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THE NUMBER OF MAXIMAL TORSION COSETS ON SUBVARIETIES OF TORI

CÉSAR MARTÍNEZ

ABSTRACT. We obtain a sharp bound on the number of maximal torsion cosets in a subvariety of the complex algebraic torus \mathbb{G}_m^n .

1. INTRODUCTION

Let \mathbb{G}_m^n be the complex algebraic torus. As an affine variety, we identify \mathbb{G}_m^n with the Zariski open subset $x_1 \cdots x_n \neq 0$ of $\mathbb{A}_{\mathbb{C}}^n$ with the usual multiplication

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = (x_1 y_1, \dots, x_n y_n).$$

Therefore, a torsion point of \mathbb{G}_m^n is an n -tuple of roots of unity. Let V be a subvariety of \mathbb{G}_m^n , we denote by $\overline{V}_{\text{tor}}$ the Zariski closure of the torsion points contained in V .

Motivated by the conjectures of Mordell and Manin-Mumford, Lang conjectured that $\overline{V}_{\text{tor}}$ is a finite union of torsion cosets, i.e. translates by torsion points of algebraic subtori of \mathbb{G}_m^n [Lan83, p.220]. This was proved by Ihara, Serre and Tate for $\dim(V) = 1$ [Lan83, Theorem 6.1] and by Laurent for higher dimensions [Lau84, Théorème 2]. This conjecture was originally formulated in a more general setting, which was proved by McQuillan [McQ95]. The following step is, then, to find an upper bound for the number of torsion varieties in V .

We say that a torsion coset contained in V is maximal if it is maximal with respect to the inclusion. It was already proved by Laurent [Lau84] that, if V is defined over \mathbb{K} by a set of polynomials of degree at most d and height at most M , the number of maximal torsion cosets in V and the degree of them is effectively bounded in terms of d , n , M and $[\mathbb{K} : \mathbb{Q}]$. Later, Bombieri and Zannier [BZ95], following the work of Zhang [Zha95], showed that both, the number of maximal torsion cosets and the degree of their defining polynomials, can be bounded just in terms of n and d .

Furthermore, Schmidt [Sch96] obtained an explicit upper bound for the number of maximal torsion cosets in V . This bound was then improved with a result of Evertse [Eve99], bounding the number of maximal torsion cosets by

$$(11d)^{n^2} \binom{n+d}{d}^{3 \binom{n+d}{d}^2}.$$

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Meanwhile, Ruppert [Rup93] and Sarnak and Adams [SA94] presented algorithms to determine the torsion subvariety of some variety $V \subset \mathbb{G}_m^n$. In particular, Ruppert's approach treats first the case $\dim(V) = 1$ where, given (d_1, \dots, d_n) the multi-degree of $V \subset (\mathbb{P}^1)^n$, he is able to bound the number of isolated torsion points in V by $22 \max(d_i) \min(d_i)$. Afterwards, he extends his algorithm to some specific varieties in higher dimension, which leads him to formulate the following conjecture:

Conjecture 1.1. *Let $f \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ of multi-degree (d_1, \dots, d_n) , the number of isolated torsion points on the hypersurface $V(f) \subset \mathbb{G}_m^n$ is bounded by $c_n d_1 \cdots d_n$, where c_n is some constant depending only on n .*

Beukers and Smyth [BS02] reconsidered the problem for $n = 2$ from a similar point of view to Ruppert's one, being able to improve the bound in this case. Given $f \in \overline{\mathbb{Q}}[X, Y]$ a polynomial, they bound the number of torsion points of $V(f)$ by $22 \text{vol}_2(\Delta)$, where Δ denotes the Newton polytope of f . This leads Aliev and Smyth [AS12] to strengthen the original conjecture of Ruppert as follows:

Conjecture 1.2. *Let $f \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$, the number of isolated torsion points on the hypersurface $V(f) \subset \mathbb{G}_m^n$ is bounded by $c_n \text{vol}_n(\Delta)$, where c_n is a constant depending only on n and Δ is the Newton polytope of f .*

Aliev and Smyth [AS12] extended Beukers and Smyth's algorithm to higher dimensions and obtained a bound far from the conjecture. For $f \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ of degree d , they bound the number of maximal torsion cosets in V by

$$c_1(n) d^{c_2(n)},$$

where $c_1(n) = n^{\frac{3}{2}(2+n)5^n}$ and $c_2(n) = \frac{1}{16}(49 \cdot d^{n-2} - 4n - 9)$.

For sparse representations of polynomials, Leroux [Ler12] obtained an algorithm to find the maximal torsion cosets in V . As a consequence, he is able to bound the number of maximal torsion cosets in V in terms of the number of non-zero coefficients of the defining polynomials of V . For dense polynomials the bound has similar order to the one of Aliev and Smyth [AS12].

Another approach to this problem has been to conceive it as a particular case of the study of the number of algebraic points of small height. That is because the torsion points on V can be thought of as the points of height 0 on V . Thereby, some upper bounds on the number of subvarieties of small height can be translated to our setting. This is how Amoroso and Viada obtained the following bound (see [AV09, Corrolary 5.4]). Let V be a subvariety of \mathbb{G}_m^n of codimension k and defined by polynomials of degree at most δ . Let

$$\theta = \delta(200n^5 \log(n^2 \delta))^{(n-k)n(n-1)}.$$

Then the number of maximal torsion cosets in V is bounded by θ^n . Note that δ can be replaced by the degree of V in the case that V is a hypersurface (as in the statement of the conjecture). This way, we can see that this bound is close to the one conjectured by Ruppert, up to a logarithmic factor.

In this article we combine the approach of Ruppert and Aliev and Smyth with the methods of Amoroso and Viada to prove both, Ruppert's and Aliev and Smyth's conjectures. Our first main result is the following:

Theorem 1.3. *Let $V \subset \mathbb{G}_m^n$ be a variety of dimension d and defined by polynomials of degree at most δ . For $j = 0, \dots, d$, let V_{tor}^j be the j -equidimensional part of $\overline{V_{\text{tor}}}$. Then*

$$\deg(V_{\text{tor}}^j) \leq c_n \delta^{n-j}$$

for every $j = 0, \dots, d$, where

$$c_n = ((2n - 1)(n - 1)(2^{2n} + 2^{n+1} - 2))^{nd}.$$

As before, δ can be replaced by the degree of V in the case that V is a hypersurface. Using John's theorem [Joh48, Theorem III], we are able to translate this result to prove our conjectures in Theorem 5.3. A direct consequence of this theorem is the following result:

Corollary 1.4. *Let $f \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ and let $\Delta \subset \mathbb{R}^n$ be a convex body such that $\text{supp}(f) \subset \Delta$. Then the number of isolated torsion points on the hypersurface $V(f) \subset \mathbb{G}_m^n$ is bounded by $\tilde{c}_n \text{vol}_n(\Delta)$, where $\tilde{c}_n = ((2n - 1)(n - 1)(2^n + 2^{n+1} - 2))^{n(n-1)} 2^n n^{2n} \omega_n^{-1}$ and ω_n is the n -volume of the n -sphere.*

Let f be of multidegree (d_1, \dots, d_n) , we can take $\Delta = [0, d_1] \times \dots \times [0, d_n]$ and prove Ruppert's conjecture. Moreover, taking Δ as the Newton polytope of f proves Aliev and Smyth's conjecture.

To discard the logarithmic error term in [AV09, Theorem 1.2], we reformulate the main theorems of Amoroso and Viada so that they suit our particular case of torsion subvarieties.

In order to do that, we extend first the argument introduced by Beukers and Smyth in [BS02] to a more algebraic setting. In Proposition 3.2 we get, for any irreducible subvariety V of \mathbb{G}_m^n , another variety $V' \subset \mathbb{G}_m^n$ with the same dimension and similar degree, such that $\overline{V_{\text{tor}}}$ lies in the intersection $V \cap V' \subsetneq V$. Moreover, this V' can be obtained explicitly from our initial V .

Next, in Theorem 3.4, we use the Hilbert function to consider, instead of the subvariety V' , a hypersurface Z satisfying that $\overline{V_{\text{tor}}} \subset V \cap Z \subsetneq V$. To do that, we rely on both, an upper and a lower bound for the Hilbert function; being the upper bound a result of Chardin [Cha89] and the lower bound a result of Chardin and Philippon [CP99]. By using this bounds, we obtain Lemma 2.8, which serves as bridge between V' and Z and is, therefore, the key element in our proof of Theorem 3.4.

Afterwards, we present two induction theorems, Theorem 4.1 and Theorem 1.3, which are the equivalent of [AV09, Theorems 2.2 and 1.2] in our case. We can observe that Theorem 1.3 already proves Aliev and Smyth's conjecture for the dense case.

Finally, we use John's theorem and Proposition 5.2 to include a transformation of any convex body Δ into a homothecy of the standard n -simplex of comparable n -volume. By doing this, we are able to translate Theorem 1.3 and obtain Theorem 5.3. As a consequence to this, we obtain Corollary 1.4 which proves both conjectures.

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2. PRELIMINARIES

2.1. Homomorphisms and subgroups of algebraic tori. Let $\mathbb{G}_m^n = (\mathbb{C}^\times)^n$ be the *multiplicative group of algebraic torus* of dimension n . A point $(x_1, \dots, x_n) \in \mathbb{G}_m^n$ is alternatively denoted by \mathbf{x} . In particular, $\mathbf{1} = (1, \dots, 1)$ represents the identity element. Given $\mathbf{x} \in \mathbb{G}_m^n$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ we denote

$$\mathbf{x}^\boldsymbol{\lambda} = x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

Moreover, given $S \subset \mathbb{G}_m^n$ any subset we denote

$$\mathbf{x} \cdot S = \{\mathbf{x} \cdot \mathbf{y} \mid \mathbf{y} \in S\}.$$

If the context is clear, we write simply $\mathbf{x}S$.

We call *homomorphism* an algebraic group homomorphism $\varphi : \mathbb{G}_m^{n_1} \rightarrow \mathbb{G}_m^{n_2}$. There is a bijection between integer matrices $\mathcal{M}_{n_2 \times n_1}(\mathbb{Z})$ and homomorphisms $\text{Hom}(\mathbb{G}_m^{n_1}, \mathbb{G}_m^{n_2})$ defined as follows. Let $M \in \mathcal{M}_{n_2 \times n_1}(\mathbb{Z})$ and let $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{n_2} \in \mathbb{Z}^{n_1}$ be the row vectors of M , then

$$\begin{aligned} \varphi_M : \mathbb{G}_m^{n_1} &\longrightarrow \mathbb{G}_m^{n_2} \\ \mathbf{x} &\longmapsto (\mathbf{x}^{\boldsymbol{\lambda}_1}, \dots, \mathbf{x}^{\boldsymbol{\lambda}_{n_2}}) \end{aligned}$$

defines the corresponding homomorphism. In particular, for any $l \in \mathbb{Z}$, we define the *multiplication* by l as the following endomorphism:

$$\begin{aligned} [l] : \mathbb{G}_m^n &\longrightarrow \mathbb{G}_m^n \\ (x_1, \dots, x_n) &\longmapsto (x_1^l, \dots, x_n^l) \end{aligned}$$

which corresponds to the diagonal matrix $l\text{Id} \in \mathcal{M}_{n \times n}(\mathbb{Z})$.

We denote by ζ_k a primitive k -th root of unity, for any $k \in \mathbb{N}_{>0}$, and by

$$\mu_k = \{\zeta \in \mathbb{G}_m \mid \zeta^k = 1\}$$

the subgroup of k -th roots of unity. In particular, we denote by

$$\mu_\infty = \bigcup_{k \in \mathbb{N}_{>0}} \mu_k$$

the subgroup of roots of unity in \mathbb{G}_m . Therefore,

$$\mu_\infty^n = \{\boldsymbol{\xi} \in \mathbb{G}_m^n \mid [k]\boldsymbol{\xi} = \mathbf{1} \text{ for some } k \in \mathbb{N}_{>0}\}$$

is the subgroup of the *torsion points* of \mathbb{G}_m^n and $\mu_k^n = \{\boldsymbol{\xi} \in \mathbb{G}_m^n \mid [k]\boldsymbol{\xi} = \mathbf{1}\}$ is the subgroup of k -torsion points of \mathbb{G}_m^n . For a subvariety $V \subset \mathbb{G}_m^n$, we denote by $V_{\text{tor}} = V \cap \mu_\infty^n$ the set of torsion points on V and by $\overline{V_{\text{tor}}}$ its Zariski closure in \mathbb{G}_m^n . We call $\overline{V_{\text{tor}}}$ the *torsion subvariety* of V .

By *torsion coset* we understand a subvariety $\boldsymbol{\omega}H \subset \mathbb{G}_m^n$, where H is an irreducible algebraic subgroup of \mathbb{G}_m^n and $\boldsymbol{\omega}$ a torsion point. Let V be a subvariety of \mathbb{G}_m^n , then we say that a torsion coset $\boldsymbol{\omega}H$ is maximal in V if $\boldsymbol{\omega}H \subset V$ and it is not contained in any other torsion coset in V . Observe that $\overline{V_{\text{tor}}}$ corresponds to the torsion cosets in V , hence we can write it

$$\overline{V_{\text{tor}}} = \bigcup_{\substack{\boldsymbol{\omega}H \subset V \\ \text{torsion coset}}} \boldsymbol{\omega}H.$$

In fact, it is enough to take the maximal torsion cosets in V in the index of the union.

Let Λ be a subgroup of \mathbb{Z}^n . We denote by $\Lambda^{\text{sat}} = (\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^n$ the *saturation* of Λ and we call $[\Lambda^{\text{sat}} : \Lambda]$ the *index* of Λ . We say that Λ is *saturated* if $[\Lambda^{\text{sat}} : \Lambda] = 1$. We define the algebraic subgroup of \mathbb{G}_m^n associated to Λ as follows

$$H_\Lambda = \{\mathbf{x} \in \mathbb{G}_m^n \mid \mathbf{x}^\lambda = 1, \lambda \in \Lambda\}.$$

The following result allows us to understand the relation between lattices in \mathbb{Z}^n and algebraic subgroups of \mathbb{G}_m^n .

Theorem 2.1. *The map $\Lambda \mapsto H_\Lambda$ is a bijection between subgroups of \mathbb{Z}^n and algebraic subgroups of \mathbb{G}_m^n . A subgroup H_Λ is irreducible if and only if Λ is saturated. Moreover, for any two subgroups Λ and Λ' of \mathbb{Z}^n we have that $H_\Lambda H_{\Lambda'} = H_{\Lambda \cap \Lambda'}$.*

Proof. See [BG06, Proposition 3.2.7 and Theorem 3.2.19]. □

Corollary 2.2. *Let H be a subgroup of \mathbb{G}_m^n of dimension $n - r$, then there exists a surjective homomorphism*

$$\varphi : \mathbb{G}_m^n \longrightarrow \mathbb{G}_m^r$$

such that $\text{Ker}(\varphi) = H$.

Proof. By Theorem 2.1, there exists a unique lattice $\Lambda \subset \mathbb{Z}^n$ such that

$$H = H_\Lambda = \{\mathbf{x} \in \mathbb{G}_m^n \mid \mathbf{x}^\lambda = 1, \lambda \in \Lambda\}.$$

Take the saturated subgroup $\Lambda^\perp = \{\mathbf{x} \in \mathbb{Z}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{y} \in \Lambda\}$, so H_{Λ^\perp} is irreducible, that is $H_{\Lambda^\perp} \simeq \mathbb{G}_m^r$. By Theorem 2.1 we have that $\mathbb{G}_m^n = H_{\{0\}} = H \cdot H_{\Lambda^\perp}$ and φ can be obtained as the following composition of homomorphisms:

$$\varphi : \mathbb{G}_m^n = H \cdot H_{\Lambda^\perp} \longrightarrow H_{\Lambda^\perp} \xrightarrow{\simeq} \mathbb{G}_m^r.$$

□

Let V be a variety in \mathbb{G}_m^n , we define the stabilizer of V as

$$\text{Stab}(V) = \{\xi \in \mathbb{G}_m^n \mid \xi V = V\}.$$

In particular, $\text{Stab}(V)$ is an algebraic subgroup of \mathbb{G}_m^n . By means of Corollary 2.2 we are able to reduce V , via a homomorphism, to a variety with trivial stabilizer.

Corollary 2.3. *Let $V \subset \mathbb{G}_m^n$ be a variety. Then there exists a homomorphism $\varphi : \mathbb{G}_m^n \rightarrow \mathbb{G}_m^r$ such that $r = \text{codim}(\text{Stab}(V))$ and $\text{Ker}(\varphi) = \text{Stab}(V)$. Moreover, φ satisfies*

- (i) $\text{Stab}(\varphi(V)) = \{\mathbf{1}\}$;
- (ii) $\varphi^{-1}(\varphi(V)) = V$;
- (iii) $\varphi^{-1}(\boldsymbol{\eta})V = \boldsymbol{\eta}_0 V$, for every $\boldsymbol{\eta} \in \mathbb{G}_m^r$ and for any $\boldsymbol{\eta}_0 \in \varphi^{-1}(\boldsymbol{\eta})$.

An extra remark should be made regarding the relation between the torsion cosets and the stabilizer of V . For any torsion coset $\boldsymbol{\omega}H$ in V , we have that $\text{Stab}(V) \cdot \boldsymbol{\omega}H$ is a union torsion cosets in V . In particular, every maximal torsion coset in V has dimension at least $\dim(\text{Stab}(V))$.

2.2. Hilbert function. Let V be a variety in \mathbb{G}_m^n . We define the *degree of definition* of V , $\delta(V)$, as the minimal degree δ such that V is the intersection of hypersurfaces of degree at most δ . We also define the *degree of incomplete definition* of V , $\delta_0(V)$, as the minimal degree δ_0 such that there exists an intersection X of hypersurfaces of degree at most δ_0 such that any irreducible component of V is a component of X . As a direct consequence of the definition, if V is equidimensional, we have the following inequalities

$$\delta_0(V) \leq \delta(V) \leq \deg(V).$$

Let V be a subvariety of \mathbb{G}_m^n and let the closure of V in \mathbb{P}^n be defined by the homogeneous radical ideal I in $\overline{\mathbb{Q}}[\mathbf{x}]$. For $\nu \in \mathbb{N}$, we denote by $H(V; \nu)$ the Hilbert function $\dim(\overline{\mathbb{Q}}[\mathbf{x}]/I)_\nu$.

The following upper bound for the Hilbert function, is a theorem of Chardin [Cha89].

Theorem 2.4. *Let $V \subseteq \mathbb{G}_m^n$ be an equidimensional variety of dimension $d = n - k$ and let $\nu \in \mathbb{N}$. Then*

$$H(V; \nu) \leq \binom{\nu + d}{d} \deg(V).$$

On the other hand, as a consequence of a result of Chardin and Phillipon [CP99, Corollaire 3] on Castelnuovo's regularity, we have the following lower bound for the Hilbert function:

Theorem 2.5. *Let $V \subseteq \mathbb{G}_m^n$ be an equidimensional variety of dimension $d = n - k$ and $m = k(\delta_0(V) - 1)$. Then, for any integer $\nu > m$, we have*

$$H(V; \nu) \geq \binom{\nu + d - m}{d} \deg(V).$$

In order to use these results later, we need effective upper bounds for $\delta_0(V')$ when V' is a specific type of equidimensional variety. Let us recall first an easy lemma for δ .

Lemma 2.6. *Let X_1, \dots, X_t be subvarieties of \mathbb{G}_m^n . Then*

$$\delta\left(\bigcup_{i=1}^t X_i\right) \leq \sum_{i=1}^t \delta(X_i).$$

Proof. It is enough to prove it for $t = 2$. Let X_1 be defined by the equations f_1, \dots, f_r with $\deg(f_i) \leq \delta(X_1)$ and equivalently let X_2 be defined g_1, \dots, g_s with $\deg(g_i) \leq \delta(X_2)$. Then $X_1 \cup X_2$ is defined by the equations $f_i g_j$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. \square

To state a similar lemma for δ_0 , we must consider more specific varieties. The following is a variation of [AV12, Lemma 2.5].

Lemma 2.7. *Let V be an irreducible subvariety of \mathbb{G}_m^n defined over \mathbb{K} . Let $T \subset \mu_\infty^n \times \text{Gal}(\mathbb{K}/\mathbb{Q})$ be a finite family with t elements. Then*

$$\delta_0\left(\bigcup_{(\mathbf{g}, \phi) \in T} \mathbf{g}V^\phi\right) \leq t\delta_0(V).$$

Proof. We say that an irreducible variety $W \subset \mathbb{G}_m^n$ is imbedded in a variety $X \subset \mathbb{G}_m^n$ if $W \subset X$ but W is not an irreducible component of X .

By definition of $\delta_0(V)$, there exists a variety X such that V is an irreducible component of X and $\delta_0(V) = \delta(X)$.

Let $G \subset \mu_\infty^n \times \text{Gal}(\mathbb{K}/\mathbb{Q})$ be the group generated by T and let $S = \{(\mathbf{g}, \phi) \in G \mid \mathbf{g}V^\phi \text{ is imbedded in } X\}$. Consider

$$\tilde{X} = X \cap \left(\bigcap_{(\mathbf{g}, \phi) \in S} \mathbf{g}^{-1}X^{\phi^{-1}} \right).$$

We have that V is an irreducible component of \tilde{X} and $\delta(\tilde{X}) = \delta(X) = \delta_0(V)$. Moreover, no $\mathbf{g}V^\phi$ is imbedded in \tilde{X} , for $(\mathbf{g}, \phi) \in G$. Assume by contradiction that there is a $\mathbf{g}V^\phi$ imbedded in \tilde{X} . We are going to show that then $(\mathbf{g}^n, \phi^n) \in S$ for every $n \in \mathbb{N}_{>0}$; in particular, taking $n = \text{lcm}(\text{ord}(\mathbf{g}), \text{ord}(\phi))$ we will have $(\mathbf{1}, \text{Id}) \in S$ which is a contradiction. Since $\tilde{X} \subset X$, $\mathbf{g}V^\phi$ is imbedded in X and so $(\mathbf{g}, \phi) \in S$. By induction, we suppose $(\mathbf{g}^n, \phi^n) \in S$ for some $n \geq 1$. Then $\tilde{X} \subset \mathbf{g}^{-n}X^{\phi^{-n}}$ and so $\mathbf{g}V^\phi$ is imbedded in $\mathbf{g}^{-n}V^{\phi^{-n}}$ which implies $(\mathbf{g}^{n+1}, \phi^{n+1}) \in S$.

Next we define

$$Y = \bigcup_{(\mathbf{g}, \phi) \in T} \mathbf{g}\tilde{X}^\phi.$$

Then $\bigcup_{(\mathbf{g}, \phi) \in T} \mathbf{g}V^\phi \subset Y$ and, by lemma 2.6, $\delta(Y) \leq t\delta(\tilde{X}) = t\delta_0(V)$. Moreover, no $\mathbf{g}V^\phi$ is imbedded in Y , for $(\mathbf{g}, \phi) \in T$. Assume by contradiction that there is a $(\mathbf{g}, \phi) \in T$ such that $\mathbf{g}V^\phi$ is imbedded in Y . Then, there exists some $(\mathbf{g}_0, \phi_0) \in T$ such that $\mathbf{g}V^\phi$ is imbedded in $\mathbf{g}_0\tilde{X}^{\phi_0}$. Thus $\mathbf{g}_0^{-1}\mathbf{g}V^{\phi_0^{-1}\phi}$ is imbedded in \tilde{X} which contradicts the definition of \tilde{X} since $(\mathbf{g}_0^{-1}\mathbf{g}, \phi_0^{-1}\phi) \in G$. □

Let $V \subset \mathbb{G}_m^n$. We say that V is *minimally defined* over \mathbb{K} , if \mathbb{K} is the minimal Galois extension of \mathbb{Q} such that V is defined over \mathbb{K} .

If \mathbb{K} is an abelian extension, by the Kronecker-Weber theorem we have that \mathbb{K} is contained in a cyclotomic extension of \mathbb{Q} . In fact, there is a unique minimal cyclotomic extension $\mathbb{Q}(\zeta_N)$ containing \mathbb{K} [Nar04, Theorem 4.27(v)]. If $N \equiv 2 \pmod{4}$, then $\mathbb{Q}(\zeta_N) = \mathbb{Q}(\zeta_{N/2})$. Therefore, we can always choose $N \not\equiv 2 \pmod{4}$.

The following lemma is a key ingredient in the proof of Theorem 3.4.

Lemma 2.8. *Let $V \subseteq \mathbb{G}_m^n$ be an irreducible variety of dimension $d = n - k$, minimally defined over \mathbb{K} . Let $\phi \in \text{Gal}(\mathbb{K}/\mathbb{Q})$ and let $e \in \mu_\infty^n$.*

- (a) *If $eV^\phi \neq V$, then there exists a homogeneous polynomial F of degree at most $2k(2d+1)\delta_0(V)$ such that $F \equiv 0$ in eV^ϕ and $F \not\equiv 0$ in V .*
- (b) *If $[2]^{-1}(eV^\phi) \neq V$, then there exists a homogeneous polynomial G of degree at most $2^nk(2d+1)\delta_0(V)$ such that $G \equiv 0$ in $[2]^{-1}(eV^\phi)$ and $G \not\equiv 0$ in V .*

Proof.

- (a) Since V is an irreducible variety, eV^ϕ is also irreducible and of the same degree. Then, using Theorem 2.4 we get, for any $\nu \in \mathbb{N}$,

$$H(eV^\phi; \nu) \leq \binom{\nu + d}{d} \deg(V).$$

On the other hand, let $V' = V \cup eV^\phi$. This is d -equidimensional of degree $2 \deg(V)$. Thereby using Theorem 2.5 we have, for any $\nu > m$,

$$H(V'; \nu) \geq \binom{\nu + d - m}{d} 2 \deg(V),$$

where $m = k(\delta_0(V') - 1)$. By Lemma 2.7, $m \leq 2k\delta_0(V)$.

Taking $\nu = m(2d + 1)$ we obtain:

$$\binom{\nu + d}{d} \binom{\nu + d - m}{d}^{-1} \leq \left(1 + \frac{m}{\nu - m}\right)^d = \left(1 + \frac{1}{2d}\right)^d \leq e^{1/2} < 2.$$

Hence, $H(eV^\phi; \nu) < H(V'; \nu)$.

This means that there exists a homogeneous polynomial F of degree ν such that $F \equiv 0$ on eV^ϕ and $F \not\equiv 0$ on $V' = eV^\phi \cup V$, in particular $F \not\equiv 0$ on V . Moreover, $\deg(F) = \nu \leq 2k(2d + 1)\delta_0(V)$, which proves (a).

(b) Let $W = [2]^{-1}(eV^\phi)$. Since W is d -equidimensional of degree $2^k \deg(V)$, by using Theorem 2.4 we get, for any $\nu \in \mathbb{N}$,

$$H(W; \nu) \leq \binom{\nu + d}{d} 2^k \deg(V).$$

On the other hand, let $E = \{e_0 \in \mu_\infty^n \mid e_0^2 \in \text{Stab}(V)\}$ and let $W' = E \cdot V$. This variety is also d -equidimensional and of degree $2^r \deg(V)$, for some $k < r \leq n$, that is because $E/\text{Stab}(V) \simeq \mu_2^r$ (see Corollary 2.3, $r = \text{codim}(\text{Stab}(V))$) and $E = \varphi^{-1}(\mu_2^r)$. Thereby, using Proposition 2.5 we have, for any $\nu > m$,

$$H(W'; \nu) \geq \binom{\nu + d - m}{d} 2^r \deg(V),$$

where $m = k(\delta_0(W') - 1)$. By Lemma 2.7, $m \leq 2^n k \delta_0(V)$.

Taking $\nu = m(2d + 1)$, we obtain:

$$\binom{\nu + d}{d} \binom{\nu + d - m}{d}^{-1} \leq e^{1/2} < 2^{r-k}.$$

Hence, $H(W; \nu) < H(W'; \nu)$.

This means that there exists a homogeneous polynomial \tilde{G} of degree ν such that $\tilde{G} \equiv 0$ on $W = [2]^{-1}(eV^\phi)$ and $\tilde{G} \not\equiv 0$ on $W' = E \cdot V$, in particular there exists $e_0 \in E$ such that $\tilde{G} \not\equiv 0$ on e_0V .

Let $G(x) = \tilde{G}(e_0^{-1}x)$. We have that $G \equiv 0$ on $e_0[2]^{-1}(eV)$. By definition of E , $e_0 \in \text{Stab}([2]^{-1}(eV))$. Hence, $G \equiv 0$ on $[2]^{-1}(eV)$. We also have that $G \not\equiv 0$ on $[2]e_0 \cdot V$. Since $[2]e_0 \in \text{Stab}(V)$, that means $G \not\equiv 0$ on V . Moreover, $\deg(G) = \nu \leq 2^n k(2d + 1)$, which proves (b). □

3. ALGEBRAIC INTERPOLATION

We generalize [BS02, Lemma 1] to general dimension n and to any abelian extension of \mathbb{Q} with the following result.

Lemma 3.1. *Let $V \subset \mathbb{G}_m^n$ be a variety defined over $\mathbb{Q}(\zeta_N)$, with $N \not\equiv 2 \pmod{4}$, and let $\omega \in V$ be a torsion point.*

1. If $4 \nmid N$, then one of the following is true:
 - (a) there exists $\boldsymbol{\eta} \in \mu_2^n \setminus \{\mathbf{1}\}$ such that $\boldsymbol{\eta} \cdot \boldsymbol{\omega} \in V$; or
 - (b) there exists $\boldsymbol{\eta} \in \mu_2^n$ such that $\boldsymbol{\eta} \cdot [2]\boldsymbol{\omega} \in V^\sigma$, where $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ maps $\zeta_N \mapsto \zeta_N^2$.
2. If $N = 4N'$, then one of the following is true:
 - (c) there exists $\boldsymbol{\eta} \in \mu_2^n \setminus \{\mathbf{1}\}$ such that $\boldsymbol{\eta} \cdot \boldsymbol{\omega} \in V$; or
 - (d) there exists $\boldsymbol{\eta} \in \mu_2^n$ such that $\boldsymbol{\eta} \cdot \boldsymbol{\omega} \in V^\tau$, where $\tau \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ maps $\zeta_N \mapsto \zeta_N^{1+2N'}$.

Note first that the case $\mathbb{K} = \mathbb{Q}$ is included in case 1, corresponding to $N = 1$ (and so σ is the identity).

Proof. Let l be the order of $\boldsymbol{\omega}$, in particular $\boldsymbol{\omega} \in \mathbb{Q}(\zeta_l)$.

1. Let $M = \text{lcm}(N, l)$. We distinguish 3 cases.
 - (i) If $l = 4l'$, then $M = 4M'$. We have $\text{gcd}(1 + 2M', M) = 1$ and thus we can take $\tau \in \text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ mapping $\zeta_M \mapsto \zeta_M^{1+2M'}$. Since $2M' \equiv 2l' \pmod{l}$, we have that τ maps $\zeta_l \mapsto \zeta_l^{1+2l'}$. On the other hand, $2M' \equiv 0 \pmod{N}$ because N is odd (in fact $N \mid M'$), therefore τ maps $\zeta_N \mapsto \zeta_N$. Hence $V^\tau = V$ and $[1 + 2l']\boldsymbol{\omega} \in V$. Therefore (a) holds choosing $\boldsymbol{\eta} = [2l']\boldsymbol{\omega} \in \mu_2^n \setminus \{\mathbf{1}\}$.
 - (ii) If $l = 2l'$ with $2 \nmid l'$, then $M = 2M'$ with $2 \nmid M'$. We have $\text{gcd}(2 + M', M) = 1$ and thus we can extend σ to a Galois automorphism in $\text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$, mapping $\zeta_M \mapsto \zeta_M^{2+M'}$ (this extends σ because N is odd and so $N \mid M'$). Since $M' \equiv l' \pmod{l}$, we have that σ maps $\zeta_l \mapsto \zeta_l^{2+l'}$. Hence $[2 + l']\boldsymbol{\omega} \in V^\sigma$. Therefore (b) holds choosing $\boldsymbol{\eta} = [l']\boldsymbol{\omega} \in \mu_2^n \setminus \{\mathbf{1}\}$.
 - (iii) If $2 \nmid l$, then $2 \nmid M$ and σ can be extended to a Galois automorphism in $\text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ mapping $\zeta_M \mapsto \zeta_M^2$. In particular, σ maps $\zeta_l \mapsto \zeta_l^2$, hence $[2]\boldsymbol{\omega} \in V^\sigma$. Therefore (b) holds choosing $\boldsymbol{\eta} = \mathbf{1}$.
2. Let $M = 4M' = \text{lcm}(N, l)$ and $\tilde{\tau}$ be an automorphism in $\text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ mapping $\zeta \mapsto \zeta^{2M'+1}$. We distinguish 2 cases.
 - (i) If $N \mid 2M'$, then $l \nmid 2M'$ (otherwise, we would have $\text{lcm}(N, l) = 2M'$). Since $2M' \equiv 2l' \pmod{l}$, we have that $\tilde{\tau}$ maps $\zeta_l \mapsto \zeta_l^{1+2l'}$. On the other hand, $2M' \equiv 0 \pmod{N}$, therefore $\tilde{\tau}$ maps $\zeta_N \mapsto \zeta_N$. Hence $V^{\tilde{\tau}} = V$ and $[1 + 2M']\boldsymbol{\omega} \in V$. Therefore (c) holds choosing $\boldsymbol{\eta} = [2l']\boldsymbol{\omega} \in \mu_2^n \setminus \{\mathbf{1}\}$.
 - (ii) If $N \nmid 2M'$, then $2N' \equiv 2M' \pmod{N}$. So, we have that $\tilde{\tau}$ maps $\zeta_N \mapsto \zeta_N^{1+2N'}$ and thus $\tilde{\tau}|_{\mathbb{Q}(\zeta_N)} = \tau$. Therefore (d) holds choosing $\boldsymbol{\eta} = [2M']\boldsymbol{\omega} \in \mu_2^n$.

□

As a consequence of this lemma, for any irreducible variety V we can find another variety V' containing the torsion subvariety of V but not containing V .

Proposition 3.2. *Let $V \subset \mathbb{G}_m^n$ be an irreducible variety, minimally defined over \mathbb{K} , of dimension $d = n - k$, $\overline{V_{\text{tor}}} \neq V$ and let $\varphi : \mathbb{G}_m^n \rightarrow \mathbb{G}_m^r$ be a homomorphism such that $\text{Stab}(V) = \text{Ker}(\varphi)$, $k < r \leq n$, where r is the codimension of $\text{Stab}(V)$ (which exists by Corollary 2.2).*

1. If \mathbb{K} is abelian and $\mathbb{Q}(\zeta_N)$ is a cyclotomic extension of \mathbb{K} , with $4 \nmid N$, then

$$\overline{V_{\text{tor}}} \subset V' = \bigcup_{\eta \in \mu_2^r \setminus \{1\}} (\varphi^{-1}(\eta)V) \cup \bigcup_{\eta \in \mu_2^r} [2]^{-1}(\varphi^{-1}(\eta)V^\sigma),$$

where $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, mapping $\zeta_N \mapsto \zeta_N^2$, and $V' \cap V \subsetneq V$.

2. If \mathbb{K} is abelian and $\mathbb{Q}(\zeta_N)$ is its minimal cyclotomic extension, with $N = 4N'$, then

$$\overline{V_{\text{tor}}} \subset V' = \bigcup_{\eta \in \mu_2^r \setminus \{1\}} (\varphi^{-1}(\eta)V) \cup \bigcup_{\eta \in \mu_2^r} (\varphi^{-1}(\eta)V^\tau),$$

where $\tau \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, mapping $\zeta_N \mapsto \zeta_N^{1+2N'}$, and $V' \cap V \subsetneq V$.

3. If \mathbb{K} is not abelian, then $\overline{V_{\text{tor}}} \subset V \cap V^\varsigma \subsetneq V$, for any $\varsigma \in \text{Gal}(\mathbb{K}/\mathbb{Q}^{\text{ab}} \cap \mathbb{K})$ such that $\varsigma \neq \text{Id}$.

Note that the V' in the proposition are finite unions of varieties. That is because, using Corollary 2.3(iii), it suffices to take, for each $\eta \in \mu_2^r$, just one preimage $\eta_0 \in \varphi^{-1}(\eta)$ instead of the the whole $\varphi^{-1}(\eta)$ in the definition of V' .

Proof.

1. Let $V_{\text{tor}} = \{\omega' \in V \mid \omega' \text{ torsion point}\}$, therefore, it is enough to see that $V_{\text{tor}} \subset V'$ to prove $\overline{V_{\text{tor}}} \subset V'$. To show that, we take $\omega' \in S$, then $\varphi(\omega')$ is a torsion point in $\varphi(V)$. Since $\varphi(V)$ is defined over $\mathbb{Q}(\zeta_N)$, with N odd, we can apply point 1 in Lemma 3.1 to $\varphi(V)$, hence one of the following is true:
 - (a) There exists $\eta \in \mu_2^r \setminus \{1\}$ such that $\varphi(\omega') \in \eta\varphi(V)$, hence $\omega' \in \varphi^{-1}(\eta)V$. Moreover, by definition of φ we have that $\varphi^{-1}(\eta) \notin \text{Stab}(V)$ and therefore $(\varphi^{-1}(\eta)V) \cap V \subsetneq V$.
 - (b) There exists $\eta \in \mu_2^r$ such that $[2](\varphi(\omega')) \in \eta\varphi(V)^\sigma$, and therefore $\omega' \in [2]^{-1}(\varphi^{-1}(\eta)V^\sigma)$. Moreover, we have that $[2]^{-1}(\varphi^{-1}(\eta)V^\sigma) \cap V \subsetneq V$. To prove that, assume $V \subset [2]^{-1}(\varphi^{-1}(\eta)V^\sigma)$. Thus, since for every $\eta_0 \in \mu_2^r$ we have $\varphi^{-1}(\eta_0) \in \text{Stab}([2]^{-1}(\varphi^{-1}(\eta)V^\sigma))$, we obtain

$$\bigcup_{\eta_0 \in \mu_2^r} \varphi^{-1}(\eta_0)V \subset [2]^{-1}(\varphi^{-1}(\eta)V^\sigma).$$

Since $\varphi^{-1}(\mu_2^r) \cap \text{Stab}(V) = \{1\}$ (see (a) above) and all the translates of V have the same stabilizer, we have that $\deg(\bigcup \varphi^{-1}(\eta_0)V) = 2^r \deg(V)$. On the other hand, $\deg([2]^{-1}(\varphi^{-1}(\eta)V^\sigma)) = 2^k \deg(V)$ (see [Hin88, Lemme 6(i)]) and since $k < r$, this leads to a contradiction.

Thereby, $\omega' \in V'$ for every $\omega' \in V_{\text{tor}}$ and thus $\overline{V_{\text{tor}}} \subset V'$. Moreover, $V' \cap V \subsetneq V$.

2. The proof follows similar to the previous one, using point 2 in Lemma 3.1. Note that, in this case, we need the minimality of N to guarantee that $V^\tau \neq V$ and, as a consequence, that $V \cap V^\tau \neq V$.
3. Since every torsion coset is defined over \mathbb{Q}^{ab} , it is invariant by ς . Hence, for every torsion coset $\omega H \subset V$, we have that $\omega H \subset V^\varsigma$ and $V \neq V^\varsigma$, due to the minimality of \mathbb{K} .

□

Remark 3.3. We can consider $\mathbb{K} = \mathbb{C}$ as the field of definition of V . Then it follows equivalently to the case when \mathbb{K} is not abelian to prove that

$$\overline{V_{\text{tor}}} \subset V \cap V^\varsigma \subsetneq V,$$

for any $\varsigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}^{\text{ab}})$ such that $\varsigma \neq \text{Id}$.

The following theorem is a specialization of [AV09, Theorem 1.2] to torsion subvarieties. Keeping the notation of the proposition above, note that $\varphi^{-1}(\boldsymbol{\eta})^2$ lies in the stabilizer of V , for any $\boldsymbol{\eta} \in \mu_\infty^r$. This is a fundamental so that we can use Lemma 2.8 in the proof of the following theorem.

Theorem 3.4. *Let $V \subset \mathbb{G}_m^n$ be an irreducible variety of dimension $d = n - k$ and not a torsion coset. Let*

$$\theta_0 = \theta_0(V) = k(2^{2n} + 2^{n+1} - 2)(2d + 1)\delta_0(V).$$

Then $\overline{V_{\text{tor}}}$ is contained in a hypersurface Z of degree at most θ_0 , which does not contain V ; i.e. $\overline{V_{\text{tor}}} \subset V \cap Z \subsetneq V$.

Proof. Let V be minimally defined over \mathbb{K} . To prove this theorem, we distinguish three cases, according to Proposition 3.2.

1. If \mathbb{K} is abelian and $\mathbb{Q}(\zeta_N)$ is a cyclotomic extension of \mathbb{K} , with $4 \nmid N$, then by point 1 in Proposition 3.2 we have that

$$\overline{V_{\text{tor}}} \subset V' = \bigcup_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} (\varphi^{-1}(\boldsymbol{\eta})V) \cup \bigcup_{\boldsymbol{\eta} \in \mu_2^r} [2]^{-1}(\varphi^{-1}(\boldsymbol{\eta})V^\sigma),$$

where $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, mapping $\zeta_N \mapsto \zeta_N^2$ and $V \cap V' \subsetneq V$.

To prove the theorem we find, for each component of V' , a hypersurface containing it, but not containing V . To conclude, it is enough to take Z as the union of these hypersurfaces and the only thing left to check is the degree of Z .

For any $\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}$, by setting $\mathbf{e} = \varphi^{-1}(\boldsymbol{\eta})$ and $\phi = \text{Id}$, we can use (a) of Lemma 2.8 and we obtain a homogeneous polynomial $F_\boldsymbol{\eta}$ of degree at most $2k(2d + 1)\delta_0(V)$ such that $F_\boldsymbol{\eta} \equiv 0$ on $\varphi^{-1}(\boldsymbol{\eta})V$ and $F_\boldsymbol{\eta} \not\equiv 0$ on V .

On the other hand, for any $\boldsymbol{\eta} \in \mu_2^r$, by setting $\mathbf{e} = \varphi^{-1}(\boldsymbol{\eta})$ and $\phi = \sigma$, we can use (b) of Lemma 2.8 and we obtain a homogeneous polynomial $G_\boldsymbol{\eta}$ of degree at most $2^n k(d + 1)\delta_0(V)$ such that $G_\boldsymbol{\eta} \equiv 0$ on $[2]^{-1}(\varphi^{-1}(\boldsymbol{\eta})V)$ and $G_\boldsymbol{\eta} \not\equiv 0$ on V .

Therefore, if we take

$$Z = V \left(\prod_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} F_\boldsymbol{\eta} \cdot \prod_{\boldsymbol{\eta} \in \mu_2^r} G_\boldsymbol{\eta} \right),$$

we have

$$\deg(Z) \leq \sum_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} 2k(2d + 1)\delta_0(V) + \sum_{\boldsymbol{\eta} \in \mu_2^r} 2^n k(2d + 1)\delta_0(V) \leq \theta_0$$

and $\overline{V_{\text{tor}}} \subset V \cap V' \subset V \cap Z \subsetneq V$.

2. If \mathbb{K} is abelian and $\mathbb{Q}(\zeta_N)$ is its minimal cyclotomic extension, with $N = 4N'$, then by point 2 in Proposition 3.2 we have

$$\overline{V_{\text{tor}}} \subset V' = \bigcup_{\boldsymbol{\eta} \in \mu_2^r \setminus \{\mathbf{1}\}} (\varphi^{-1}(\boldsymbol{\eta})V) \cup \bigcup_{\boldsymbol{\eta} \in \mu_2^r} (\varphi^{-1}(\boldsymbol{\eta})V^\tau),$$

where $\tau \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, mapping $\zeta_N \mapsto \zeta_N^{1+2N'}$ and $V \cap V' \subsetneq V$. We proceed as before.

For any $\eta \in \mu_2^r \setminus \{1\}$, by setting $e = \varphi^{-1}(\eta)$ and $\phi = \text{Id}$, we can use (a) of Lemma 2.8 and we obtain a homogeneous polynomial F_η of degree at most $2k(2d+1)\delta_0(V)$ such that $F_\eta \equiv 0$ on $\varphi^{-1}(\eta)V$ and $F_\eta \not\equiv 0$ on V .

On the other hand, for any $\eta \in \mu_2^r$, by setting $e = \varphi^{-1}(\eta)$ and $\phi = \tau$, we can use again (a) of Lemma 2.8 and we obtain a homogeneous polynomial F'_η of degree at most $2k(2d+1)\delta_0(V)$ such that $F'_\eta \equiv 0$ on $\varphi^{-1}(\eta)V^\tau$ and $F'_\eta \not\equiv 0$ on V .

Therefore, if we take

$$Z = V\left(\prod_{\eta \in \mu_2^r \setminus \{1\}} F_\eta \cdot \prod_{\eta \in \mu_2^r} F'_\eta\right),$$

we have

$$\deg(Z) \leq \sum_{\eta \in \mu_2^r \setminus \{1\}} 2k(2d+1)\delta_0(V) + \sum_{\eta \in \mu_2^r} 2k(2d+1)\delta_0(V) \leq \theta_0$$

and $\overline{V_{\text{tor}}} \subset V \cap V' \subset V \cap Z \subsetneq V$.

3. If \mathbb{K} is not abelian, by point 3 in Proposition 3.2, we have that $\overline{V_{\text{tor}}} \subset V \cap V^\varsigma$ for any $\varsigma \in \text{Gal}(\mathbb{K}/\mathbb{Q}^{\text{ab}} \cap \mathbb{K})$ such that $\varsigma \neq \text{Id}$.

By setting $e = 1$ and $\phi = \varsigma$ we can use (a) in Lemma 2.8 and we obtain a homogeneous polynomial F of degree at most $2k(2d+1)\delta_0(V)$ such that $F \equiv 0$ on V^ς and $F \not\equiv 0$ on V . Therefore, if we take $Z = V(F)$, then we have that $\deg(Z) \leq \theta_0$ and $\overline{V_{\text{tor}}} \subset V \cap V^\varsigma \subset V \cap Z \subsetneq V$.

If V is not defined over an extension of \mathbb{Q} , using Remark 3.3 it follows analogous to point 3 that the theorem also holds in this case. \square

4. INDUCTION THEOREMS

The following theorems correspond to Theorem 2.2 and Theorem 1.2 in [AV09]. Their proofs follow similarly to the ones of Amoroso and Viada. For the convenience of the reader, we reproduce the proofs.

Theorem 4.1. *Let $V_0 \subset V_1$ subvarieties of \mathbb{G}_m^n of codimension k_0 and k_1 respectively and V_0 irreducible. Let*

$$\theta = \theta(V_1) = ((2n-1)k_0(2^{2n} + 2^{n+1} - 2))^{k_0 - k_1 + 1} \delta(V_1).$$

Then either

- (a) *there exists a torsion coset B such that $V_0 \subseteq B \subseteq V_1$; or*
- (b) *there exists a hypersurface Z of degree at most θ such that $V_0 \not\subseteq Z$ and $\overline{V_{0,\text{tor}}} \subseteq Z$.*

Proof. We assume the statement is false, that is

- (a') V_0 is not contained in any torsion coset $B \subset V_1$; and
- (b') for every hypersurface Z satisfying $\overline{V_{0,\text{tor}}} \subset Z$, we also have that $V_0 \subset Z$.

We define, for $r = 0, \dots, k_0 - k_1 + 1$,

$$D_r = ((2n-1)k_0(2^{2n} + 2^{n+1} - 2))^r \delta(V_1),$$

and we build a chain of varieties

$$X_0 = V_1 \supseteq \cdots \supseteq X_{k_0 - k_1 + 1}$$

such that, for every $r = 0, \dots, k_0 - k_1 + 1$, the following holds:

- (i) $V_0 \subseteq X_r$,
- (ii) each irreducible component of X_r containing V_0 has codimension at least $r + k_1$, and
- (iii) $\delta(X_r) \leq D_r$.

If this holds for $r = k_0 - k_1 + 1$, we arrive to a contradiction (because we would have a component of X_r of codimension at least $k_0 + 1$ containing V_0).

We build the chain by recursion:

- For $r = 0$, $X_0 = V_1$ satisfies the properties.
- For $r + 1 > 1$, we assume we have already constructed X_r . Let W_1, \dots, W_t be the irreducible components of X_r such that

$$V_0 \subset W_j \Leftrightarrow 1 \leq j \leq s.$$

The property (i) guaranties $s > 0$ and, together with property (ii), says $r + k_1 \leq \text{codim}(W_j) \leq k_0$ for $1 \leq j \leq s$.

For every $j = 1, \dots, s$ we have $\delta_0(W_j) \leq \delta(X_r) \leq D_r \leq \theta$ and $V_0 \subseteq W_j \subset V_1$, with $\text{codim}(W_j) = k$. Hence, by hypothesis (a'), W_j is not a torsion coset and we can apply Theorem 4.1 to W_j . Let Z_j be the hypersurface of degree at most $(2d + 1)k(2^{2k} + 2^{k+1} - 2)\delta_0(W_j) \leq (2n - 1)k_0(2^{2k_0} + 2^{k_0+1} - 2)D_r = D_{r+1}$ such that $\overline{W_{j,\text{tor}}} \subseteq W_j \cap Z_j \subsetneq W_j$. Since $V_0 \subseteq W_j$, $\overline{V_{0,\text{tor}}} \subseteq \overline{W_{j,\text{tor}}} \subset Z_j$ and, together with the fact that $\deg(Z_j) \leq D_{r+1} \leq \theta$, hypothesis (b') guaranties $V_0 \subset Z_j$.

So

$$V_0 \subset \bigcap_{j=1}^s Z_j,$$

and we define

$$X_{r+1} = X \cap \bigcap_{j=1}^s Z_j.$$

In particular, $V_0 \subseteq X_{r+1}$ and property (i) is satisfied. Moreover, property (iii) is also satisfied, because

$$\delta(X_{r+1}) = \max\{\delta(X_r), \deg Z_1, \dots, \deg Z_s\} \leq \max\{D_r, D_{r+1}\} = D_{r+1}.$$

By taking $W'_j = W_j \cap Z_1 \cap \cdots \cap Z_s$ for every $j = 1, \dots, t$ we have

$$X_{r+1} = W'_1 \cup \cdots \cup W'_s \cup W'_{s+1} \cup W'_t.$$

For $j = 1, \dots, s$, we have that every irreducible component of W'_j has codimension at least $\text{codim}(W_j) + 1 \geq r + k_1 + 1$; and for $j = s + 1, \dots, t$ $V_0 \not\subseteq W_j$ and thus V_0 is not contained in any irreducible component of W'_j . This shows that property (ii) is satisfied. □

Theorem 1.3. *Let $V \subset \mathbb{G}_m^n$ be a variety of dimension d . For $j = 0, \dots, d$, let V_{tor}^j be the j -equidimensional part of $\overline{V_{\text{tor}}}$. Then*

$$\deg(V_{\text{tor}}^j) \leq c_{n,j} \delta(V)^{n-j}$$

for every $i = 0, \dots, d$, where

$$c_{n,j} = ((2n-1)(n-1)(2^{2n} + 2^{n+1} - 2))^{d(n-j)}.$$

Proof. If $\dim(V) = 0$, $\text{card}(\overline{V_{\text{tor}}}) \leq \deg(V) \leq \delta(V)^n$ and we are done. Hence, we suppose $\dim(V) > 0$.

Let

$$\theta = \theta(V) = ((2n-1)(n-1)(2^{2n} + 2^{n+1} - 2))^d \delta(V).$$

Observe that $c_{n,j} \delta(V)^{n-j} = \theta^{n-j}$.

Let $V = X^d \cup \dots \cup X^0$, where X^j represents the j -equidimensional part of V for every j . We have $\overline{V_{\text{tor}}} = V_{\text{tor}}^d \cup \dots \cup V_{\text{tor}}^0$ and we build the family $V_{\text{tor}}^d, \dots, V_{\text{tor}}^0$ recursively as follows:

CLAIM. For every $r = d, \dots, 0$ there exist an r -equidimensional varietie Y^r , such that

- (i) $\overline{V_{\text{tor}}} \subseteq V_{\text{tor}}^d \cup \dots \cup V_{\text{tor}}^{r+1} \cup Y^r \cup X^{r-1} \cup \dots \cup X^0$;
- (ii) $\sum_{i=d}^{r+1} \theta^{i-r} \deg(V_{\text{tor}}^i) + \deg(Y^r) \leq \sum_{i=d}^r \theta^{i-r} \deg(X^i)$.

If the claim holds for $r = 0$, we have $V_{\text{tor}}^0 \subset Y^0$, by assertion (i). Moreover, assertion (i) also guaranties that $V_{\text{tor}}^r \subset Y^r$ which, using assertion (ii), implies

$$\sum_{i=d}^r \theta^{i-r} \deg(V_{\text{tor}}^i) \leq \sum_{i=d}^r \theta^{i-r} \deg(X^i)$$

A result of Philippon [Phi95, Corollaire 5] (with $m = n$ and $S = \mathbb{P}^n$) shows that, for $\theta \geq \delta(V)$, we have

$$\sum_{i=r}^d \theta^{i-r} \deg(X^i) \leq \theta^r.$$

Hence, setting $r = 0$, we obtain

$$\sum_{i=d}^0 \theta^i \deg(V_{\text{tor}}^i) \leq \theta^n$$

and the inequalities of the statement follow trivially.

It remains to proof the claim. We build the family as follows.

- For $r = d$, we take $Y^d = X^d$ and the claim holds.
- Let $d \geq r > 0$. We assume that we have Y^r satisfying the claim. If Y^r has a component which is imbedded in $V_{\text{tor}}^d \cup \dots \cup V_{\text{tor}}^{r+1}$ or a component that does not intersect $\overline{V_{\text{tor}}}$, we descart it (this won't have any effect on the veracity of our claim). Next, let

$$Y^r = V_{\text{tor}}^r \cup W_1 \cup \dots \cup W_s$$

be the decomposition of Y^r such that V_{tor}^r is the union of all torsion cosets B of dimension r which are components of Y^r , and W_1, \dots, W_s are the rest of irreducible components of Y^r .

For every $j = 1, \dots, s$, W_j satisfies the following.

Remark. There does not exist any torsion coset B such that $W_j \subseteq B \subseteq V$.

Proof. If a torsion coset B as such exists, $B \subset \overline{V_{\text{tor}}}$ and $\dim(B) \geq r$. Therefore

$$W_j \subseteq B \subseteq V_{\text{tor}}^d \cup \cdots \cup V_{\text{tor}}^r,$$

which contradicts the definition of V_{tor}^r or the fact that no component of Y^r is imbedded in $V_{\text{tor}}^d \cup \cdots \cup V_{\text{tor}}^{r+1}$. \square

We apply Theorem 4.1 to $V_0 = W_j$ and $V_1 = V$, where $k_0 = n - r \leq n - 1$ and $k_1 = n - d$. Conclusion (a) of the theorem cannot be true, due to the previous remark; hence there exists a hypersurface Z_j of degree at most θ such that $\overline{W_{j,\text{tor}}} \subset W_j \cap Z_j \subsetneq W_j$. Krull's Hauptsatz implies that $W_j \cap Z_j$ is either empty or an $(r - 1)$ -equidimensional variety. This allows us to define Y^{r-1} as follows:

$$Y^{r-1} = X^{r-1} \cup \bigcup_{j=1}^s (W_j \cap Z_j).$$

By the construction of Y^{r-1} , assertion (i) of our claim is satisfied for $r - 1$.

Moreover, by Bézout's theorem, the following inequality holds

$$\deg(Y^{r-1}) \leq \theta \sum_{j=1}^s \deg(W_j) + \deg(X^{r-1}).$$

Since $Y^r = V_{\text{tor}}^r \cup W_1 \cup \cdots \cup W_s$, we have

$$\deg(Y^{r-1}) \leq \theta (\deg(Y^r) - \deg(V_{\text{tor}}^r)) + \deg(X^{r-1}).$$

Additioning $\sum_{i=r}^d \theta^{i+1-r} \deg(V_{\text{tor}}^i)$ to both sides of the inequality, we obtain

$$\begin{aligned} \sum_{i=r}^d \theta^{i+1-r} \deg(V_{\text{tor}}^i) + \deg(Y^{r-1}) &\leq \sum_{i=r}^d \theta^{i+1-r} \deg(V_{\text{tor}}^i) \\ &\quad + \theta (\deg(Y^r) - \deg(V_{\text{tor}}^r)) + \deg(X^{r-1}) \\ &= \theta \left(\sum_{i=r+1}^d \theta^{i-r} \deg(V_{\text{tor}}^i) + \deg(Y^r) \right) \\ &\quad + \deg(X^{r-1}). \end{aligned}$$

By the induction, we have $\sum_{i=r+1}^d \theta^{i-r} \deg(V_{\text{tor}}^i) + \deg(Y^r) \leq \sum_{i=r}^d \theta^{i-r} \deg(X^i)$. Therefore

$$\theta \left(\sum_{i=r}^d \theta^{i-r} \deg(V_{\text{tor}}^i) + \deg(Y^r) \right) + \deg(X^{r-1}) \leq \sum_{i=r-1}^d \theta^{i+1-r} \deg(X^i),$$

proving that assertion (ii) of our claim holds for $r - 1$. \square

If V is a hypersurface we can replace δ by $\deg(V)$. Observe then that this result is close to the conjectures.

Remark. Following the theorems presented by Amoroso and Viada [AV09], we could obtain, for each maximal torsion coset ωH in V , $\delta_0(H) \leq \theta$. However, we have the following sharper bound:

$$\delta(H) \leq n\delta(V),$$

which is a result of Bombieri and Gubler [BG06, Theorem 3.3.8].

Remark. In Theorem 1.3 we could give a more precise bound, depending on the field of definition of our variety V . To understand this, first observe that the varieties V' we obtain in Proposition 3.2 are defined over the same field as V . Hence, in Theorem 3.4 we could consider changing the definition of θ_0 , depending on the field of definition of V .

In the case that V is not defined over \mathbb{Q}^{ab} , Theorem 3.4 remains true for

$$\theta_0 = 2k(2d + 1)\delta_0(V).$$

Using this definition of θ_0 in the induction theorems, we can improve the bound in Theorem 1.3 for this case. Hence, if V is not defined over \mathbb{Q}^{ab} , the number of maximal torsion cosets in V is bounded by θ^n , where

$$\theta = (2(2n - 1)(n - 1))^{n-k}\delta(V).$$

In the case that V is defined over \mathbb{Q}^{ab} , this does not change significantly our bound since the order of n in the constant would remain essentially the same.

5. PROOF OF THE CONJECTURES

To prove the conjectures we are going to consider a more general setting and then transform the result in Theorem 1.3 to this setting. Both conjectures will follow as corollaries to this general result.

The following is a result of John [Joh48, Theorem III] which allows us to compare the volume of any convex polytope Δ with the volume of the ellipsoid of smallest volume containing Δ .

Theorem 5.1. *If E is the ellipsoid of smallest volume containing a set S in \mathbb{R}^n , then the ellipsoid $\frac{1}{n}E$ is contained in the convex hull of S , $\text{conv}(S)$.*

An ellipsoid E is determined by an invertible matrix $M \in \text{GL}_n(\mathbb{R})$ and a vector $\mathbf{v} \in \mathbb{R}^n$ so that

$$E = MB_n + \mathbf{v},$$

where B_n is the n -dimensional unit ball with center $\mathbf{0}$. In particular, the volume of E is determined by M since $\text{vol}_n(E) = \det(M)\omega_n$, where ω_n is the n -volume of n -sphere.

In the following result we consider a convex polytope Δ and the ellipsoid of smallest volume containing it. As a consequence of John's Theorem, we are able to approximate homothecies of M^{-1} and \mathbf{v} by an invertible integer matrix M_l and an integer vector \mathbf{v}_l to include a translate of $M_l\Delta - \mathbf{v}_l$ in an expansion of the standard simplex $\mathcal{S}_n = \{\mathbf{t} \in (\mathbb{R}_{\geq 0})^n \mid \|\mathbf{t}\|_1 \leq 1\}$ with comparable volume.

Proposition 5.2. *Let Δ be a convex polytope with integer vertexes. For any $l \in \mathbb{N}_{>0}$, there exists a non-singular integer matrix M_l and an integer vector $\boldsymbol{\tau}_l$ such that*

$$M_l\Delta + \boldsymbol{\tau}_l \subset 2n(l + n \text{diam}_1(\Delta) + n)\mathcal{S}_n,$$

and

$$(1) \quad \lim_{l \rightarrow +\infty} l^{-n} \det(M_l) \geq \omega_n n^{-n} \text{vol}_n(\Delta)^{-1}.$$

Proof. Replacing Δ with a translate by an integer vector, we can always assume that Δ lies in $(\mathbb{R}_{\geq 0})^n$ and that $\Delta \cap \{x_i = 0\} \neq \emptyset$, for every $i = 0, \dots, n$. Thus for any matrix $N \in \mathcal{M}_{n \times n}(\mathbb{R})$ with maximum norm $\|N\| \leq 1$, we have $N\Delta \subset n \text{diam}_1(\Delta)B_n$.

Let E be the ellipsoid of smallest volume containing Δ . Let $M \in \text{GL}(\mathbb{R})$ and $\mathbf{v} \in \mathbb{R}^n$ be the corresponding matrix and vector such that $M B_n + \mathbf{v} = E$. In particular, we have that B_n is the ellipsoid of smallest volume containing $M^{-1}\Delta - \mathbf{v}$.

Next, take $M_l \in \text{GL}(\mathbb{Z})$ and $\mathbf{v}_l \in \mathbb{Z}^n$ to be the integer approximations of lM^{-1} and $l\mathbf{v}$ respectively such that

$$\begin{aligned} lM^{-1} &= M_l + M' & \|M'\| &< 1 \\ l\mathbf{v} &= \mathbf{v}_l + \mathbf{v}' & \|\mathbf{v}'\| &< 1 \end{aligned}$$

where $\|\cdot\|$ denotes the maximum norm.

We take

$$\boldsymbol{\tau}_l = (l + n \text{diam}_1(\Delta) + n)\mathbf{1} - \mathbf{v}_l.$$

We now proceed to bound the domain where $M_l\Delta + \boldsymbol{\tau}_l$ lies. To do that, we develop as follows

$$M_l\Delta + \boldsymbol{\tau}_l = l(M^{-1}\Delta - \mathbf{v}) - M'\Delta + \mathbf{v}' + (l + n \text{diam}_1(\Delta) + n)\mathbf{1}.$$

We have $l(M^{-1}\Delta - \mathbf{v}) \subset lB_n$ and $\mathbf{v}' \in nB_n$. Moreover, since $\|M'\| \leq 1$, $M'\Delta \subset n \text{diam}_1(\Delta)B_n$. Putting it all together we obtain

$$\begin{aligned} M_l\Delta + \boldsymbol{\tau}_l &\subset (l + n \text{diam}_1(\Delta) + n)B_n + (l + n \text{diam}_1(\Delta) + n)\mathbf{1} \\ &\subset 2n(l + n \text{diam}_1(\Delta) + n)\mathcal{S}_n. \end{aligned}$$

It is left to prove (1). First observe that $\lim_{l \rightarrow +\infty} l^{-n} \det(M_l) = \det(M)^{-1}$. By John's result (Theorem 5.1), we have that $E \subset n\Delta$. Therefore

$$\text{vol}_n(E) = \omega_n \det(M) \leq n^n \text{vol}_n(\Delta)$$

which proves the inequality. \square

By using this proposition we extend the result in Theorem 1.3. To do so, we need to introduce a different notion of degree. Let $\Delta \subset \mathbb{R}^n$ be a convex polytope with integral vertexes and let $\psi_\Delta : \mathbb{G}_m^n \rightarrow \mathbb{P}^{N-1}$, with $N = \text{card}(\Delta \cap \mathbb{Z}^n)$, be the morphism mapping $\mathbf{t} \mapsto (\mathbf{t}^\alpha)_{\alpha \in \Delta \cap \mathbb{Z}^n}$. For any variety $V \subset \mathbb{G}_m^n$, we define the Δ -degree of V as

$$\text{deg}_\Delta(V) = \text{deg}(\psi_\Delta(V)).$$

In particular, if V is of dimension d we can take general polynomials f_1, \dots, f_d with support $\text{supp}(f) \subset \Delta$ and

$$\text{deg}_\Delta(V) = \text{card}(V \cap V(f_1) \cap \dots \cap V(f_d)).$$

Theorem 5.3. *Let $V \subset \mathbb{G}_m^n$ be a variety of dimension d and defined by polynomials with support in the convex polytope Δ . For $j = 0, \dots, d$ let V_{tor}^j be the j -equidimensional part of $\overline{V_{\text{tor}}}$. Then*

$$\text{deg}_\Delta(V_{\text{tor}}^j) \leq \tilde{c}_{n,j} \text{vol}_n(\Delta)$$

for every $j = 0, \dots, d$, where

$$\tilde{c}_{n,j} = ((2n-1)(n-1)(2^n + 2^{n+1} - 2))^{(n-1)(n-j)} 2^n n^{2n} \omega_n^{-1},$$

where ω_n is the volume of the n -sphere.

Proof. Let M_l and τ_l be as in Proposition 5.2. Let $\varphi : \mathbb{G}_m^n \rightarrow \mathbb{G}_m^n$ be the endomorphism defined by M_l , mapping $\mathbf{x} \mapsto \mathbf{x}^{M_l}$.

By Proposition 5.2, for any polynomial f with support $\text{supp}(f) \subset \Delta$, we have

$$\text{supp}(f(\mathbf{x}^{M_l}) \cdot \mathbf{x}^{\tau_l}) \subset 2n(l + n \text{diam}_1(\Delta) + n)\mathcal{S}_n.$$

Let $W = \varphi^{-1}(V)$. Now W can be defined by polynomials of degree at most $2n(l + \text{diam}_1(\Delta) + n)$. Moreover, for every $j = 0, \dots, d$, we have that $\varphi^{-1}(V_{\text{tor}}^j) = W_{\text{tor}}^j$.

Let us fix j . By Theorem 1.3 we have the following inequality:

$$(2) \quad \deg(W_{\text{tor}}^j) \leq c_{n,j} 2n(l + n \text{diam}_1(\Delta) + n)^{n-j}.$$

We proceed to compare $\deg(W_{\text{tor}}^j)$ and $\deg_{\Delta}(V_{\text{tor}}^j)$. To do this, take f_1, \dots, f_j generic polynomials such that $\text{supp}(f_i) \subset \Delta$ and so

$$\deg_{\Delta}(V_{\text{tor}}^j) = \text{card}(V_{\text{tor}}^j \cap V(f_1) \cap \dots \cap V(f_j)).$$

Since $f \circ \varphi^{-1} = f(\mathbf{x}^{M_l})$ and $f(\mathbf{x}^{M_l}) \cdot \mathbf{x}^{\tau_l}$ define the same variety,

$$\varphi^{-1}(V_{\text{tor}}^j \cap V(f_1) \cap \dots \cap V(f_j)) = W_{\text{tor}}^j \cap V(f_1(\mathbf{x}^{M_l}) \cdot \mathbf{x}^{\tau_l}) \cap \dots \cap V(f_j(\mathbf{x}^{M_l}) \cdot \mathbf{x}^{\tau_l})$$

and, because f_1, \dots, f_j are generic, by Bézout we have that

$$\begin{aligned} \text{card}(W_{\text{tor}}^j \cap V(f_1(\mathbf{x}^{M_l}) \cdot \mathbf{x}^{\tau_l}) \cap \dots \cap V(f_j(\mathbf{x}^{M_l}) \cdot \mathbf{x}^{\tau_l})) \\ \leq \deg(W_{\text{tor}}^j) 2n(l + n \text{diam}_1(\Delta) + n)^j. \end{aligned}$$

Since $\text{card}(\varphi^{-1}(\mathbf{x})) = \det(M_l)$ for any point $\mathbf{x} \in \mathbb{G}_m^n$, we have the following inequality:

$$\begin{aligned} \det(M_l) \deg_{\Delta}(V_{\text{tor}}^j) &= \text{card}(\varphi^{-1}(V_{\text{tor}}^j \cap V(f_1) \cap \dots \cap V(f_j))) \\ &\leq \deg(W_{\text{tor}}^j) 2n(l + n \text{diam}_1(\Delta) + n)^j. \end{aligned}$$

Combining this with (2), we obtain

$$(3) \quad \deg_{\Delta}(V_{\text{tor}}^j) \leq c_{n,j} (2n(l + n \text{diam}_1(\Delta) + n))^n \det(M_l)^{-1}.$$

By Proposition 5.2, we have that

$$\lim_{l \rightarrow +\infty} l^n \det(M_l)^{-1} \leq n^n \omega_n^{-1} \text{vol}_n(\Delta).$$

Hence, taking the limit for $l \rightarrow +\infty$ in (3) we get

$$\deg_{\Delta}(V_{\text{tor}}^j) \leq c_{n,j} 2^n n^{2n} \omega_n^{-1} \text{vol}_n(\Delta).$$

□

Note that $\deg_{\Delta}(V_{\text{tor}}^0) = \deg(V_{\text{tor}}^0)$ and so Corolary 1.4 is just a particular case of Theorem 5.3.

Let $V \subset \mathbb{G}_m^n$ be a hypersurface given by a polynomial $f \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$. If we take $\Delta = [0, d_1] \times \dots \times [0, d_n]$ where (d_1, \dots, d_n) is the multidegree of f , Theorem 5.3 for $j = 0$ proves Ruppert's conjecture. A slightly better result can be obtained applying Theorem

1.3 directly to the hypersurface W defined by $f(x_1^{D_1}, \dots, x_n^{D_n})$ with $D_i = d_1 \cdots d_n / d_i$. In this case we obtain that

$$V_{\text{tor}}^0 \leq n^n c_{n,0} d_1 \cdots d_n.$$

On the other hand, if we take $\Delta = \text{conv}(\text{supp}(f))$, Theorem 5.3 proves Aliev and Smyth's conjecture.

Moreover, by comparing the bound in Theorem 1.3 and the bound in Theorem 5.3 for the dense case ($\text{conv}(\text{supp}(f)) = \text{deg}(f)\mathcal{S}_n$), we can observe that they differ only by a multiplying factor $2^n n^{2n} \omega_n$ which does not increase the order of the constant.

6. EXAMPLE

We present a final example to show that, in Theorem 1.3, the exponent of d is optimal and the constant c_n must depend on n . To do this, we need first a result of Conway and Jones on vanishing sums of roots of unity. Let us define for $m \in \mathbb{N}_{>0}$:

$$\Psi(m) := 2 + \sum_{\substack{p|m \\ p \text{ prime}}} (p - 2).$$

Then we have:

Theorem 6.1 ([CJ76]). *Let ξ_1, \dots, ξ_N be N roots of unity. Let $a_1, \dots, a_N \in \mathbb{Z}$ such that $S = a_1 \xi_1 + \dots + a_N \xi_N = 0$ and minimal (i.e. there are no non-trivial vanishing subsums of S). Let*

$$m = \text{lcm}(\text{ord}(\xi_2/\xi_1), \dots, \text{ord}(\xi_N/\xi_1)).$$

Then m is squarefree and $\Psi(m) \leq N$.

Example. First of all, let p_1, \dots, p_n be n different primes such that $p_i > 2n$ for all $i = 1, \dots, n$. In particular, we will have that $\Psi(p_j p_k) > 2n$ for every different $j, k = 1, \dots, n$. Let W be the variety defined by

$$g(X_1, \dots, X_n) = \zeta_{p_1} + \cdots + \zeta_{p_n} + X_1 + \cdots + X_n.$$

We claim that, if $\omega \in W_{\text{tor}}$, then

$$\{\omega_1, \dots, \omega_n\} = \{-\zeta_{p_1}, \dots, -\zeta_{p_n}\}.$$

Proof. Take $\omega \in \mu_\infty^n$ such that $g(\omega) = 0$ and consider

$$S = g(\omega) = \zeta_{p_1} + \cdots + \zeta_{p_n} + \omega_1 + \cdots + \omega_n.$$

Let $S = S_1 + \cdots + S_t$ be a decomposition of S in minimal subsums, such that $S_i = 0$ for every $i = 1, \dots, r$. If, up to reordering, $t = n$ and $S_i = \zeta_{p_i} + \omega_i$, we are done.

Suppose that this is not the case. Hence, there exists a minimal vanishing non-trivial subsum S' with at least three elements. Without loss of generality, we can assume that S' has ζ_{p_j} and ζ_{p_k} as summands, for some different $j, k = 1, \dots, n$. Taking m' as in Theorem 6.1, we have that $p_j p_k \mid m'$ and so $\Psi(m') \geq \Psi(p_j p_k) > 2n$. On the other hand, since S' is a minimal sum with at most $2n$ summands and Theorem 6.1 affirms $\Psi(m') \leq 2n$ which contradicts the fact that $\Psi(m') > 2n$. Therefore, a vanishing subsum as S' does not exist and our claim is proved. \square

Since our claim holds, we have

$$W_{\text{tor}} = \{\omega \in \mu_{\infty}^n \mid \{\omega_1, \dots, \omega_n\} = \{-\zeta_{p_1}, \dots, -\zeta_{p_n}\}\}.$$

So $\overline{W_{\text{tor}}} = W_{\text{tor}}$ is a discrete ensemble with $n!$ elements.

It is enough to take $V = [d_1, \dots, d_n]^{-1}(W)$, which is the hypersurface in \mathbb{G}_m^n defined by

$$f(X_1, \dots, X_n) = \zeta_{p_1} + \dots + \zeta_{p_n} + X_1^d + \dots + X_n^d.$$

Then, we have that $\overline{V_{\text{tor}}} = [d]^{-1}(\overline{W_{\text{tor}}})$ which is the following discrete ensemble:

$$\overline{V_{\text{tor}}} = \left\{ \omega \in \mathbb{G}_m^n \mid \{\omega_1^d, \dots, \omega_n^d\} = \{-\zeta_{p_1}, \dots, -\zeta_{p_n}\} \right\}.$$

Hence, the number of isolated torsion points in V is $n!d^n$. In this case this is, in fact, the number of maximal torsion cosets in V .

By considering the homomorphism $[d_1, \dots, d_n] : \mathbb{G}_m^n \rightarrow \mathbb{G}_m^n$, mapping $(t_1, \dots, t_n) \mapsto (t_1^{d_1}, \dots, t_n^{d_n})$, instead of simply $[d]$, we would obtain $V = V(f)$ with

$$f(x_1, \dots, x_n) = \zeta_{p_1} + \dots + \zeta_{p_n} + X_1^{d_1} + \dots + X_n^{d_n}.$$

In this case, the number of isolated torsion points in V is $n!d_1 \cdots d_n$ which shows the effectiveness of the bound in terms of the multidegree.

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