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GEODESIC GROWTH OF RIGHT-ANGLED COXETER GROUPS BASED ON TREES

LAURA CIOBANU & ALEXANDER KOLPAKOV

ABSTRACT. In this paper we exhibit two infinite families of trees $\{T_n^1\}_{n \geq 17}$ and $\{T_n^2\}_{n \geq 17}$ on n vertices, such that T_n^1 and T_n^2 are non-isomorphic, co-spectral, and the right-angled Coxeter groups (RACGs) based on T_n^1 and T_n^2 have the same geodesic growth with respect to the standard generating set. We then show that the spectrum of a tree does is not sufficient to determine the geodesic growth of the RACG based on that tree, by providing two infinite families of trees $\{S_n^1\}_{n \geq 11}$ and $\{S_n^2\}_{n \geq 11}$, on n vertices, such that S_n^1 and S_n^2 are non-isomorphic, co-spectral, and the right-angled Coxeter groups (RACGs) based on S_n^1 and S_n^2 have distinct geodesic growth.

Asymptotically, as $n \rightarrow \infty$, each set T_n^i , or S_n^i , $i = 1, 2$, has the cardinality of the set of all trees on n vertices. Our proofs are constructive and use two families of trees previously studied by B. McKay and C. Godsil.

2000 Mathematics Subject Classification: 20E08, 20F65.

Key words: geodesic growth, Coxeter group, RACG, regular languages.

*“And then he had a Clever Idea. He would
go up very quietly to the Six Pine Trees ...”
A.A. Milne, Winnie-the-Pooh*

1. INTRODUCTION

The *geodesic growth function* of a group G with respect to a finite generating set S counts, for each positive integer n , the number of geodesics of length n starting at the identity 1_G in the Cayley graph of G with respect to S . The *geodesic growth series* is the formal power series that takes the values of the geodesic growth function as its coefficients (see Definition 2.4).

The groups that we consider in this paper, right-angled Coxeter groups, or RACGs, are known to have a regular language of geodesics with respect to the standard generating sets, and therefore rational geodesic growth series (see [12] or [3, Theorem 4.8.3] for proofs of these facts). Our goal here is to obtain more specific data concerning the geodesic growth of RACGs. Namely, we are interested in extracting information about the geodesic growth series from the defining graph of the group. It is known that non-isomorphic graphs define non-isomorphic RACGs [14], and that non-isomorphic RACGs can have equal geodesic growth [1]. However, we are interested in knowing how much the similarities or differences between two defining graphs influence the geodesic growth of the corresponding RACGs.

Two RACGs $G_i = G(\Gamma_i)$, $i = 1, 2$, with non-isomorphic defining graphs Γ_i , may have equal standard growth (see Definition 2.4 (2)); this can be determined by computing the f -polynomials of the graphs Γ_i [5, Proposition 17.4.2]. However, their geodesic growth exhibits more subtle properties [1], and in general it is not known which graph theoretic conditions

completely determine the geodesic growth of a RACG. The examples of non-isomorphic graphs Γ_i defining RACGs with equal geodesic growth in [1] are degree-regular and have cycles. In this paper we consider the case when the Γ_i 's are trees. Although these are some of the simplest classes of graphs, we already encounter a phenomenon that shows a great difference between the behaviour of the standard growth and that of the geodesic growth. The standard growth of each RACG G_i is determined solely by the number of vertices (or edges) in the respective tree Γ_i , while the geodesic growth can be distinct even for two co-spectral trees. Recall that two graphs are *co-spectral* if the characteristic polynomials of their adjacency matrices are the same.

In this paper we count and compute with the help of an automaton generating the geodesic language in a RACG based on a tree T . This automaton reflects some of the path information from T . In general, a good deal of combinatorial information about T can be extracted from its spectrum. However, we encounter two rather different behaviors: on one hand we produce (infinitely many) pairs of trees T_1 and T_2 which are non-isomorphic, co-spectral and the RACGs $G(T_i)$, $i = 1, 2$, based on them have the same geodesic growth; on the other hand we obtain (infinitely many) pairs of trees S_1 and S_2 which are non-isomorphic and co-spectral, and whose respective RACGs $G(S_i)$ have distinct geodesic growth. This shows that the spectrum of a tree alone does not determine the geodesic growth of the RACG based on that particular tree.

The following two theorems, proved in Sections 3 and 5, respectively, are the main results of the paper:

Theorem 1.1. *There exist two families of trees $\mathcal{T}_n^1 = \{T_1^1, T_2^1, \dots\}$ and $\mathcal{T}_n^2 = \{T_1^2, T_2^2, \dots\}$ on $n \geq 17$ vertices such that for all $i \geq 1$:*

- (1) T_i^1 and T_i^2 are not isomorphic, but co-spectral, and
- (2) the RACGs $G(T_i^1)$ and $G(T_i^2)$ have equal geodesic growth series.

Theorem 1.1 thus answers positively Question 1 in [1, Section 8].

Theorem 1.2. *There exist two families of trees $\mathcal{S}_n^1 = \{S_1^1, S_2^1, \dots\}$ and $\mathcal{S}_n^2 = \{S_1^2, S_2^2, \dots\}$ on $n \geq 11$ vertices such that for all $i \geq 1$:*

- (1) S_i^1 and S_i^2 are not isomorphic, but co-spectral, and
- (2) the RACGs $G(S_i^1)$ and $G(S_i^2)$ have distinct geodesic growth series.

We note that McKay showed in [13] that the trees T_i^1 and T_i^2 in Theorem 1.1 are simultaneously co-spectral, have co-spectral complements and co-spectral line graphs. On the contrary, the trees S_i^1 and S_i^2 from Theorem 1.2 might have line graphs with distinct spectra. Moreover, by a result of McKay [13], in both Theorem 1.1 and Theorem 1.2 the cardinality of each family \mathcal{T}_n^i and \mathcal{S}_n^i tends asymptotically to the cardinality of the set Υ_n of all trees on n vertices:

$$(1) \quad \frac{\text{card } \mathcal{T}_n^i}{\text{card } \Upsilon_n} \rightarrow 1 \quad \text{and} \quad \frac{\text{card } \mathcal{S}_n^i}{\text{card } \Upsilon_n} \rightarrow 1, \quad \text{as } n \rightarrow \infty, \quad i = 1, 2.$$

The two theorems above, together with McKay's results, lead us to the following conjecture:

Conjecture 1.3. *If two trees are simultaneously co-spectral, have co-spectral complements and co-spectral line graphs, then they have the same geodesic growth.*

Section 4 contains details about the computation of several kinds of geodesics, and their numbers, for the trees introduced in Section 3.

The authors have created a Python code Monty¹ [11], described in detail in the Appendix, which performs the computations needed for the results in this paper by using either SAGE standard routines for symbolic computation of determinants and rational expressions of growth series, or the Berlekamp-Massey algorithm for restoring the rational expression for a growth series from a sufficient number of its coefficients, as a more efficient approach. Our Python code constructs finite-state automata that accept the geodesic languages in RACGs based on triangle-free graphs (in this case, trees), and then proceeds to determine the respective growth series.

2. DEFINITIONS AND NOTATION

Let S be a finite set and S^* the free monoid on S . We identify S^* with the set of words over S , that is, finite sequences of elements of S . We use $| \cdot |$ to denote word length.

Let $G = \langle S \rangle$ be a group generated by S . For an element g of G , denote by $g|_S$ the word length of g with respect to S . Given $w \in S^*$, we denote by \bar{w} the image of w in G under the natural projection $S^* \rightarrow G$.

Definition 2.1. A word w over an alphabet S is *geodesic* in $G = \langle S \rangle$ if $|w| = |\bar{w}|_S$. The set of geodesics in G with respect to S will be denoted by $Geo(G)$ or $Geo(S)$.

Let $\Gamma = \Gamma(G, S)$ be a simple (no loops, no multiple edges) graph with vertex set $V(\Gamma) = S$ (or simply V) and edge set $E(\Gamma)$ (or simply E), where $E \subseteq V \times V$. The RACG based on Γ is given by the presentation

$$\langle s \in S \mid s^2 = 1 \ \forall s \in S, \text{ and } (ss')^2 = 1, \forall \{s, s'\} \in E \rangle.$$

It is easy to see that for any two involutions s and s' the relation $(ss')^2 = 1$ implies $ss' = s's$. This leads to another possible presentation for RACGs: $\langle s \in S \mid s^2 = 1, ss' = s's, \forall \{s, s'\} \in E \rangle$.

In the present paper we use the same letters for the vertices of Γ and the corresponding generators of the group $G(\Gamma)$.

The *star* of a vertex $v \in V$ in Γ , denoted by $St_\Gamma(v)$, or $St(v)$ if the ambient graph is clear in the given context, is the set of vertices in Γ that are adjacent to v . That is,

$$St_\Gamma(v) = \{w \in V \mid \{v, w\} \in E\}.$$

We now describe a finite deterministic automaton that recognises geodesics in RACGs (see [10] for definitions of languages and automata). We define such automata in the standard way, as quintuples $(Q, \Sigma, \delta, q_0, F)$, where Q is the finite set of *states*, Σ the input alphabet, δ the *transition function*, q_0 the *initial* or *start* state, and F the set of *final* or *accepting* states. The following definition is a simplified version of Proposition 4.1 in [1].

Definition 2.2. Let $\Gamma = (V, E)$ be a tree. The deterministic finite state automaton recognising the geodesics in $G(\Gamma)$ is $A = (Q, S, \delta, \{\emptyset\}, F)$, where the set of states is $Q = \{\emptyset\} \cup V \cup E \cup \{\rho\}$, ρ is the unique “fail” state, and $\{\emptyset\}$ is the start state. The input alphabet is $S = V$, the set of accept states F is $\emptyset \cup V \cup E$, and the transition function $\delta : Q \times S \rightarrow Q$ is given by

- (1) $\delta(\sigma, s) = (St(s) \cap \sigma) \cup \{s\}$, for $s \notin \sigma$;
- (2) $\delta(\sigma, s) = \rho$, otherwise.

¹this is not the given name of the code, which would be obviously too posh for such a petty thing, but a reference name, which is seemingly good for any Python code.

Definitions 2.3. Any set L of words over an alphabet Σ gives rise to a *strict growth function* $f_L : \mathbb{N} \rightarrow \mathbb{N}$, defined by

$$f_L(n) := |\{W \in L \mid |W| = n\}|.$$

Definitions 2.4. Let G be a group generated by S . Then we define the following:

- (1) The *geodesic growth function* $f_{Geo(G)} : \mathbb{N} \rightarrow \mathbb{N}$ is given by

$$f_{Geo(G)}(r) := f_{Geo(G,S)}(r) = |\{w \in S^* \mid |w| = |\bar{w}|_S = r\}|,$$

and the *geodesic growth series* of G equals

$$\mathcal{G}_{(G,S)}(t) = \sum_{r=0}^{\infty} f_{Geo(G)}(r) t^r.$$

- (2) The *spherical (standard) growth function* $\sigma_{(G,S)} : \mathbb{N} \rightarrow \mathbb{N}$ is given by

$$\sigma_{(G,S)}(r) = |\{g \in G \mid |g|_S = r\}|,$$

and the *spherical (standard) growth series* of G equals

$$\Sigma_G(t) := \Sigma_{(G,S)}(t) = \sum_{r=0}^{\infty} \sigma_{(G,S)}(r) t^r.$$

We recall that the *f-polynomial* of a graph Γ is the generating function for the number of cliques (that is, complete subgraphs) of size i in Γ : $f(t) = f_0 + f_1 t + f_2 t^2 + \dots$, where f_i is the number of i -cliques in Γ . We consider the empty set to be a clique on zero vertices, and therefore $f_0 = 1$. We remark that the spherical (standard) growth function of a RACG is determined by the f -polynomial of its defining graph [5, Proposition 17.4.2]. We want to contrast this to the fact that the geodesic growth function is not uniquely determined by the f -polynomial, or even by the spectrum of the defining graph, as the following sections show.

3. RACGS WITH EQUAL GEODESIC GROWTH

In this section we prove Theorem 1.1 by giving an explicit construction of the families of trees \mathcal{T}_n^i , $i = 1, 2$.

Definition 3.1. We define the *coalescence* $\tau \cdot \sigma$ of two rooted trees τ and σ to be the tree which results from merging τ and σ at their roots. The tree $\tau \cdot \sigma$ has as vertex set the union of the vertex sets of τ and σ , and as root the identification of the roots of τ and σ , as shown in Fig. 1.

Consider the trees T_1 and T_2 , both rooted at 0, first described by McKay in [13], as shown in Fig. 2. In [13] the following fact is proved:

Theorem 3.2. *Let T be a rooted tree with at least two vertices and with root labelled 0. Then the trees $\Gamma_i = T \cdot T_i$, $i = 1, 2$, are not isomorphic, but are co-spectral. Also, their complements $\bar{\Gamma}_i$ and line graphs $L(\Gamma_i)$, $L(\bar{\Gamma}_i)$, $\bar{L}(\bar{\Gamma}_i)$, $i = 1, 2$, are respectively co-spectral.*

Now let $G_1 := G(T_1)$ and $G_2 := G(T_2)$ be the RACGs associated to the trees T_1 and T_2 defined above. First note that since the trees T_1 and T_2 are isomorphic as graphs, the groups G_1 and G_2 are isomorphic. However, T_1 and T_2 are not isomorphic as rooted trees.

Let τ be a tree with n vertices. Fix a labelling $\{0, \dots, n-1\}$ of the vertices of τ , and suppose that 0 represents τ 's root. Define $\Gamma_1 = \tau \cdot T_1$ and $\Gamma_2 = \tau \cdot T_2$ to be the trees obtained as the coalescence of τ with T_i at vertex 0, and let $G(\Gamma_1)$ and $G(\Gamma_2)$ be the RACGs based on Γ_1 and

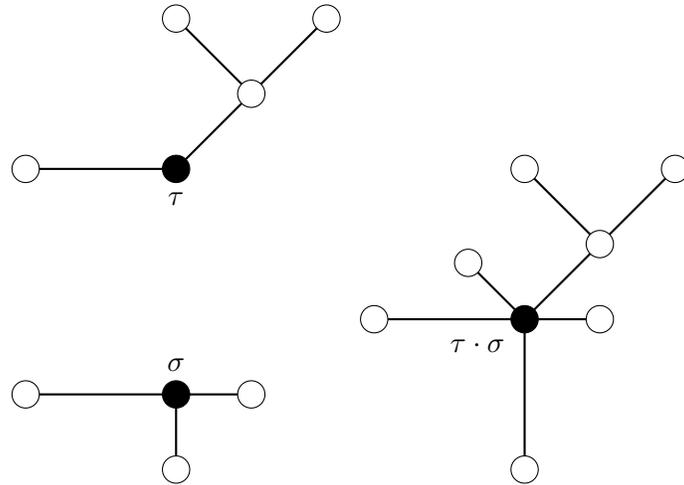


FIGURE 1. Trees τ and σ with marked roots (on the left) and their coalescence $\tau \cdot \sigma$ (on the right)

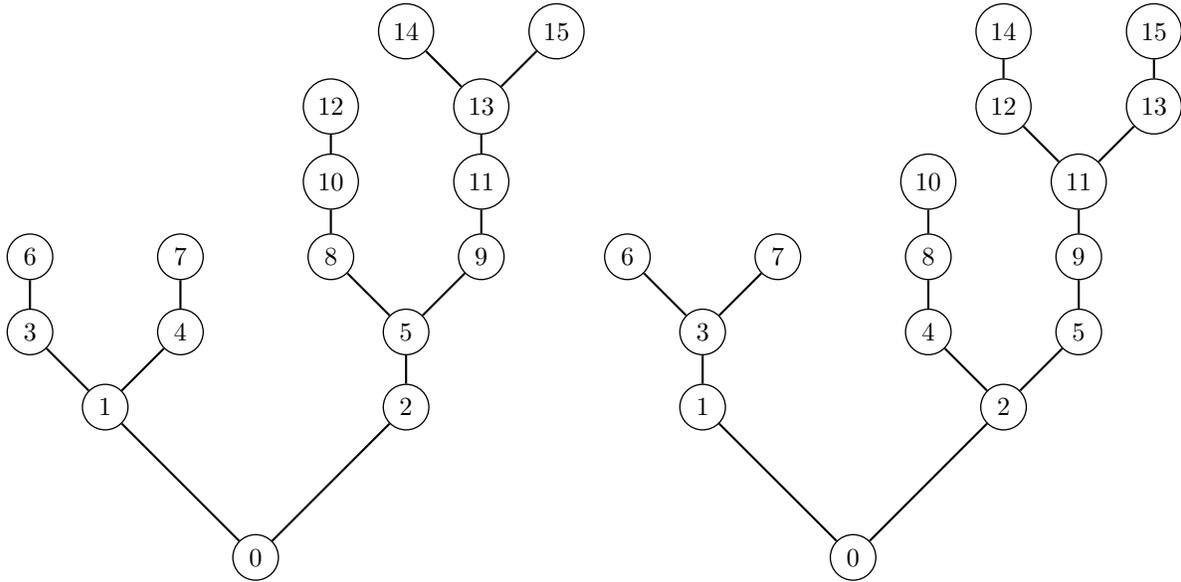


FIGURE 2. McKay's rooted trees: T_1 on the left, and T_2 on the right

Γ_2 , respectively. Since Γ_1 and Γ_2 are non-isomorphic, $G(\Gamma_1)$ and $G(\Gamma_2)$ are non-isomorphic, as well.

The following lemma is the key ingredient of Theorem 1.1.

Lemma 3.3. *The groups $G(\Gamma_1)$ and $G(\Gamma_2)$ have the same geodesic growth series.*

In order to simplify the exposition in the proof of Lemma 3.3 we use the notation below.

Notation 3.4. Let G be a group with generating set T containing the letter 0. Denote the set of words in $Geo(T)$ starting with 0 by $Geo_0(T)$, the set of words in $Geo(T)$ ending in 0 by $Geo^0(T)$, and the set of words in $Geo(T)$ ending and starting with 0 by $Geo_0^0(T)$.

Proof. (of Lemma 3.3) Notice that $f_{Geo(\Gamma_1)}(r)$ (respectively $f_{Geo(\Gamma_2)}(r)$) is equal to the number of all words of length r in Γ_1^* (respectively, Γ_2^*) minus the number of those words of length r in Γ_1^* that are not geodesics. We denote the number of non-geodesics by $\overline{f_{Geo(\Gamma_1)}}(r)$ (and $\overline{f_{Geo(\Gamma_2)}}(r)$, respectively). Since $|\Gamma_1| = |\Gamma_2|$, clearly $f_{Geo(\Gamma_1)}(r) = f_{Geo(\Gamma_2)}(r)$ if and only if $\overline{f_{Geo(\Gamma_1)}}(r) = \overline{f_{Geo(\Gamma_2)}}(r)$.

We now show that $\overline{f_{Geo(\Gamma_1)}}(r) = \overline{f_{Geo(\Gamma_2)}}(r)$ for all $r \geq 1$. Any word in Γ_1^* can be written as $w_1 u_1 w_2 \dots u_n$, where $u_i \in (\tau \setminus \{0\})^*$ and $w_i \in T_1^*$. A non-geodesic word w in Γ_1^* belongs to one of the following sets (or is of the type), depending on its form:

- A : w contains non-geodesics $u_i \in (\tau \setminus \{0\})^*$ or $w_j \in T_1^*$, or both, for some $1 \leq i, j \leq n$, or
- B : all w_i and u_i are geodesic on their respective alphabets, and there exists $1 \leq j < n$ such that $0u_j 0$ is a subword of w with $u_j \in (\text{St}_{\Gamma_1}(0) \cap \tau)^*$, or
- C : all w_i and u_i are geodesic on their respective alphabets, and w contains a subword of the form $s0s$, where $s \in (\text{St}_{\Gamma_1}(0) \cap \tau)^*$.

Notice that the set of non-geodesics is then $A \cup B \cup C$, where $A \cap (B \cup C) = \emptyset$ and $B \cap C \neq \emptyset$.

The number of words in A depends only on the geodesic growth series of τ and T_1 , and thus it will be equal to the number of words of type A in Γ_2 .

The computations in Section 4 show that the following identities hold: $f_{Geo(T_1)}(r) = f_{Geo(T_2)}(r)$, $f_{Geo_0(T_1)}(r) = f_{Geo_0(T_2)}(r)$, $f_{Geo^0(T_1)}(r) = f_{Geo^0(T_2)}(r)$ and $f_{Geo_0^0(T_1)}(r) = f_{Geo_0^0(T_2)}(r)$ for all $r \geq 1$, as a result of (2). This means that there is a length-preserving bijection ϕ_0 between $Geo_0(T_1)$ and $Geo_0(T_2)$, i.e. for each geodesic $w = 0v \in Geo_0(T_1)$ there is a geodesic $w' = 0v' = \phi_0(w) \in Geo_0(T_2)$, and $|w| = |w'|$. Analogously, there is a length-preserving bijection $\phi^0 : Geo^0(T_1) \rightarrow Geo^0(T_2)$, and a bijection $\phi_0^0 : Geo_0^0(T_1) \rightarrow Geo_0^0(T_2)$. This means there is a length-preserving bijection $\phi : Geo_0(T_1) \cup Geo^0(T_1) \rightarrow Geo_0(T_2) \cup Geo^0(T_2)$ between those geodesics starting or ending with 0, for which we have that $\phi|_{Geo_0} = \phi_0$, $\phi|_{Geo^0} = \phi^0$ and $\phi|_{Geo_0^0} = \phi_0^0$. By the computations in (2) we have that $f_{(Geo_0 \cup Geo^0)(T_1)}(r) = f_{(Geo_0 \cup Geo^0)(T_2)}(r)$ by the standard formula for the cardinality of the union of two sets, and since $f_{Geo(T_1)}(r) = f_{Geo(T_2)}(r)$, we also have that $f_{Geo \setminus (Geo_0 \cup Geo^0)(T_1)}(r) = f_{Geo \setminus (Geo_0 \cup Geo^0)(T_2)}(r)$. Thus, there is a length-preserving bijection ψ between $Geo \setminus (Geo_0 \cup Geo^0)(T_1)$ and $Geo \setminus (Geo_0 \cup Geo^0)(T_2)$.

The bijection ϕ can be extended to the set $Geo(T_1)$ of all geodesics on T_1 by letting $\phi(w) = \psi(w)$ for all $w \in Geo \setminus (Geo_0 \cup Geo^0)(T_1)$, and then furthermore extended to $Geo(T_1) \cup Geo(\tau \setminus \{0\})$ by letting $\phi(w) = w$ for all $w \in Geo(\tau \setminus \{0\})$.

Then ϕ provides a bijection between the words of type B in Γ_1^* and the words of type B in Γ_2^* . To see this, associate to each $w_1 u_1 w_2 \dots u_n$ the word $\phi(w_1) \phi(u_1) \dots \phi(w_n) \phi(u_n) = \phi(w_1) u_1 \dots \phi(w_n) u_n$. Then w_i ends in 0, w_{i+1} starts with 0, and $u_i \in \text{St}_\tau(0)^*$ if and only if $\phi(w_i)$ ends in 0, $\phi(w_{i+1})$ starts with 0, and $\phi(u_i) = u_i \in \text{St}_\tau(0)^*$, by definition.

It is immediate to see that ϕ also provides a bijection between the words of type C , and between the words of type $B \cap C$ in Γ_1 and Γ_2 , respectively. Thus, there is a length-preserving bijection between the words of type $A \cup B \cup C$ (i.e. all non-geodesic words) in Γ_1 and Γ_2 . This concludes the proof of the lemma. \square

Proof. (of Theorem 1.1) Let $\Upsilon_k = \{\tau_1, \tau_2, \dots\}$ be the set of non-isomorphic trees on $k \geq 2$ vertices. For $i = 1, 2$ the two families of trees $\mathcal{T}_n^i = \{\tau \cdot T_i \mid \tau \in \Upsilon_k\}$, $k = n - 15$, satisfy Theorem 1.1.

From Theorem 3.2 we already know that $\Gamma_1 = \tau \cdot T_1$ and $\Gamma_2 = \tau \cdot T_2$ are co-spectral, for any $\tau \in \Upsilon_k$. By Lemma 3.3 the groups $G(\Gamma_1)$ and $G(\Gamma_2)$ have the same geodesic growth series. \square

Theorem 3.5 (Lemma 4.3 in [13]). *Let $p_i(n)$, $i = 1, 2$, be the proportion of trees on n vertices that have T_i as a limb. Then $p_1(n) = p_2(n)$ for all n and $\lim_{n \rightarrow \infty} p_i(n) = 1$.*

From Theorem 3.5, we obtain that $\frac{\text{card } \mathcal{T}_n^i}{\text{card } \mathcal{T}_n} \rightarrow 1$, as $n \rightarrow \infty$.

4. COMPUTING THE NUMBERS OF SPECIAL GEODESICS

In this section we prove, by concrete computations, the equalities between the numbers of special geodesics required in the proof of Lemma 3.3.

Let T_1 and T_2 be as in Figure 2, and recall Notation 3.4. For each group $G_i = G(T_i)$, $i = 1, 2$, we construct a finite automaton A_i accepting the geodesic language of G_i , as described in Definition 2.2, and with the help of A_i we compute the growth series

$$\begin{aligned} \gamma_{G_1}(t) &= \sum_{r \geq 0} f_{Geo(T_1)}(r) t^r, \quad {}_0\gamma_{G_1}(t) = \sum_{r \geq 0} f_{Geo_0(T_1)}(r) t^r, \\ {}^0\gamma_{G_1}(t) &= \sum_{r \geq 0} f_{Geo^0(T_1)}(r) t^r, \quad \text{and } {}^0_0\gamma_{G_1}(t) = \sum_{r \geq 0} f_{Geo^0_0(T_1)}(r) t^r. \end{aligned}$$

We also compute the analogous growth series $\gamma_{G_2}(t)$, ${}_0\gamma_{G_2}(t)$, ${}^0\gamma_{G_2}(t)$ and ${}^0_0\gamma_{G_2}(t)$ for G_2 , where the series coefficients are given by the sequences $Geo(T_2)(r)$, $Geo(T_2)_0(r)$, $Geo(T_2)^0(r)$ and $Geo(T_2)^0_0(r)$, respectively.

Our computations, which we elaborate upon below, show that

$$(2) \quad \gamma_{G_1}(t) = \gamma_{G_2}(t), \quad {}_0\gamma_{G_1}(t) = {}_0\gamma_{G_2}(t), \quad {}^0\gamma_{G_1}(t) = {}^0\gamma_{G_2}(t) \text{ and } {}^0_0\gamma_{G_1}(t) = {}^0_0\gamma_{G_2}(t).$$

These identities prove the equality of the corresponding numbers of geodesics.

Let A be a deterministic finite-state automaton with accepting states q_i , $i = 0, 1, \dots, N$, where q_0 is the start state and the “fail” state is denoted by q . Let $M = M(A)$ be the transition matrix of the automaton A . Computing the generating function $\gamma_A(t)$ of A is a standard technique, and the formula for $\gamma_A(t)$ is

$$(3) \quad \gamma_A(t) = \frac{e^T M w}{\det(I - tM)},$$

where I is the $N \times N$ identity matrix, and $e = (1, 0, \dots, 0)^T$ and $w = (1, 1, \dots, 1)^T$ are two vectors in \mathbb{Z}^N .

Now let A_i be the deterministic finite-state automata accepting the language of geodesics in G_i , $i = 1, 2$, and let $M_i = M(A_i)$ be the transition matrix of A_i . Then the geodesic growth series $\gamma_{G_i}(t)$ is a rational function (determined by the equality (3)), and the coefficients of its numerator and denominator can be easily computed, see [7]. The Python code Monty [11], described in detail in the Appendix, may perform the above computation either by finding the symbolic determinant $\det(I - tM_i)$ (usually slow), or by applying the Berlekamp-Massey algorithm (a faster one). This Python code starts by creating the finite-state automaton A_i , given a triangle-free graph T_i (in this case, a tree), and then proceeds to determining $\gamma_{G_i}(t)$.

Since G_1 and G_2 are isomorphic, the equality $\gamma_{G_1}(t) = \gamma_{G_2}(t)$ is immediate. Here we provide an explicit formula for these identical growth series. The output of Monty for both trees T_1 and T_2 consists of two finite-state automata A_i which are isomorphic, as one only rennumbers the vertices of T_2 in order to obtain T_1 . Each A_i has 32 states and 466 transition arrows, and

the corresponding growth series are

$$\begin{aligned} \gamma_{G_1}(t) = \gamma_{G_2}(t) &= (1+t)(1+2t-2t^3-4t^4-t^5)(1+5t+10t^2+9t^3-5t^4-26t^5-34t^6 \\ &- 22t^7-t^8+7t^9+4t^{10})(1-8t-85t^2-243t^3-222t^4+332t^5+1194t^6+1349t^7+132t^8 \\ &- 1510t^9-2008t^{10}-1088t^{11}+28t^{12}+359t^{13}+170t^{14}+15t^{15})^{-1}. \end{aligned}$$

Now we compute the functions ${}_0\gamma_{G_i}(t)$, $i = 1, 2$, which will turn out to be equal, as well. However, in this case the equality is not known to hold beforehand: even though the groups G_1 and G_2 are isomorphic, the image of vertex 0 in T_1 under the canonical isomorphism does not correspond to vertex 0 in T_2 . We shall explicitly compute the generating function ${}_0\gamma_{G_i}(t)$ for the number of geodesic words starting with 0 accepted by each A_i , $i = 1, 2$, and then compare these functions. Suppose that the start state of each A_i is $q_0 = \emptyset$. Let the corresponding transition function δ_i be so that $\delta_i(q_0, 0) = q_{k_i} = \{0\}$. If a word w labels a path from q_{k_i} to an accept state, then the word $0w$ is a geodesic word in G_i starting with 0. Thus, we have to compute the generating function ${}_0\alpha_{G_i}(t)$ for the number of words starting at q_{k_i} and ending at an accept state:

$${}_0\alpha_{G_i}(t) = \frac{e^T M_i w}{\det(I - tM_i)},$$

where $e = (0, \dots, 0, \underbrace{1}_{k_i}, 0, \dots, 0)^T$, and $w = (1, 1, \dots, 1)^T$. Then we use the fact that ${}_0\gamma_{G_i}(t) = t \cdot {}_0\alpha_{G_i}(t)$. By symmetry, we get ${}_0\gamma_{G_i}(t) = {}_0\gamma_{G_i}(t)$, $i = 1, 2$.

By using Monty we obtain

$$\begin{aligned} {}_0\gamma_{G_1}(t) = {}_0\gamma_{G_2}(t) &= t(1+t)(1+2t-2t^3-4t^4-t^5)(1+4t+4t^2-3t^3-9t^4-5t^5+3t^6 \\ &+ t^7-3t^8-3t^9)(1-8t-85t^2-243t^3-222t^4+332t^5+1194t^6+1349t^7+132t^8 \\ &- 1510t^9-2008t^{10}-1088t^{11}+28t^{12}+359t^{13}+170t^{14}+15t^{15})^{-1}. \end{aligned}$$

Finally, it remains to compute the growth series ${}_0\gamma_{G_i}(t)$.

There are two kinds of words forming disjoint subsets of the geodesic language of G_i that we are interested in:

- (I) the words w starting at the accept state $q_{k_i} = \delta(q_0, 0) = \{0\}$ of A_i and coming back to it: then the geodesic word $0w$ starts and ends with a “0” (since there are only 0-transitions leading to $q_{k_i} = \{0\}$ and no 0-transition coming out of q_{k_i}), by Definition 2.2 (i).
- (II) the words w starting at the state $q_{k_i} = \{0\}$ and ending at a state q_{l_i} such that $\delta(q_{l_i}, 0) = q_{m_i} \neq q_{k_i}$ and q_{m_i} is an accept state: then the word $0w0$ will be a geodesic word starting and ending with a “0”.

Let ${}_0\alpha_{G_i}(t)$ be the generating function for the words of type (I), and ${}_0\beta_{G_i}(t)$ be that for the words of type (II). Then, ${}_0\gamma_{G_i}(t) = t \cdot {}_0\alpha_{G_i}(t) + t^2 \cdot {}_0\beta_{G_i}(t)$. We have that

$${}_0\alpha_{G_i}(t) = \frac{e^T M_i w}{\det(I - tM_i)},$$

with $e = w = (0, \dots, 0, \underbrace{1}_{k_i}, 0, \dots, 0)^T$.

Analogously,

$${}_0\beta_{G_i}(t) = \frac{e^T M_i w}{\det(I - tM_i)},$$

with $e = (0, \dots, 0, \underbrace{1}_{k_i}, 0, \dots, 0)^T$, and w having a “1” at position l_i for all states q_{l_i} as described above, and zeroes at all other places.

By using Monty we obtain

$$\begin{aligned} {}^0_0\gamma_{G_1}(t) = {}^0_0\gamma_{G_2}(t) = & t \cdot (1 - 5t - 94t^2 - 374t^3 - 456t^4 + 955t^5 + 4275t^6 + 5652t^7 - 1617t^8 \\ & - 16773t^9 - 24255t^{10} - 7337t^{11} + 26583t^{12} + 45100t^{13} + 26181t^{14} - 12789t^{15} - 34553t^{16} \\ & - 24957t^{17} - 3147t^{18} + 8130t^{19} + 6288t^{20} + 1398t^{21} - 458t^{22} - 284t^{23} - 24t^{24}) \cdot (1 + 3t \\ & + 2t^2 - 3t^3 - 9t^4 - 8t^5 + 4t^7 + 3t^8 - t^9)^{-1} (1 - 8t - 85t^2 - 243t^3 - 222t^4 + 332t^5 + 1194t^6 \\ & + 1349t^7 + 132t^8 - 1510t^9 - 2008t^{10} - 1088t^{11} + 28t^{12} + 359t^{13} + 170t^{14} + 15t^{15})^{-1}. \end{aligned}$$

Thus, the numbers of special geodesics in G_1 and G_2 coincide.

5. RACGS WITH DIFFERENT GEODESIC GROWTH

In this section we give a proof of Theorem 1.2. Our construction will be analogous to that in Section 3, although we shall use different trees, S_1 and S_2 , in order to construct the families $S_n^i = \{\tau \cdot S_i | \tau \in \Upsilon_k\}$, $k = n - 9$. Namely, we will use the rooted trees in Fig. 3.

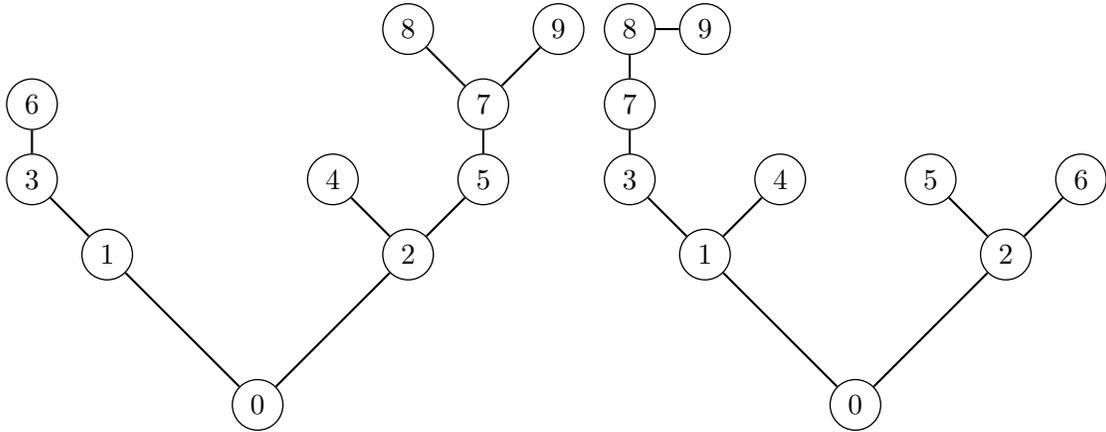


FIGURE 3. Godsil’s rooted trees: S_1 on the left, and S_2 on the right

We set $\Gamma_1 = \tau \cdot S_1$ and $\Gamma_2 = \tau \cdot S_2$, where τ is the rooted tree depicted in Fig. 4. Then the following holds:

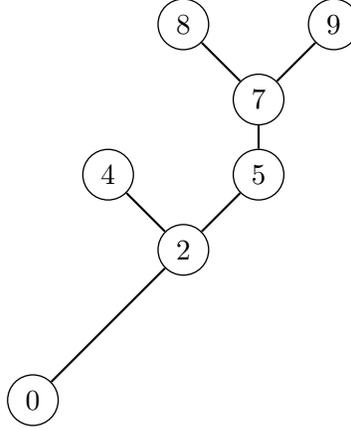
Theorem 5.1 (page 27 of [8]). *The trees Γ_1 and Γ_2 are not isomorphic, but are co-spectral and have co-spectral complements. Their line graphs $L(\Gamma_i)$, $L(\overline{\Gamma_i})$ and $\overline{L(\Gamma_i)}$ are not necessarily co-spectral.*

Proof. The trees Γ_1 and Γ_2 are not isomorphic, since Γ_1 has τ as a limb more times than Γ_2 .

The characteristic polynomials of τ , S_i and Γ_i (*i.e.* the characteristic polynomials of the adjacency matrices of these graphs) are related by

$$\phi_{\Gamma_i}(t) = \phi_{\tau}(t) \cdot \phi_{S_i \setminus \{0\}}(t) + \phi_{\tau \setminus \{0\}}(t) \cdot \phi_{S_i}(t) - t \cdot \phi_{\tau \setminus \{0\}}(t) \cdot \phi_{S_i \setminus \{0\}}(t),$$

according to [13, Lemma 2.2 (i)]. Given that the trees S_1 and S_2 are isomorphic as graphs (though not as rooted trees), we obtain that Γ_1 and Γ_2 share the same characteristic polynomial (*i.e.* are co-spectral).

FIGURE 4. The rooted tree τ with root 0

The fact that $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$ are co-spectral follows from the above and [13, Theorem 3.1 (i)].

Now, by letting τ be the line graph on two vertices we obtain that $L(\Gamma_1)$, $L(\overline{\Gamma_1})$ and $\overline{L(\Gamma_1)}$ have different characteristic polynomials from $L(\Gamma_2)$, $L(\overline{\Gamma_2})$ and $\overline{L(\Gamma_2)}$, respectively. \square

Lemma 5.2. *Let τ be a rooted tree with at least two vertices, and let $\Gamma_1 = \tau \cdot S_1$ and $\Gamma_2 = \tau \cdot S_2$. The RACGs $G(\Gamma_1)$ and $G(\Gamma_2)$ have distinct geodesic growth series.*

Proof. We will prove that $f_{Geo(\Gamma_1)}(10) \neq f_{Geo(\Gamma_2)}(10)$. We use the same argument as in the proof of Lemma 3.3, that is, $f_{Geo(\Gamma_1)}(r) = f_{Geo(\Gamma_2)}(r)$ if and only if $\overline{f_{Geo(\Gamma_1)}}(r) = \overline{f_{Geo(\Gamma_2)}}(r)$, where $\overline{f_{Geo(\Gamma_i)}}(r)$ denotes the numbers of non-geodesics of length r , $r \geq 1$. In this proof we use the result of our explicit computations with Monty, which shows that $f_{Geo_0(\Gamma_1)}(r) = f_{Geo_0(\Gamma_2)}(r)$ for $r < 8$, but $f_{Geo_0(\Gamma_1)}(8) = f_{Geo_0(\Gamma_1)}(8) = 8919523$ for G_1 and $f_{Geo_0(\Gamma_2)}(8) = f_{Geo_0(\Gamma_2)}(8) = 8919522$ for G_2 .

Any word in Γ_1^* has the form $w_1 u_1 w_2 \dots u_n$, where $u_i \in (\tau \setminus \{0\})^*$, $w_i \in S_1^*$, and w_i, u_i non-empty except for perhaps w_1 and u_n . A non-geodesic word w in Γ_1^* belongs to one of the following sets (or is of the type), depending on its form:

- (A) it either contains non-geodesics $w_i \in (\tau \setminus \{0\})^*$ or $u_j \in S_1^*$, or both, for some $1 \leq i, j \leq n$,
or
- (B) all w_i and u_i are geodesic on their respective alphabets, and there exists $1 \leq j < n$ such that $0u_j 0$ is a subword of w and $u_j \in (\text{St}_{\Gamma_1}(0) \cap \tau)^*$.
- (C) all w_i and u_i are geodesic on their respective alphabets, and w contains a subword of the form $s0s$, where $s \in (\text{St}_{\Gamma_1}(0) \cap \tau)^*$.

For the remaining discussion we only consider words w of length 10, and call the number of non-empty subwords u_i or w_i the *syllable length* of w . The number words of type (A) depends only on the geodesic growth series of τ and S_1 , and thus it will be equal to the number of words of type (A) in Γ_2 . Notice that if the syllable length of w is ≤ 2 there are no words of type (B) or (C), so we obtain the same numbers of words in Γ_1 and Γ_2 . If the syllable length of w is > 4 then all w_i, u_i are shorter than or equal to 6, and our computations show that the numbers of special geodesics of length less than or equal to 6 in Γ_1 and Γ_2 coincide. Thus, a discrepancy may appear only when the syllable length is 3 or 4. The words of type (B) or (C) with syllable length 4 have the form $w = w_1 u_1 w_2 u_2$ (or $w = u_1 w_2 u_2 w_3$), where w_1 ends in 0

and w_2 starts with 0, or $w_2 = 0$. But in this case $|w_1|, |w_2| \leq 7$, and our computations show that $f_{Geo_0(\Gamma_1)}(r) = f_{Geo_0(\Gamma_2)}(r)$, up to $r = 7$.

Next, we consider the non-geodesics w of syllable length 3. If they have the form $w = u_1 w_1 u_2$, then $w_1 = 0$, and the numbers of geodesics u_i is the same since they are all written over the same alphabet determined by the tree τ , so no discrepancy in the numbers of non-geodesics occurs. Thus, it remains to count the words of the form $w = w_1 u_1 w_2$, where $|w_1| \leq 8$, w_1 ends in 0, $|u_1| \geq 1$, $u_1 \in (\text{St}_{\Gamma_1}(0) \cap \tau)^*$, and w_2 starts with 0, $|w_2| \leq 8$. Again, if $|w_i| \leq 7$ we obtain the same numbers of non-geodesics. However, a discrepancy occurs when $|w_1| = 8$ or $|w_2| = 8$.

Indeed, the number of such words is $2(f_{Geo_0(\Gamma_1)}(8) \deg_{\tau}(0))$ in Γ_1^* and $2(f_{Geo_0(\Gamma_2)}(8) \deg_{\tau}(0))$ in Γ_2^* . By using Monty, we obtained $f_{Geo_0(\Gamma_1)}(8) = f_{Geo^0(\Gamma_1)}(8) = 8919523$ for G_1 , but $f_{Geo_0(\Gamma_2)}(8) = f_{Geo^0(\Gamma_2)}(8) = 8919522$ for G_2 , so the numbers of non-geodesics in these two groups are distinct. \square

Now we can finish the proof of our second main result.

Proof. (of Theorem 1.2) This is a straightforward consequence of Theorem 5.1 and Lemma 5.2. \square

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APPENDIX

In this section we give the Python code “Monty” that we used in our computations, with comments and remarks. Its on-line copy [11] can be downloaded as a SAGE worksheet from the second author’s web-page.

We begin by defining the automaton A that recognizes the language of geodesics of a given RACG G whose defining graph Γ is given as input.

```

1 def Automaton(t):
2 # takes a triangle-free graph, returns the automaton recognising
3 # the resp. RACG as a digraph
4 CliqueComplex = t.clique_complex(); C = list();
5 for s in CliqueComplex.faces().values():
6     for f in s: C.append(set(f));
7 n = len(C); a = DiGraph(); a.add_vertices(range(n));
8 for i in range(n):
9     for v in t.vertices():
10        if not(C[i].issuperset([v])):
11            st = set(t.vertex_boundary([v])).union([v]);
12            d = set([v]).union(st.intersection(C[i]));
13            k = C.index(d);

```

```

14         a.add_edge((i,k));
15     a.set_vertices({i : C[i] for i in range(n)})
16     return a;

```

Given the automaton A , we then compute the growth function of its accepted language (in this case, the geodesic language for G with respect to S).

```

1 def GrowthFunc(a):
2     # takes a geodesic automaton, returns the resp. (geodesic) growth function
3     am = a.adjacency_matrix();
4     n = am.nrows();
5     R = FractionField(PolynomialRing(QQ, 't'));
6     R.inject_variables();
7     m = diagonal_matrix([1]*n) - t*am;
8     M = (1/m.det())*m.adjoint();
9     ee = [0]*(n-1);
10    ee.append(1);
11    e = vector(ee);
12    w = vector([1]*n);
13    func = e*M*w;
14    return func.numerator().factor()/func.denominator().factor();

```

A clique c in the defining graph Γ of the RACG $G = G(\Gamma)$ corresponds to a state q_c in the automaton A . We need the following auxiliary function in order to determine the index of q_c represented as a vertex of the digraph A (the automaton) created by the procedure `Automaton`.

```

1 def Index(a, c):
2     # takes a geodesic automaton 'a', a clique 'c' in the resp. defining graph,
3     # returns the vertex of the automaton 'a' corresponding to 'c'
4     l = None;
5     for v in a.vertices():
6         if a.get_vertex(v) == c:
7             l = v;
8     return l;

```

Below we compute the growth function ${}_0\alpha(t)_G$ as described in the proof of Theorem 1.1.

```

1 def GrowthFuncStart0(a):
2     # takes a geodesic automaton, returns the growth function for geodesic words
3     # starting at the state q, where  $\delta(\text{Start}, 0) = q$ 
4     am = a.adjacency_matrix();
5     n = am.nrows();
6     R = FractionField(PolynomialRing(QQ, 't'));
7     R.inject_variables();
8     m = diagonal_matrix([1]*n) - t*am;
9     M = (1/m.det())*m.adjoint();
10    ind = Index(a, set([0]));
11    ee = [0]*n;
12    ee[ind] = 1;
13    e = vector(ee);
14    w = vector([1]*n);

```

```

15     func = e*M*w;
16     return func.numerator().factor()/func.denominator().factor();

```

Now we define a function that takes as input the geodesic automaton A for a RACG $G = G(\Gamma)$, a list of cliques $l = [c_0, c_1, \dots, c_k]$ in the respective defining graph Γ and returns the growth function for geodesic words starting at the state $q = \delta(q_0, 0)$ that bring A to any of the states described by the cliques in l .

```

1  def GrowthFuncStart0End(a, l):
2  # takes an automaton and a list of cliques 'l' in the resp. defining graph
3  # as input, returns the growth function for geodesic words starting at
4  # state q, where  $\delta(\text{Start}, 0) = q$ , and ending at any of the states
5  # corresponding to cliques in 'l'
6      am = a.adjacency_matrix();
7      n = am.nrows();
8      R = FractionField(PolynomialRing(QQ, 't'));
9      R.inject_variables();
10     m = diagonal_matrix([1]*n) - t*am;
11     M = (1/m.det())*m.adjoint();
12     ee = [0]*n;
13     ind = Index(a, set([0]));
14     ee[ind] = 1;
15     e = vector(ee);
16     ww = [0]*n;
17     for c in l:
18         ind = Index(a, set(c));
19         ww[ind] = 1;
20     w = vector(ww);
21     func = e*M*w;
22     return func.numerator().factor()/func.denominator().factor();

```

By using a suitable list of cliques l we can compute the functions ${}_0^0\alpha(t)_G$ and ${}_0^0\beta(t)_G$. Namely, in the proof of Theorem 1.1, we find

```

1  # the function  ${}_0^0\alpha(t)_{G_1}$ 
2  a001 = GrowthFuncStart0End(a1, [[0]]);
3  # the function  ${}_0^0\alpha(t)_{G_2}$ 
4  a002 = GrowthFuncStart0End(a2, [[0]]);
5  # the function  ${}_0^0\beta(t)_{G_1}$ 
6  b001 = GrowthFuncStart0End(a1, [[1], [1,3], [1,4], [2], [2,5]]);
7  # the function  ${}_0^0\beta(t)_{G_2}$ 
8  b002 = GrowthFuncStart0End(a2, [[1], [1,3], [2], [2,4], [2,5]]);

```

The list $l = [[1], [1,3], [1,4], [2], [2,5]]$ above contains the cliques of Γ_1 corresponding to the accept states q of the geodesic automaton A_1 for $G_1 = G(\Gamma_1)$ such that $\delta(p, 0) = q$, for a state p . The list $l = [[1], [1,3], [2], [2,4], [2,5]]$ contains the cliques of Γ_2 with analogous properties, corresponding to the states of the geodesic automaton A_2 for the RACG $G_2 = G(\Gamma_2)$.

The above described Python procedures are also used to perform the computations in the proof of Theorem 1.2.

The on-line version of Monty [11] contains a variation of the `GrowthFunc` procedure, called `GrowthFuncBM`, that uses the Berlekamp-Massey algorithm for faster computing.

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