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A discontinuous isoperimetric profile for a complete Riemannian manifold

Stefano Nardulli and Pierre Pansu

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ABSTRACT: The first known example of a complete Riemannian manifold whose isoperimetric profile is discontinuous is given.

RESUMÉ : On construit le premier exemple connu d' une variété riemannienne complète dont le profil isopérimétrique est discontinu.

1 Introduction

1.1 The problem

Let M be a Riemannian manifold. We are concerned with the continuity of the isoperimetric profile of M . Given $0 < v < \text{vol}(M)$, consider all domains, i.e. smooth compact codimension 0 submanifolds in M with volume v . Define $I_M(v)$ as the least upper bound of the boundary areas of such domains. In this way, one gets a function $I_M : (0, \text{vol}(M)) \rightarrow \mathbf{R}_+$ called the *isoperimetric profile* of M .

Question 1 *When is the isoperimetric profile a continuous function ?*

The answer is yes when M is compact, Lemme 6.2 of [Gal88]. S. Gallot's proof uses techniques of metric geometry. In the compact case, alternative proofs, based on the direct method of the calculus of variations, can be found in books like [AFP00], [Mor09], [Mag12].

The latter argument has been extended to the case of complete manifolds with $C^{2,\alpha}$ -bounded geometry, see Theorem 1 of [FN15b] and Theorem 2.2 of [FN15a].

¹S. Nardulli, Instituto de Matemática, Universidade Federal de Rio de Janeiro

²P. Pansu, Univ Paris-Sud, Laboratoire de Mathématiques d'Orsay, Orsay, F-91405

³CNRS, Orsay, F-91405.

If one assumes existence of isoperimetric regions of every volume, one can weaken bounded geometry assumptions. It suffices to assume a lower bound on Ricci curvature and on the volumes of balls of radius 1, see Theorem 4.1 [FN15b]. In our opinion, it remains an open question whether the noncollapsing assumption (lower bound on the volumes of balls) can be removed or not, see Question 3 below.

The isoperimetric profile is continuous also when the volume of M is finite; a proof of this fact can be found in Corollary 2.4 of [NR15].

When the ambient manifold is a non-compact homogeneous space, Hsiang showed that its isoperimetric profile is a non-decreasing and absolutely continuous function [[Hsi92], Lemma 3, Thm. 6].

In a recent paper, [Rit15], Manuel Ritoré showed that a complete Riemannian manifold possessing a strictly convex Lipschitz continuous exhaustion function has continuous and nondecreasing isoperimetric profile. Hadamard manifolds and complete non-compact manifolds with strictly positive sectional curvature belong to this class. This shows that earlier attempts to construct counterexamples using pieces of increasing negative curvature are doomed to fail. An example of a manifold with density with discontinuous isoperimetric profile has been described by Adams, Morgan and Nardulli in Prop. 2 of [AMN13].

For more informations about the literature on the continuity of the isoperimetric profile the reader should consult the introduction of [Rit15] and references therein.

1.2 The result

Theorem 1 *There exists a connected non compact 3-dimensional Riemannian manifold M such that I_M is a discontinuous function.*

The proof is a modification of the treatment of Riemannian manifolds with density by Adams, Morgan and Nardulli, an account of which can be found in Frank Morgan's blog, [AMN13].

Start with a disjoint union of Riemannian manifolds $N = \coprod_n M_n$ such that $\text{vol}(M_n) = 1 + \tau_n$ where $\tau_n > 0$ tends to 0. Then $I_N(1 + \tau_n) = 0$. Assume that, for all n , $I_{M_n}(1) = I_{M_n}(\tau_n) \geq 1$. Then it is not too hard to show that $I_N(1) \geq 1$. Connecting M_n to M_{n+1} with a very thin tube produces a connected Riemannian manifold M for which $I_N(1 + \epsilon_n)$ tends to 0. Again, it is not too hard to show that $I_M(1) > 0$. Therefore I_M is discontinuous.

Thus the key input is the sequence of Riemannian manifolds M_n with $\text{vol}(M_n)$ bounded and $I_{M_n}(\tau_n)$ bounded below. Adams, Morgan and Nardulli indulged themselves in introducing densities. They took for M_n a tiny round sphere with a high constant density. Since volumes and boundary areas rescale differently, one can achieve $I_{M_n}(\epsilon_n) \geq 1$. Instead, we use nilmanifolds equipped with metrics which converge (up to rescaling) to a single Carnot-Carathéodory metric. The Carnot-Carathéodory isoperimetric profile established in [Pan83] gives a uniform lower bound for the isoperimetric profiles of such metrics.

A similar construction certainly works in any dimension ≥ 3 .

Question 2 *Does there exist a 2-dimensional Riemannian manifold whose isoperimetric profile is discontinuous?*

Question 3 *Does a manifold with Ricci curvature bounded below and admitting isoperimetric regions in every volume, have a continuous isoperimetric profile?*

2 Isoperimetry in nilmanifolds

2.1 Isoperimetry in the Heisenberg group

The Heisenberg group \mathbf{H} is the group of real upper triangular unipotent 3×3 matrices,

$$\mathbf{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y, z \in \mathbf{R} \right\}.$$

Putting integer entries produces the discrete subgroup $\mathbf{H}_{\mathbf{Z}} \subset \mathbf{H}$. Let $dx, dy, \theta = dz - xdy$ be a basis of left-invariant forms. Let

$$g_{\epsilon} = dx^2 + dy^2 + \frac{1}{\epsilon^2}\theta^2.$$

This is a left-invariant Riemannian metric on \mathbf{H} . As ϵ tends to 0, the distance d_{ϵ} associated to g_{ϵ} converges to the *Carnot-Carathéodory distance*

$$d_c(p, q) = \inf\{\text{length}(\gamma) ; \gamma(0) = p, \gamma(1) = q, \gamma^*\theta = 0\}.$$

- d_c has Hausdorff dimension 4, with spherical 4-dimensional measure proportional to Haar measure $\mathcal{S}^4 = dx dy dz$.
- Smooth surfaces S in \mathbf{H} have Hausdorff dimension 3, with spherical 3-dimensional measure proportional to the measure denoted by \mathcal{S}^3 to be described soon (the proportionality constants are universal and will be ignored in the sequel).
- Smooth curves which are transverse to the contact structure $\ker(\theta)$ have Hausdorff dimension 2, with spherical 2-dimensional measure \mathcal{S}^2 given by integration of θ (up to sign).
- Smooth curves tangent to $\ker(\theta)$ have Hausdorff dimension 1, with spherical 1-dimensional measure \mathcal{S}^1 being length.

\mathcal{S}^3 is locally the product $\mathcal{S}^1 \otimes \mathcal{S}^2$. Specifically, let $d\ell$ denote a (locally defined away from points where S is tangent to $\ker(\theta)$) unit 1-form on S whose kernel is orthogonal to the trace of $\ker(\theta)$ on the tangent plane to S . Then, up to a sign, \mathcal{S}^3 is obtained by integrating $d\ell \wedge \theta$.

The Heisenberg isoperimetric inequality ([Pan83]) states that for all smooth domains $\Omega \subset \mathbf{H}$,

$$\mathcal{S}^3(\partial\Omega) \geq \mathcal{S}^4(\Omega)^{3/4}, \quad (1)$$

up to a universal constant that we ignore again.

Here is an alternate description of \mathcal{S}^3 . Let $d\text{area}_\epsilon$ denote the area induced by Riemannian metric g_ϵ . The 1-form θ restricts to a 1-form on S , we denote its g_ϵ -norm by $|\theta|_S|_\epsilon$. Then \mathcal{S}^3 has density $|\theta|_S|_\epsilon$ with respect to g_ϵ -area,

$$d\mathcal{S}^3 = |\theta|_S|_\epsilon d\text{area}_\epsilon.$$

Since $|\theta|_\epsilon = \epsilon$, $|\theta|_S|_\epsilon \leq \epsilon$, therefore $\mathcal{S}^3(S) \leq \epsilon \text{area}_\epsilon(S)$. On the other hand, the Riemannian volume element of g_ϵ is $\frac{1}{\epsilon} dx dy dz$. This shows that the Heisenberg isoperimetric inequality (1) implies a lower bound on the isoperimetric profile of (\mathbf{H}, g_ϵ) for all $\epsilon > 0$,

$$I_{(\mathbf{H}, g_\epsilon)}(v) \geq \frac{1}{\epsilon^{1/4}} v^{3/4}. \quad (2)$$

This is asymptotically sharp for large volumes, but not for small volumes, where the correct asymptotics is $v^{2/3}$. Never mind, it is the dependance on ϵ which is most important here.

We shall not directly use inequality (2). Instead, we shall rely on inequality (1) to study the Carnot-Carathéodory isoperimetric profile of a quotient of \mathbf{H} . Only at the very end shall we return to Riemannian geometry.

2.2 Nilmanifolds

\mathbf{H} possesses group automorphisms $\delta_t(x, y, z) = (tx, ty, tz)$. Let $\Gamma_t = \delta_t(\mathbf{H}\mathbf{Z})$ and $N_t = \Gamma_t \backslash \mathbf{H}$ be the quotient manifold. It inherits quotient metrics g_ϵ , yielding Riemannian nilmanifolds $N_{t,\epsilon}$ of total volume equal to $\frac{t^4}{\epsilon}$. But it also inherits a Carnot-Carathéodory metric that depends only on t . Our first goal is to show that the Carnot-Carathéodory isoperimetric profile of N_t satisfies an inequality similar to (1). Note that δ_t induces a homothetic map of N_1 onto N_t , so it suffices to work with one single compact space N_1 . The volume of N_1 is $\mathcal{S}^4(N_1) = 1$.

Theorem 2 *There exists a constant c such that the Carnot-Carathéodory isoperimetric profile of N_1 satisfies $I_{(N_1, d_c)}(v) \geq c \min\{v, 1 - v\}^{3/4}$. In other words, if $\Omega \subset N_1$ is a smooth domain of volume less than $1/2$,*

$$\mathcal{S}^3(\partial\Omega) \geq c \mathcal{S}^4(\Omega)^{3/4}.$$

The method consists in cutting domains of N_1 into pieces that lift to covering spaces. Ultimately, pieces lift to \mathbf{H} where one can apply (1). This covers cases where volume is smaller than some universal constant v_0 . To treat domains with volume $\geq v_0$, we apply a compactness result due to [FSSC01] (see also [LR03]).

2.3 Reduction to pillars

A first step is to cut domains into pieces called *pillars* that lift to a $\mathbf{Z} \oplus \mathbf{Z}$ covering space Z of N_1 .

Definition 1 Let ζ denote the center of \mathbf{H}_Z . Let us call *pillar* a subset of $Z = \zeta \setminus \mathbf{H}$ whose projection to $\mathbf{H}/[\mathbf{H}, \mathbf{H}] = \mathbf{R}^2$ is contained in a unit square. Denote by PI_Z the pillar profile of Z , i.e.

$$PI_Z(v) = \inf\{\mathcal{S}^3(\partial P); P \text{ a pillar, } \mathcal{S}^4(P) = v\}.$$

Proposition 2 (Reduction to pillars) *The pillar profile of Z bounds the profile of N_1 from below, with an error term,*

$$I_{(N_1, d_c)}(v) \geq PI_Z(v) - 4v.$$

Proof. The coordinate functions x and y on \mathbf{H} pass to a quotient $N_1 \rightarrow \mathbf{Z} \setminus \mathbf{R}$. For $u = (s, s') \in (\mathbf{Z} \setminus \mathbf{R})^2$, let

$$G_u = \{p \in N_1; x(p) = s \text{ or } y(p) = s'\}.$$

This is the union of two surfaces, each of which is a level set of one of the functions x or y . The complement of G_u has a cyclic fundamental group that maps isomorphically onto ζ .

Let Ω be a domain in N_1 . By the coarea formula,

$$\mathcal{S}^4(\Omega) = \int_{\mathbf{Z} \setminus \mathbf{R}} \mathcal{S}^3(x^{-1}(s) \cap \Omega) ds.$$

This coarea formula follows from the fact that the volume element (viewed as a 4-form) splits,

$$d\mathcal{S}^4 = dx \wedge dy \wedge \theta = dx \wedge d\mathcal{S}^3,$$

since $dy \wedge \theta = d\mathcal{S}^3$ along the fibers of x (one can take $d\ell = dy$ globally).

The same inequality holds with x replaced with y . This shows that there exists $u = (s, s') \in (\mathbf{Z} \setminus \mathbf{R})^2$ such that

$$\mathcal{S}^3(x^{-1}(s) \cap \Omega) \leq \mathcal{S}^4(\Omega), \quad \mathcal{S}^3(y^{-1}(s') \cap \Omega) \leq \mathcal{S}^4(\Omega),$$

and thus

$$\mathcal{S}^3(G_u \cap \Omega) \leq 2\mathcal{S}^4(\Omega).$$

The complement $\Omega \setminus G_u$ lifts to the cyclic covering space Z . Pick some lift. Its closure P is a pillar. Indeed, on P , the real valued functions x and y take values in intervals of length 1. The boundary of P consists of a part that isometrically and injectively maps to $\partial\Omega$, and of a part that maps 2-1 to $G_u \cap \Omega$. Therefore

$$\mathcal{S}^3(\partial P) \leq \mathcal{S}^3(\partial\Omega) + 2\mathcal{S}^3(G_u \cap \Omega) \leq \mathcal{S}^3(\partial\Omega) + 4\mathcal{S}^4(\Omega).$$

If $\mathcal{S}^4(\Omega) = v$, this shows that

$$I_{(N_1, d_c)}(v) \geq PI_Z(v) - 4v. \blacksquare$$

2.4 Treatment of pillars

Proposition 3 (Treatment of pillars) *The profile of \mathbf{H} bounds the pillar profile of Z from below, with an error term,*

$$PI_Z(v) \geq I_{\mathbf{H}}(v) - 8v.$$

Proof. Let $P \subset Z$ be a pillar. We can assume that its projection to \mathbf{R}^2 is contained in $\{0 \leq x \leq 1\}$. Its inverse image \tilde{P} in \mathbf{H} is a ζ -invariant subset with small projection in \mathbf{R}^2 . Again, we cut \tilde{P} into logs of height 1 using level sets of the z function. This time, we split the volume element as

$$d\mathcal{S}^4 = dx \wedge dy \wedge dz = dz \wedge (dx \wedge dy) = dz \wedge \frac{1}{|x|} d\mathcal{S}^3 \geq dz \wedge d\mathcal{S}^3.$$

We have used the expression $d\mathcal{S}^3 = |x| dx dy$ for the measure induced on horizontal planes $\{z = s\}$. On such surfaces, one can take $d\ell = dx$ globally, whence $d\mathcal{S}^3 = \pm dx \wedge \theta = |x| dx dy$. The coarea formula gives

$$\begin{aligned} \mathcal{S}^4(P) &= \mathcal{S}^4(\tilde{P} \cap \{0 \leq z \leq 1\}) \\ &= \int_0^1 \left(\int_{\tilde{P} \cap \{z=s\}} \frac{1}{|x|} d\mathcal{S}^3 \right) ds \\ &\geq \int_0^1 \mathcal{S}^3(\tilde{P} \cap \{z = s\}) ds. \end{aligned}$$

There exists $s \in [0, 1]$ such that

$$\mathcal{S}^3(\tilde{P} \cap \{z = s\}) \leq \mathcal{S}^4(P).$$

Set $\Omega' = \tilde{P} \cap \{s \leq z \leq s+1\}$. Then

$$\mathcal{S}^3(\partial\Omega') \leq \mathcal{S}^3(\partial P) + 2\mathcal{S}^4(P).$$

If P has volume v , this leads to

$$PI_Z(v) \geq I_{\mathbf{H}}(v) - 2v.$$

2.5 Profile of (N_1, d_c)

Proposition 4 (Carnot-Carathéodory isoperimetric inequality for small volumes) *If $v \leq 12^{-4}$,*

$$I_{(N_1, d_c)}(v) \geq \frac{1}{2}v^{3/4}.$$

Proof. Combined with Propositions 2 and 3, the Heisenberg isoperimetric inequality (1) yields

$$I_{(N_1, d_c)}(v) \geq v^{3/4} - 4v - 2v = v^{3/4}(1 - 6v^{1/4}) \geq \frac{1}{2}v^{3/4},$$

since $v \leq 12^{-4}$. ■

2.6 Proof of Theorem 2

There is a notion of Carnot-Carathéodory perimeter, an appropriate topology for which \mathcal{S}^4 is continuous and the perimeter (which coincides with \mathcal{S}^3 for smooth domains) lower semi-continuous, and a compactness theorem for sets of bounded perimeter in a compact Carnot manifold, [FSSC01]. This implies that the Carnot-Carathéodory isoperimetric profile $I_{(N_1, d_c)}$ is positive on $(0, 1)$ and lower semi-continuous. Therefore, there exists $\eta > 0$ such that $I_{(N_1, d_c)} \geq \eta$ on $[12^{-4}, 1 - 12^{-4}]$. Set $c = \min\{\frac{1}{2}, 2^{3/4}\eta\}$. Then $I_{(N_1, d_c)}(v) \geq \eta = c(\frac{1}{2})^{3/4} \geq cv^{3/4}$ for every $v \in [12^{-4}, \frac{1}{2}]$. On the other hand, Proposition 4 shows that $I_{(N_1, d_c)}(v) \geq cv^{3/4}$ for all $v \in [0, 12^{-4}]$. ■

Note that the proof does not provide an effective constant c .

2.7 Riemannian profile

Corollary 3 *Let $N_{t,\epsilon}$ denote the quotient $(\delta_t(\mathbf{H}_\mathbf{Z})) \setminus \mathbf{H}$ equipped with the Riemannian metric induced by g_ϵ . The isoperimetric profile of $N_{t,\epsilon}$ satisfies*

$$I_{N_{t,\epsilon}}(v) \geq \frac{c}{\epsilon^{1/4}} \min\{v, \frac{t^4}{\epsilon} - v\}^{3/4}.$$

Proof. The homothetic map $N_1 \rightarrow N_t$ induced by the automorphism δ_t transports the inequality of Theorem 2 to N_t without any change but the fact that $\mathcal{S}^4(N_t) = t^4$ replaces 1. The Riemannian volume element of $N_{t,\epsilon}$ is $\frac{1}{\epsilon}\mathcal{S}^4$, the Riemannian area induced on surfaces satisfies $\epsilon \text{ area} \geq \mathcal{S}^3$. This leads to the indicated dependence on ϵ in the isoperimetric profile of $N_{t,\epsilon}$. ■

3 Proof of Theorem 1

3.1 The case of a disjoint union of nilmanifolds

Proposition 5 *Let $\tau_n = \frac{1}{n}$, $\epsilon_n = \tau_n^3$ and $t_n = \tau_n^{3/4}(1 + \tau_n)^{1/4}$. Let $N = \coprod_n N_{t_n, \epsilon_n}$. Then, for all $v \in [\frac{1}{16}, 1]$, $I_N(v) \geq \frac{c}{8}$, where c is the constant of Theorem 2.*

Proof. By construction, $\text{vol}(N_{t_n, \epsilon_n}) = 1 + \tau_n$. Let Ω be a domain in N with $\text{vol}(\Omega) = v$. Write $\Omega = \coprod_n \Omega_n$ where $\Omega_n \subset N_{t_n, \epsilon_n}$ has volume v_n , $\sum_{n=1}^\infty v_n = v$.

If some v_n satisfies $v_n \geq \frac{1}{2}(1 + \tau_n)$, then

$$\begin{aligned} \text{area}(\partial\Omega_n) &\geq \frac{c}{\epsilon_n^{1/4}}(1 + \tau_n - v_n)^{3/4} \\ &\geq \frac{c}{\epsilon_n^{1/4}}\tau_n^{3/4} = c, \end{aligned}$$

so

$$\text{area}(\partial\Omega) \geq c \tag{3}$$

in this case.

Otherwise, for all $n \geq 1$,

$$\text{area}(\partial\Omega_n) \geq \frac{c}{\epsilon_n^{1/4}} v_n^{3/4} \geq c v_n^{3/4}.$$

We use the concavity inequality

$$a^\alpha + b^\alpha \geq (a + b)^\alpha,$$

valid for all $0 \leq \alpha \leq 1$, $a \geq 0$ and $b \geq 0$. This gives

$$\begin{aligned} \text{area}(\partial\Omega) &= \sum_{n=1}^{\infty} \text{area}(\partial\Omega_n) \\ &\geq c \sum_{n=1}^{\infty} v_n^{3/4} \\ &\geq c \left(\sum_{n=1}^{\infty} v_n \right)^{3/4} = \left(\frac{1}{16} \right)^{3/4} c = \frac{c}{8}. \blacksquare \end{aligned}$$

3.2 Connecting manifolds

Proof. We construct a noncompact manifold that has the shape of an infinite pearl necklace, adjusting suitable parameters carefully. Let $0 < \tau_n < 1$ be the sequence of positive real numbers chosen in the proof of Proposition 5. Pick another sequence of volumes $w_n < 1$, such that

$$\sum_n w_n < \frac{1}{2}, \tag{4}$$

and a sequence of areas $a_n > 0$ such that

$$\sum_n a_n < \frac{c}{16}, \tag{5}$$

where c is the constant of Theorem 2.

The manifolds N_{t_n, ϵ_n} that we want to connect to obtain our counterexample M , are like in Proposition 5, in particular we retain here that $V(N_{t_n, \epsilon_n}) = 1 + \tau_n$. Take two small disjoint balls $B_{n,1}, B_{n,2}$ inside N_{t_n, ϵ_n} whose boundaries have total area $\leq a_n$. Arrange that $B_{n,2}$ and $B_{n+1,1}$ be nearly isometric. Put $\tilde{N}_n := N_{t_n, \epsilon_n} \setminus (B_{n,1} \cup B_{n,2})$.

Consider tubes or cylinders T_n of the form $T_n := (S^2(1) \times [0, 1], g_n)$, where the metrics g_n are chosen in such a way that $V(g_n) \leq w_n$ and they glue together into a smooth metric on the connected sum $M_n := \tilde{N}_n \# T_n$ where the gluing is done along $i_n(S^2(1) \times \{0\}) \cong \partial B_{n,2}$. Now consider

$$(M, g) := M_1 \# M_2 \# \cdots \# M_n \# M_{n+1} \# \cdots \tag{6}$$

where M_n and M_{n+1} are glued together along the boundaries $i_n(S^2(1) \times \{1\}) \cong \partial B_{(n+1),1}$, where $i_n : T_n \rightarrow M$ is the isometric embedding associated to our construction.

We show that the right limit $I_M(1+)$ vanishes. Consider domains $D_n := \tilde{N}_n$, we get $V(D_n) = 1 + \tau_n - \tilde{v}'_{n-1} - \tilde{v}'_n = 1 + \alpha_n$, with $\alpha_n \rightarrow 0$, $\varepsilon'_n := A(\partial D_n) = A_g(\partial B_{n,2} \dot{\cup} \partial B_{n+1,1}) \rightarrow 0$. This implies readily

$$0 \leq \lim_{n \rightarrow +\infty} I_M(1 + \alpha_n) \leq \lim_{n \rightarrow +\infty} A(\partial D_n) = 0. \quad (7)$$

We show that $I_M(1) > 0$. Let Ω be a domain in M such that $V(\Omega) = 1$. Write $\tilde{\Omega} := \bigcup \tilde{\Omega}_n$, where $\tilde{\Omega}_n := \Omega \cap \tilde{N}_n$. Then

$$V(\tilde{\Omega}) \geq 1 - \sum_n w_n \geq \frac{1}{2}.$$

According to Proposition 5,

$$A(\partial \tilde{\Omega}) \geq \frac{c}{8}.$$

Since, for all n ,

$$\partial \tilde{\Omega}_n = ((\partial \Omega) \cap \tilde{N}_n) \dot{\cup} (\Omega \cap \partial \tilde{N}_n),$$

$$A(\partial \tilde{\Omega}_n) - A((\partial \Omega) \cap \tilde{N}_n) \leq A_g(\partial B_{n,2} \dot{\cup} \partial B_{n,1}) \leq a_n,$$

thus

$$A(\partial \Omega) \geq A(\partial \tilde{\Omega}) - \sum_n a_n \geq \frac{c}{8} - \frac{c}{16} = \frac{c}{16}.$$

This show that $I_M(1) \geq \frac{c}{16}$.

This concludes the proof of Theorem 1. ■

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Stefano Nardulli
Instituto de Matemática
Universidade Federal de Rio de Janeiro (Brazil)
`nardulli@im.ufrj.br`
<http://www.im.ufrj.br/nardulli>

Pierre Pansu
Laboratoire de Mathématiques d'Orsay, UMR 8628 du C.N.R.S.
Bâtiment 425, Université Paris-Sud - 91405 Orsay (France)
`Pierre.Pansu@math.u-psud.fr`
<http://www.math.u-psud.fr/~pansu>