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# A note on the adaptive estimation of the differential entropy by wavelet methods

Christophe Chesneau

*Laboratoire de Mathématiques Nicolas Oresme,  
Université de Caen BP 5186, F 14032 Caen Cedex, FRANCE. e-mail:  
christophe.chesneau@unicaen.fr*

**Abstract:** In this note we consider the estimation of the differential entropy of a probability density function. We propose a new adaptive estimator based on a plug-in approach and wavelet methods. We prove that it attains fast rates of convergence under the mean  $\mathbb{L}_p$  error,  $p \geq 1$ , for a wide class of functions. A key result in our development is a new upper bound for the mean  $\mathbb{L}_p$  error,  $p \geq 1$ , of a general version of the plug-in estimator. This upper bound may be of independent interest.

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## 1. Introduction

The entropy is a measure of uncertainty which plays a fundamental role in information theory and engineering sciences. In this paper, we focus our attention on the concept of differential entropy originally introduced by Shannon (1948). We investigate the estimation of the differential entropy of a probability density function  $f : [0, 1]^d \rightarrow [0, \infty)$ ,  $d \geq 1$ , defined by

$$H = - \int_{[0,1]^d} f(\mathbf{x}) \log(f(\mathbf{x})) d\mathbf{x}. \quad (1.1)$$

The literature on the estimation of  $H$  is very extensive, see, e.g., Beirlant *et al.* (1997) and the references cited therein. Among the existing estimation methods, we consider a plug-in integral estimator of the form :

$$\hat{H} = - \int_{\hat{A}} \hat{f}(\mathbf{x}) \log(\hat{f}(\mathbf{x})) d\mathbf{x},$$

where  $\hat{f}$  denotes an estimator for  $f$  and  $\hat{A} \subseteq \{\mathbf{x} \in [0, 1]^d; \hat{f}(\mathbf{x}) > 0\}$ . Such plug-in integral estimator was introduced by Dmitriev and Tarasenko (1973) with a kernel density estimator. They showed its strong consistency. Other aspects of this estimator have been studied by Prakasa Rao (1983), Mokkadem (1989), Györfi and van der Meulen (1990, 1991) and Mason (2003). Recent developments can be found in, e.g., Bouzebda and Elhattab (2009, 2010, 2011).

The contributions of this paper are twofold. Firstly, we establish a new general upper bound for the mean  $\mathbb{L}_p$  error of  $\hat{H}$ , i.e.,  $R(\hat{H}, H) = \mathbb{E}(|\hat{H} -$

$H|^p$ ), expressed in terms of mean integrated  $\mathbb{L}_{2p}$  error of  $\hat{f}$ , i.e.,  $R_*(\hat{f}, f) = \mathbb{E} \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x} \right)$ . The obtained bound illustrates the fact that more  $\hat{f}$  is efficient under the mean integrated  $\mathbb{L}_{2p}$  error, more  $\hat{H}$  is efficient under the mean  $\mathbb{L}_p$  error. The advantage of this result is its great flexibility on the model and the estimation method for  $\hat{f}$ . This result can also be viewed as an extension of the mean  $\mathbb{L}_p$  error of  $\hat{H}$  obtained by Mokkadem (1989) for the standard density model and kernel method.

Secondly, we introduce a new integral estimator  $\hat{H}$  based on a multidimensional hard thresholding wavelet estimator for  $\hat{f}$ . Such wavelet estimator  $\hat{f}$  was introduced by Donoho *et al.* (1996) and Delyon and Juditsky (1996). It has the advantages of being adaptive ; its construction does not depend on the smoothness of  $f$ , and efficient under the mean integrated  $\mathbb{L}_q$  error, with  $q \geq 1$ . Applying our general upper bound, we prove that  $\hat{H}$  attains fast rates of convergence under mild assumptions on  $f$  ; it is supposed that  $f$  belongs to a wide function sets : the Besov balls. Consequences of this results is the convergences  $\mathbb{L}_p$  and a.s. of our estimator. To the best of our knowledge, in this statistical context,  $\hat{H}$  is the first adaptive estimator for  $H$  based on wavelets.

The rest of this paper is organized as follows. In the next section, we present an upper bound for the mean  $\mathbb{L}_p$  error of  $\hat{H}$ . Section 3 is devoted to our wavelet estimator and its performances in terms of rates of convergence under the mean  $\mathbb{L}_p$  error over Besov balls. The proofs are postponed to Section 4.

## 2. A general upper bound

### 2.1. Notations and assumptions

We define the  $\mathbb{L}_p([0, 1]^d)$ -spaces with  $p \geq 1$  by

$$\mathbb{L}_p([0, 1]^d) = \left\{ h : [0, 1]^d \rightarrow \mathbb{R}; \int_{[0,1]^d} |h(\mathbf{x})|^p d\mathbf{x} < \infty \right\}.$$

We formulate the following boundedness assumptions on  $f$  :

(A1) There exists a constant  $c_* > 0$  such that

$$\inf_{\mathbf{x} \in [0,1]^d} f(\mathbf{x}) \geq c_*.$$

(A2) There exists a constant  $C_* > 0$  such that

$$\sup_{\mathbf{x} \in [0,1]^d} f(\mathbf{x}) \leq C_*.$$

These assumptions are satisfied by a wide family of probability density functions. They have ever been used in the context of estimating the differential entropy (see, for instance, (Beirlant *et al.*, 1997, conditions T2 and P2)).

## 2.2. Result

In this section, we adopt a general estimation setting : let  $\hat{f} : [0, 1]^d \rightarrow \mathbb{R}$  be an estimator of  $f$  constructed from random vectors defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Various estimation methods can be found in, e.g., Tsybakov (2004). Suppose that **(A1)** is satisfied. We study the following plug-in integral estimator for  $H$  (1.1) :

$$\hat{H} = - \int_{\hat{A}} \hat{f}(\mathbf{x}) \log(\hat{f}(\mathbf{x})) d\mathbf{x}, \quad (2.1)$$

where

$$\hat{A} = \left\{ \mathbf{x} \in [0, 1]^d; \hat{f}(\mathbf{x}) \geq \frac{c_*}{2} \right\}.$$

Such plug-in integral estimator was introduced by Dmitriev and Tarasenko (1973) with a kernel density estimator (and a different  $\hat{A}$ ). Related results can be found in Beirlant *et al.* (1997) and the references cited therein. In particular, Mokkadem (1989) has explored the mean  $\mathbb{L}_p$  error of  $\hat{H}$  for the standard density model and a kernel estimator (with a different  $\hat{A}$ ).

Without the specification of the model and for any estimator  $\hat{f}$  for  $f$ , Proposition 2.1 establishes a general upper bound for the mean  $\mathbb{L}_p$  error of  $\hat{H}$  in terms of the mean integrated  $\mathbb{L}_{2p}$  error of  $\hat{f}$ .

**Proposition 2.1.** *Let  $p \geq 1$ . Suppose that **(A1)** and **(A2)** are satisfied and  $\hat{f} \in \mathbb{L}_{2p}([0, 1]^d)$ . Let  $\hat{H}$  be defined by (2.1) and  $H$  be defined by (1.1). Then we have the following upper bound for the mean  $\mathbb{L}_p$  error of  $\hat{H}$  :*

$$\begin{aligned} & \mathbb{E}(|\hat{H} - H|^p) \\ & \leq K \left( \sqrt{\mathbb{E} \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x} \right)} + \mathbb{E} \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x} \right) \right), \end{aligned}$$

where

$$K = 2^{p-1} \max \left( \left( \frac{3C_*}{c_*} + \left( \frac{2C_*}{c_*} + 1 \right) \max(|\log(c_*)|, |\log(C_*)|) \right)^p, \left( \frac{3}{c_*} \right)^p \right).$$

Proposition 2.1 illustrates the intuitive idea that more  $\hat{f}$  is efficient in terms of mean integrated  $\mathbb{L}_{2p}$  error, more  $\hat{H}$  is efficient in terms of mean  $\mathbb{L}_p$  error. The obtained bound has the advantage of enjoying a great flexibility on the model and the choice of  $\hat{f}$ .

In order to highlight this flexibility, one can consider the standard density model :  $f$  is the common probability density function of  $n$  *iid*  $[0, 1]^d$ -valued random vectors,  $d \geq 1$ ,  $\mathbf{X}_1, \dots, \mathbf{X}_n$ , or with no *iid* assumption, or  $f$  can be a probability density function emerging from a more sophisticated density model as the convolution one (see, e.g., Caroll and Hall (1988), Devroye (1989) and Fan (1991)) . . . . On the other hand, one can consider several type of estimators as kernel, spline, Fourier series or wavelet series, as soon as they enjoy good mean integrated  $\mathbb{L}_{2p}$  error properties.

**Remark 2.1.** If  $n$  is such that  $c_* \geq 1/\log(n)$ , one can define  $\hat{H}$  (2.1) by replacing  $c_*$  in  $\hat{A}$  by  $1/\log(n)$  and Proposition 2.1 is still valid with  $1/\log(n)$  instead of  $c_*$ , implying that  $K \leq C(\log(n))^p$ .

In the rest of the study we focus our attention on a nonlinear wavelet estimator having the features to be adaptive and efficient under the mean integrated  $\mathbb{L}_{2p}$  error for a wide class of functions  $f$ .

### 3. Adaptive wavelet estimator

Before introducing our main estimator, let us present some basics on wavelets and the considered function spaces characterizing the unknown smoothness of  $f$ ; the Besov balls.

#### 3.1. Wavelet bases on $[0, 1]$

We consider an orthonormal wavelet basis generated by dilations and translations of the scaling and wavelet functions  $\phi$  and  $\psi$  from the Daubechies family  $\text{db}_{2R}$ , with  $R \geq 1$  (see Daubechies (1992)). We define the scaled and translated version of  $\phi$  and  $\psi$  by

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Then, with an appropriate treatment at the boundaries, there exists an integer  $\tau$  satisfying  $2^\tau \geq 2R$  such that, for any integer  $j_* \geq \tau$ , the collection

$$\{\phi_{j_*,k}, k \in \{0, \dots, 2^{j_*} - 1\}; \psi_{j,k}; j \in \mathbb{N} - \{0, \dots, j_* - 1\}, k \in \{0, \dots, 2^j - 1\}\},$$

forms an orthonormal basis of  $\mathbb{L}_2([0, 1])$ . See Meyer (1992), Daubechies (1992), Cohen *et al.* (1993) and Mallat (2009).

#### 3.2. Wavelet tensor product bases on $[0, 1]^d$

We use compactly supported tensor product wavelet bases on  $[0, 1]^d$  based on the Daubechies family. Their construction is recall below. For any  $\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d$ , we set

$$\Phi(\mathbf{x}) = \prod_{v=1}^d \phi(x_v),$$

and

$$\Psi_u(\mathbf{x}) = \begin{cases} \psi(x_u) \prod_{\substack{v=1 \\ v \neq u}}^d \phi(x_v) & \text{for } u \in \{1, \dots, d\}, \\ \prod_{v \in A_u} \psi(x_v) \prod_{v \notin A_u} \phi(x_v) & \text{for } u \in \{d+1, \dots, 2^d - 1\}, \end{cases}$$

where  $(A_u)_{u \in \{d+1, \dots, 2^d-1\}}$  forms the set of all non void subsets of  $\{1, \dots, d\}$  of cardinality greater or equal to 2.

For any integer  $j$  and any  $\mathbf{k} = (k_1, \dots, k_d)$ , we consider

$$\begin{aligned}\Phi_{j,\mathbf{k}}(\mathbf{x}) &= 2^{jd/2} \Phi(2^j x_1 - k_1, \dots, 2^j x_d - k_d), \\ \Psi_{j,\mathbf{k},u}(\mathbf{x}) &= 2^{jd/2} \Psi_u(2^j x_1 - k_1, \dots, 2^j x_d - k_d), \text{ for any } u \in \{1, \dots, 2^d - 1\}.\end{aligned}$$

Let  $D_j = \{0, \dots, 2^j - 1\}^d$ . Then, with an appropriate treatment at the boundaries, there exists an integer  $\tau$  such that the collection

$$\{\Phi_{\tau,\mathbf{k}}, \mathbf{k} \in D_\tau; (\Psi_{j,\mathbf{k},u})_{u \in \{1, \dots, 2^d-1\}}, j \in \mathbb{N} - \{0, \dots, \tau - 1\}, \mathbf{k} \in D_j\}$$

forms an orthonormal basis of  $\mathbb{L}_2([0, 1]^d)$ .

For any integer  $j_*$  such that  $j_* \geq \tau$ , a function  $h \in \mathbb{L}_2([0, 1]^d)$  can be expanded into a wavelet series as

$$h(\mathbf{x}) = \sum_{\mathbf{k} \in D_{j_*}} \alpha_{j_*,\mathbf{k}} \Phi_{j_*,\mathbf{k}}(\mathbf{x}) + \sum_{u=1}^{2^d-1} \sum_{j=j_*}^{\infty} \sum_{\mathbf{k} \in D_j} \beta_{j,\mathbf{k},u} \Psi_{j,\mathbf{k},u}(\mathbf{x}), \quad \mathbf{x} \in [0, 1]^d,$$

where

$$\alpha_{j,\mathbf{k}} = \int_{[0,1]^d} h(\mathbf{x}) \Phi_{j,\mathbf{k}}(\mathbf{x}) d\mathbf{x}, \quad \beta_{j,\mathbf{k},u} = \int_{[0,1]^d} h(\mathbf{x}) \Psi_{j,\mathbf{k},u}(\mathbf{x}) d\mathbf{x}. \quad (3.1)$$

### 3.3. Besov balls

Let  $M > 0$ ,  $s \in (0, R)$ ,  $p \geq 1$  and  $q \geq 1$ . We say that a function  $h$  in  $\mathbb{L}_r([0, 1]^d)$  belongs to  $\mathbf{B}_{r,q}^s(M)$  if, and only if, there exists a constant  $M^* > 0$  (depending on  $M$ ) such that the associated wavelet coefficients (3.1) satisfy

$$\left( \sum_{j=\tau}^{\infty} \left( 2^{j(s+d/2-d/r)} \left( \sum_{u=1}^{2^d-1} \sum_{\mathbf{k} \in D_j} |\beta_{j,\mathbf{k},u}|^r \right)^{1/r} \right)^q \right)^{1/q} \leq M^*.$$

In this expression,  $s$  is a smoothness parameter and  $p$  and  $q$  are norm parameters. Besov spaces include many traditional smoothness spaces as the standard Hölder and Sobolev balls. See Meyer (1992), Härdle *et al.* (1998) and Mallat (2009).

### 3.4. Wavelet estimation

Let  $\mathbf{X}_1, \dots, \mathbf{X}_n$  be  $n$  iid  $[0, 1]^d$ -valued random vectors,  $d \geq 1$ , with common probability function  $f$ . We aim to estimate the differential entropy of  $f$  :

$$H = - \int_{[0,1]^d} f(\mathbf{x}) \log(f(\mathbf{x})) d\mathbf{x},$$

from  $\mathbf{X}_1, \dots, \mathbf{X}_n$ . Under **(A1)**, we consider the following estimator for  $H$  :

$$\hat{H} = - \int_{\hat{A}} \hat{f}(\mathbf{x}) \log(\hat{f}(\mathbf{x})) d\mathbf{x}, \quad (3.2)$$

where

$$\hat{A} = \left\{ \mathbf{x} \in [0, 1]^d; \hat{f}(\mathbf{x}) \geq \frac{c_*}{2} \right\}$$

and  $\hat{f}$  is the following hard thresholding wavelet estimator for  $f$  :

$$\hat{f}(\mathbf{x}) = \sum_{\mathbf{k} \in D_\tau} \hat{\alpha}_{\tau, \mathbf{k}} \Phi_{\tau, \mathbf{k}}(\mathbf{x}) + \sum_{u=1}^{2^d-1} \sum_{j=j_*}^{\infty} \sum_{\mathbf{k} \in D_j} \hat{\beta}_{j, \mathbf{k}, u} \mathbf{1}_{\{|\hat{\beta}_{j, \mathbf{k}, u}| \geq \kappa \lambda_n\}} \Psi_{j, \mathbf{k}, u}(\mathbf{x}), \quad (3.3)$$

where

$$\hat{\alpha}_{j, \mathbf{k}} = \frac{1}{n} \sum_{i=1}^n \Phi_{j, \mathbf{k}}(\mathbf{X}_i), \quad \hat{\beta}_{j, \mathbf{k}, u} = \frac{1}{n} \sum_{i=1}^n \Psi_{j, \mathbf{k}, u}(\mathbf{X}_i),$$

$j_1$  is the resolution level satisfying  $2^{dj_1} = [n/\log(n)]$  (the integer part of  $n/\log(n)$ ),  $\mathbf{1}$  is the indicator function,  $\kappa$  is a large enough constant and  $\lambda_n$  is the threshold

$$\lambda_n = \sqrt{\frac{\log(n)}{n}}.$$

This estimator was introduced by Donoho *et al.* (1996) for  $d = 1$  and generalized to the multidimensional case by Delyon and Juditsky (1996). The central idea is to estimate only the wavelet coefficients with a high magnitude because they contain all the necessary informations inherent to  $f$ . The others, less important, are suppressed instead of being estimated in order to avoid the cumulation of superfluous errors in their estimation.

This estimator is adaptive ; its construction does not depend on the unknown smoothness of  $f$ .

Let us mention that  $\hat{\alpha}_{j, \mathbf{k}}$  and  $\hat{\beta}_{j, \mathbf{k}, u}$  are unbiased estimators for the wavelet coefficients  $\alpha_{j, \mathbf{k}}$  and  $\beta_{j, \mathbf{k}, u}$  respectively. They also satisfied powerful moment inequalities and concentration inequalities. Further details on wavelet estimation can be found in, e.g., Antoniadis (1997), Härdle *et al.* (1998) and Vidakovic (1999).

Theorem 3.1 below investigates the rates of convergence attained by  $\hat{H}$  under the mean  $\mathbb{L}_p$  error over Besov balls for  $f$ .

**Theorem 3.1.** *Let  $p \geq 1$ . Suppose that **(A1)** and **(A2)** are satisfied. Let  $\hat{H}$  be (3.2). Suppose that  $f \in \mathbf{B}_{r, q}^s(M)$  with  $s > d/r$ ,  $r \geq 1$  and  $q \geq 1$ . Then there exists a constant  $C > 0$  such that, for  $n$  large enough,*

$$\mathbb{E}(|\hat{H} - H|^p) \leq C \varphi_n(p),$$

where

$$\varphi_n(p) = \begin{cases} \left( \frac{\log(n)}{n} \right)^{sp/(2s+d)}, & \text{for } 2rs > d(2p-r), \\ \left( \frac{\log(n)}{n} \right)^{(s-d/r+d/(2p))p/(2s-2d/r+d)}, & \text{for } 2rs < d(2p-r), \\ \left( \frac{\log(n)}{n} \right)^{(s-d/r+d/(2p))p/(2s-2d/r+d)} (\log(n))^{\max(2p-r/q, 0)}, & \text{for } 2rs = d(2p-r). \end{cases}$$

The proof of Theorem 3.1 is based on Proposition 2.1 and a result on the rates of convergence of  $\hat{f}$  under the mean integrated  $\mathbb{L}_{2p}$  error. Among the consequences of Theorem 3.1, note that :

- $\hat{H}$  converge to  $H$  under the mean  $\mathbb{L}_p$  error, i.e.,  $\lim_{n \rightarrow \infty} \mathbb{E}(|\hat{H} - H|^p) = 0$ ,
- under some restriction on  $s, r$  and  $q$ , one can find  $p$  such that, for any  $\epsilon > 0$ , by the Markov inequality,  $\sum_{n=1}^{\infty} \mathbb{P}(|\hat{H} - H| \geq \epsilon) \leq \epsilon^{-p} \sum_{n=1}^{\infty} \varphi_n(p) < \infty$  (convergent Bertrand series). Therefore  $\hat{H}$  converge to  $H$  a.s. by the Borel-Cantelli lemma.

**Remark 3.1.** As in Remark 2.1, if  $n$  is such that  $c_* \geq 1/\log(n)$ , one can define  $\hat{H}$  (3.2) by replacing  $c_*$  in  $\hat{A}$  by  $1/\log(n)$  and Theorem 3.1 is still valid with the rate of convergence  $(\log(n))^p \varphi_n(p)$ .

#### 4. Proofs

**Proof of Proposition 2.1.** We have

$$\hat{H} - H = - \int_{\hat{A}} (\hat{f}(\mathbf{x}) \log(\hat{f}(\mathbf{x})) - f(\mathbf{x}) \log(f(\mathbf{x}))) d\mathbf{x} + \int_{\hat{A}^c} f(\mathbf{x}) \log(f(\mathbf{x})) d\mathbf{x}.$$

The triangular inequality yields

$$|\hat{H} - H| \leq F + G, \quad (4.1)$$

where

$$F = \int_{\hat{A}} |\hat{f}(\mathbf{x}) \log(\hat{f}(\mathbf{x})) - f(\mathbf{x}) \log(f(\mathbf{x}))| d\mathbf{x}, \quad G = \int_{\hat{A}^c} f(\mathbf{x}) |\log(f(\mathbf{x}))| d\mathbf{x}.$$

*Upper bound for G.* Observe that, thanks to **(A1)**, we have

$$\begin{aligned} \hat{A}^c &= \{\mathbf{x} \in [0, 1]^d; \hat{f}(\mathbf{x}) < c_*/2\} \subseteq \{\mathbf{x} \in [0, 1]^d; f(\mathbf{x}) - \hat{f}(\mathbf{x}) > c_*/2\} \\ &\subseteq \{\mathbf{x} \in [0, 1]^d; |\hat{f}(\mathbf{x}) - f(\mathbf{x})| > c_*/2\}. \end{aligned} \quad (4.2)$$

By **(A1)** and **(A2)**, (4.2) and the Markov inequality, we obtain

$$G \leq C_* \max(|\log(c_*)|, |\log(C_*)|) \int_{\hat{A}^c} d\mathbf{x} \leq C_o \int_{[0, 1]^d} |\hat{f}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x}, \quad (4.3)$$



where  $C_o = 2C_* \max(|\log(c_*)|, |\log(C_*)|)/c_*$ .

*Upper bound for  $F$ .* Note that

$$\begin{aligned} & \hat{f}(\mathbf{x}) \log(\hat{f}(\mathbf{x})) - f(\mathbf{x}) \log(f(\mathbf{x})) \\ &= \log(f(\mathbf{x}))(\hat{f}(\mathbf{x}) - f(\mathbf{x})) + f(\mathbf{x})(\log(\hat{f}(\mathbf{x})) - \log(f(\mathbf{x}))) \\ &+ (\hat{f}(\mathbf{x}) - f(\mathbf{x}))(\log(\hat{f}(\mathbf{x})) - \log(f(\mathbf{x}))). \end{aligned} \quad (4.4)$$

Moreover, using the inequality :  $|\log(x)| \leq |1/x - 1| + |x - 1|$ ,  $x > 0$ , for any  $\mathbf{x} \in \hat{A}$  (implying  $\hat{f}(\mathbf{x}) \geq c_*/2 > 0$ ), we have

$$\begin{aligned} & |\log(\hat{f}(\mathbf{x})) - \log(f(\mathbf{x}))| \\ &= \left| \log\left(\frac{\hat{f}(\mathbf{x})}{f(\mathbf{x})}\right) \right| \leq \left| \frac{f(\mathbf{x})}{\hat{f}(\mathbf{x})} - 1 \right| + \left| \frac{\hat{f}(\mathbf{x})}{f(\mathbf{x})} - 1 \right| \\ &= \left( \frac{1}{\hat{f}(\mathbf{x})} + \frac{1}{f(\mathbf{x})} \right) |\hat{f}(\mathbf{x}) - f(\mathbf{x})| \leq \frac{3}{c_*} |\hat{f}(\mathbf{x}) - f(\mathbf{x})|. \end{aligned} \quad (4.5)$$

It follows from (4.4), the triangular inequality and (4.5) that

$$\begin{aligned} & |\hat{f}(\mathbf{x}) \log(\hat{f}(\mathbf{x})) - f(\mathbf{x}) \log(f(\mathbf{x}))| \\ &\leq |\log(f(\mathbf{x}))| |\hat{f}(\mathbf{x}) - f(\mathbf{x})| + f(\mathbf{x}) |\log(\hat{f}(\mathbf{x})) - \log(f(\mathbf{x}))| \\ &+ |\hat{f}(\mathbf{x}) - f(\mathbf{x})| |\log(\hat{f}(\mathbf{x})) - \log(f(\mathbf{x}))| \\ &\leq \left( |\log(f(\mathbf{x}))| + \frac{3}{c_*} f(\mathbf{x}) \right) |\hat{f}(\mathbf{x}) - f(\mathbf{x})| + \frac{3}{c_*} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^2. \end{aligned} \quad (4.6)$$

By **(A1)** and **(A2)**, we obtain

$$\begin{aligned} & |\hat{f}(\mathbf{x}) \log(\hat{f}(\mathbf{x})) - f(\mathbf{x}) \log(f(\mathbf{x}))| \\ &\leq C_{oo} |\hat{f}(\mathbf{x}) - f(\mathbf{x})| + \frac{3}{c_*} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^2, \end{aligned}$$

where  $C_{oo} = \max(|\log(c_*)|, |\log(C_*)|) + 3C_*/c_*$ .

Hence

$$F \leq C_{oo} \int_{[0,1]^d} |\hat{f}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} + \frac{3}{c_*} \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^2 d\mathbf{x}. \quad (4.7)$$

Combining (4.1), (4.3) and (4.7), we have

$$|\hat{H} - H| \leq C' \int_{[0,1]^d} |\hat{f}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} + \frac{3}{c_*} \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^2 d\mathbf{x},$$

where

$$C' = C_{oo} + C_o = \frac{3C_*}{c_*} + \left( \frac{2C_*}{c_*} + 1 \right) \max(|\log(c_*)|, |\log(C_*)|).$$

The inequality :  $|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p)$ ,  $(x, y) \in \mathbb{R}^2$ , implies that

$$\begin{aligned} & |\hat{H} - H|^p \\ & \leq 2^{p-1} \left( (C')^p \left( \int_{[0,1]^d} |\hat{f}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \right)^p + \right. \\ & \quad \left. \left( \frac{3}{c_*} \right)^p \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^2 d\mathbf{x} \right)^p \right) \\ & \leq K \left( \left( \int_{[0,1]^d} |\hat{f}(\mathbf{x}) - f(\mathbf{x})| d\mathbf{x} \right)^p + \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^2 d\mathbf{x} \right)^p \right), \end{aligned}$$

where

$$K = 2^{p-1} \max \left( \left( \frac{3C_*}{c_*} + \left( \frac{2C_*}{c_*} + 1 \right) \max(|\log(c_*)|, |\log(C_*)|) \right)^p, \left( \frac{3}{c_*} \right)^p \right).$$

The Hölder inequality applied two times gives

$$|\hat{H} - H|^p \leq K \left( \sqrt{\int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x}} + \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x} \right).$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} & \mathbb{E}(|\hat{H} - H|^p) \\ & \leq K \left( \sqrt{\mathbb{E} \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x} \right)} + \mathbb{E} \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x} \right) \right). \end{aligned}$$

The proof of Proposition 2.1 is complete.  $\square$

**Proof of Theorem 3.1.** First of all, let us present a result on the rates of convergence of  $\hat{f}$  (3.3) under the mean  $\mathbb{L}_\theta$  error over Besov balls.

**Theorem 4.1** (Delyon and Juditsky (1996) & Kerkyacharian and Picard (2000)). *Suppose that (A2) holds. Let  $\theta \geq 1$  and  $\hat{f}$  be (3.3). Suppose that  $f \in \mathbf{B}_{r,q}^s(M)$  with  $s > d/r$ ,  $r \geq 1$  and  $q \geq 1$ . Then there exists a constant  $C > 0$  such that*

$$\mathbb{E} \left( \int_{[0,1]^d} |\hat{f}(\mathbf{x}) - f(\mathbf{x})|^\theta d\mathbf{x} \right) \leq C \Psi_n(\theta),$$

where

$$\Psi_n(\theta) = \begin{cases} \left(\frac{\log(n)}{n}\right)^{s\theta/(2s+d)}, & \text{for } 2rs > d(\theta - r), \\ \left(\frac{\log(n)}{n}\right)^{(s-d/r+d/\theta)\theta/(2s-2d/r+d)}, & \text{for } 2rs < d(\theta - r), \\ \left(\frac{\log(n)}{n}\right)^{(s-d/r+d/\theta)\theta/(2s-2d/r+d)} (\log(n))^{\max(\theta-r/q, 0)}, & \text{for } 2rs = d(\theta - r). \end{cases}$$

Theorem 4.1 can be proved using similar arguments to (Kerkycharian and Picard, 2000, Theorem 5.1) for a bound of the mean integrated  $\mathbb{L}_\theta$  error of  $\hat{f}$  and (Delyon and Juditsky, 1996, Theorem 1) for the determination of the rates of convergence.

It follows from Proposition 2.1 and Theorem 3.1 with  $\theta = 2p$  that, for any  $f \in \mathbf{B}_{r,q}^s(M)$  with  $s > d/r$ ,  $r \geq 1$  and  $q \geq 1$ , and for  $n$  large enough,

$$\begin{aligned} & \mathbb{E}(|\hat{H} - H|^p) \\ & \leq K \left( \sqrt{\mathbb{E} \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x} \right)} + \mathbb{E} \left( \int_{[0,1]^d} (\hat{f}(\mathbf{x}) - f(\mathbf{x}))^{2p} d\mathbf{x} \right) \right) \\ & \leq K \max(\sqrt{C}, C) \left( \sqrt{\Psi_n(2p)} + \Psi_n(2p) \right) \leq 2K \max(\sqrt{C}, C) \sqrt{\Psi_n(2p)} \\ & = C_* \varphi_n(p), \end{aligned}$$

with  $C_* = 2K \max(\sqrt{C}, C)$ .

This ends the proof of Theorem 3.1.  $\square$

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