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Dynamics of Anisotropic Universes

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Abstract. We present a general study of the dynamical properties of Anisotropic Bianchi Universes in the context of Einstein General Relativity. Integrability results using Kovalevskaya exponents are reported and connected to general knowledge about Bianchi dynamics. Finally, dynamics toward singularity in Bianchi type VIII and IX universes are showed to be equivalent in some precise sence.

Keywords: General relativity, Dynamical systems, Anisotropic universes, Analytical methods, Numerical methods

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HOMOGENEOUS UNIVERSE AND BIANCHI MODELS

Considering the usual synchronous frame of General Relativity ¹ $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{ij} dx^i dx^j - dt^2$. A Universe is said homogeneous when there exists an isometry group wich preserves the infinitesimal spacial lenght $dl^2 = \tilde{g}_{ij} dx^i dx^j$. A characterization of the isometry group is possible writing structure constants

$$C_{ab}^c = (\partial_i e_j^c - \partial_j e_i^c) e_a^i e_b^j \quad (1)$$

where $dx^i = e_j^i dy^j$. Constants C_{ab}^c are tensorial, low components are antisymetrics, and follow the Jacobi rule :

$$C_{ab}^e C_{ec}^d + C_{bc}^e C_{ea}^d + C_{ca}^e C_{eb}^d = 0 \quad (2)$$

Decomposing $C_{ab}^c = \varepsilon_{abd} N^{dc} + \delta_b^c A_a - \delta_a^c B_b$, where N^{ab} is a symetric tensor, one can show that equivalence classes of homogeneous universes are equivalence classes of N^{ab} with $N^{ab} A_b = 0$. Without less of generality, the symmetry of N^{ab} allows us to write

$$N^{ab} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix} \quad \text{and} \quad A_b = [a, 0, 0] \quad (3)$$

Models split then into Class A with $a = 0$ and Class B with $a \neq 0$ and can be arranged in the well known Bianchi models

	n_1	n_2	n_3	a	Model
0 is a triple eigenvalue of N	0	0	0	0	B_I
	0	0	0	$\neq 0$	B_V
0 is a double eigenvalue of N	1	0	0	0	B_{II}
	0	1	0	$\neq 0$	B_{IV}
0 is a simple eigenvalue of N	1	1	0	0	B_{VII_o}
	0	1	1	$\neq 0$	B_{VII_a}
	1	-1	0	0	B_{VI_o}
	0	1	-1	$\neq 1$	B_{VI_a}
	0	1	-1	1	B_{III}
0 is not an eigenvalue of N	1	1	1	0	B_{IX}
	1	1	-1	0	B_{VIII}

¹ without contrary indications greek indexes run from 0 to 3, latin indexes from 1 to 3, metric signature is $(-, +, +, +)$, ε and δ are respectively the completely antisymmetric Levi-Cevita tensor and the Kronecker symbol

EINSTEIN EQUATIONS

Following [1], one writes $ds^2 = \gamma(\tau) \omega^i \omega^j - N^2(\tau) d\tau^2$ with $\gamma(\tau) = \text{diag} [e^{A_1(\tau)}, e^{A_2(\tau)}, e^{A_3(\tau)}]$ and $dt = N(\tau) d\tau$. The so called invariant differential forms basis ω^i are linear combinations of dx_i with exponential or trigonometric coefficient in x_i (See [1]). Finally, τ and $N(\tau)$ are respectively the conformal time and the lapse function.

BKL Formalism

This formalism was introduced in the 70's by [3]. Filling Universe by a barotropic fluid with pressure P and energy density ρ such that $P = (\Gamma - 1)\rho$, taking $N^2(\tau) = V^2 = e^{A_1+A_2+A_3}$ for the lapse function, some algebra then gives from Einstein equations, the equations for the dynamics of Bianchi Universes

$$\begin{cases} 0 &= E_c + E_p + E_m = H \\ \chi(2-\Gamma)V^{2-\Gamma} &= A_1'' + (n_1 e^{A_1})^2 - (n_2 e^{A_2} - n_3 e^{A_3})^2 \\ \chi(2-\Gamma)V^{2-\Gamma} &= A_2'' + (n_2 e^{A_2})^2 - (n_3 e^{A_3} - n_1 e^{A_1})^2 \\ \chi(2-\Gamma)V^{2-\Gamma} &= A_3'' + (n_3 e^{A_3})^2 - (n_1 e^{A_1} - n_2 e^{A_2})^2 \end{cases} \quad (4)$$

where $\chi = 8\pi Gc^{-4}$, $E_c = \frac{1}{2} \sum_{i \neq j=1}^3 A_i' A_j'$, $E_p = \sum_{i \neq j=1}^3 n_i n_j e^{A_i + A_j} - \sum_{i=1}^3 n_i^2 e^{2A_i}$, $E_m = -4\chi\rho V^2$ and a $'$ stands for $d/d\tau$

Hamiltonian formalism

This formalism was introduced by [4], at almost the same time than the precedent. It consists to diagonalize the quadratic form E_c by introducing new variables such that

$$M := \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \quad \begin{aligned} \mathbf{q} &:= [q_1 \ q_2 \ q_3]^T = M [A_1 \ A_2 \ A_3]^T \\ \mathbf{p} &:= [p_1 \ p_2 \ p_3]^T = M [A_1' \ A_2' \ A_3']^T = \mathbf{q}' \end{aligned} \quad (5)$$

Dynamical equations for Bianchi Universes then becomes

$$q'_{1,2} = -\frac{\partial H}{\partial p_{1,2}}, \quad p'_{1,2} = -\frac{\partial H}{\partial q_{1,2}} \quad \text{and} \quad q'_3 = \frac{\partial H}{\partial p_3}, \quad p'_3 = -\frac{\partial H}{\partial q_3} \quad (6)$$

where $H = \frac{1}{2} \langle \mathbf{p}, \mathbf{p} \rangle + \sum_{i=1}^7 k_i e^{\langle \mathbf{a}_i, \mathbf{q} \rangle}$ with the following products

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \quad \langle \mathbf{x}, \mathbf{y} \rangle := +x_1 y_1 + x_2 y_2 + x_3 y_3 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{y} \rangle := -x_1 y_1 - x_2 y_2 + x_3 y_3 \quad (7)$$

and the constants $k_1 := 2n_1 n_2$, $k_2 := 2n_1 n_3$, $k_3 := 2n_2 n_3$, $k_4 := -n_1^2$, $k_5 := -n_2^2$, $k_6 := -n_3^2$, $k_7 = -4\rho_0 \chi$. The set of vectors $\mathbf{a}_{i=1, \dots, 7}$ is highly symmetric (see Fig. 1): It allows to use algebraic techniques based on Lie algebra in some dynamical treatments of this problem.

INTEGRABILITY OF BIANCHI MODELS

Kovalevskaya stuff and autosimilar ODE

If an ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \text{with} \quad \mathbf{x} \in \mathbb{R}^n \quad \text{and} \quad \dot{\ } \equiv d/dt \quad (8)$$

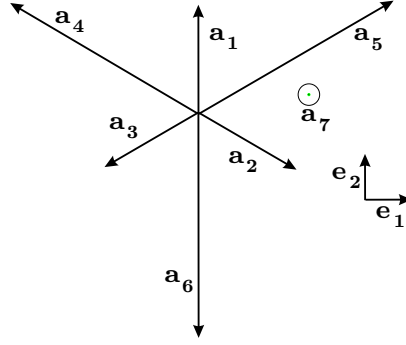


FIGURE 1. Projection of \mathbf{a}_i vectors on $(\mathbf{e}_1, \mathbf{e}_2)$ plane

admits a self-similar solution

$$\tilde{\mathbf{x}} = [c_1 (t - t_o)^{-g_1}, \dots, c_n (t - t_o)^{-g_n}]^T \quad (9)$$

with the weight vector \mathbf{g} and the constant vector \mathbf{c} lying respectively \mathbb{Z}^n and \mathbb{R}^n . One can then prove that the linearized system around $\tilde{\mathbf{x}}$, admits too an auto-similar solution

$$\mathbf{z} = [d_1 (t - t_o)^{k_1 - g_1}, \dots, d_n (t - t_o)^{k_n - g_n}]^T \quad (10)$$

where \mathbf{d} is a constant vector, and, this time the Kovalevskaya vector \mathbf{k} , wch components are the so-called Kovalevskaya exponents, lies \mathbb{C}^n . In practice, one can compute this vector : it is easy to show that Kovalevskaya exponents are eigenvalues of $K := D[\mathbf{f}(\mathbf{x})](\mathbf{c}) + \text{diag}(\mathbf{g})$. A theorem by Poincaré shows that each component of the non linear general solution of equation (8) is on the form

$$x_i(t) \propto (t - t_o)^{-g_i} S[(t - t_o)^{k_1}, \dots, (t - t_o)^{k_n}] \quad (11)$$

where $S[\cdot]$ stands for a multiple series. This result is at the basis of a sufficient condition (known as Yoshida's Theorem see [5],[6]) : if all the Kovalevskaya exponents are in \mathbb{Q} , then the system is algebraically integrable.

Kowalevskaya exponents and Bianchi Universes

Such a work was Pioneering by [7], applied for the first time to some special cases by [8], and more recently, developed with some imprecisions and incompletude by [9]. We present here results obtained in a precise way in [10]. Introducing new variables

$$\{\mathbf{q}, \mathbf{p}\} \mapsto \{\mathbf{u}, \mathbf{v}\} \quad \text{where} \quad \begin{cases} \mathbf{u} \in \mathbb{R}^7, u_{i=1, \dots, 7} := \langle \mathbf{a}_i, \mathbf{p} \rangle \\ \mathbf{v} \in \mathbb{R}^7, v_{i=1, \dots, 7} := \exp(\mathbf{a}_i, \mathbf{q}) \end{cases} \quad (12)$$

Einstein's Hamiltonian equations become

$$\forall i = 1, \dots, 7 \quad \begin{cases} v'_i = u_i v_i \\ u'_i = \sum_{j=1}^7 W_{ij} v_j \end{cases} \quad \text{with } W_{ij} := -k_j \langle \mathbf{a}_i, \mathbf{a}_j \rangle \quad (13)$$

This dynamical system admits an autosimilar solution $\tilde{\mathbf{x}} = [\lambda t^{-1}, \mu t^{-2}]^T$ where the constants $[\lambda, \mu] \in \mathbb{R}^7 \times \mathbb{R}^7$ are solutions of the algebraic system

$$\forall i = 1, \dots, 7 \quad \begin{cases} \sum_{j=1}^7 W_{ij} \mu_j = -\lambda_i \\ \lambda_i \mu_i = -2\mu_i \end{cases} \quad (14)$$

Taking into account that $\text{Rank}(W) = 3$, it exists 45 distincts non trivial solution for system (14), and then 45 sets of 14 Kovalevskaya exponents for all Bianchi Universes. Analysis of such sets let us claim that :

- In vacuum or with stiff matter ($\Gamma = 2$), excepted B_{IX} and B_{VIII} , all other class A Bianchi models have fractional Kovalevskaya exponents.
- With non stiff matter ($0 \leq \Gamma < 2$) class A Bianchi models have at least one real or complex Kovalevskaya exponent, except for B_I with fractional values of Γ and B_{II} with Γ fractional in $\left[0, \frac{11+\sqrt{73}}{3} \simeq 0.82\right]$

These integrability indications are confortred by [12] for B_{VIII} using Morales-Ruis Theory and by [11] using Painlevé's analysis for B_{IX} .

EXACT SOLUTIONS

B_I dynamics : The fundamental state

In vacuum, the general solution of B_I model could be explicited. The line element is

$$ds^2 = t^{2p_1} dx_1^2 + t^{2p_2} dx_2^2 + t^{2p_3} dx_3^2 - dt^2 \quad (15)$$

where

$$\exists ! u \in [1, +\infty[\text{ such that } \begin{cases} p_1 = -\Omega u / (1 + u + u^2) \\ p_2 = \Omega(1 + u) / (1 + u + u^2) \\ p_3 = \Omega u(1 + u) / (1 + u + u^2) \end{cases} \quad (16)$$

The real Ω is directly proportional to Universe Volume variation wich is constant in B_I . Toward singularity ($t \rightarrow 0$), x_1 expands when x_2 and x_3 contracts, all with a constant exponential rate. Any couple $[u, \Omega] \in [1, +\infty[\times \mathbb{R}$ associated to an order of axis defines a Kasner state.

B_{II} dynamics : One Kasner transition

As noted by [3], B_{II} dynamics corresponds to a transition between 2 asymptotic Kasner states :

$$\left. \begin{array}{c} [u, \Omega] \ (\square \triangle \diamond) \\ p_1 < p_2 < p_3 \\ \square \quad \triangle \quad \diamond \\ \text{Initial Kasner State : } t \rightarrow +\infty \end{array} \right] \rightsquigarrow \left[\begin{array}{l} [u - 1, \Omega(1 - 2p_1)] \ (\triangle \diamond \square) \text{ if } u \geq 2 \\ [(u - 1)^{-1}, \Omega(1 - 2p_1)] \ (\triangle \square \diamond) \text{ if } u < 2 \\ \text{Final Kasner State : } t \rightarrow 0 \end{array} \right] \quad (17)$$

Conjectures (partially proven)

As indicated by the rigorous proof by [16], all classes of Bianchi models converge generically toward a simple Kasner state or a closure of Kasner states when $t \rightarrow 0$. Numerical analysis, more precisely Billiard analogy (see next section), let us think that :

- B_{VI_0} and B_{VII_0} dynamics correspond to a finite number of Kasner transitions
- B_{VIII} and B_{IX} dynamics correspond to an infinite number of Kasner transitions

BILLIARD ANALOGY

Those works was pioneered by [4], and more recently by [13] or [15]. Introducing the super-time \tilde{t} such that $d\tilde{t} = V^{1/3} dt$, a "mass" $m = V^{4/3}$ and an "energy" $E = (dV/dt)^2/2V^{2/3}$, dynamical equation (6) in vacuum becomes

$$\left\{ \begin{array}{l} \frac{dq_{1,2}}{d\tilde{t}} = \frac{p_{1,2}}{m} = \frac{\partial E}{\partial q_{1,2}} \\ \frac{dp_{1,2}}{d\tilde{t}} = -\frac{\partial \xi}{\partial q_{1,2}} = \frac{\partial E}{\partial p_{1,2}} \end{array} \right. \quad \text{with} \quad \begin{array}{l} E = \frac{p_1^2 + p_2^2}{2m} - \xi(\mathbf{q}), \quad \xi(\mathbf{q}) = \sum_{i=1}^6 k_i e^{(\pi(\mathbf{a}_i) \cdot \mathbf{q})} \\ \mathbf{q} \in \mathbb{R}^2, \pi : \text{normal projector onto } (\mathbf{e}_1, \mathbf{e}_2) \text{ plane} \end{array} \quad (18)$$

Toward singularity, $E \rightarrow +\infty$ and $m \rightarrow 0$. When we go through time backward, equations (18) are ones of a ball with a decreasing mass m and with an increasing energy. The exponential nature of the potential ξ allows a precise description of the dynamics, namely, the billiard analogy. When ξ is exponentially negligible, solutions of (18) are almost straight lines in phase plane (q_1, q_2) : these are almost Kasner states in BKL representation. When ξ cannot be neglected anymore, it is mainly due to one of its exponential terms, equation of motion – for a linear combination y of (q_1, q_2) – takes a form equivalent to²

$$\frac{d^2 y}{dx^2} = -k^2 e^y \quad \text{with} \quad y(0) = \left. \frac{dy}{dx} \right|_{x=0} = 0 \quad (19)$$

This equation could be solved analytically in

$$y(x) = \ln \left[1 - \text{th}^2 \left(\frac{kx}{\sqrt{2}} \right) \right] = -2 \ln \left[\text{ch} \left(\frac{kx}{\sqrt{2}} \right) \right] \quad (20)$$

It shows clearly a transition between the two Kasner states associated to the straight lines which are the asymptotes of this solution (see Fig. 2). This generalized B_{II} transition corresponds to a bounce of a ball in an amazing billiard. As a matter of fact, when $p_1^2 + p_2^2 = 0$, all ball's energy is concentrated in the potential term ξ , this situation corresponds to the maximum of the y function plotted in Fig. 2: this is a bounce. It happens effectively against the isocontour line $\xi = \xi(\mathbf{q}^*)$ where $d\mathbf{q}^*/d\tilde{t} = 0$. As we go toward singularity ($\tilde{t} \rightarrow 0$) these lines move outward as one can see in Fig. 3. The Universe ball then moves in an expanding billiard. It is easy to prove that ball's velocity norm $|d\mathbf{q}/d\tilde{t}|$ is always greater than the one of cushions which is the energy variation (see [18] for instance). This property remains true when barotropic matter fills Universe provided that $\Gamma < 2$. The amplitude of the bounce depends on k : the largest k is, the smallest is the bounce angle between the two asymptotic linear regimes. Between the two ideal cases (quasi Kasner state and bounce) the parameter k is dynamical, its variations remains negligible during each phase.

DYNAMICAL PROPERTIES OF BIANCHI BILLIARDS

In the billiard formulation, dynamics of Bianchi Universe is fully understood considering isocontours of the potential ξ which are presented on Fig. 3. Excluding B_{IX} and B_{VIII} all isocontours are open curves. Hence after a finite number of bounces, the ball representing Universe find a hole and go away to reach the asymptotic regime solution of equation (19) which is a Kasner state. By opposite, for the two closed curves potential, there is no hole in the billiard, cushion always expands less quickly than the ball, and there is an infinite sequence of B_{II} transitions toward singularity. Although B_{VIII} cushions seems open along a vertical channel, the bounce process makes the ball which enter the channel goes back. As sketched on Fig. 2, in the channel $k \sim q_2$, then as the ball sinks into this region, q_2 grows and bounces are more and more pinched. By consequence, as the cushion is not strictly vertical, the ball finally goes back and leaves the channel. These results are surely well known and one remaining question could be: Is B_{VIII} dynamics equivalent in some precise sense to B_{IX} one? In terms of complexity the two dynamics seems equivalent as they have exactly the same set of Kovalevskaya exponents. Let us show that they are also equivalent by the fact that they present the same kind of transition to chaos, and they have the same kind of attractor.

² The x variable represents the supertime \tilde{t} such that when $\tilde{t} \sim x$ one has $\xi \sim -k^2 e^y$

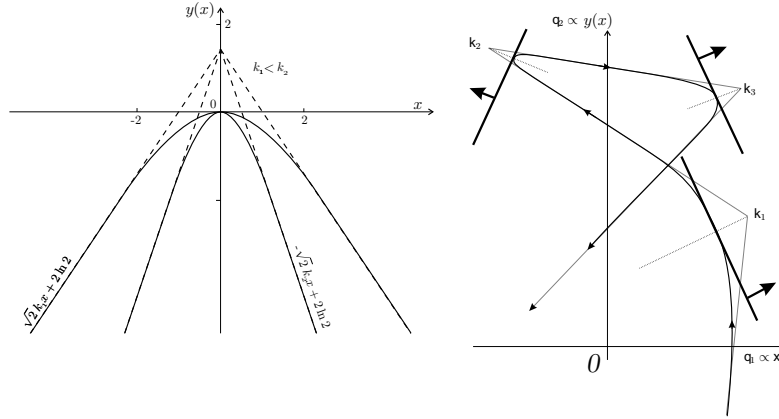


FIGURE 2. Reflexion on a straight cushion (left) and in a corner (right)

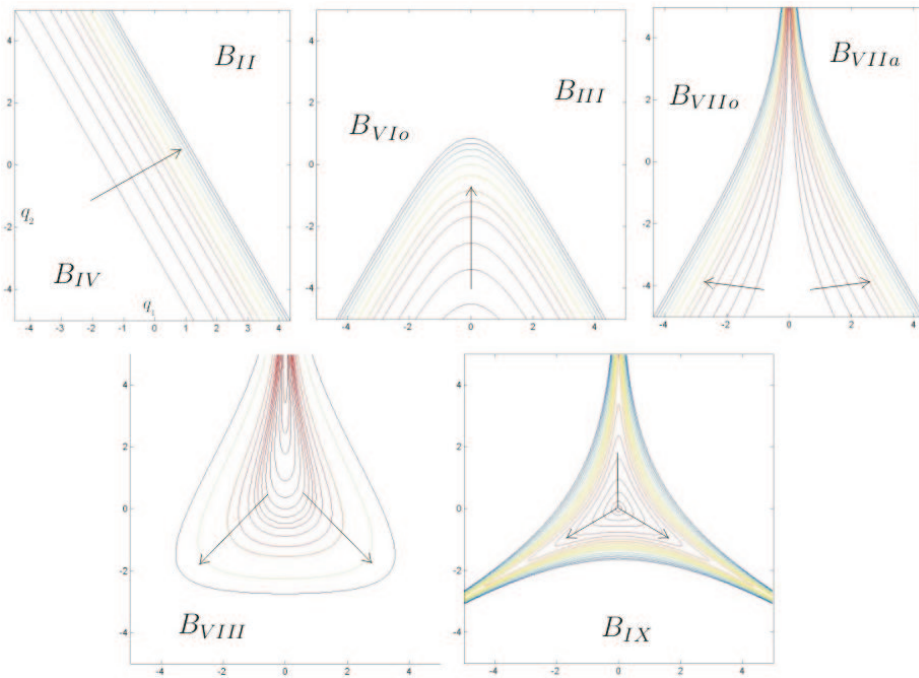


FIGURE 3. Isocontours of potential ξ for all distinct Bianchi Universe in vacuum. Arrows indicate the increasing values of ξ

B_{IX} and B_{VIII} Poincaré map analysis

Detailed analysis show that Bianchi Universe volume is a regular variable. It is the only variable attached to cushions expansion in the billiard formalism. Therefore, the complexity of the dynamics is contained in the cushions's shape. In order to study this complexity in the two relevant cases which are B_{IX} and B_{VIII} Universes, we have studied dynamical properties of temporal sections of the hamiltonian E involving in equation (18). As V , t , \tilde{t} and τ are all related in a bijective way, by temporal section we mean sections of the dynamics described by equation (18) with $m = cst$ and $E = cst$. For convenience we use always a ball with unit mass, we focus attention under Poincaré maps associated to growing values of E . Such maps are obtained by considering the intersection between an orbit (a solution of the 4D differential system) and a fixed plane, namely $\Pi = \{q_2 = 0, p_2 = 0\}$ for maps plotted in Fig. 4. The greater are values of the energy E , the closer from singularity sections are. Analysis of such maps is clear using KAM Theory : When periodic or quasiperiodic orbits correspond to Poincaré sections disposed along curves, chaotic orbits sections

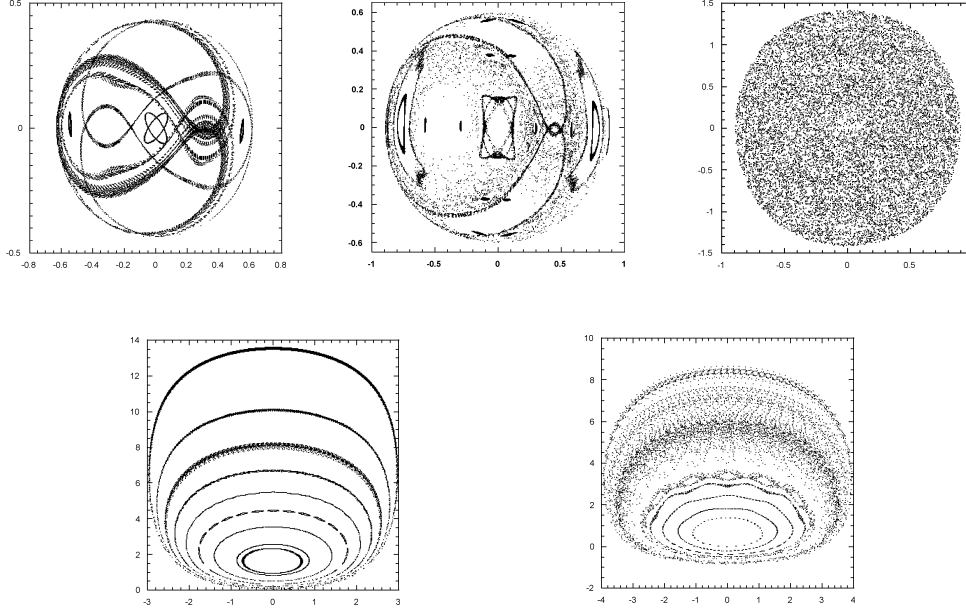


FIGURE 4. Poincaré maps toward singularity of temporal sections of $m = cst$ and $E = cst$ of B_{IX} (top : $m = 1$ and $E = -2.8, -2.1, 0$ from left to right) and B_{VIII} (bottom : $m = 1$ and $E = 5.0, 10.0$ from left to right) dynamics. For each map a set of 9 homogeneously distributed initial conditions have been chosen.

fill dense regions. We can then conclude that far from singularity orbits are regular in both B_{IX} and B_{VIII} . Approaching singularity, a transition to chaos seems to appear in the dynamics of both these closed cushions expanding billiards. In order to produce a comparative measure of this chaos we propose to produce an analysis base on B_{IX} and B_{VIII} Universes truncated dynamics.

B_{IX} and B_{VIII} fractal attractors

As suggested by all previous indicators (Kovalevskaya exponents, Poincaré maps), we suspect B_{VIII} dynamics to possess the same kind of attractor exhibited initially by [17] for B_{IX} . In order to confirm this suspicion we have applied the truncated dynamical technique used by [17] in the context of Hamiltonian formalism described above.

Initial conditions we used are $\mathbf{q}_o = [0, -\sqrt{6}\ln 2, \Omega]$ and $\mathbf{p}_o = [\cos \theta, \sin \theta, \omega]$ where the constraint $H = 0$ is fulfilled provided that

$$\Omega = \frac{3}{\sqrt{6}} \ln \left[\pm \frac{(1 - \omega^2)}{2^4 (1 \mp 2)} \right] \quad (21)$$

The upper and lower signs are associated respectively to B_{IX} and B_{VIII} Universes. The two free parameters ω and θ fixe respectively the initial variation rate of Univers volume (or the initial escape speed of cushions in the billiard formalism) and the orientation of the initial velocity vector. These initial conditions are represented on Fig. 5. The numerical integration of the dynamics from these initial conditions shows an unlimited sequence of B_{II} transitions between Kasner states (see formula (17)). We stop numerical integration when the Kasnerian u parameter is greater than a critical value (for maps presented on Fig. 6 we have used $u_{exit} = 8$). When the dynamic is stopped, we affect to the corresponding point in the $\theta - \omega$ plane a color which correspond to the expanding axis of stopped Kasner state (Red $\leftrightarrow A_1$, Blue $\leftrightarrow A_2$ and Green $\leftrightarrow A_3$). We have studied the $[0, \pi/2] \times [-2, -3]$ portion of the $\theta - \omega$ plane, projected on a regular 500×500 grid. The corresponding maps are presented in Fig. 6. From this map we can compute the 3 Hausdorff dimensions associated to each color set. We can resume these three numbers into one by averaging each dimension weighted by the surface proportion occupied by the corresponding color. We then obtain a mean Hausdorff dimension of each map which is $d = 1.6976 \pm 8.9 \times 10^{-3}$ for B_{VIII} 's map and $d = 1.7141 \pm 8.5 \times 10^{-3}$ for B_{IX} 's

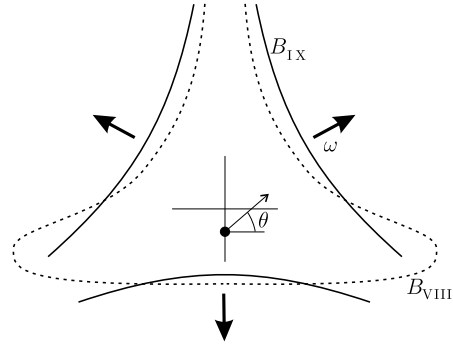


FIGURE 5. Initial conditions θ and ω for fractals maps represented on Fig. 6

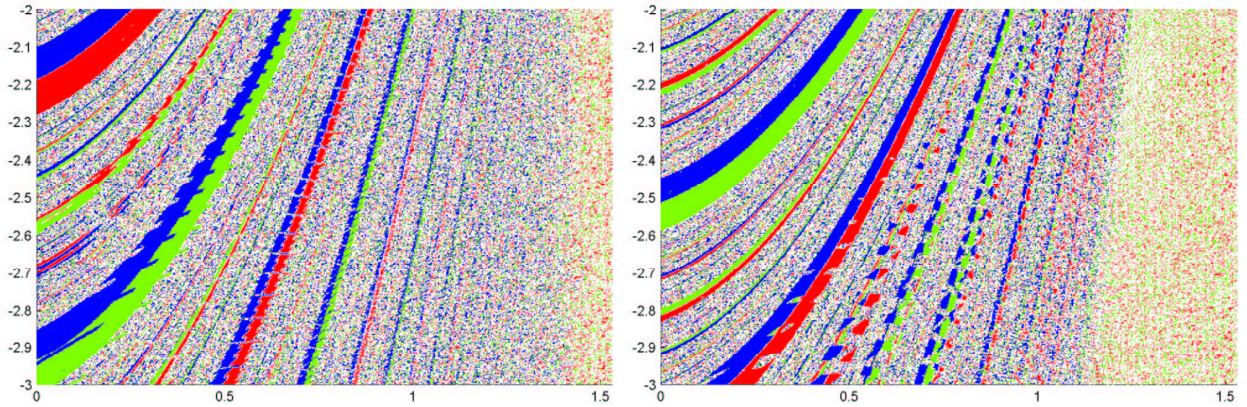


FIGURE 6. B_{VIII} (left) and B_{IX} (right) fractal maps in the $[0, \pi/2] \times [-2, -3]$ portion of the $\theta - \omega$ plane

map. These two numbers are then numerically indistinguishable. In this sense we claim that B_{VIII} and B_{IX} universes are dynamically equivalent

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