On the stability of a gaseous sphere against non-radial perturbations

Jérôme Perez, Jean-Jacques Aly

Mon. Not. R. Astron. Soc. (1992) **259,** 95–103

On the stability of a gaseous sphere against non-radial perturbations

Jean-Jacques Aly[1] and Jérôme Pérez[1,2]

[1] *Service d'Astrophysique, CE Saclay, 91191 Gif sur Yvette Cedex, France*
[2] *ETCA/CREA, 16 bis av. Prieur de la côte d'or, 94114 Arcueil Cedex, France*

Accepted 1992 May 5. Received 1992 May 5; in original form 1991 December 30

ABSTRACT

We present a simplified proof of the Antonov–Lebovitz theorem, asserting that any spherical barotropic star having a mass density decreasing monotonically outwards and vanishing at its surface is stable to all non-radial perturbations. We also develop a simple argument showing in a straightforward way a related but somewhat weaker result, according to which any such star is stable if and only if it is stable to radial perturbations. Extension of these results to a star with a non-decreasing specific entropy distribution is also briefly discussed.

Key words: instabilities – celestial mechanics, stellar dynamics – stars: interiors.

1 INTRODUCTION

In the sixties, a quite remarkable stability result was obtained independently by Antonov (1962), Lebovitz (1965), and Lynden-Bell & Sanitt (1969) ('stability' is understood in this paper to mean 'linear stability'). It may be stated as follows. Any spherical barotropic star having a mass density decreasing monotonically outwards and vanishing at its surface is stable to all the purely non-radial perturbations – i.e. to all the perturbations for which the associated variation of the gravitational potential has a zero mean value over any sphere concentric with the star. This general theorem is referred to as the 'Antonov–Lebovitz Theorem' (ALT) in the monograph by Binney & Tremaine (1987, hereafter BT), where it is given a clear presentation (pp. 300, 687). Besides its obvious importance in the theory of stellar structure, it is also of great interest when one studies the properties of a spherical cluster of stars interacting together through their mean gravitational field (collisionless approximation). It allows a demonstration of the stability to all non-radial perturbations of a cluster described by a distribution function depending only on the energy of a star and decreasing with it ['Antonov's Second Law' in BT, p. 306; see also Antonov (1962), Lynden-Bell & Sanitt (1969), Kandrup & Sygnet (1985)].

In the first part of this paper (Sections 2 and 3), we revisit ALT and demonstrate a new proof based on Chandrasekhar's energy principle (e.g. BT, p. 300). The method we use consists of constructing a positive lower bound on the second variation of the energy just by appealing to some tricky algebraic transformations and to two standard inequalities (Schwartz's and Wirtinger's, a derivation of which is given in Appendix A). Our proof turns out to be much simpler and more transparent than the earlier ones, in which it was shown in a first step that the eigenfunctions of some operator are stable modes, and in a second step that they form a complete set in the space of the admissible perturbations [actually, this result was taken for granted by the authors quoted above and proved only later by Eisenfeld (1969)]. In particular, our approach allows us to bypass completely this difficult completeness problem.

In the second part of this paper (Section 4), we develop a simple argument showing that a star as considered above is stable if and only if it is stable to the radial perturbations. Although leading to a result somewhat weaker than ALT, our argument sheds some light on the reasons why this latter holds. It is based on the technique of symmetrization of a function, which has already been used in the study of spherical clusters of stars – in an intuitive way by Antonov (1962), Lynden-Bell & Wood (1968), Ipser (1974) and certainly others, and in a more formal way by Aly (1989).

It must be noted that we have made the barotropy assumption only so as not to obscure the presentation with too complicated technical details. Our results are still true under the much less restrictive assumption – actually the one made in Lebovitz's (1965) original paper – that the specific entropy of the gas is a non-decreasing function of r (Schwarzschild's criterion for the absence of convection). This point is briefly discussed in Appendix B.

2 STATEMENT OF THE PROBLEM

2.1 The equilibrium

Consider an isolated spherically symmetric barotropic star of mass $M\,(<\infty)$, radius $R\,(<\infty)$, and centre O in static equilibrium in the inertial Cartesian frame $(O, \hat{x}, \hat{y}, \hat{z})$. Then the mass density ρ_0 of the gas, its thermal pressure p_0 (imposed to vanish at the surface of the star), and the gravitational potential ψ_0 (imposed to vanish at infinity) are functions only of the distance r $(0 \le r < \infty)$ to the origin and they are related by (e.g. Chandrasekhar 1958)

$$p_0' = -\rho_0 \psi_0', \tag{1}$$

$$\Delta\psi_0 = \frac{1}{r^2}(r^2\psi_0')' = 4\pi G\rho_0, \tag{2}$$

$$p_0 = g(\rho_0), \tag{3}$$

where a prime indicates a derivative with respect to r, G is the gravitational constant, and $g(\rho)$ is a given function of $\rho \ge 0$ satisfying

$$g(0) = 0 \qquad \text{and} \qquad \dot{g} \triangleq \frac{\mathrm{d}g}{\mathrm{d}\rho}(\rho) > 0, \qquad \rho > 0. \tag{4}$$

(Here, $\triangleq$ means 'equal by definition'.) As is well known, ρ_0 and p_0 are strictly decreasing functions of r for $0 < r < R$ and vanish for $r \ge R$, while ψ_0' is a positive quantity for $r > 0$ and decreases as r^{-2} at infinity. Then we have in particular,

$$\rho_0'(r) < 0 \qquad \text{for} \qquad 0 < r < R \qquad \text{and} \qquad \rho_0'(r) = 0 \qquad \text{for} \qquad r > R, \tag{5}$$

$$\psi_0' > 0 \qquad \text{for} \qquad 0 < r. \tag{6}$$

On the other hand, differentiating (2) with respect to r, we obtain

$$\Delta\psi_0' = 2\frac{\psi_0'}{r^2} + 4\pi G\rho_0' = 2\frac{\psi_0'}{r^2} - 4\pi G|\rho_\psi|\psi_0', \tag{7}$$

where we have set, taking (5) and (6) into account,

$$|\rho_\psi| = -\rho_\psi \triangleq -\frac{\mathrm{d}\rho_0}{\mathrm{d}\psi_0} = -\frac{\rho_0'}{\psi_0'} \ge 0 \tag{8}$$

(clearly, $|\rho_\psi| > 0$ for $0 < r < R$, and $|\rho_\psi| = 0$ for $R < r$).

We shall use hereafter the notation $\psi[\rho, \boldsymbol{r}]$ (or $\psi[\rho]$ for short) to represent the general solution of Poisson's equation $\Delta\psi = 4\pi G\rho$ in $\mathbb{R}^3$, with $\lim_{r\to\infty} \psi = 0$, i.e. we set

$$\psi[\rho, \boldsymbol{r}] = -G\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\,\mathrm{d}\boldsymbol{r}' \tag{9}$$

(unless otherwise stated, an integral with respect to $\mathrm{d}\boldsymbol{r}$ is assumed to be taken over $\mathbb{R}^3$).

2.2 Chandrasekhar's energy principle

The 'potential energy' of an arbitrary self-gravitating configuration (equilibrium or not), characterized by a mass density $\rho(\boldsymbol{r})$ and a pressure $p(\boldsymbol{r})$ related to $\rho(\boldsymbol{r})$ by the barotropic law $p = g(\rho)$, is given by

$$W[\rho] \triangleq \int \left\{ U(\rho(\boldsymbol{r})) + \frac{1}{2}\rho(\boldsymbol{r})\,\psi[\rho, \boldsymbol{r}] \right\} \mathrm{d}\boldsymbol{r}, \tag{10}$$

where

$$U(\rho) = \rho\int_0^\rho \frac{g(\tau)}{\tau^2}\,\mathrm{d}\tau \tag{11}$$

is the thermal energy per unit of volume of the gas. In particular, the potential energy of the equilibrium described in Section 2.1 is just given by $W[\rho_0]$.

Let us then perturb the previous equilibrium by applying a Lagrangian displacement $\boldsymbol{\xi}(\boldsymbol{r})$ to each element of the fluid, the entropy being assumed to be locally conserved during that process. In the resulting state, the mass density and the gravitational potential can be written [by introducing a standard system of spherical coordinates $(O, \hat{r}, \hat{\theta}, \hat{\phi})$]

$$\rho(r, \theta, \phi) = \rho_0(r) + \rho_1(r, \theta, \phi) + \ldots, \tag{12}$$

$$\psi(r, \theta, \phi) = \psi_0(r) + \psi_1(r, \theta, \phi) + \ldots, \tag{13}$$

with ρ_1 and ψ_1 of first order in $\boldsymbol{\xi}$ and being related to each other by the perturbed Poisson equation

$$\Delta\psi_1 = 4\pi G\rho_1, \qquad \text{or} \qquad \psi_1 = \psi[\rho_1]. \tag{14}$$

Because of the assumption on the conservation of the entropy, the barotropic law $p = g(\rho)$ holds in the final configuration too, and we can consider the variation $W[\rho] - W[\rho_0]$ of the potential energy resulting from the deformation of the equilibrium. Clearly, this quantity vanishes to first order in ρ_1 owing to the hydrostatic equation (1), and it is written to second order:

$$\delta^2 W[\rho_1] = \frac{1}{2}\int_D \left(\frac{\rho_1^2}{|\rho_\psi|} + \rho_1\psi[\rho_1]\right) \mathrm{d}\boldsymbol{r}, \tag{15}$$

where $D \triangleq \{\boldsymbol{r} : |\boldsymbol{r}| < R\}$ is the sphere occupied by the unperturbed fluid. The form of the first term in the integrand is a consequence of the relation

$$\ddot{U}(\rho_0) = \frac{\dot{g}(\rho_0)}{\rho_0} = |\rho_\psi|^{-1}, \tag{16}$$

where the first equality just results from (11) while the second one holds because of (1) and (4).

Chandrasekhar's energy principle (e.g. BT, p. 300) asserts that the star is linearly stable with respect to all the adiabatic perturbations if and only if

$$\delta^2 W[\rho_1] > 0 \tag{17}$$

for any $\rho_1 \neq 0$ compatible with total mass conservation and invariance of the position of the centre of mass, i.e. for any $\rho_1 \neq 0$ satisfying

$$\int_D \rho_1 \mathrm{d}\boldsymbol{r} = 0, \tag{18}$$

$$\int_D \rho_1 \boldsymbol{r}\, \mathrm{d}\boldsymbol{r} = 0. \tag{19}$$

Note that (9), along with the constraint (18), implies

$$r^2\psi_1 \underset{r\to\infty}{\simeq} r^3|\nabla\psi_1|\underset{r\to\infty}{\simeq}\mathscr{O}(1), \tag{20}$$

while the second condition (actually not quoted explicitly in BT) allows one to eliminate the irrelevant variations $\rho_1 \neq 0$ resulting from a mere small global translation of the star, for which $\delta^2 W[\rho_1] = 0$ while obviously no physical instability is implied. It is worth noticing that the other usual class of neutral perturbations, the so-called 'interchanges', is automatically overlooked in the above formulation of the energy principle in terms of ρ_1 rather than $\boldsymbol{\xi}$. Such perturbations correspond to displacements of the form $\boldsymbol{\xi} = \rho_0^{-1}\nabla \times \boldsymbol{A}$, with $\boldsymbol{A}$ some arbitrary vector, and they do not change the mass density distribution as they have $\rho_1 = -\nabla\cdot(\rho_0\boldsymbol{\xi}) = 0$ (for instance, purely toroidal displacements are clearly of this type). Then a star, if it is shown to be stable by the criterion above, should be said more correctly to be 'neutrally stable'. All of this is, however, sufficiently well known to make a detailed discussion unnecessary here.

We now introduce the decomposition for $f_1 = \rho_1$ or ψ_1:

$$f_1(r, \theta, \phi) = \overline{f_1}(r) + \delta f_1(r, \theta, \phi), \tag{21}$$

where $(\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi)$

$$\overline{f_1}(r) = \frac{1}{4\pi}\int f_1(r, \theta, \phi)\, \mathrm{d}\Omega, \tag{22}$$

and thus

$$\int \delta f_1(r,\,\theta,\,\phi)\,\mathrm{d}\Omega = 0. \tag{23}$$

It follows from (14), (19) and (20) that

$$\Delta\overline{\psi_1} = 4\pi G\overline{\rho_1}, \qquad \text{or} \qquad \overline{\psi_1} = \psi[\overline{\rho_1}], \tag{24}$$

$$\Delta\delta\psi_1 = 4\pi G\delta\rho_1, \qquad \text{or} \qquad \delta\psi_1 = \psi[\delta\rho_1], \tag{25}$$

$$r^2\overline{\psi_1}_{r\to\infty} \simeq r^2\delta\psi_{1\,r\to\infty} \simeq r^3|\nabla\overline{\psi_1}|_{r\to\infty} \simeq r^3|\nabla\delta\psi_1|_{r\to\infty} \simeq \mathscr{O}(1), \tag{26}$$

$$\int_D \delta\rho_1 \boldsymbol{r}\,\mathrm{d}\boldsymbol{r} = 0. \tag{27}$$

Substituting (21) into (15) and using (22), we can rewrite the stability criterion in the form

$$\delta^2 W[\rho_1] = \delta_{\mathrm{r}}^2 W[\overline{\rho_1}] + \delta_{\mathrm{nr}}^2 W[\delta\rho_1] > 0, \tag{28}$$

where

$$\delta_{\mathrm{r}}^2 W[\overline{\rho_1}] \triangleq \frac{1}{2}\int_D \left(\frac{\overline{\rho_1}^2}{|\rho_\psi|} + \overline{\rho_1}\,\overline{\psi_1}\right)\mathrm{d}\boldsymbol{r}, \tag{29}$$

$$\delta_{\mathrm{nr}}^2 W[\delta\rho_1] \triangleq \frac{1}{2}\int_D \left(\frac{\delta\rho_1^2}{|\rho_\psi|} + \delta\rho_1\delta\psi_1\right)\mathrm{d}\boldsymbol{r}. \tag{30}$$

The ALT (BT, p. 687) then states that

$$\delta_{\mathrm{nr}}^2 W[\delta\rho_1] > 0 \tag{31}$$

for any $\delta\rho_1 \neq 0$ satisfying (23) and (27). Our goal now is to give a new proof of (31) which is simpler than the earlier ones. For convenience, we shall assume from now on that $\delta\rho_1 = 0$ for $r > R$ [this does not entail any loss of generality, as $\delta\rho_1$ just appears in the integral (30) which is taken over D].

3 A NEW PROOF OF THE ANTONOV–LEBOVITZ THEOREM

Using the perturbed Poisson equation (25) and Schwartz's inequality, we obtain (remembering that $\delta\rho_1 = 0$ and $|\rho_\psi| = 0$ for $r > R$)

$$\left(\int \frac{|\nabla\delta\psi_1|^2}{4\pi G}\,\mathrm{d}\boldsymbol{r}\right)^2 = \left(\int \delta\rho_1\delta\psi_1\,\mathrm{d}\boldsymbol{r}\right)^2 \leq \int \frac{|\delta\rho_1|^2}{|\rho_\psi|}\,\mathrm{d}\boldsymbol{r}\cdot\int |\rho_\psi|\,\delta\psi_1|^2\,\mathrm{d}\boldsymbol{r}, \tag{32}$$

with equality if and only if

$$\delta\rho_1 = -\lambda^{-1}|\rho_\psi|\,\delta\psi_1, \tag{33}$$

for some constant λ. Whence, injecting (32) into (30),

$$2\delta_{\mathrm{nr}}^2 W[\delta\rho_1] \geq \left(\int \frac{|\nabla\delta\psi_1|^2}{4\pi G}\,\mathrm{d}\boldsymbol{r}\right)\left(\int |\rho_\psi|\,\delta\psi_1|^2\,\mathrm{d}\boldsymbol{r}\right)^{-1}\left(\int \frac{|\nabla\delta\psi_1|^2}{4\pi G}\,\mathrm{d}\boldsymbol{r} - \int |\rho_\psi|\,\delta\psi_1|^2\,\mathrm{d}\boldsymbol{r}\right). \tag{34}$$

We now set

$$\delta\psi_1(r,\,\theta,\,\phi) \triangleq \psi_0'(r)\,h(r,\,\theta,\,\phi), \tag{35}$$

with h being clearly bounded over $\mathbb{R}^3$ [in particular, $h_r \underset{\infty}{\simeq} \mathcal{O}(1)$ owing to $\psi_0' \simeq \delta\psi_1 \simeq r^{-2}$ when $r \to \infty$]. Then we have, using Gauss' theorem:

$$\int |\nabla \delta\psi_1|^2 \mathrm{d}\boldsymbol{r} = \int \{|\nabla\psi_0'|^2 h^2 + \psi_0' \nabla\psi_0' \cdot \nabla h^2 + |\psi_0'|^2 |\nabla h|^2\} \mathrm{d}\boldsymbol{r}$$

$$= \int \{|\nabla\psi_0'|^2 h^2 - h^2 \nabla \cdot (\psi_0' \nabla\psi_0') + |\psi_0'|^2 |\nabla h|^2\} \mathrm{d}\boldsymbol{r}$$

$$= \int \{-h^2 \psi_0' \Delta\psi_0' + |\psi_0'|^2 |\nabla h|^2\} \mathrm{d}\boldsymbol{r}, \tag{36}$$

whence, making use of (7),

$$\int |\nabla \delta\psi_1|^2 \mathrm{d}\boldsymbol{r} = \int |\psi_0'|^2 \left(\left| \frac{\partial h}{\partial r} \right|^2 + |\nabla_S h|^2 - \frac{2h^2}{r^2} \right) \mathrm{d}\boldsymbol{r} + 4\pi G \int |\rho_\psi| \, \delta\psi_1|^2 \mathrm{d}\boldsymbol{r}, \tag{37}$$

where $\nabla_S \triangleq \nabla - \hat{\boldsymbol{r}}\,\partial/\partial r$. On the other hand, we note that, because of (23), h has zero average value over a sphere S_r of centre O and arbitrary radius r. Therefore h satisfies the general inequality (referred to sometimes as Wirtinger's inequality; see appendix)

$$\int \frac{2}{r^2} |h(r, \theta, \phi)|^2 \mathrm{d}\Omega \leq \int |\nabla_S h(r, \theta, \phi)|^2 \mathrm{d}\Omega \quad \forall r \in]0, \infty[, \tag{38}$$

with equality if and only if

$$h(\boldsymbol{r}) = h_0(r)\, \boldsymbol{r} \cdot \hat{\boldsymbol{a}} \tag{39}$$

for some fixed unit vector $\hat{\boldsymbol{a}}$. Combining (34), (37) and (38), we thus eventually obtain

$$2\delta_{\mathrm{nr}}^2 W \geq \frac{1}{4\pi G} \int |\psi_0'|^2 \left| \frac{\partial h}{\partial r} \right|^2 \mathrm{d}\boldsymbol{r} \cdot \left(1 + \frac{\int |\psi_0'|^2 |\partial h/\partial r|^2 \mathrm{d}\boldsymbol{r}}{4\pi G \int |\rho_\psi| \cdot |\delta\psi_1|^2 \mathrm{d}\boldsymbol{r}} \right) \geq 0. \tag{40}$$

Let us now suppose that $\delta_{\mathrm{nr}}^2 W = 0$. Because of (40), this implies that $\partial h/\partial r = 0$ and that the two inequalities (32) and (38) we have used to bound $\delta_{\mathrm{nr}}^2 W$ from below reduce to equalities. Then, in particular, (39) holds, and we must have

$$\delta\psi_1 = \varepsilon \hat{\boldsymbol{a}} \cdot \nabla\psi_0, \tag{41}$$

for some $\hat{\boldsymbol{a}}$ and some constant ε. Whence, by (25) and (2):

$$\delta\rho_1 = \varepsilon \hat{\boldsymbol{a}} \cdot \nabla \frac{\Delta\psi_0}{4\pi G} = \varepsilon \hat{\boldsymbol{a}} \cdot \nabla\rho_0 = \rho_\psi \delta\psi_1, \tag{42}$$

a relation which turns out to be just of the form (33) (with $\lambda = 1$). To be admissible, however, this $\delta\rho_1$ has to satisfy the constraint (27), i.e. we must have (using Gauss' theorem)

$$\int \delta\rho_1 \boldsymbol{r} \mathrm{d}\boldsymbol{r} = -\varepsilon \hat{\boldsymbol{a}} M = 0, \tag{43}$$

and therefore $\varepsilon = 0$ and $\delta\rho_1 = 0$ [physically, the variations $\delta\psi_1$ and $\delta\rho_1$ given by (41) and (42), respectively, result from a mere constant displacement $\boldsymbol{\xi} = \varepsilon\hat{\boldsymbol{a}}$ of the whole star, which is clearly forbidden by the constraint of invariance of the centre of mass position]. This argument implies at once that (31) indeed holds true for any non-zero $\delta\rho_1$ satisfying (27).

The proof we have just reported differs in several important respects from earlier ones. For instance, let us compare it in detail with that given in BT (p. 687) (our discussion also applying, with some minor changes, to all the other variants).

(i) In BT, it is assumed that the minimum of $\delta_{\mathrm{nr}}^2 W[\delta\rho_1]$ over the set of admissible perturbations – satisfying the arbitrarily chosen normalization constraint $\int |\rho_\psi|^{-1} |\delta\rho_1|^2 \mathrm{d}\boldsymbol{r} = 2K$, taken into account later on by means of a Lagrange multiplier λ – is actually reached for some $\delta\rho_1^-$ belonging to that set, the corresponding $\delta\psi_1^-$ thus solving – for some value λ^- of λ – the associated Euler–Lagrange boundary value problem:

$$-\Delta\delta\psi_1 + \frac{4\pi G}{\lambda} |\rho_\psi| \, \delta\psi_1 = 0 \qquad \text{in} \qquad \mathbb{R}^3, \tag{44}$$

$$r^2\,\delta\psi_1 \underset{r\to\infty}{=} \mathcal{O}(1), \tag{45}$$

completed by (23) and a normalization condition deriving from that imposed on $\delta\rho_1$. It is easily checked that

$$\delta^2_{\rm nr}W[\delta\rho_1] \geq \delta^2_{\rm nr}W[\delta\rho_1^-] = (1-\lambda^-)\,K,$$

and therefore the equilibrium should be necessarily stable if $\lambda^- < 1$.

That $\delta^2_{\rm nr}W[\delta\rho_1]$ reaches its minimum at some admissible $\delta\rho_1^-$ is, however, not completely obvious [there are many instances in physics of minimization problems which have no solutions; see, e.g. Aly (1992) for an example of a functional which, although being bounded from below, does not reach its minimum over the set of admissible functions, in which, however, the associated Euler–Lagrange equation has a unique solution], and some justification is needed. As pointed out to us by the referee, it appears here that the gap may be filled by appealing to a completeness theorem proven by Eisenfeld (1969). Although we have not considered this point in detail, it seems that Eisenfeld's result (which relies on the original formulation of the problem by Lebovitz 1965) implies that (44)–(45) admit a complete system of eigenfunctions $\{\delta\psi_1^{(i)}\}$, and we have $\delta\psi_1^- = \delta\psi_1^{(1)}$ and $\lambda^- = \lambda_1 \triangleq \max\{\lambda_i\}$. Then we can write, for an arbitrary perturbation:

$$\delta\psi_1 = \sum c_i\,\delta\psi_1^{(i)}, \tag{46}$$

and

$$\delta^2_{\rm nr}W[\delta\rho_1] = \sum |c_i|^2(1-\lambda_i)\,K \geq (1-\lambda_1)\sum |c_i|^2 K = \delta^2_{\rm nr}W[\delta\rho_1^{(1)}], \tag{47}$$

and stability of $\delta\rho_1^{(1)}$ indeed implies general stability. In our proof, on the contrary, we do not have to deal with the problem of the existence of a minimizer $\delta\rho_1^-$: we just use our two inequalities to get a lower bound on $\delta^2_{\rm nr}W[\delta\rho_1]$ which applies to any $\delta\rho_1$.

(ii) BT's calculation following their (5C-5) (our equation 44) and ours following (35) certainly bear some apparent similarities. They are, however, quite different, as BT establish ALT by proving the positivity of $\delta^2_{\rm nr}W[\delta\rho_1^-]$, thus using in a crucial way the equation defining $\delta\rho_1^-$, while we stay at a completely general level throughout our argument, which applies directly to an arbitrary admissible $\delta\psi_1$. This way of proceeding turns out to lead to many simplifications (for instance, we do not have to deal with a matching problem at the surface of D as in BT) and allows us at the same time to identify precisely the reason why the RHS of (34) is necessarily positive: this is just due to inequality (38) which holds on every sphere S_r, a point somewhat hidden in BT's proof [and more so in that of Lynden-Bell & Sannit (1969), which apparently rests on arguments about the radial structure of the solutions of (44)–(45)].

4 AN ALTERNATIVE APPROACH

Let us now develop a somewhat different approach to the non-radial stability problem. For this, we first introduce the set of functions (conserving the notations of the previous sections)

$$\mathbb{H} \triangleq \left\{\rho\,|\,\rho(\boldsymbol{r})\geq 0;\quad \int \rho(\boldsymbol{r})\,{\rm d}\boldsymbol{r} = M;\quad \int \rho(\boldsymbol{r})\,\boldsymbol{r}\,{\rm d}\boldsymbol{r} = 0\right\}, \tag{48}$$

to which belongs the equilibrium density ρ_0.

By effecting a so-called Schwartz's symmetrization (e.g. Lieb 1977; Mossino 1984), we can construct from any ρ its 'spherically symmetric decreasing rearrangement' $\rho^*(r)$, which is the essentially unique non-increasing, non-negative function of r which has the same 'distribution function' as ρ (i.e. we have ${\rm meas}\{\boldsymbol{r}\,|\,\rho(\boldsymbol{r})>\tau\} = {\rm meas}\{\boldsymbol{r}\,|\,\rho^*(r)>\tau\}$ for any $\tau\geq 0$, with meas A denoting the volume of the subset A of $\mathbb{R}^3$). ρ^* is well known to satisfy the following.

(i) For any sufficiently regular function G,

$$\int G(\rho^*)\,{\rm d}\boldsymbol{r} = \int G(\rho)\,{\rm d}\boldsymbol{r}. \tag{49}$$

Taking in particular $G(t)=t$, we conclude immediately that ρ^* belongs to $\mathbb{H}$ too (the condition $\int \rho^*\,\boldsymbol{r}\,{\rm d}\boldsymbol{r}=0$ being trivially satisfied owing to the symmetry of ρ^*).

(ii) Riez's inequality:

$$-\int \rho\psi[\rho]\,{\rm d}\boldsymbol{r} = G\int \frac{\rho(\boldsymbol{r})\,\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,{\rm d}\boldsymbol{r}\,{\rm d}\boldsymbol{r}' \leq G\int \frac{\rho^*(r)\,\rho^*(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,{\rm d}\boldsymbol{r}\,{\rm d}\boldsymbol{r}' = -\int \rho^*\,\psi[\rho^*]\,{\rm d}\boldsymbol{r}, \tag{50}$$

with equality if and only if $\rho=\rho^*$ (Lieb 1977).

Physically, (49) and (50) appear to be quite simple to interpret. ρ^* can be considered as being obtained from ρ by moving in an incompressible way the fluid elements of mass $\rho\,\mathrm{d}\boldsymbol{r}$, rearranging them in a symmetric way, the heavier of two elements being placed nearer the centre. During that process, $G(\rho)\,\mathrm{d}\boldsymbol{r}$ is thus conserved, whence (49), while the potential energy is clearly decreased, whence (50).

From (49) with $G = U$ and (50), we can assert that

$$W[\rho] - W[\rho_0] \geq W[\rho^*] - W[\rho_0], \tag{51}$$

where W is defined by (10) and $W[\rho]$ is therefore the energy of a configuration related to the equilibrium by an entropy-conserving transform.

Let us now assume that $\rho_1 = \rho - \rho_0$ is a small perturbation of ρ_0. Thus, as we have $\rho_0^* = \rho_0$ because of (5), $(\rho^*)_1 = \rho^* - \rho_0$ is also a small perturbation of ρ_0, and we can write, developing each member of (51) up to the second order, and using (16):

$$\int_D \left\{ \frac{|\rho_1|^2}{|\rho_\psi|} + \rho_1 \psi[\rho_1] \right\} \mathrm{d}\boldsymbol{r} \geq \int_D \left\{ \frac{|(\rho^*)_1|^2}{|\rho_\psi|} + (\rho^*)_1 \psi[(\rho^*)_1] \right\} \mathrm{d}\boldsymbol{r}, \tag{52}$$

with equality if and only if $(\rho^*)_1 = \rho_1$. If the star is stable wih respect to any radial admissible perturbation, Chandrasekhar's energy principle thus implies that

$$\delta^2 W[\rho_1] \geq \delta^2 W[(\rho^*)_1] \geq 0, \tag{53}$$

with the second inequality being strict if $(\rho^*)_1 \neq 0$, and therefore

$$\delta^2 W[\rho_1] > 0, \tag{54}$$

for any admissible $\rho_1 \neq 0$, as $\delta^2 W[\rho_1] = 0$ would imply $\rho_1 = (\rho^*)_1 = 0$. A star thus appears to be stable against arbitrary perturbations – in particular against non-radial ones – if it is radially stable. This result is slightly weaker than ALT but it has been obtained with almost no calculations and it certainly allows us to understand better this theorem.

5 CONCLUSION

We have given in this paper a proof of ALT which turns out to be much simpler than those established earlier, which involved a lot of algebraic computation (e.g. BT, p. 687) or complicated considerations on the radial structure of the eigenmodes (Lynden-Bell & Sanitt 1969) and required a difficult proof of eigenfunction completeness. In fact, we have shown that ALT is essentially a simple consequence of the standard Wirtinger's inequality, which can be applied to $\delta\psi_1(r, \theta, \phi)$ over the sphere S_r (for all r) because of the imposed constraint '$\delta\psi_1$ has zero mean value over S_r'. We have also proved in a straightforward way a related result (which can be considered in fact as a corollary of ALT) according to which it is only needed to establish the stability of a barotropic star with $\rho_0'(r) < 0$ to prove its radial stability. This theorem has been obtained by using in a formal way the powerful symmetrization technique.

On the other hand, we show (Appendix B) that all these results are still true in the case when the specific entropy of the star is no longer imposed to be constant, but is more generally allowed to be a non-decreasing function of r.

ACKNOWLEDGMENT

We thank the referee for his comments on the manuscript and for having pointed out to us that the completeness problem discussed in Section 3 had been solved by Eisenfeld.

REFERENCES

Aly J. J., 1989, MNRAS, 241, 15
Aly J. J., 1992, Phys. Fluids B, in press.
Antonov V. A., 1992, Vest. Leningrad Univ., 7, 135
Aubin T., 1982, Nonlinear Analysis on Manifolds. Monge-Ampère Equations. Springer Verlag, Berlin
Binney J., Tremaine S., 1987, Galactic Dynamics. Princeton Univ. Press, Princeton NJ
Chandrasekhar S., 1958, An Introduction to the Study of Stellar Structure. Dover, New York
Courant R., Hilbert D., 1962, Methods of Mathematical Physics, Interscience, New York
Eisenfeld J., 1969, J. Math. An. Applic., 26, 357
Ipser J. R., 1974, ApJ, 193, 463
Kandrup H., Sygnet J. F., 1985, ApJ, 298, 27
Lebovitz N. R., 1965, ApJ, 142, 229
Lieb E. H., 1977, Stud. Appl. Math., 57, 93
Lynden-Bell D., Sannit N., 1969, MNRAS, 143, 167

Lynden-Bell D., Wood R., 1968, MNRAS, 138, 495
Mossino J., 1984, Inégalités Isopérimétriques et Applications en Physique. Hermann, Paris

APPENDIX A: WIRTINGER'S INEQUALITY

For the convenience of the reader, we give a simple proof of inequality (38), which is of quite general use. Let us then consider a function $f(r, \theta, \phi)$ with zero average value $\overline{f(r)}$ over the sphere S_r. Owing to the completeness of the spherical harmonics and the standard relations of orthogonality (e.g. Courant & Hilbert 1962),

$$\int_{S_r} \nabla_S Y^l_m \cdot \nabla_S Y^{l'}_{m'} \, d\Omega = \frac{l(l+1)}{r^2} \int_{S_r} Y^l_m \cdot Y^{l'}_{m'} \, d\Omega = \frac{l(1+1)}{r^2} \delta^l_{l'} \delta^m_{m'} \tag{A1}$$

(which results at once from the equation $-\Delta_s Y^l_m = l(l+1) r^{-2} Y^l_m$, where $\Delta_S = \nabla_S \cdot \nabla_S$), we can write

$$f(r, \theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} c^l_m(r) \, Y^l_m(\theta, \phi), \tag{A2}$$

and

$$\int_{S_r} |\nabla_S f|^2 \, d\Omega = \sum_{l \geq 1; m} |c^l_m(r)|^2 \frac{l(l+1)}{r^2} \geq \frac{2}{r^2} \sum_{l \geq 1; m} |c^l_m(r)|^2 = \frac{2}{r^2} \int_{S_r} |f|^2 \, d\Omega. \tag{A3}$$

Note that if $\overline{f(r)} \neq 0$, we can apply the previous inequality to $f(r, \theta, \phi) - \overline{f(r)}$.

In fact, (38) is a particular case of a general inequality valid on any closed regular surface S. For any f defined on S and averaging to zero over it, one has (Aubin 1982, corollary 4.3, p. 102)

$$\int_S |f|^2 \, d\sigma \leq \frac{1}{\lambda_1^2} \int_S |\nabla_S f|^2 \, d\sigma, \tag{A4}$$

where $d\sigma$ and ∇_S denote the surface element and the surface gradient on S and λ_1^2 is the smallest eigenvalue of the problem:

$$-\Delta_S f_i = \lambda_i^2 f_i \qquad \text{on} \qquad S, \tag{A5}$$

$$\int_S f_i \, d\sigma = 0. \tag{A6}$$

The equality holds in (A4) if and only if $f = f_1$ (of course up to a multiplicative constant). When $S = S_r$, $\lambda_1^2 = 2/r^2$ and (38) is recovered.

APPENDIX B: THE CASE OF A NON-BAROTROPIC STAR

In this appendix, we consider a class of equilibria more general than the one introduced in Section 2.1. Instead of assuming the specific entropy s_0 of the gas to be constant throughout the volume of the star (barotropic condition), we allow this quantity to be an arbitrary non-decreasing function of r, i.e. we just impose

$$s_0'(r) \geq 0 \qquad \text{for} \qquad 0 \leq r \leq R. \tag{B1}$$

(B1) may be considered as a form of Schwarzschild's criterion for the absence of convection.

Let us then consider some admissible equilibrium characterized by the functions $(\rho_0, p_0, s_0, \psi_0)$. The second variation of the energy resulting from an entropy-conserving perturbation has two parts (Lebovitz 1965). The first one is just given by (15), and thus, by the results of Section 3, it is positive for any non-radial perturbation generating a variation ρ_1 of the mass density satisfying the constraints (23) and (27). The second part is always non-negative because of (B1) (it is equal to the integral of $s_0'(r) \tilde{\rho}_1^2$ – up to some factors – with $\tilde{\rho}_1$ the *Lagrangian* variation of the mass density). Therefore, ALT is also valid for this class of equilibria and our proof is still relevant here.

Let us now show how it is possible to extend the argument of Section 4 to our more general equilibrium $(\rho_0, p_0, s_0, \psi_0)$. We first introduce the set of configurations

$$\mathbb{F} \triangleq \left\{ (\rho, s) \,|\, \rho \in \mathbb{H}; \quad \text{supp } s \subset \text{supp } \rho; \quad \int_{\{s < \tau\}} \rho \, d\mathbf{r} = m_0(\tau) \, \forall \tau \right\}, \tag{B2}$$

where $\mathbb{H}$ is defined by (48), 'supp' stands for 'support' (for an arbitrary function f we have $\mathrm{supp}\, f \triangleq \{\boldsymbol{r} |\ f(\boldsymbol{r}) \neq 0\}$), and the function m_0 is defined by

$$m_0(\tau) \triangleq \int_{\{s_0 < \tau\}} \rho_0 \, \mathrm{d}\boldsymbol{r}. \tag{B3}$$

A configuration of $\mathbb{F}$ is thus a given spatial distribution of mass and entropy, with the entropy vanishing where there is no matter and having a 'distribution function' with respect to the measure $\rho \, \mathrm{d}\boldsymbol{r}$ equal to that of s_0 with respect to $\rho_0 \mathrm{d}\boldsymbol{r}$. Clearly, $\mathbb{F}$ contains all the configurations accessible from the equilibrium by an arbitrary adiabatic transform. The energy of a couple (ρ, s) of $\mathbb{F}$ is given by

$$W[\rho, s] \triangleq \int \rho u(\rho, s) \, \mathrm{d}\boldsymbol{r} + \frac{1}{2} \int \rho \psi[\rho] \, \mathrm{d}\boldsymbol{r}, \tag{B4}$$

with the specific internal energy u being quite generally an increasing function of both its arguments, and being assumed here to satisfy

$$u(\rho_2, s_2) + u(\rho_1, s_1) \geq u(\rho_2, s_1) + u(\rho_1, s_2) \tag{B5}$$

whenever $\rho_2 \geq \rho_1$ and $s_2 \geq s_1$ [this condition is not very restrictive; it is satisfied if $\partial^2 u / \partial\rho \, \partial s \geq 0$ or, equivalently, if the coefficient of thermal expansion of the gas $\alpha \triangleq -\rho^{-1} (\partial\rho / \partial T)_p \geq 0$, with T the temperature of the gas].

Given any (ρ, s) of $\mathbb{F}$, we construct a new configuration (ρ^*, s_*) as follows: ρ^* is the rearrangement of ρ defined in Section 4, while s_* is the unique non-decreasing function of r such that

$$\int_{\{s_* < \tau\}} \rho^* \, \mathrm{d}\boldsymbol{r} = m_0(\tau). \tag{B6}$$

Clearly, (ρ^*, s_*) belongs to $\mathbb{F}$ too and it has a gravitational energy lower than that of (ρ, s) (equal if $\rho = \rho^*$). Moreover, because of assumption (B5), its internal energy is not larger than that of (ρ, s) either. This can be shown in a formal way, but it will be enough here to present an intuitive argument. In fact, we can consider the transform $(\rho, s) \rightarrow (\rho^*, s_*)$ as a two-step process. In the first step, we rearrange the fluid elements to get the mass distribution ρ^*, the mass and the entropy of each element being conserved, and the internal energy thus keeping a constant value. Thus, in a second step, we get s_* by taking couples of fluid elements having the same mass δm, and characterized by (r_2, ρ_2^*, s_2) and (r_1, ρ_1^*, s_1), respectively, with $r_2 > r_1$, $\rho_2^* \leq \rho_1^*$ and $s_2 \leq s_1$, and by exchanging the values of s between them – which does not increase the internal energy due to (B5). Therefore, we can associate with any non-radially symmetric distribution of $\mathbb{F}$ a radial one belonging to the same set and not having a larger energy.

If the equilibrium we consider realizes a relative minimum of $W[\rho, s]$ in the subset of $\mathbb{F}$ containing the radial configurations, it is thus clear that it will have the same property in the whole $\mathbb{F}$ – an arbitrary (ρ, s) in a neighbourhood of (ρ_0, s_0) giving rise to a (ρ^*, s_*) near (ρ_0, s_0) too, as $(\rho_0^*, s_{0*}) = (\rho_0, s_0)$. Thus we can say: any equilibrium statisfying (B1) is stable – as in the barotropic case – if it is radially stable.