S}$ a.s., since Y is (the first component of) the solution to the RBSDE with barrier ξ. We show that $Y_{\tau^*_S} \leq \xi_{\tau^*_S}$ a.s. by using the assumption of right-upper semicontinuity on the process ξ; the arguments are similar to those used in the proof of part (i) of Lemma 4.1 and are left to the reader.
Let us prove the second statement. By definition of τ^*_S, we have that for a.e. $\omega \in \Omega$, $Y_t(\omega) > \xi_t(\omega)$ on $[S(\omega), \tau^*_S(\omega)[$; hence, for a.e. ω, the trajectory $A^c(\omega)$ is constant on $[S(\omega), \tau^*_S(\omega)[$ and even on the closed interval $[S(\omega), \tau^*_S(\omega)]$ due to the continuity. On the other hand, due to the assumption of

l.u.s.c. on the process ξ, we have $A(\omega) = A^c(\omega)$ for a.e. ω (see Theorem 3.4). Thus, for a.e. ω, $A(\omega)$ is constant on $[S(\omega), \tau_S^*(\omega)]$. We show that $C_{t^-}(\omega)$ is constant on $[S(\omega), \tau_S^*(\omega)]$ by the same arguments as those of the proof of part (ii) of Lemma 4.1. We conclude by using Remark 4.8. □

Proposition 4.2 *Let f be a Lipschitz driver. Let $(\xi_t, 0 \leq t \leq T)$ be a left-limited r.u.s.c. and l.u.s.c. process in $\mathcal{S}^2$. The stopping time τ_S^* defined in* (4.33) *is optimal for problem* (4.29)*, that is $Y_S = \mathcal{E}^f_{S,\tau_S^*}(\xi_{\tau_S^*})$ a.s.*

Proof: The result is a consequence of the previous Lemma 4.2 and the "if part" of the optimality criterion from Proposition 4.3. □

Definition 4.4 *Let $(\xi_t, 0 \leq t \leq T)$ be a process in $\mathcal{S}^2$ and let f be a Lipschitz driver. A process $(Y_t, 0 \leq t \leq T)$ in $\mathcal{S}^2$ is said to be the $\mathcal{E}^f$-* Snell envelope *of ξ if it is the smallest strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ.*

We show the following property:

Proposition 4.5 *Let $T > 0$ be the terminal time. Let $(\xi_t, 0 \leq t \leq T)$ be a left-limited r.u.s.c. process in $\mathcal{S}^2$ and let f be a Lipschitz driver. Let (Y, Z, A, C) be the solution to the reflected BSDE with parameters (ξ, f) as in Definition 3.1. The process Y is the $\mathcal{E}^f$-Snell envelope of ξ.*

Proof: By Proposition A.5 in the Appendix, the process Y is a strong $\mathcal{E}^f$-supermartingale. Moreover, since Y is (the first component of) the solution to the reflected BSDE with parameters (f, ξ), it is greater than or equal to ξ (cf. Def. 3.1).
It thus remains to show the minimality property. Let Y' be another $\mathcal{E}^f$-supermartingale greater than or equal to ξ. Let $S \in \mathcal{T}_{0,T}$. For each $\tau \in \mathcal{T}_{S,T}$, we have

$$Y'_S \geq \mathcal{E}^f_{S,\tau}(Y'_\tau) \geq \mathcal{E}^f_{S,\tau}(\xi_\tau) \quad a.s.\,,$$

where the second inequality follows from the inequality $Y' \geq \xi$ and the monotonicity property of $\mathcal{E}^f$ with respect to the terminal condition. By taking the supremum over $\tau \in \mathcal{T}_{S,T}$, we get

$$Y'_S \geq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) = Y_S \quad a.s., \tag{4.34}$$

where the last equality follows from the above characterization theorem (Theorem 4.2). The desired result follows. □

5 Additional results

5.1 $\mathcal{E}^f$-Mertens decomposition of $\mathcal{E}^f$-strong supermartingales

We now show an $\mathcal{E}^f$-Mertens decomposition for $\mathcal{E}^f$-strong supermartingales, which generalizes Mertens decomposition to the case of f-expectations.
We first show the following lemma.

Lemma 5.1 *Let $(Y_t) \in \mathcal{S}^2$ be a strong $\mathcal{E}^f$-supermartingale (resp. $\mathcal{E}^f$-submartingale). Then, (Y_t) is right upper-semicontinuous (resp. right lower-semicontinuous).*

Proof: Suppose that (Y_t) is a strong $\mathcal{E}^f$-supermartingale. Let $\tau \in \mathcal{T}_{0,T}$ and let (τ_n) be a nonincreasing sequence of stopping times such that $\lim_{n\to+\infty} \tau_n = \tau$ a.s. and for all $n \in I\!N$, $\tau_n > \tau$ a.s. on $\{\tau < T\}$. Suppose that $\lim_{n\to+\infty} Y_{\tau_n}$ exists a.s. We note that the random variable $\lim_{n\to+\infty} Y_{\tau_n}$ is $\mathcal{F}_\tau$-measurable (as the filtration is right-continuous). Let us show that

$$Y_\tau \geq \lim_{n\to+\infty} Y_{\tau_n} \quad \text{a.s.}$$

Since (Y_t) is a strong $\mathcal{E}^f$-supermartingale and the sequence (τ_n) is nonincreasing, we have, for all $n \in I\!N$, $\mathcal{E}^f_{\tau,\tau_n}(Y_{\tau_n}) \leq \mathcal{E}^f_{\tau,\tau_{n+1}}(Y_{\tau_{n+1}}) \leq Y_\tau$. We deduce that the sequence of random variables $(\mathcal{E}^f_{\tau,\tau_n}(Y_{\tau_n}))_{n\in I\!N}$ is nondecreasing (hence, converges a.s.) and its limit (in the a.s. sense) satisfies $Y_\tau \geq \lim_{n\to+\infty} \uparrow \mathcal{E}^f_{\tau,\tau_n}(Y_{\tau_n})$ a.s. This observation, combined with the continuity property of BSDEs with respect to terminal time and terminal condition (cf. [26, Prop. A.6]) gives

$$Y_\tau \geq \lim_{n\to+\infty} \mathcal{E}^f_{\tau,\tau_n}(Y_{\tau_n}) = \mathcal{E}^f_{\tau,\tau}(\lim_{n\to+\infty} Y_{\tau_n}) = \lim_{n\to+\infty} Y_{\tau_n} \quad \text{a.s.}$$

This result, together with a result of the general theory of processes (cf. [5, Prop. 2, page 300]), ensures that the optional process (Y_t) is right-upper semicontinuous.

□

Theorem 5.2 ($\mathcal{E}^f$-Mertens decomposition) *Let (Y_t) be a process in $\mathcal{S}^2$. Let f be a Lipschitz driver.*
The process (Y_t) is a strong $\mathcal{E}^f$-supermartingale (resp. $\mathcal{E}^f$-submartingale) if and only if there exists a nondecreasing (resp. non increasing) right-continuous predictable process A in $\mathcal{S}^2$ with $A_0 = 0$ and a nondecreasing (resp. nonincreasing) right-continuous adapted purely discontinuous process C in $\mathcal{S}^2$ with $C_{0-} = 0$, as well as a predictable process $Z \in I\!H^2$, such that

$$-dY_t = f(t, Y_t, Z_t)dt + dA_t + dC_{t^-} - Z_t dW_t.$$

Proof: The "if part" has been shown in Proposition A.5 of the Appendix. Let us show the "only if" part. Suppose that (Y_t) is a strong $\mathcal{E}^f$-supermartingale. Let $S \in \mathcal{T}_{0,T}$. Since (Y_t) is a strong $\mathcal{E}^f$-supermartingale, it follows that for each $\tau \in \mathcal{T}_{S,T}$, we have $Y_S \geq \mathcal{E}^f_{S,\tau}(Y_\tau)$ a.s. We thus get

$$Y_S \geq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(Y_\tau) \quad a.s.$$

Now, by definition of the essential supremum, $Y_S \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(Y_\tau)$ a.s. (since $S \in \mathcal{T}_{S,T}$). The previous two inequalities imply

$$Y_S = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(Y_\tau) \quad a.s.$$

By our characterization theorem (Theorem 4.2) combined with Remark 4.9, the process (Y_t) coincides with the solution of the reflected BSDE associated with the obstacle (Y_t) (which is r.u.s.c. by Lemma 5.1). The result follows. □

Remark 5.10 *It follows from the previous theorem that strong $\mathcal{E}^f$-supermartingales and strong $\mathcal{E}^f$-submartingales have left and right limits.*

5.2 Comparison theorem for RBSDEs

Theorem 5.3 (Comparison) *Let ξ^1, ξ^2 be square-integrable barriers. Let f^1 and f^2 be Lipschitz drivers. Suppose that*

- $\xi_t^2 \leq \xi_t^1$, $0 \leq t \leq T$ *a.s.*
- $f^2(t,y,z) \leq f^1(t,y,z)$, *for all* $(y,z) \in \mathbf{R}^2$ $dP \otimes dt - a.s.$

Let (Y^i, Z^i, A^i, C^i) be the solution of the RBSDE associated with (ξ^i, f^i) , $i = 1, 2$. Then,

$$Y_t^2 \leq Y_t^1, \ \forall t \in [0,T] \ a.s.$$

Proof: Let $S \in \mathcal{T}_{0,T}$. For $i = 1, 2$ and for each $\tau \in \mathcal{T}_{S,T}$, let us denote by $\mathcal{E}^{f^i}$ the conditional (non-linear) expectation associated with driver f^i. By the comparison theorem for BSDEs, for each τ in $\mathcal{T}_{S,T}$ we have

$$\mathcal{E}^{f^2}_{S,\tau}(\xi_\tau^2) \leq \mathcal{E}^{f^1}_{S,\tau}(\xi_\tau^1) \ a.s.$$

By taking the essential supremum over τ in $\mathcal{T}_{S,T}$ and by using Theorem 4.2, we get

$$Y_S^2 = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^{f^2}_{S,\tau}(\xi_\tau^2) \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^{f^1}_{S,\tau}(\xi_\tau^1) = Y_S^1 \quad \text{a.s.}$$

Since this inequality holds for each $S \in \mathcal{T}_{0,T}$, the result follows. □

6 Further extensions

The previous results can be extended to a framework where, in addition to the Brownian motion W, there is also an independent Poisson random measure. More precisely, we place ourselves in the following framework: Let $(U, \mathcal{U})$ be a measurable space equipped with a σ-finite positive measure ν. Let $N(dt, du)$ be a Poisson random measure with compensator $dt \otimes \nu(du)$. Let $\tilde{N}(dt, du)$ be its compensated process. The filtration $I\!F = \{\mathcal{F}_t : t \in [0,T]\}$ corresponds here to the (complete) natural filtration associated with W and N.
We denote by L^2_ν the set of measurable functions $\ell : U \to \mathbf{R}$ such that $\|\ell\|^2_\nu := \int_U |\ell(u)|^2 \nu(du) < +\infty$. Let $I\!H^2_\nu$ be the set of processes l which are *predictable*, that is, measurable
$l : (\Omega \times [0,T] \times U, \ \mathcal{P} \otimes \mathcal{U}) \to (\mathbf{R}\ , \mathcal{B}(\mathbf{R}))$; $\quad (\omega, t, u) \mapsto l_t(\omega, u)$ with

$$\|l\|^2_{I\!H^2_\nu} := E\left[\int_0^T \|l_t\|^2_\nu \, dt\right] < \infty.$$

In this framework the notions of driver and Lipschitz driver are defined as follows.

Definition 6.1 (Driver, Lipschitz driver) *A function f is said to be a* driver *if*

- $f : [0,T] \times \Omega \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$
  $(\omega, t, y, z, k) \mapsto f(\omega, t, y, z, k)$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ *measurable,*

- $f(\cdot, 0, 0, 0) \in I\!H^2$.

A driver f is called a Lipschitz driver *if moreover there exists a constant $K \geq 0$ such that $dP \otimes dt$-a.s. , for each (y_1, z_1, k_1), (y_2, z_2, k_2),*

$$|f(\omega, t, y_1, z_1, k_1) - f(\omega, t, y_2, z_2, k_2)| \leq K(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_\nu).$$

The notion of a solution to the reflected BSDE associated with driver f and obstacle ξ is now defined as follows.

Definition 6.2 *A process (Y, Z, k, A, C) is said to be a solution to the reflected BSDE associated with driver f and obstacle ξ if*

$(Y, Z, k(\cdot), A, C) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{S}^2 \times \mathcal{S}^2$

$$Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, k_s)ds + A_T - A_t + C_{T^-} - C_{t^-} - \int_t^T Z_s dW_s - \int_t^T \int_U k_s(u)\tilde{N}(ds, du),$$

where the equality holds (as before) for all $t \in [0,T]$ a.s.,

with $Y \geq \xi$ (up to an evanescent set) and where A and C satisfy the same conditions as in the Brownian case.

It can be shown that the results of Section 3 still hold in this setting.

We recall that in this extended framework the monotonicity property of f-conditional expectations is not automatically satisfied. We are thus led to making the following assumption on the driver f, which ensures the nondecreasing property of $\mathcal{E}^f(\cdot)$ by the comparison theorem for BSDEs with jumps (cf. [26, Thm. 4.2]).

Assumption 6.1 *Assume that $dP \otimes dt$-a.s for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,*

$$f(t, y, z, k_1) - f(t, y, z, k_2) \geq \langle \theta_t^{y,z,k_1,k_2} \, , \, k_1 - k_2 \rangle_\nu,$$

with

$$\theta : [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \to L^2_\nu \, ; \, (\omega, t, y, z, k_1, k_2) \mapsto \theta_t^{y,z,k_1,k_2}(\omega, \cdot)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes d\nu(u)$-a.s. , for each (y, z, k_1, k_2) $\in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\theta_t^{y,z,k_1,k_2}(u) \geq -1 \quad \text{and} \quad |\theta_t^{y,z,k_1,k_2}(u)| \leq \psi(u), \tag{6.35}$$

where $\psi \in L^2_\nu$.

This framework is appropriate for generalizing the results of Section 4.

A Appendix

The following observation is given for the convenience of the reader.

Proposition A.3 *Let Y be a cadlag process. Then, $\sup_{t\in[0,T]} Y_t$ is a random variable and we have*

$$\sup_{t\in[0,T]} Y_t = \operatorname*{ess\,sup}_{t\in[0,T]} Y_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} Y_\tau.$$

The following definition can be found in [7, Appendix 1, Def.1].

Definition A.1 *Let $(Y)_{t\in[0,T]}$ be an optional process. We say that Y is a strong (optional) supermartingale if*

- *Y_τ is integrable for all $\tau \in \mathcal{T}_{0,T}$ and*
- *$Y_S \geq E[Y_\tau \mid \mathcal{F}_S]$ a.s., for all $S, \tau \in \mathcal{T}_{0,T}$ such that $S \leq \tau$ a.s.*

We recall a decomposition of strong optional supermartingales, known as *Mertens decomposition* (cf. [7, Theorem 20, page 429, combined with Remark 3(b), page 205], or [8, Prop. 2.26], or [20, Theorem 4, page 527]).

Theorem A.1 (Mertens decomposition) *Let $\overline{\overline{Y}}$ be a strong optional supermartingale of class (D). There exists a unique uniformly integrable martingale (M_t), a unique predictable right-continuous nondecreasing process (A_t) with $A_0 = 0$ and $E[A_T] < \infty$, and a unique right-continuous adapted nondecreasing process (C_t), which is purely discontinuous, with $C_{0-} = 0$ and $E[C_T] < \infty$, such that*

$$Y_t = M_t - A_t - C_{t-}, \ \ 0 \leq t \leq T \ \text{ a.s.} \tag{A.36}$$

Moreover, the process A is continuous if and only if $\overline{\overline{Y}}$ is left-continuous along stopping times in expectation (i.e. for any non-decreasing sequence of stopping times (S_n), we have $\lim_{n\to\infty} E[\overline{\overline{Y}}_{S_n}] = E[\overline{\overline{Y}}_S]$, where $S := \lim_{n\to\infty} S_n$).

The following remark can be found in [7, Appendix 1, Thm.20, equalities (20.2)].

Remark A.11 *We have $\Delta C_t = Y_t - Y_{t+}$ and $\Delta A_S = Y_{S-} - {}^pY_S$ for all predictable stopping times $S \in \mathcal{T}_{0,T}$, where ${}^pY$ is the predictable projection of Y.*

The following result can be found in [7].

Theorem A.2 (Dellacherie-Meyer) *Let A be a non-decreasing predictable process. Let U be the* potential *of the process A, i.e. $U_t := E[A_T|F_t] - A_t$ for all $t \in [0,T]$. Assume that there exists a non-negative $\mathcal{F}_T$-measurable random variable X such that $U_S \leq E[X|\mathcal{F}_S]$ a.s. for all $S \in \mathcal{T}_{0,T}$. Then,*

$$E[A_T^2] \leq cE[X^2], \tag{A.37}$$

where $c > 0$ is a positive constant.

Proof: For the proof of the result the reader is referred to Paragraph 18 in [7, Appendix 1] generalizing Theorem VI.99 of the same reference to the case of a non-decreasing process which is not necessarily right-continuous nor left-continuous. □

For a given strong optional supermartingale Y, we define $\overline{\overline{A}}$ by $\overline{\overline{A}}_t := A_t + C_{t-}$, where A and C are the two nondecreasing processes of Mertens decomposition of Y from equation (A.36). The process $\overline{\overline{A}}$ is sometimes referred to as the *Mertens process* associated with Y.
By using the previous theorem, we obtain the following integrability property for the Mertens process.

Corollary A.1 *Let Y be a strong optional supermartingale of class (D) such that: for all $S \in \mathcal{T}_{0,T}$, $|Y_S| \leq E[X|\mathcal{F}_S]$ a.s., where X is a non-negative $\mathcal{F}_T$-measurable random variable. Let $\overline{\overline{A}}$ be the Mertens process associated with Y. There exists a positive constant c such that*

$$E\left[\left(\overline{\overline{A}}_T\right)^2\right] \leq cE[X^2], \tag{A.38}$$

Proof: Let $S \in \mathcal{T}_{0,T}$. From Mertens decomposition, we have $Y_S = M_S - \overline{\overline{A}}_S$ a.s. and $Y_T = M_T - \overline{\overline{A}}_T$ a.s. Taking conditional expectations in the second equation gives $E[Y_T|\mathcal{F}_S] = E[M_T|\mathcal{F}_S] - E[\overline{\overline{A}}_T|\mathcal{F}_S]$ a.s. By substracting this equation from the first, we obtain $Y_S - E[Y_T|\mathcal{F}_S] = M_S - E[M_T|\mathcal{F}_S] + E[\overline{\overline{A}}_T|\mathcal{F}_S] - \overline{\overline{A}}_S$ a.s. Thus, $Y_S - E[Y_T|\mathcal{F}_S] = E[\overline{\overline{A}}_T|\mathcal{F}_S] - \overline{\overline{A}}_S$ a.s., where we have used the fact that M is a martingale, together with the optional sampling theorem. From this equality we easily get

$$|E[\overline{\overline{A}}_T|\mathcal{F}_S] - \overline{\overline{A}}_S| = |Y_S - E[Y_T|\mathcal{F}_S]| \leq |Y_S| + E[|Y_T||\mathcal{F}_S] \text{ a.s.} \tag{A.39}$$

By using this observation and the assumption, we get the following upper bound for the potential at time S of the process $\overline{\overline{A}}$:

$$|E[\overline{\overline{A}}_T|\mathcal{F}_S] - \overline{\overline{A}}_S| \leq E[X|\mathcal{F}_S] + E[E[X|\mathcal{F}_T]|\mathcal{F}_S] = E[2X|\mathcal{F}_S] \text{ a.s.}$$

By applying Theorem A.2, we obtain the desired conclusion. □

The following proposition is recalled for the convenience of the reader.

Proposition A.4 *Let X and Y be two optional processes such that $X_S \geq Y_S$ a.s. for all $S \in \mathcal{T}_{0,T}$. Then, $X \geq Y$ up to an evanescent set.*

Proof: The result is a direct consequence of the optional section theorem (cf. [6, Theorem IV.84]). The details of the proof are analogous to those of the proof of Theorem IV.86 of [6] and are left to the reader. □

Let us recall the change of variables formula for optional semimartingales which are not necessarily cad. The result can be seen as a generalization of the classical Itô formula and can be found in [15, Theorem 8.2] (cf. also [20, Chapter VI, Section 3, page 538]). We recall the result in our framework in which the underlying filtered probability space satisfies the usual conditions.

Theorem A.3 (Gal'chouk-Lenglart) *Let $n \in \mathbb{N}$. Let X be an n-dimensional optional semimartingale, i.e. $X = (X^1, \ldots, X^n)$ is an n-dimensional optional process with decomposition $X^k = X_0^k + M^k + A^k + B^k$, for all $k \in \{1, \ldots, n\}$, where M^k is a (cadlag) local martingale, A^k is a right-continuous process of finite variation such that $A_0 = 0$, and B^k is a left-continuous process of finite variation which is purely discontinuous and such that $B_0 = 0$. Let F be a twice continuously differentiable function on $\mathbf{R}^n$. Then, almost surely, for all $t \geq 0$,*

$$\begin{aligned} F(X_t) = F(X_0) &+ \sum_{k=1}^{n} \int_{]0,t]} D^k F(X_{s-}) d(A^k + M^k)_s \\ &+ \frac{1}{2} \sum_{k,l=1}^{n} \int_{]0,t]} D^k D^l F(X_{s-}) d < M^{kc}, M^{lc} >_s \\ &+ \sum_{0<s\leq t} \left[F(X_s) - F(X_{s-}) - \sum_{k=1}^{n} D^k F(X_{s-}) \Delta X_s^k \right] \\ &+ \sum_{k=1}^{n} \int_{[0,t[} D^k F(X_s) d(B^k)_{s+} \\ &+ \sum_{0\leq s<t} \left[F(X_{s+}) - F(X_s) - \sum_{k=1}^{n} D^k F(X_s) \Delta_+ X_s^k \right], \end{aligned}$$

where D^k denotes the differentiation operator with respect to the k-th coordinate, and M^{kc} denotes the continuous part of M^k.

Corollary A.2 *Let Y be a one-dimensional optional semimartingale with decomposition $Y = Y_0 + M + A + B$, where M, A, and B are as in the above theorem. Let $\beta > 0$. Then, almost surely, for all $t \geq 0$,*

$$\begin{aligned} \mathrm{e}^{\beta t} Y_t^2 = Y_0^2 &+ \int_{]0,t]} \beta \, \mathrm{e}^{\beta s} Y_s^2 ds + 2 \int_{]0,t]} \mathrm{e}^{\beta s} Y_{s-} d(A + M)_s \\ &+ \int_{]0,t]} \mathrm{e}^{\beta s} \, d < M^c, M^c >_s \\ &+ \sum_{0<s\leq t} \mathrm{e}^{\beta s} (Y_s - Y_{s-})^2 + \int_{[0,t[} 2 \, \mathrm{e}^{\beta s} Y_s d(B)_{s+} + \sum_{0\leq s<t} \mathrm{e}^{\beta s} (Y_{s+} - Y_s)^2. \end{aligned}$$

Proof: It suffices to apply the change of variables formula from Theorem A.3 with $n = 2$, $F(x, y) = xy^2$, $X_t^1 = \mathrm{e}^{\beta t}$, and $X_t^2 = Y_t$. Indeed, by applying Theorem A.3 and by noting that the local martingale

part and the purely discontinuous part of X^1 are both equal to 0, we obtain

$$\begin{aligned}
\mathrm{e}^{\beta t}\, Y_t^2 = Y_0^2 &+ \int_{]0,t]} \beta\, \mathrm{e}^{\beta s}\, Y_s^2 ds + 2\int_{]0,t]} \mathrm{e}^{\beta s}\, Y_{s-} d(A+M)_s \\
&+ \frac{1}{2}\int_{]0,t]} 2\,\mathrm{e}^{\beta s}\, d < M^c, M^c >_s \\
&+ \sum_{0<s\leq t} \mathrm{e}^{\beta s}\left(Y_s^2 - (Y_{s-})^2 - 2Y_{s-}(Y_s - Y_{s-})\right) \\
&+ \int_{[0,t[} 2\,\mathrm{e}^{\beta s}\, Y_s d(B)_{s+} + \sum_{0\leq s<t} \mathrm{e}^{\beta s}\left((Y_{s+})^2 - (Y_s)^2 - 2Y_s(Y_{s+} - Y_s)\right).
\end{aligned}$$

The desired expression follows as $Y_s^2 - (Y_{s-})^2 - 2Y_{s-}(Y_s - Y_{s-}) = (Y_s - Y_{s-})^2$ and $(Y_{s+})^2 - (Y_s)^2 - 2Y_s(Y_{s+} - Y_s) = (Y_{s+} - Y_s)^2$. □

Proposition A.5 *Let f be a Lipschitz driver. Let A be a nondecreasing (resp. nonincreasing) right-continuous predictable process in $\mathcal{S}^2$ with $A_0 = 0$ and let C be a nondecreasing (resp. nonincreasing) right-continuous adapted purely discontinuous process in $\mathcal{S}^2$ with $C_{0-} = 0$.*

Let $(Y, Z) \in \mathcal{S}^2 \times \mathbb{H}^2$ satisfy

$$-dY_t = f(t, Y_t, Z_t)dt + dA_t + dC_{t-} - Z_t dW_t, \tag{A.40}$$

in the sense that, for each $\tau \in \mathcal{T}_{0,T}$, the equality

$$Y_\tau = Y_T + \int_\tau^T f(s, Y_s, Z_s)ds + A_T - A_\tau + C_{T-} - C_{\tau-} - \int_\tau^T Z_s dW_s$$

holds almost-surely. Then the process (Y_t) is a strong $\mathcal{E}^f$-supermartingale (resp $\mathcal{E}^f$-submartingale).

Proof: We address the case where A and C are nondecreasing. Let $\tau, \theta \in \mathcal{T}_0$ be such that $\tau \leq \theta$ a.s. Let us show that $Y_\tau \geq \mathcal{E}^f_{\tau,\theta}(Y_\theta)$ a.s.
We denote by (X, π) the solution to the BSDE associated with driver f, terminal time θ, and terminal condition Y_θ; then $\mathcal{E}^f_{\tau,\theta}(Y_\theta) = X_\tau$ a.s. (by definition of $\mathcal{E}^f$).

Set $\bar{Y}_t = Y_t - X_t$ and $\bar{Z}_t = Z_t - \pi_t$. Then

$$-d\bar{Y}_t = h_t dt + dA_t + dC_{t-} - \bar{Z}_t dW_t, \quad \bar{Y}_\theta = 0,$$

where $h_t := f(t, Y_{t-}, Z_t) - f(t, X_{t-}, \pi_t)$. By the same arguments as those of the proof of the comparison theorem for BSDEs (cf. Theorem 2.2 in [11]), we have

$$h_t = \delta_t \bar{Y}_{t-} + \beta_t \bar{Z}_t,\ 0 < t \leq T, \quad dP \otimes dt - \text{a.e.}, \tag{A.41}$$

where δ and β are predictable bounded processes (which can be expressed as increment rates of f with respect to y and z).

Let Γ be the unique solution to the following forward SDE

$$d\Gamma_s = \Gamma_s \left[\delta_s ds + \beta_s dW_s\right], \;\; \Gamma_0 = 1. \tag{A.42}$$

Suppose for a while that we have shown

$$\bar{Y}_\tau \Gamma_\tau = E\left[\int_\tau^\theta \Gamma_s \left(dA_s + dC_s\right) \mid \mathcal{F}_\tau\right], \qquad \text{a.s.} \tag{A.43}$$

Then, since $\Gamma_s \geq 0$, we have $\bar{Y}_\tau \Gamma_\tau \geq 0$ a.s. Since $\Gamma_\tau > 0$ a.s., we have $\bar{Y}_\tau \geq 0$, that is $Y_\tau \geq X_\tau = \mathcal{E}_{\tau,\theta}(Y_\theta)$ a.s. , which is the desired result.
It remains to show (A.43). We use that $\bar{Y}$ is a strong optional semimartingale with decomposition $\bar{Y} = M^1 + A^1 + B^1$, where $M^1_t = \int_0^t \bar{Z}_s dW_s$, $A^1_t := -\int_0^t h_s ds - A_s$, and $B^1_t := -C_{t-}$, and we apply the generalized change of variables formula from Theorem A.3 with $n := 2$, $X^1 := \bar{Y}$, $X^2 := \Gamma$, and $F(x^1, x^2) := x^1 x^2$. We obtain

$$d(\overline{Y}_s \Gamma_s) = \Gamma_s(\bar{Z}_s + \bar{Y}_{s-}\beta_s)dW_s + \Gamma_s(\bar{Y}_{s-}\delta_s + \beta_s \bar{Z}_s - h_s)ds - \Gamma_s dA_s - \Gamma_s dC_s.$$

By using the equality (A.41), we obtain

$$d(\bar{Y}_s \Gamma_s) = \Gamma_s(\bar{Z}_s + \bar{Y}_{s-}\beta_s)dW_s - \Gamma_s(dA_s + dC_s).$$

Now, the stochastic integral in the above equation is a martingale (since $\Gamma \in S^2$, $\bar{Z} \in I\!H^2$, $\bar{Y} \in \mathcal{S}^2$, and (β_t) is bounded). By integrating between τ and θ and by taking the conditional expectation, we derive (A.43). The proof is thus complete. □

We end this Appendix with the proof of Proposition 2.1. We make a remark which will be used in the proof.

Remark A.12 *Let (ϕ_n) be a non-negative sequence of optional processes. We assume that the sequence $(\phi^n_t(\omega))$ is non-decreasing for all $t \in [0,T]$, for all $\omega \in \Omega$. We set $\phi_t(\omega) := \lim_n \phi^n_t(\omega) = \sup_n \phi^n_t(\omega)$. Then, we have $\lim_n \uparrow E(\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} |\phi^n_\tau|^2) = E(\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} |\phi_\tau|^2)$. The result is a consequence of the monotone convergence theorem, combined with the following observation:*
$\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} \sup_n |\phi^n_\tau|^2 = \sup_n \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} |\phi^n_\tau|^2$ *a.s.*

Proof of Proposition 2.1:
We first note that $|||\cdot|||_{\mathcal{S}^2}$ is a norm on the space of optional processes. The positive homogeneity and the triangular inequality are easy to check. Suppose now that $\phi \in \mathcal{S}^2$ is such that $|||\phi|||_{\mathcal{S}^2} = 0$. Then, $\operatorname{ess\,sup}_{S \in \mathcal{T}_{0,T}} |\phi_S|^2 = 0$ a.s., which implies that $|\phi_S|^2 = 0$ a.s. for all $S \in \mathcal{T}_{0,T}$. By applying the optional section theorem (cf. also Prop. A.4), we obtain that $\phi = 0$ up to an evanescent set. We conclude that $|||\cdot|||_{\mathcal{S}^2}$ is a norm on $\mathcal{S}^2$.
Let us prove that the space $(\mathcal{S}^2, |||\cdot|||_{\mathcal{S}^2})$ is complete. The main steps of the proof are similar to those of the proof of the completeness of the space $(L^2, \|\cdot\|_{L^2})$. Let (ϕ^n) be a Cauchy sequence in $\mathcal{S}^2$ for the norm $|||\cdot|||_{\mathcal{S}^2}$. By standard arguments, we extract a subsequence $(\phi^{n_k})_{k \in I\!N}$ such that

$$|||\phi^{n_{k+1}} - \phi^{n_k}|||_{\mathcal{S}^2} \leq \frac{1}{2^k}, \text{ for all } k \in I\!N. \tag{A.44}$$

For $n \in I\!N$, for $t \in [0,T]$, for $\omega \in \Omega$, we set $g_t^n(\omega) := \sum_{k=1}^n |\phi_t^{n_{k+1}}(\omega) - \phi^{n_k}(\omega)|$. We note that $g^n \in \mathcal{S}^2$ for all $n \in I\!N$. Moreover, the sequence $(|||g^n|||_{\mathcal{S}^2})$ is bounded. Indeed, by the triangular inequality and property (A.44), we have $|||g^n|||_{\mathcal{S}^2} \leq \sum_{k=1}^n \left|\left|\left|\phi_t^{n_{k+1}} - \phi^{n_k}\right|\right|\right|_{\mathcal{S}^2} \leq \sum_{k=1}^n \frac{1}{2^k} \leq \sum_{k=1}^\infty \frac{1}{2^k} = 1$. We set $g_t(\omega) := \lim_n \uparrow g_t^n(\omega)$, for all $t \in [0,T]$, for all $\omega \in \Omega$ (the limit exists in $[0,+\infty]$ as the sequence $(g_t^n(\omega))_n$ is non-negative non-decreasing). Being the limit of optional processes, the process g is optional. Moreover, $g \in \mathcal{S}^2$. Indeed, by using Remark A.12, we have $|||g|||_{\mathcal{S}^2} = \lim_n \uparrow |||g^n|||_{\mathcal{S}^2}$; as the sequence $(|||g^n|||_{\mathcal{S}^2})$ is bounded by 1, we get $|||g|||_{\mathcal{S}^2} \leq 1$.
Let us now show that $\lim_n |||g - g^n|||_{\mathcal{S}^2} = 0$. For all $n \geq 1$, we have $g - g^n = \lim_{p\to\infty} \left(\sum_{k=n+1}^{n+p} |\phi_t^{n_{k+1}} - \phi^{n_k}|\right)$. By using this and Remark A.12, we get

$$|||g - g^n|||_{\mathcal{S}^2} = ||| \lim_{p\to\infty} \big(\sum_{k=n+1}^{n+p} |\phi^{n_{k+1}} - \phi^{n_k}|\big)|||_{\mathcal{S}^2} = \lim_{p\to\infty} ||| \sum_{k=n+1}^{n+p} |\phi^{n_{k+1}} - \phi^{n_k}|||_{\mathcal{S}^2}.$$

Now, for all $p \geq 1$, for all $n \geq 1$, we have

$$||| \sum_{k=n+1}^{n+p} |\phi^{n_{k+1}} - \phi^{n_k}|||_{\mathcal{S}^2} \leq \sum_{k=n+1}^{n+p} |||\phi^{n_{k+1}} - \phi^{n_k}|||_{\mathcal{S}^2} \leq \sum_{k=n+1}^{n+p} \frac{1}{2^k} \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k}.$$

Therefore, for all $n \geq 1$, $|||g - g^n|||_{\mathcal{S}^2} \leq \sum_{k=n+1}^\infty \frac{1}{2^k}$. By letting n go to ∞, we obtain $\lim_n |||g - g^n|||_{\mathcal{S}^2} = 0$. On the other hand, for all $m \geq l \geq 2$,

$$|\phi_t^{n_m}(\omega) - \phi_t^{n_l}(\omega)| \leq g_t(\omega) - g_t^{l-1}(\omega). \tag{A.45}$$

Hence, $(\phi_t^{n_m}(\omega))$ is a Cauchy sequence in $\mathbf{R}$; we set $\phi_t(\omega) := \lim_{m\to\infty} \phi_t^{n_m}(\omega)$. By letting m go to ∞ in (A.45), we obtain $|\phi_t(\omega) - \phi_t^{n_l}(\omega)| \leq g_t(\omega) - g_t^{l-1}(\omega)$. From this inequality we get $\phi \in \mathcal{S}^2$ and $|||\phi - \phi^{n_l}|||_{\mathcal{S}^2} \leq \left|\left|\left|g - g^{l-1}\right|\right|\right|_{\mathcal{S}^2}$. Letting now l go to ∞ gives $|||\phi - \phi^{n_l}|||_{\mathcal{S}^2} \underset{l\to\infty}{\longrightarrow} 0$, which concludes the proof.

□

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