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Multi-resurgence of formal solutions of linear meromorphic differential systems

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Abstract

In this paper, we consider a linear meromorphic differential system at the origin. For any of its levels ρ , we prove with the factorization theorem that the Borel transforms of its ρ -reduced formal solutions are resurgent and we give a complete description of all their singularities. Then, restricting ourselves to some special geometric configurations of the singular points of these Borel transforms, we make explicit formulæ relating the Stokes multipliers of level ρ of the given system to some connection constants in the Borel plane. So, we generalize the results already obtained by M. Loday-Richaud and the author for systems with a unique level and for the lowest and highest levels of systems with multi-levels. As an illustration, we develop one example.

Keywords. Linear differential system, multisummability, Stokes phenomenon, Stokes multipliers, resurgence, singularities, connection constants

AMS subject classification. 34M03, 34M30, 34M35, 34M40

1 Introduction

All along the article, we consider a linear meromorphic differential system (in short, a differential system or a system) of dimension $n \geq 2$ at the origin $0 \in \mathbb{C}$ of the form

$$x^{r+1} \frac{dY}{dx} = A(x)Y \tag{A}$$

where $r \geq 1$ is a positive integer and where $A(x) \in M_n(\mathbb{C}\{x\})$ is a $n \times n$ -analytic matrix at 0 such that $A(0) \neq 0$. Using a finite algebraic extension $x \mapsto x^\nu$ with $\nu \in \mathbb{N}^*$ and a meromorphic gauge transformation $Y \mapsto T(x)Y$

with a suitable polynomial matrix $T(x)$ in x and $1/x$ if needed, we can always assume (see [5]) that system (A) admits as formal fundamental solution at 0 a matrix of the form $\tilde{Y}(x) = \tilde{F}(x)x^L e^{Q(1/x)}$ with

(N₁) $\tilde{F}(x) \in M_n(\mathbb{C}[[x]])$ a formal power series in x satisfying $\tilde{F}(x) = I_n + O(x^r)$, where I_n denotes the identity matrix of size n ,

(N₂) $L = \bigoplus_{j=1}^J (\lambda_j I_{n_j} + J_{n_j})$, where J is an integer ≥ 2 , the eigenvalues λ_j satisfy $0 \leq \operatorname{Re}(\lambda_j) < 1$ and where

$$J_{n_j} = \begin{cases} 0 & \text{if } n_j = 1 \\ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} & \text{if } n_j \geq 2 \end{cases}$$

is an irreducible Jordan block of size n_j ,

(N₃) $Q(1/x)$ a diagonal matrix of the form

$$Q\left(\frac{1}{x}\right) = \bigoplus_{j=1}^J q_j\left(\frac{1}{x}\right) I_{n_j}$$

where the $q_j(1/x)$ are polynomials in $1/x$ of degree $\leq r$ and without constant terms.

Recall that normalizations (N₁) and (N₂) guarantee the unicity of $\tilde{F}(x)$ as formal series solution of the homological system (A_H) associated with system (A) (see [5]).

Under the hypothesis that system (A) has the *unique level* $r \geq 1$ (see definition 2.1 below for the exact definition of levels), M. Loday–Richaud and the author investigated in [12] (case $r = 1$) and [23] (case $r \geq 2$) the *resurgence* of the Borel transforms of the *r-reduced series* (= sub-series of terms r by r) of $\tilde{F}(x)$ and displayed a *complete* description of *all* their singularities. Then, as an application, they stated some *Stokes-to-connection formulæ* making explicit the *Stokes multipliers* of system (A) in terms of some *connection constants* in the Borel plane, providing thus an efficient tool for the effective calculation of the *Stokes–Ramis matrices* of system (A).

When system (A) has multi-levels $r_1 < \dots < r_p$, these results were generalized later to the *lowest* [22] and *highest* [21] levels by respectively considering, on one hand, the r_1 -reduced series and r_p -reduced series of $\tilde{F}(x)$ and, on the other hand, the lowest and highest levels' Stokes-Ramis matrices.

In the present paper, we propose to extend the results above to any level r_k of system (A). To do that, we shall proceed similarly as the approach developed in [22] for the lowest level by first showing that the study of level r_k can always be reduced to the study of the highest level of a convenient system. This point, which is central in our present approach, is based on the factorization theorem of $\tilde{F}(x)$ [9, 19, 20] (see section 2, theorem 2.7 below) and on a block-diagonalisation theorem allowing to write system (A) on a convenient block-diagonal form (section 3.2, theorem 3.6). Using that and the results of [21], we then prove that the Borel transforms $\widehat{F}^{[k;u]}(\tau)$, $u = 0, \dots, r_k - 1$, of the r_k -reduced series of $\tilde{F}(x)$ are *resurgent* (section 3.3, theorem 3.10) and we give a *complete* description of *all* their singularities (section 3.4, theorem 3.15). In next section 4, we restrict our study to some special geometric configurations of singular points of the $\widehat{F}^{[k;u]}$'s; then, for such configurations, we display *connection-to-Stokes formulæ of level r_k* relating the Stokes multipliers of level r_k of $\tilde{F}(x)$ to the connection constants of the $\widehat{F}^{[k;u]}$'s in the Borel plane (theorem 4.11). As an illustration of these formulæ, we develop one example (section 4.3).

2 Preliminaries

Split the matrix $\tilde{F}(x)$ into J column-blocks fitting to the Jordan block-structure of matrix L (for $\ell = 1, \dots, J$, the matrix $\tilde{F}^{\bullet;\ell}(x)$ has n_ℓ columns):

$$\tilde{F}(x) = \begin{bmatrix} \tilde{F}^{\bullet;1}(x) & \tilde{F}^{\bullet;2}(x) & \dots & \tilde{F}^{\bullet;J}(x) \end{bmatrix}.$$

The aim of this section is to briefly recall some basic definitions/results about the summation theory and to introduce some notations we are needed in the sequel.

2.1 Some definitions and notations

Given a pair (q_j, q_ℓ) such that $q_j \neq q_\ell$, we denote

$$(q_j - q_\ell) \left(\frac{1}{x} \right) = -\frac{\alpha_{j,\ell}}{x^{r_{j,\ell}}} + o\left(\frac{1}{x^{r_{j,\ell}}} \right), \quad \alpha_{j,\ell} \neq 0.$$

Definition 2.1 (Levels, Stokes values and anti-Stokes directions of $\tilde{F}^{\bullet;\ell}(x)$)

Let $j, \ell \in \{1, \dots, J\}$ such that $q_j \not\equiv q_\ell$.

- The degree $r_{j,\ell}$ is called a *level* of $\tilde{F}^{\bullet;\ell}(x)$.
- The coefficient $\alpha_{j,\ell}$ is called a *Stokes value* of level $r_{j,\ell}$ of $\tilde{F}^{\bullet;\ell}(x)$.
- The directions of maximal decay of $e^{(q_j - q_\ell)(1/x)}$, i.e., the $r_{j,\ell}$ directions $\arg(\alpha_{j,\ell})/r_{j,\ell} \pmod{2\pi/r_{j,\ell}}$ along which $-\alpha_{j,\ell}/x^{r_{j,\ell}}$ is real negative, are called *anti-Stokes directions* of level $r_{j,\ell}$ of $\tilde{F}^{\bullet;\ell}(x)$.

Note that a Stokes value (resp. an anti-Stokes direction) of $\tilde{F}^{\bullet;\ell}(x)$ may be with several levels. Note also that the denomination “*anti-Stokes directions*” is not universal: sometimes, one calls such directions “*Stokes directions*”.

Notation 2.2 The set $\mathcal{R}^{(\ell)} := \{r_1^{(\ell)} < \dots < r_{p_\ell}^{(\ell)}\}$ with $p_\ell \geq 1$ denotes the set of all levels of $\tilde{F}^{\bullet;\ell}(x)$.

Note that, according to normalization (N_3), all the levels $r_k^{(\ell)}$ are integer; one refers sometimes this case as *the unramified case*.

Note also that, for all ℓ , we have $r_{p_\ell}^{(\ell)} \leq r$ the rank of system (A). Actually, if there exists ℓ such that $r_{p_\ell}^{(\ell)} < r$, then $r_{p_\ell}^{(\ell)} < r$ for all $\ell \in \{1, \dots, J\}$ and polynomials q_j have the same degree r and the same terms of highest degree. One then reduces to the case $r_{p_\ell}^{(\ell)} = r$ by means of a change of unknown vector of the form $Y = Ze^{q(1/x)}$ with a convenient polynomial $q(1/x) \in x^{-1}\mathbb{C}[x^{-1}]$. Recall that such a change does not affect levels or Stokes-Ramis matrices of system (A).

Definition 2.3 (Levels, Stokes values and anti-Stokes directions of $\tilde{F}(x)$)

We call

- *level* of $\tilde{F}(x)$ (or of system (A)) any level of the $\tilde{F}^{\bullet;\ell}(x)$'s,
- *Stokes value* of $\tilde{F}(x)$ (or of system (A)) any Stokes value of the $\tilde{F}^{\bullet;\ell}(x)$'s,
- *anti-Stokes direction* of $\tilde{F}(x)$ (or of system (A)) any anti-Stokes direction of the $\tilde{F}^{\bullet;\ell}(x)$'s.

Notation 2.4 The set $\mathcal{R} := \{r_1 < \dots < r_p\}$ with $p \geq 1$ denotes the set of all levels of $\tilde{F}(x)$ (or of system (A)).

We clearly have $\mathcal{R} = \bigcup_{\ell=1}^J \mathcal{R}^{(\ell)}$ and $r_p = r$.

When $p = 1$, system (A) is said to be *with the unique level* r . Recall that such a system was already investigated in great details in [12] (case $r = 1$) and [23] (case $r \geq 2$). Henceforth, we suppose from now on $p \geq 2$, *i.e.*, system (A) has at least two levels. Note however that some column-blocks $\tilde{F}^{\bullet;\ell}(x)$ may have the unique level r , *i.e.*, $p_\ell = 1$ and $\mathcal{R}^{(\ell)} = \{r\}$.

2.2 Multisummability

◁ **Multisummability of $\tilde{F}(x)$.** The multisummability of formal power series in $\mathbb{C}[[x]]$ was investigated by many authors and several multisummation process based on various methods such as asymptotic, cohomology, integral operators, etc... were built [2–4, 6, 9, 16, 18]. Of course, all these process provide a *same* and *unique* multisum (see [11] for instance). In this article, we shall use either of these process depending on our needs.

Notation 2.5 Given a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $\underline{k} := (k_1 < \dots < k_s)$ a s -tuple of positive numbers, we denote by

- $\mathbb{C}\{x\}_{\underline{k};\theta}$ the set of \underline{k} -summable formal series in direction θ ,
- $s_{\underline{k};\theta}(\tilde{h})(x)$ the \underline{k} -sum of $\tilde{h}(x) \in \mathbb{C}\{x\}_{\underline{k};\theta}$ in direction θ .

Recall that $s_{\underline{k};\theta}(\tilde{h})(x)$ defines an analytic function $1/k_1$ -Gevrey asymptotic to $\tilde{h}(x)$ on a germ of sector with vertex 0, bisected by θ and opening larger than π/k_s ¹. In particular, $\tilde{h}(x)$ is a $1/k_1$ -Gevrey formal series (denoted below by $\tilde{h}(x) \in \mathbb{C}[[x]]_{1/k_1}$), *i.e.*, its formal Borel transform $\tilde{\mathcal{B}}_{k_1}(\tilde{h})$ of level k_1 is analytic at the origin $0 \in \mathbb{C}$. Recall also that, for $\underline{k} := (k)$, the set $\mathbb{C}\{x\}_{\underline{k};\theta}$ coincides with the set $\mathbb{C}\{x\}_{k;\theta}$ of classical k -Borel-Laplace-summable formal series in direction θ [17]. We also denote by

- $\mathbb{C}\{x\}_{\underline{k}}$ the set of \underline{k} -summable formal series, *i.e.*, the set of \underline{k} -summable formal series in all directions but finitely many.

Note that $\mathbb{C}\{x\} \subset \mathbb{C}\{x\}_{\underline{k}}$ for any \underline{k} .

Back to $\tilde{F}(x)$, one has the following classical theorem:

¹When opening is $< 2\pi$, the sector can be seen as a sector of $\mathbb{C} \setminus \{0\}$; otherwise, it must be considered as a sector of the Riemann surface $\tilde{\mathbb{C}} := \widetilde{\mathbb{C} \setminus \{0\}}$ of the logarithm.

Theorem 2.6 ([4, 6, 9, 16, 18])

1. Multisummability of $\tilde{F}(x)$.

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a non anti-Stokes direction of $\tilde{F}(x)$.

Let $\underline{r} := (r_1 < \dots < r_p)$ be the p -tuple of all the levels of $\tilde{F}(x)$.

Then, $\tilde{F}(x) \in \mathbb{C}\{x\}_{\underline{r};\theta}$.

2. Multisummability of $\tilde{F}^{\bullet;\ell}(x)$.

Let $\theta^{(\ell)} \in \mathbb{R}/2\pi\mathbb{Z}$ a non anti-Stokes direction of $\tilde{F}^{\bullet;\ell}(x)$.

Let $\underline{r}^{(\ell)} := (r_1^{(\ell)} < \dots < r_{p_\ell}^{(\ell)})$ the p_ℓ -tuple of all the levels of $\tilde{F}^{\bullet;\ell}(x)$.

Then, $\tilde{F}^{\bullet;\ell}(x) \in \mathbb{C}\{x\}_{\underline{r}^{(\ell)};\theta^{(\ell)}}$.

◁ **Factorization theorem.** The factorization theorem 2.7 below tells us that $\tilde{F}(x)$ can be written essentially uniquely as a product of r_k -summable formal series $\tilde{F}_{r_k}(x)$ for the different levels r_k of $\tilde{F}(x)$. It was first proved by J.-P. Ramis in [19, 20] by using a technical way based on Gevrey estimates. A quite different proof based on Stokes cocycles and mainly algebraic was given later by M. Loday–Richaud in [9]. Both proofs are nonconstructive. However, as we shall see in section 3, this theorem provides sufficient informations to allow us to investigate the resurgence and the singularities of the Borel transforms of the r_k -reduced series of $\tilde{F}(x)$.

Theorem 2.7 (Factorization theorem, [9, 19, 20])

Let $\mathcal{R} = \{r_1 < r_2 < \dots < r_p = r\}$ denote the set of levels of $\tilde{F}(x)$ ².

Then, $\tilde{F}(x)$ can be factored in $\tilde{F}(x) = \tilde{F}_{r_p}(x)\dots\tilde{F}_{r_2}(x)\tilde{F}_{r_1}(x)$ where, for all $k = 1, \dots, p$, $\tilde{F}_{r_k}(x) \in M_n(\mathbb{C}[[x]])$ is a r_k -summable formal series with singular directions the anti-Stokes directions of level r_k of $\tilde{F}(x)$.

This factorization is essentially unique: let $\tilde{F}(x) = \tilde{G}_{r_p}(x)\dots\tilde{G}_{r_2}(x)\tilde{G}_{r_1}(x)$ be another decomposition of $\tilde{F}(x)$; then, there exist $p - 1$ invertible matrices $P_{r_1}(x), \dots, P_{r_{p-1}}(x) \in GL_n(\mathbb{C}\{x\}[x^{-1}])$ with meromorphic entries at 0 such that $\tilde{G}_{r_1} = P_{r_1}\tilde{F}_{r_1}$, $\tilde{G}_{r_k} = P_{r_k}\tilde{F}_{r_k}P_{r_{k-1}}^{-1}$ for $k = 2, \dots, p - 1$ and $\tilde{G}_{r_p} = \tilde{F}_{r_p}P_{r_{p-1}}^{-1}$. In particular, we can always choose \tilde{F}_{r_k} so that $\tilde{F}_{r_k}(x) = I_n + O(x^r)$ for all $k = 1, \dots, p$ ³.

Notation 2.8 Given a level $\rho \in \mathcal{R}$ of $\tilde{F}(x)$, we denote by

²Recall that we suppose $p \geq 2$ in this paper.

³Actually, such conditions, like the initial condition $\tilde{F}(x) = I_n + O(x^r)$, allow us to have “good” normalizations for the r_k -reduced series and thus to simplify future calculations (see sections 3.3 and 3.4 below).

- $\rho^- := (r_1 < \dots < \rho)$ the tuple of levels of \mathcal{R} which are $\leq \rho$,
- ρ^{+1} the level of \mathcal{R} immediately greater than ρ when $\rho < r$,
- $\rho^+ := (\rho^{+1} < \dots < r)$ the tuple of levels of \mathcal{R} which are $> \rho$, with the convention $\rho^+ = +\infty$ when $\rho = r$,
- $\tilde{F}_{\rho^-}(x)$ the sub-product of $\tilde{F}(x)$ defined by $\tilde{F}_{\rho^-}(x) := \tilde{F}_{\rho}(x) \dots \tilde{F}_{r_1}(x)$,
- $\tilde{F}_{\rho^+}(x)$ the sub-product of $\tilde{F}(x)$ defined by $\tilde{F}_{\rho^+}(x) := \tilde{F}_r(x) \dots \tilde{F}_{\rho^{+1}}(x)$ with the convention $\tilde{F}_{\rho^+}(x) = \tilde{F}_{+\infty}(x) = I_n$ when $\rho = r$.

Note that, following [18, Lem. 7], $\tilde{F}_{\rho^-}(x) \in \mathbb{C}\{x\}_{\rho^-; \theta}$ and $\tilde{F}_{\rho^+}(x) \in \mathbb{C}\{x\}_{\rho^+; \theta}$ for any non anti-Stokes direction θ of $\tilde{F}(x)$.

Let us now consider the matrix

$$A_\rho(x) := \tilde{F}_{\rho^+}(x)^{-1} A(x) \tilde{F}_{\rho^+}(x) - x^{r+1} \tilde{F}_{\rho^+}(x)^{-1} \frac{d\tilde{F}_{\rho^+}}{dx}(x)$$

of the system obtained from system (A) by the formal gauge transformation $Y \mapsto \tilde{F}_{\rho^+}(x)Y$. Then [9], $A_\rho(x)$ is analytic at 0 and the matrix $\tilde{Y}_{\rho^-}(x) := \tilde{F}_{\rho^-}(x)x^L e^{Q(1/x)}$ is a formal fundamental solution of system

$$x^{r+1} \frac{dY}{dx} = A_\rho(x)Y. \quad (A_\rho)$$

Note that system (A_ρ) and matrix $\tilde{Y}_{\rho^-}(x)$ coincide with system (A) and matrix $\tilde{Y}(x)$ when $\rho = r$. Note also that all systems (A_ρ) have same levels $r_1 < r_2 < \dots < r_p$ as system (A) and that all matrices $\tilde{Y}_{\rho^-}(x)$ have same normalizations as $\tilde{Y}(x)$.

When $\rho < r$, the structure of $A_\rho(x)$ will be precised in theorem 3.6 below. In particular, we shall show that $A_\rho(x)$ (and, consequently, $\tilde{F}_{\rho^-}(x)$) can always be chosen with a convenient “block-diagonal form”.

3 Main results

Since any of the J column-blocks $\tilde{F}^{\bullet; \ell}(x)$ can be positionned at the first place by means of a convenient permutation P on the columns of $\tilde{Y}(x)$ and since this same permutation acting on the rows of $\tilde{Y}(x)$ allows to keep initial normalizations of $\tilde{Y}(x)$ ⁴, we can restrict ourselves, without loss of generality,

⁴The new formal fundamental solution reads $P\tilde{Y}(x)P = P\tilde{F}(x)P x^{P^{-1}LP} e^{P^{-1}Q(1/x)P}$ with $P\tilde{F}(x)P = I_n + O(x^r)$.

to the study of the first column-block $\tilde{F}^{\bullet;1}(x)$ which we denote below by $\tilde{f}(x)$. Note that the size of $\tilde{f}(x)$ is $n \times n_1$. Note also that $\tilde{f}(x) = I_{n,n_1} + O(x^r)$, where I_{n,n_1} denotes the first n_1 columns of the identity matrix I_n .

3.1 Setting the problem

In addition to normalizations $(N_1) - (N_3)$ of $\tilde{Y}(x)$, we suppose that

$$(N_4) \quad \lambda_1 = 0 \text{ and } q_1 \equiv 0,$$

conditions that can always be fulfilled by means of the change of unknown vector $Y = x^{\lambda_1} e^{q_1(1/x)} Z$. Doing that, the levels $r_1^{(1)} < \dots < r_{p_1}^{(1)}$ of $\tilde{f}(x)$ (see definition 2.1 and notation 2.2) are the degrees of nonzero polynomials q_j of Q . To simplify notations, we denote them below by $\rho_1 < \dots < \rho_{p_1}$. Recall that $\rho_{p_1} = r$ the highest level of $\tilde{F}(x)$.

Notation 3.1 To simplify calculations below, we suppose from now on that matrix Q reads on the form

$$Q = Q_1 \oplus \dots \oplus Q_{p_1}$$

where

- Q_1 is a diagonal matrix whose entries are all the polynomials q_j of degree $\leq \rho_1$, *i.e.*, all the polynomials $q_j \equiv 0$ (in particular, q_1) and all the polynomials q_j of degree ρ_1 ,
- for all $k \geq 2$, Q_k is a diagonal matrix whose entries are all the polynomials q_j of degree ρ_k and whose the leading term $\mathcal{Q}_k := x^{\rho_k} Q_k|_{x=0}$ has a block-decomposition of the form

$$\bigoplus_{\ell=1}^{s_k} Q_{k,\ell} I_{m_{k,\ell}} \quad ; \quad Q_{k,\ell} \in \mathbb{C} \setminus \{0\} \text{ and } Q_{k,\ell} \neq Q_{k,\ell'} \text{ if } \ell \neq \ell'.$$

Note that decomposition of Q can always be fulfilled by means of a convenient permutation acting both on the rows and columns with indices $\geq n_1 + 1$ of $\tilde{Y}(x)$. In particular, such a permutation does not affect normalizations $(N_1) - (N_4)$ of $\tilde{Y}(x)$ or the first place of $\tilde{f}(x)$. Note also that $Q = Q_1$ when $p_1 = 1$.

Notation 3.2 Following decomposition of Q ,

- we denote by N_k the size of the square matrix Q_k , $k = 1, \dots, p_1$,

- we split matrix L of exponents of formal monodromy like Q :

$$L = L_1 \oplus \dots \oplus L_{p_1} \quad \text{with } L_k \in M_{N_k}(\mathbb{C}).$$

For $k \in \{1, \dots, p_1\}$, we denote by $\tilde{\mathbf{f}}^{[k;u]}(t)$, with $u = 0, \dots, \rho_k - 1$, the ρ_k -reduced series of $f(x)$, *i.e.*, the sub-series of terms ρ_k by ρ_k of $\tilde{f}(x)$. Recall that these series are uniquely determined by relation

$$\tilde{f}(x) = \tilde{\mathbf{f}}^{[k;0]}(x^{\rho_k}) + x\tilde{\mathbf{f}}^{[k;1]}(x^{\rho_k}) + \dots + x^{\rho_k-1}\tilde{\mathbf{f}}^{[k;\rho_k-1]}(x^{\rho_k}).$$

Following proposition 3.3 gives us a first property of the formal Borel transforms $\hat{\mathbf{f}}^{[k;u]}(\tau) := \tilde{\mathcal{B}}_1(\tilde{\mathbf{f}}^{[k;u]})(\tau)$ of level 1.

Proposition 3.3 *Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a non anti-Stokes direction of $\tilde{f}(x)$. Let $k \in \{1, \dots, p_1\}$ and $\theta^{[k]} := \rho_k\theta$.*

- Case $k = 1$. Then, $\hat{\mathbf{f}}^{[1;u]}(\tau)$ is analytic at 0:

$$\boxed{\hat{\mathbf{f}}^{[1;u]}(\tau) \in \mathbb{C}\{\tau\}.$$

- Case $k \geq 2$. Then, $\hat{\mathbf{f}}^{[k;u]}(\tau)$ is summable in direction $\theta^{[k]}$:

$$\boxed{\hat{\mathbf{f}}^{[k;u]}(\tau) \in \mathbb{C}\{\tau\}_{\underline{\rho}^{[k]}; \theta^{[k]}}, \text{ where } \underline{\rho}^{[k]} := \left(\frac{\rho_1}{\rho_k - \rho_1}, \dots, \frac{\rho_{k-1}}{\rho_k - \rho_{k-1}} \right).$$

We denote by $\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]}(\tau)$ the sum thus defined by $\hat{\mathbf{f}}^{[k;u]}(\tau)$ in direction $\theta^{[k]}$ and by $V_0(\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]})$ the domain of definition of $\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]}(\tau)$ ⁵.

Proof. Since $\tilde{f}(x)$ is $(\rho_1, \dots, \rho_{p_1})$ -summable in direction θ with $\rho_1 \geq 1$ (see theorem 2.6), [1] tells us that $\tilde{f}(x)$ can be split into the form

$$\tilde{f}(x) = \tilde{g}_1(x) + \dots + \tilde{g}_{p_1}(x) \quad \text{with } \tilde{g}_j(x) \in \mathbb{C}\{x\}_{\rho_j; \theta} \text{ for all } j = 1, \dots, p_1.$$

Thereby, denoting by $\tilde{\mathbf{g}}_j^{[k;u]}(t)$, $u = 0, \dots, \rho_k - 1$, the ρ_k -reduced series of $\tilde{g}_j(x)$, the formal series $\tilde{\mathbf{f}}^{[k;u]}(t)$ reads as

$$\tilde{\mathbf{f}}^{[k;u]}(t) = \tilde{\mathbf{g}}_1^{[k;u]}(t) + \dots + \tilde{\mathbf{g}}_{p_1}^{[k;u]}(t) \quad \text{with } \tilde{\mathbf{g}}_j^{[k;u]}(t) \in \mathbb{C}\{t\}_{\rho_j/\rho_k; \theta^{[k]}}.$$

⁵Precisely, $V_0(\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]})$ is a disc centered at 0 if $k = 1$ and a sector with vertex 0, bisected by $\theta^{[k]}$ and opening larger than $\pi(\rho_k - \rho_{k-1})/\rho_{k-1}$ if $k \geq 2$ [18].

Hence, [3, pp. 81 and 101] implies identity

$$\widehat{\mathbf{f}}^{[k;u]}(\tau) = \widehat{\mathbf{g}}_1^{[k;u]}(\tau) + \dots + \widehat{\mathbf{g}}_{p_1}^{[k;u]}(\tau)$$

where

$$\widehat{\mathbf{g}}_j^{[k;u]}(\tau) \in \begin{cases} \mathbb{C}\{\tau\}_{\rho_j/(\rho_k - \rho_j); \boldsymbol{\theta}^{[k]}} & \text{if } \rho_j/\rho_k < 1 \\ \mathbb{C}\{\tau\} & \text{if } \rho_j/\rho_k \geq 1 \end{cases}$$

In particular, we have $\widehat{\mathbf{g}}_k^{[k;u]}(\tau) + \dots + \widehat{\mathbf{g}}_{p_1}^{[k;u]}(\tau) \in \mathbb{C}\{\tau\}$ and [18, Lem. 7] implies $\widehat{\mathbf{g}}_1^{[k;u]}(\tau) + \dots + \widehat{\mathbf{g}}_{k-1}^{[k;u]}(\tau) \in \mathbb{C}\{\tau\}_{\boldsymbol{\rho}^{[k]}; \boldsymbol{\theta}^{[k]}}$. This ends the proof. ■

The aim of section 3 is to investigate the resurgent character of functions $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]}}^{[k;u]}(\tau)$ and to give a complete description of all their singularities. Note that, since $\rho_{p_1} = r$ is the highest level of $\widetilde{F}(x)$, the case $k = p_1$ was already treated in [21]. For other cases $k \in \{1, \dots, p_1 - 1\}$, we shall see in sections 3.3 and 3.4 that their study can actually be reduced to this case of “highest level”. To do that, we shall use an approach based on factorization theorem 2.7 and on block-diagonalisation theorem 3.6 below which will allow us, on one hand, to isolate levels $\leq \rho_k$ of $\widetilde{f}(x)$ by means of the relation $\widetilde{F}(x) = \widetilde{F}_{\rho_k^+}(x)\widetilde{F}_{\rho_k^-}(x)$ and, on the other hand, to write the corresponding matrix $A_{\rho_k}(x)$ into a convenient block-diagonal form. Recall that such an approach was already used in [22] for lowest level ρ_1 .

3.2 Block-diagonalisation theorem

In this section, we fix $k \in \{1, \dots, p_1 - 1\}$. Our aim is to prove that system (A_{ρ_k}) can be written as a convenient direct sum of sub-systems allowing to isolate the levels $\leq \rho_k$ of $\widetilde{f}(x)$.

Notation 3.4 Using notations 3.1 and 3.2, we denote by

- $Q_{<d} = Q_1 \oplus \dots \oplus Q_{d-1}$, $Q_{\leq d} = Q_{<d} \oplus Q_d$ and $Q_{>d} = Q_{d+1} \oplus \dots \oplus Q_{p_1}$,
- $N_{<d} = N_1 + \dots + N_{d-1}$, $N_{\leq d} = N_{<d} + N_d$ and $N_{>d} = N_{d+1} + \dots + N_{p_1}$,
- $L_{<d} = L_1 \oplus \dots \oplus L_{d-1}$, $L_{\leq d} = L_{<d} \oplus L_d$ and $L_{>d} = L_{d+1} \oplus \dots \oplus L_{p_1}$

when sums make sense.

According to notation 3.4 above, matrix Q reads as

$$Q = Q_{\leq k} \oplus Q_{>k} \tag{3.1}$$

with $Q_{\leq k}$ (resp. $Q_{> k}$) of size $N_{\leq k}$ (resp. $N_{> k}$). Block-diagonalisation theorem 3.6 below, which is an improved version of the one stated in [22, Thm. 3.3], tells us that, up to analytic gauge transformation, system (A_{ρ_k}) can be split into a direct sum of two sub-systems fitting to the block-decomposition (3.1) of Q . In particular, it shows that matrix $A_{\rho_k}(x)$ can be reduced into a block-diagonal form $A_{\rho_k}(x) = A'_{\rho_k}(x) \oplus A''_{\rho_k}(x)$.

Theorem 3.6 stems from following technical lemma 3.5 which is based on the results of B. Malgrange proved in [13] and on Tauberian theorems due to J. Martinet and J.-P. Ramis [18].

Before stating this lemma, let us recall that a (formal) meromorphic gauge transformation $Z = T(x)W$ transforms any system

$$x^{r+1} \frac{dW}{dx} = \mathcal{A}(x)W$$

into the system

$$x^{r+1} \frac{dZ}{dx} = {}^T\mathcal{A}(x)Z \quad \text{where } {}^T\mathcal{A}(x) = T\mathcal{A}(x)T^{-1} + x^{r+1} \frac{dT}{dx} T^{-1}.$$

Lemma 3.5 *Let $d \in \{k+1, \dots, p_1\}$. Let a system*

$$x^{\rho_d+1} \frac{dW}{dx} = \mathcal{A}(x)W \quad , \quad \mathcal{A}(x) \in M_{N_{\leq d}}(\mathbb{C}\{x\}) \quad (3.2)$$

together with a formal fundamental solution at 0 of the form

$$\widetilde{W}(x) = \widetilde{H}(x)x^{L_{\leq d}}e^{Q_{\leq d}(1/x)}$$

where $\widetilde{H}(x) \in M_{N_{\leq d}}(\mathbb{C}[[x]])$ satisfies $\widetilde{H}(x) = I_{N_{\leq d}} + O(x^r)$.

Suppose that $\widetilde{H}(x)$ is summable of levels $\leq \rho_k$.

Then, there exists an invertible matrix $T(x) \in GL_{N_{\leq d}}(\mathbb{C}\{x\})$ with analytic entries at 0 such that

1. $T(x) = I_{N_{\leq d}} + O(x^r)$,
2. *the gauge transformation $Z = T(x)W$ transforms system (3.2) into a system*

$$x^{\rho_d+1} \frac{dZ}{dx} = \begin{bmatrix} \mathcal{A}'(x) & 0 \\ 0 & \mathcal{A}''(x) \end{bmatrix} Z \quad (3.3)$$

with $\mathcal{A}'(x) \in M_{N_{< d}}(\mathbb{C}\{x\})$ and $\mathcal{A}''(x) \in M_{N_d}(\mathbb{C}\{x\})$,

3. the formal fundamental solution $\tilde{Z}(x) = T(x)\tilde{W}(x)$ of system (3.3) has a block-diagonal decomposition

$$\tilde{Z}(x) = \tilde{H}'(x)x^{L_{<d}}e^{Q_{<d}(1/x)} \oplus \tilde{H}''(x)x^{L_d}e^{Q_d(1/x)}$$

where

- (a) $\tilde{H}'(x)$ and $\tilde{H}''(x)$ satisfy $\tilde{H}'(x) = \tilde{H}''(x) = I_* + O(x^r)$,
- (b) $\tilde{H}'(x)x^{L_{<d}}e^{Q_{<d}(1/x)}$ is a formal fundamental solution of system

$$x^{\rho_{d-1}+1}\frac{dZ}{dx} = \mathcal{A}'(x)Z, \quad (3.4)$$

- (c) $\tilde{H}''(x)x^{L_d}e^{Q_d(1/x)}$ is a formal fundamental solution of system

$$x^{\rho_d+1}\frac{dZ}{dx} = \mathcal{A}''(x)Z.$$

Moreover, both formal series $\tilde{H}'(x)$ and $\tilde{H}''(x)$ are summable of levels $\leq \rho_k$.

Proof. \triangleleft Since $\tilde{H}(0) = I_{N_{\leq d}}$, the matrix $\mathcal{A}(x)$ of system (3.2) reads

$$\mathcal{A}(x) = x^{\rho_d+1}\frac{dQ_{\leq d}}{dx} + x^{\rho_d}\mathcal{B}(x)$$

with $\mathcal{B}(x)$ analytic at 0. Hence, according to the block-decomposition of matrix Q (see notation 3.1), the heading term $\mathcal{A}(0) = 0_{N_{<d}} \oplus (-\rho_d Q_d)$ of $\mathcal{A}(x)$ reads

$$\mathcal{A}(0) = 0_{N_{<d}} \oplus \left(\bigoplus_{\ell=1}^{s_d} -\rho_d Q_{d,\ell} I_{m_{d,\ell}} \right)$$

with $Q_{d,\ell} \neq 0$ and $Q_{d,\ell} \neq Q_{d,\ell'}$ if $\ell \neq \ell'$. Thereby, applying [13, Thm. 1.5], there exists an invertible matrix $T_1(x) \in GL_{N_{\leq d}}(\mathbb{C}[[x]]_{1/\rho_d}[x^{-1}])$ with meromorphic $1/\rho_d$ -Gevrey entries at 0 such that the matrix ${}^{T_1}\mathcal{A}(x)$ has a block-decomposition like $\mathcal{A}(0)$. Note that the entries of ${}^{T_1}\mathcal{A}(x)$ are generally meromorphic $1/\rho_d$ -Gevrey and not convergent. Denote by $\mathcal{A}^{(\ell)}(x)$, $\ell = 0, \dots, s_d$, the blocks of ${}^{T_1}\mathcal{A}(x)$. By construction, all the sub-systems

$$x^{\rho_d+1}\frac{dW}{dx} = \mathcal{A}^{(\ell)}(x)W \quad , \quad \ell = 0, \dots, s_d,$$

have levels $< \rho_d$. Then, [13, Thm. 1.4] applies and, consequently, there exists, for all $\ell = 0, \dots, s_d$, an invertible matrix $T_2^{(\ell)}(x)$ with meromorphic

$1/\rho_d$ -Gevrey entries at 0 such that the matrix $T_2^{(\ell)}\mathcal{A}^{(\ell)}(x)$ has meromorphic entries at 0. Finally, normalizing the formal fundamental solutions of these last systems by means of convenient polynomial gauge transformations in x and $1/x$ if needed, calculations above tell us that there exists a matrix $T(x) \in GL_{N_{\leq d}}(\mathbb{C}[[x]]_{1/\rho_d}[x^{-1}])$ satisfying points 2 – 3. of lemma 3.5. Note that point 1 results from equalities

$$T(x)\tilde{H}(x) = \tilde{H}'(x) \oplus \tilde{H}''(x) = I_{N_{\leq d}} + O(x^r) \quad (3.5)$$

and from assumption $\tilde{H}(x) = I_{N_{\leq d}} + O(x^r)$.

◁ We are left to prove that $T(x)$ is analytic at 0 and that formal series $\tilde{H}'(x)$ and $\tilde{H}''(x)$ are both summable of levels $\leq \rho_k$. According to construction above, we already known that $\tilde{H}'(x)$ and $\tilde{H}''(x)$ are both summable of levels $< \rho_d$. Then, the first equality of (3.5) and hypothesis “ $\tilde{H}(x)$ summable of levels $\leq \rho_k$ ” tell us that $T(x)$ is actually both $1/\rho_d$ -Gevrey and summable of levels $< \rho_d$ (indeed, $\rho_k < \rho_d$ for all $d = k + 1, \dots, p_1$). Hence, applying Tauberian theorem [18, Prop. 7, p. 349], $T(x)$ is analytic at 0. As a result, $T(x)\tilde{H}(x)$ is still summable of levels $\leq \rho_k$ and, consequently, $\tilde{H}'(x)$ and $\tilde{H}''(x)$ are both summable of levels $\leq \rho_k$ too. This ends the proof of lemma 3.5. ■

Note that the hypothesis “ $\tilde{H}(x)$ is summable of levels $\leq \rho_k$ ” is crucial in the proof of lemma 3.5: without it, we can not prove the analyticity of $T(x)$. Note also that lemma 3.5 can be again applied to sub-system (3.4) when $d \geq k + 2\dots$ and so on as long as $d \neq k + 1$.

In the case of system (A_{ρ_k}) , an iterative application of lemma 3.5 starting with $d = p_1$ allows us to state the following result:

Theorem 3.6 (Block-diagonalisation theorem) *There exists an invertible matrix $T_k(x) \in GL_n(\mathbb{C}\{x\})$ with analytic entries at 0 such that*

1. $T_k(x) = I_n + O(x^r)$,
2. the gauge transformation $Z = T_k(x)Y$ transforms system (A_{ρ_k}) into a system

$$x^{r+1} \frac{dZ}{dx} = {}^{T_k}A_{\rho_k}(x)Z \quad ({}^{T_k}A_{\rho_k})$$

where the matrix ${}^{T_k}A_{\rho_k}(x) \in M_n(\mathbb{C}\{x\})$ has a block-diagonal decomposition like block-decomposition (3.1) of Q :

$${}^{T_k}A_{\rho_k}(x) = A'_{\rho_k}(x) \oplus A''_{\rho_k}(x)$$

with $A'_{\rho_k}(x) \in M_{N_{\leq k}}(\mathbb{C}\{x\})$ and $A''_{\rho_k}(x) \in M_{N_{> k}}(\mathbb{C}\{x\})$,

3. the formal fundamental solution $\tilde{Z}_{\rho_k^-}(x) = T_k(x)\tilde{Y}_{\rho_k^-}(x)$ of system $({}^{T_k}A_{\rho_k})$ has a block-diagonal decomposition

$$\tilde{Z}_{\rho_k^-}(x) = \tilde{F}'_{\rho_k^-}(x)x^{L_{\leq k}}e^{Q_{\leq k}(1/x)} \oplus \tilde{F}''_{\rho_k^-}(x)x^{L_{> k}}e^{Q_{> k}(1/x)}$$

where

- (a) $\tilde{F}'_{\rho_k^-}(x)$ and $\tilde{F}''_{\rho_k^-}(x)$ satisfy $\tilde{F}'_{\rho_k^-}(x) = \tilde{F}''_{\rho_k^-}(x) = I_* + O(x^r)$,
 (b) the matrix $\tilde{Y}'_{\rho_k^-}(x) := \tilde{F}'_{\rho_k^-}(x)x^{L_{\leq k}}e^{Q_{\leq k}(1/x)}$ is a formal fundamental solution of system

$$x^{\rho_k+1}\frac{dZ}{dx} = A'_{\rho_k}(x)Z, \quad (A'_{\rho_k})$$

- (c) the matrix $\tilde{Y}''_{\rho_k^-}(x) := \tilde{F}''_{\rho_k^-}(x)x^{L_{> k}}e^{Q_{> k}(1/x)}$ is a formal fundamental solution of system

$$x^{r+1}\frac{dZ}{dx} = A''_{\rho_k}(x)Z. \quad (A''_{\rho_k})$$

In particular, the matrix $T_k(x)\tilde{F}_{\rho_k^-}(x)$ has the block-decomposition

$$T_k(x)\tilde{F}_{\rho_k^-}(x) = \tilde{F}'_{\rho_k^-}(x) \oplus \tilde{F}''_{\rho_k^-}(x)$$

where $\tilde{F}'_{\rho_k^-}(x)$ and $\tilde{F}''_{\rho_k^-}(x)$ are both ρ_k^- -summable.

Remark 3.7 According to the analyticity of $T_k(x)$ and the “unicity” of factorization theorem 2.7, block-diagonalisation theorem 3.6 tells us that we can always choose as matrix $\tilde{F}_{\rho_k^-}(x)$ the matrix $T_k(x)\tilde{F}_{\rho_k^-}(x)$ and as system (A_{ρ_k}) the system $({}^{T_k}A_{\rho_k})$. This we do from now on.

Note that one of the interests of the choice of system $({}^{T_k}A_{\rho_k})$ for system (A_{ρ_k}) is that its sub-system (A'_{ρ_k}) “contains” all the levels $\leq \rho_k$ of $\tilde{f}(x)$ and has ρ_k as highest level.

Note also that block-diagonalisation theorem 3.6 and remark 3.7 above can be extended to the highest level $\rho_{p_1} = r$ of $\tilde{f}(x)$ by setting $N_{> p_1} = 0$ and $T_{p_1}(x) = I_n$. Doing that, we clearly have

$$\tilde{F}(x) = \tilde{F}_{r^-}(x) = T_{p_1}(x)\tilde{F}_{r^-}(x) = \tilde{F}'_{r^-}(x)$$

and systems (A) , (A_r) , $({}^{T_{p_1}}A_r)$ and (A'_r) coincide.

3.3 Resurgence

In this section, we shall investigate the resurgent character of functions $\widehat{f}_{\theta^{[k]}}^{[k;u]}(\tau)$ given in proposition 3.3. In particular, we shall prove a *resurgence theorem* which generalizes resurgence theorems stated by M. Loday-Richaud and the author in [12, 23] for systems with single-level and in [21, 22] for lowest and highest levels of systems with multi-levels.

3.3.1 Resurgence theorem

Recall that a *resurgent* function is an analytic function near the origin which can be analytically continued on all a convenient Riemann surface. More precisely, one has the following.

Definition 3.8 (Resurgent function) Let $\Omega \subset \mathbb{C}$ be a finite subset of \mathbb{C} containing 0. A function defined and analytic near 0 is said to be

- *resurgent with singular support $\Omega, 0$* when it can be analytically continued on all the Riemann surface \mathcal{R}_Ω defined as (the terminal end of) all homotopy classes in $\mathbb{C} \setminus \Omega$ of paths issuing from 0 and bypassing all points of Ω (only homotopically trivial paths are allowed to turn back to 0); in particular, such a function is analytic at 0 in the first sheet,
- *resurgent with singular support $\Omega, \widetilde{0}$* when it can be analytically continued on all the Riemann surface $\widetilde{\mathcal{R}}_\Omega :=$ the universal cover of $\mathbb{C} \setminus \Omega$.

We denote by $\mathcal{R}es_{\Omega,0}$ and $\mathcal{R}es_{\Omega,\widetilde{0}}$ the sets of resurgent functions with singular support $\Omega, 0$ and of resurgent functions with singular support $\Omega, \widetilde{0}$.

Recall that the difference between \mathcal{R}_Ω and $\widetilde{\mathcal{R}}_\Omega$ just lies in the fact that \mathcal{R}_Ω has no branch point at 0 in the first sheet. In particular, we have a natural injection $\mathcal{R}es_{\Omega,0} \hookrightarrow \mathcal{R}es_{\Omega,\widetilde{0}}$. Recall also that the choice of the Riemann surface $\widetilde{\mathcal{R}}_\Omega$ or \mathcal{R}_Ω only depends on the fact that the function we consider has a singular point at 0 or not.

Definition 3.9 (Resurgent function with exponential growth) Given $\kappa > 0$, a resurgent function of $\mathcal{R}es_{\Omega,0}$ (resp. $\mathcal{R}es_{\Omega,\widetilde{0}}$) is said to be *with exponential growth of order $\leq \kappa$* if it grows at most exponentially with an order $\leq \kappa$ on any bounded sector of infinity of \mathcal{R}_Ω (resp. $\widetilde{\mathcal{R}}_\Omega$).

We denote by $\mathcal{R}es_{\Omega,0}^{\leq \kappa}$ (resp. $\mathcal{R}es_{\Omega,\widetilde{0}}^{\leq \kappa}$) the set of resurgent functions of $\mathcal{R}es_{\Omega,0}$ (resp. $\mathcal{R}es_{\Omega,\widetilde{0}}$) with exponential growth of order $\leq \kappa$. As before, we have a natural injection $\mathcal{R}es_{\Omega,0}^{\leq \kappa} \hookrightarrow \mathcal{R}es_{\Omega,\widetilde{0}}^{\leq \kappa}$.

When $\kappa = 1$, any function of $\mathcal{R}es_{\Omega,0}^{\leq \kappa}$ (resp. $\mathcal{R}es_{\Omega,\tilde{0}}^{\leq \kappa}$) is said to be *summable-resurgent with singular support* $\Omega, 0$ (resp. $\Omega, \tilde{0}$). Following notations of [12, 21–23], we denote $\mathcal{R}es_{\Omega,0}^{sum}$ (resp. $\mathcal{R}es_{\Omega,\tilde{0}}^{sum}$) for $\mathcal{R}es_{\Omega,0}^{\leq 1}$ (resp. $\mathcal{R}es_{\Omega,\tilde{0}}^{\leq 1}$).

We are now able to state the main result of this section.

Theorem 3.10 (Resurgence theorem) *Let $k \in \{1, \dots, p_1\}$. Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a non anti-Stokes direction of $\tilde{f}(x)$ and $\theta^{[k]} := \rho_k \theta$. Let $\Omega_{\rho_k}^*$ be the set of Stokes values of level ρ_k of $\tilde{f}(x)$ (see definition 2.1) and $\Omega_{\rho_k} := \Omega_{\rho_k}^* \cup \{0\}$. Let $\kappa_1, \dots, \kappa_{p_1} > 0$ be the positive numbers defined by*

$$\kappa_j := \frac{\rho_j^+}{\rho_j^+ - \rho_j} \text{ for } j = 1, \dots, p_1 - 1 \text{ and } \kappa_{p_1} := 1.$$

- Case $k = 1$. Then, for all $u = 0, \dots, \rho_1 - 1$,

$$\widehat{\mathbf{f}}_{\theta^{[1]}}^{[1;u]}(\tau) \in \mathcal{R}es_{\Omega_{\rho_1},0}^{\leq \kappa_1}.$$

- Case $k \geq 2$. Then, for all $u = 0, \dots, \rho_k - 1$,

$$\widehat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]}(\tau) \in \mathcal{R}es_{\Omega_{\rho_k},\tilde{0}}^{\leq \kappa_k}.$$

Remark 3.11 When $\tilde{f}(x)$ has the unique level r (i.e., $p_1 = 1$ and so $\rho_1 = \rho_{p_1} = r$), we find again, of course, the resurgence theorem already stated by M. Loday-Richaud and the author in [12] and [23], namely

$$\widehat{\mathbf{f}}_{\theta^{[1]}}^{[1;u]}(\tau) \in \mathcal{R}es_{\Omega_{\rho_1},0}^{sum} \text{ for all } u = 0, \dots, \rho_1 - 1.$$

The proof of theorem 3.10 is developed in section 3.3.2 below. It is based on factorization theorem 2.7, block-diagonalisation theorem 3.6 and on the results of [21].

3.3.2 Proof of theorem 3.10

◁ **A fundamental identity.** Let $k \in \{1, \dots, p_1\}$. According to factorization theorem 2.7, block-diagonalisation theorem 3.6 and remark 3.7, the formal series $\tilde{f}(x)$ can be written on the form

$$\tilde{f}(x) = \tilde{F}_{\rho_k^+}(x) \tilde{f}_{\rho_k^-}(x) \quad \text{with} \quad \tilde{f}_{\rho_k^-}(x) = \begin{bmatrix} \tilde{f}'_{\rho_k^-}(x) \\ 0_{N_{>k} \times n_1} \end{bmatrix} \quad (3.6)$$

where

- $\widetilde{F}_{\rho_k^+}(x) \in M_n(\mathbb{C}[[x]])$ is a ρ_k^+ -summable formal series satisfying $\widetilde{F}_{\rho_k^+}(x) = I_n + O(x^r)$ when $k < p_1$ and $\widetilde{F}_{\rho_k^+}(x) = I_n$ when $k = p_1$,
- $\widetilde{f}'_{\rho_k^-}(x)$ denotes the first n_1 columns of $\widetilde{F}'_{\rho_k^-}(x) \in M_{N_{\leq k}}(\mathbb{C}[[x]])$,
- $0_{N_{>k} \times n_1}$ denotes the null-matrix of size $N_{>k} \times n_1$.

Note that $\widetilde{f}_{\rho_k^-}(x) = \widetilde{f}'_{\rho_k^-}(x)$ when $k = p_1$.

As before, we denote by $\widetilde{\mathbf{F}}_{\rho_k^+}^{[k;u]}(t)$ and $\widetilde{\mathbf{f}}_{\rho_k^-}{}^{\prime[k;u]}(t)$ with $u \in \{0, \dots, \rho_k - 1\}$ the ρ_k -reduced series of $\widetilde{F}_{\rho_k^+}(x)$ and of $\widetilde{f}_{\rho_k^-}(x)$. We also denote by

- $\widetilde{\mathbf{f}}^{[k]}(t) = \begin{bmatrix} \widetilde{\mathbf{f}}^{[k;0]}(t) \\ \vdots \\ \widetilde{\mathbf{f}}^{[k;\rho_k-1]}(t) \end{bmatrix} \in M_{\rho_k n, n_1}(\mathbb{C}[[t]])$ the matrix formed by the ρ_k -reduced series of $\widetilde{f}(x)$,
- $\widetilde{\mathbf{f}}_{\rho_k^-}{}^{\prime[k;u]}(t) = \begin{bmatrix} \widetilde{\mathbf{f}}_{\rho_k^-}{}^{\prime[k;u]}(t) \\ 0_{N_{>k} \times n_1} \end{bmatrix}$ and $\widetilde{\mathbf{f}}_{\rho_k^-}^{[k]}(t) = \begin{bmatrix} \widetilde{\mathbf{f}}_{\rho_k^-}^{[k;0]}(t) \\ \vdots \\ \widetilde{\mathbf{f}}_{\rho_k^-}^{[k;\rho_k-1]}(t) \end{bmatrix}$.

Then, relation (3.6) above implies relation $\widetilde{\mathbf{f}}^{[k]}(t) = \widetilde{\mathbf{F}}_{\rho_k^+}^{[k]}(t) \widetilde{\mathbf{f}}_{\rho_k^-}^{[k]}(t)$ where

$$\widetilde{\mathbf{F}}_{\rho_k^+}^{[k]}(t) := \begin{bmatrix} \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;0]}(t) & t \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;\rho_k-1]}(t) & \cdots & \cdots & t \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;1]}(t) \\ \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;1]}(t) & \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;0]}(t) & t \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;\rho_k-1]}(t) \\ \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;\rho_k-1]}(t) & \cdots & \cdots & \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;1]}(t) & \widetilde{\mathbf{F}}_{\rho_k^+}^{[k;0]}(t) \end{bmatrix}$$

is a $\left(\frac{\rho_k^+}{\rho_k}, \dots, \frac{r}{\rho_k}\right)$ -summable formal series satisfying $\widetilde{\mathbf{F}}_{\rho_k^+}^{[k]}(t) = I_{\rho_k n} + O(t)$ when $k < p_1$ and where $\widetilde{\mathbf{F}}_{\rho_k^+}^{[k]}(t) = I_{rn}$ when $k = p_1$. In particular, applying [3, p. 81], its formal Borel transform $\widehat{\mathbf{F}}_{\rho_k^+}^{[k]}(\tau)$ reads as

$$\widehat{\mathbf{F}}_{\rho_k^+}^{[k]}(\tau) = \begin{cases} \delta I_{\rho_k n} + \widehat{\mathbf{G}}_k(\tau) & \text{when } k < p_1 \\ \delta I_{rn} & \text{when } k = p_1 \end{cases},$$

where $\widehat{\mathbf{G}}_k(\tau)$ defines an entire function on all \mathbb{C} with exponential growth of order $\leq \kappa_k = \rho_k^{+1}/(\rho_k^{+1} - \rho_k)$ at infinity. Indeed, $\rho_k^{+1}/\rho_k > 1$. This brings then us to the following lemma:

Lemma 3.12 *Let $k \in \{1, \dots, p_1\}$. Then, the formal Borel transforms $\widehat{\mathbf{f}}^{[k;u]}(\tau)$ of $\widetilde{\mathbf{f}}^{[k;u]}(t)$ and the formal Borel transforms $\widehat{\mathbf{f}}_{\rho_k^-}^{[k;u]}(\tau)$ of $\widetilde{\mathbf{f}}_{\rho_k^-}^{[k;u]}(t)$ are related, for all $u = 0, \dots, \rho_k - 1$, by relation*

$$\widehat{\mathbf{f}}^{[k;u]} = \begin{bmatrix} \widehat{\mathbf{f}}_{\rho_k^-}^{[k;u]} \\ 0_{N_{>k} \times n_1} \end{bmatrix} + \mathbf{E}_{k,u} * \begin{bmatrix} \widehat{\mathbf{f}}_{\rho_k^-}^{[k;u]} \\ 0_{N_{>k} \times n_1} \end{bmatrix} \quad (3.7)$$

where $\mathbf{E}_{k,u}$ is a convenient entire function on all \mathbb{C} with exponential growth of order $\leq \kappa_k$ at infinity when $k < p_1$ and where $E_{k,u} \equiv 0$ when $k = p_1$.

◁ **Resurgence of $\widehat{\mathbf{f}}_{\rho_k^-}^{[k;u]}(\tau)$.** By construction (see block-diagonalisation theorem 3.6 and remark 3.7), the matrix $\widetilde{F}_{\rho_k^-}'(x)x^{L \leq k}e^{Q \leq k(1/x)}$ is a formal fundamental solution of a system of the form

$$x^{\rho_k+1} \frac{dY}{dx} = A'_{\rho_k}(x)Y \quad (A'_{\rho_k})$$

with a convenient matrix $A'_{\rho_k}(x) \in M_{N_{\leq k}}(\mathbb{C}\{x\})$ satisfying $A'_{\rho_k}(0) \neq 0$. In particular, one can easily check the following points:

- system (A'_{ρ_k}) has ρ_k as highest level,
- the levels of $\widetilde{f}'_{\rho_k^-}(x)$ are the levels $\leq \rho_k$ of $\widetilde{f}(x)$, namely $\rho_1 < \dots < \rho_k$,
- the Stokes values (hence, the anti-Stokes directions) of level $\ell \in \{\rho_1, \dots, \rho_k\}$ of $\widetilde{f}'_{\rho_k^-}(x)$ and $\widetilde{f}(x)$ coincide.

Thereby, choosing a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ as in theorem 3.10, it is clear that

1. θ is not an anti-Stokes direction of $\widetilde{f}'_{\rho_k^-}(x)$,
2. the $\widehat{\mathbf{f}}_{\rho_k^-}^{[k;u]}(\tau)$'s are, as the $\widehat{\mathbf{f}}^{[k;u]}(\tau)$'s, analytic at 0 if $k = 1$ and $\underline{\rho}^{[k]}$ -summable in direction $\boldsymbol{\theta}^{[k]} = \rho_k \theta$ if $k \geq 2$ (see proposition 3.3).

Hence, denoting as before by $\widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k]}}^{[k;u]}(\tau)$ the sum thus defined and applying [21], we have the following.

Proposition 3.13 ([21, Thm. 4.9]) *Let $k \in \{1, \dots, p_1\}$. Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and Ω_{ρ_k} as in theorem 3.10.*

- Case $p_1 = 1$. Then, for all $u = 0, \dots, \rho_1 - 1$:

$$\widehat{\mathbf{f}}_{\rho_1^-; \theta^{[1]}}'^{[1;u]}(\tau) \in \mathcal{R}es_{\Omega_{\rho_1}, 0}^{sum}.$$

- Case $p_1 \geq 2$. Then, for all $u = 0, \dots, \rho_k - 1$:

$$\widehat{\mathbf{f}}_{\rho_k^-; \theta^{[k]}}'^{[k;u]}(\tau) \in \mathcal{R}es_{\Omega_{\rho_k}, \tilde{0}}^{sum}.$$

We are now able to end the proof of theorem 3.10.

◁ **Conclusion.** According to lemma 3.12, functions $\widehat{\mathbf{f}}_{\theta^{[k]}}'^{[k;u]}$ and $\widehat{\mathbf{f}}_{\rho_k^-; \theta^{[k]}}'^{[k;u]}$ are defined on the same domain $V_0(\widehat{\mathbf{f}}_{\theta^{[k]}}'^{[k;u]})$ (see proposition 3.3) and are related by relation

$$\widehat{\mathbf{f}}_{\theta^{[k]}}'^{[k;u]} = \begin{bmatrix} \widehat{\mathbf{f}}_{\rho_k^-; \theta^{[k]}}'^{[k;u]} \\ 0_{N_{>k} \times n_1} \end{bmatrix} + \mathbf{E}_{k,u} * \begin{bmatrix} \widehat{\mathbf{f}}_{\rho_k^-; \theta^{[k]}}'^{[k;u]} \\ 0_{N_{>k} \times n_1} \end{bmatrix}. \quad (3.8)$$

Theorem 3.10 follows then from proposition 3.13 and from the fact that the exponential growth κ_k of \mathbf{E}_u at infinity is greater than 1 when $k < p_1$. This ends the proof.

3.4 Singularities

Resurgence theorem 3.10 above tells us in particular that the only possible singular points of $\widehat{\mathbf{f}}_{\theta^{[k]}}'^{[k;u]}(\tau)$ are 0 and the Stokes values $\omega \in \Omega_{\rho_k}^*$ of level ρ_k of $\tilde{f}(x)$. In this section, we propose to give a *complete* description of all the singularities of the $\widehat{\mathbf{f}}_{\theta^{[k]}}'^{[k;u]}(\tau)$ at the various Stokes values of $\Omega_{\rho_k}^*$. Before starting the calculations, let us recall some definitions and notations about singularities. For more precise details, we refer to [7, 14, 24].

3.4.1 Some spaces of singularities

Denote by \mathcal{O} the space of holomorphic germs at $0 \in \mathbb{C}$ and by $\tilde{\mathcal{O}}$ the space of holomorphic germs at 0 on the Riemann surface $\tilde{\mathbb{C}}$ of the logarithm. One calls any element of the quotient space $\mathcal{C} := \tilde{\mathcal{O}}/\mathcal{O}$ a *singularity at 0*. Recall that \mathcal{C} is also denoted by SING_0 by J. Écalle and al. (cf. [24] for instance). Recall also that the elements of \mathcal{C} are called *micro-functions* by B. Malgrange

[14, 15] by analogy with hyper- and micro-functions defined by Sato, Kawai and Kashiwara in higher dimensions.

The elements of \mathcal{C} are usually denoted with a nabla, like $\overset{\nabla}{\varphi}$, for a singularity of the function φ . A representative of $\overset{\nabla}{\varphi}$ in $\tilde{\mathcal{O}}$ is often denoted by $\tilde{\varphi}$ and is called *a major* of φ .

It is worth to consider the two natural maps

$$\begin{aligned} \text{can} : \tilde{\mathcal{O}} &\longrightarrow \mathcal{C} = \tilde{\mathcal{O}}/\mathcal{O} && \text{the canonical map and} \\ \text{var} : \mathcal{C} &\longrightarrow \tilde{\mathcal{O}} && \text{the variation map,} \end{aligned}$$

action of a positive turn around 0 defined by $\text{var } \overset{\nabla}{\varphi} = \tilde{\varphi}(\tau) - \tilde{\varphi}(\tau e^{-2i\pi})$, where $\tilde{\varphi}(\tau e^{-2i\pi})$ is the analytic continuation of $\tilde{\varphi}(\tau)$ along a path turning once clockwise around 0 and close enough to 0 for $\tilde{\varphi}$ to be defined all along (the result is independent of the choice of the major $\tilde{\varphi}$). The germ $\text{var } \overset{\nabla}{\varphi}$ is called *the minor* of $\overset{\nabla}{\varphi}$.

One can not multiply two elements of \mathcal{C} , but an element of \mathcal{C} and an element of \mathcal{O} : $\alpha \overset{\nabla}{\varphi} := \text{can}(\alpha \tilde{\varphi}) = \overset{\nabla}{\alpha \varphi}$ for all $\alpha \in \mathcal{O}$ and $\overset{\nabla}{\varphi} \in \mathcal{C}$.

On the other hand, one can defined a convolution product \otimes on \mathcal{C} by setting $\overset{\nabla}{\varphi}_1 \otimes \overset{\nabla}{\varphi}_2 := \text{can}(\tilde{\varphi}_1 *_u \tilde{\varphi}_2)$, where $\tilde{\varphi}_1 *_u \tilde{\varphi}_2$ is the truncated convolution product

$$(\tilde{\varphi}_1 *_u \tilde{\varphi}_2)(\tau) := \int_u^{\tau-u} \tilde{\varphi}_1(\tau - \eta) \tilde{\varphi}_2(\eta) d\eta \in \tilde{\mathcal{O}}$$

with u arbitrarily close to 0 satisfying $\tau \in]0, u[$ and $\arg(\tau - u) = \arg(\tau) - \pi$. Note that $\overset{\nabla}{\varphi}_1 \otimes \overset{\nabla}{\varphi}_2$ makes sense since it does not depend on u , nor on the choice of the majors $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$. The convolution product \otimes is commutative and associative on \mathcal{C} with unit $\delta := \text{can}\left(\frac{1}{2i\pi\tau}\right)$.

In the sequel of this article, we shall use especially the following subspaces of \mathcal{C} :

◁ *The subspace $\mathcal{C}^{\leq 1}$ of singularities for which the variation defines an entire function on all $\tilde{\mathbb{C}}$ with exponential growth of order ≤ 1 on any bounded sector of infinity. Recall that this space is isomorphic, via the Borel-Laplace transformation, to the space of analytic functions with subexponential growth at $0 \in \tilde{\mathbb{C}}$ [7, pp. 46-48]; in particular, any power t^λ with $\lambda \in \mathbb{C}$ and any exponential $e^{P(t^{1/p})}$ with $p \geq 2$ and $P(t)$ polynomial in t of degree $< p$ define singularities in $\mathcal{C}^{\leq 1}$.*

◁ *The subspace $\tilde{\mathcal{N}}il_{\Omega,0}^{\overset{\nabla}{res}; \leq \kappa}$ (resp. $\tilde{\mathcal{D}}et_{\Omega,0}^{\overset{\nabla}{res}; \leq \kappa}$) of resurgent singularities of Nilsson class (resp. of finite determination) with singular support $\Omega, 0$ (resp. $\Omega, \tilde{0}$) and exponential growth of order $\leq \kappa$ at infinity (κ denotes a positive*

number and Ω a finite subset of \mathbb{C} containing 0). Recall that these singularities are the singularities of \mathcal{C} for which the variation reads on the form

$$\sum_{\text{finite}} \varphi_{\alpha,p}(\tau) \tau^\alpha (\ln \tau)^p$$

with $\alpha \in \mathbb{C}$, $p \in \mathbb{N}$ and $\varphi_{\alpha,p}(\tau) \in \mathcal{R}es_{\Omega,0}^{\leq \kappa}$ (resp. $\varphi_{\alpha,p}(\tau) \in \mathcal{R}es_{\Omega,\tilde{0}}^{\leq \kappa}$ holomorphic on a punctured disc at 0). When $\kappa = 1$, such singularities are said *summable-resurgent* and we simply denote $\mathcal{N}il_{\Omega,0}^{\nabla s-res}$ (resp. $\mathcal{D}et_{\Omega,\tilde{0}}^{\nabla s-res}$) for $\mathcal{N}il_{\Omega,0}^{\nabla res;\leq 1}$ (resp. $\mathcal{D}et_{\Omega,\tilde{0}}^{\nabla res;\leq 1}$).

For any $\omega \in \mathbb{C}^*$, we denote by \mathcal{C}_ω the space of *singularities at ω* , *i.e.*, the space \mathcal{C} translated from 0 to ω . A function $\check{\varphi}$ is then a major of a singularity at ω if $\check{\varphi}(\omega + \tau)$ is a major of a singularity at 0. In the same way, we define the translated space $\mathcal{C}^{\leq 1}_\omega$, etc...

3.4.2 Description of singularities

Let $k \in \{1, \dots, p_1\}$ and $u \in \{0, \dots, \rho_k - 1\}$. The behavior of $\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]}(\tau)$ at any of its singular points $\omega \in \Omega_{\rho_k}^*$ depends, of course, on the sheet of the Riemann surface where we are, *i.e.*, it depends on the “homotopic class” of the path γ of analytic continuation followed from any point $a \neq 0$ of $V_0(\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]})$ ⁶ to a neighborhood of ω . Note in particular that “homotopic class” implies that the behavior of $\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]}(\tau)$ does not depend on the choice of a .

We denote below by $\hat{\mathbf{f}}_{\theta^{[k];\omega,\gamma}^{[k;u]}}^{\nabla}$ the singularity of $\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]}(\tau)$ defined by the analytic continuation of $\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]}(\tau)$ along the path γ . Before starting the calculations, let us first introduce the key notion of *front of a singularity* [21–23].

◁ **Front of a singularity.** Let $\omega \in \Omega_{\rho_k}^*$. We call *front of level ρ_k of ω* the set of all the polynomials $q_j(1/x)$ of $Q(1/x)$ with leading term $-\omega/x^{\rho_k}$. We denote it by $Fr_{\rho_k}(\omega)$ and we have

$$Fr_{\rho_k}(\omega) := \left\{ -\frac{\omega}{x^{\rho_k}} + q_{\omega,k,\ell} \left(\frac{1}{x} \right) ; \ell = 1, \dots, s_k \right\}$$

where s_k is an integer ≥ 1 and where all the $q_{\omega,k,\ell}(1/x)$ are polynomials in $1/x$ with degree $< \rho_k$ and without constant term.

⁶See proposition 3.3 for the exact definition of $V_0(\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]})$.

Following [21], ω (hence, its corresponding singularity $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k];\omega,\gamma}}^{\nabla[k;u]}$) is said to be with a *good front* when $Fr_{\rho_k}(\omega)$ is a singleton (case $s_k = 1$) and with a *bad front* otherwise (case $s_k \geq 2$). Note that ω has *always* a good front when ρ_k is the smallest level of $\widetilde{F}(x)$ (see [22]).

In the special case where ω has a good front, we simply denote $q_{\omega,k}$ for $q_{\omega,k,1}$. Then,

$$Fr_{\rho_k}(\omega) = \left\{ -\frac{\omega}{x^{\rho_k}} + q_{\omega,k} \left(\frac{1}{x} \right) \right\}$$

and we more precisely say that ω (and its corresponding singularity too) has a *good monomial front* when $q_{\omega,k} \equiv 0$ and a *good nonmonomial front* otherwise.

Let us now turn to the study of singularities.

◁ **Description of singularities.** We proceed similarly to the proof of resurgence theorem 3.10 by first studying the singularities of functions $\widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k]}}^{\prime[k;u]}(\tau)$, then by applying identity (3.8). The structure of these singularities (see proposition 3.14 below) stems straightaway from [21, Thm. 4.24] and from properties of system (A'_{ρ_k}) to which the $\widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k]}}^{\prime[k;u]}(\tau)$'s are intimately related (see section 3.3.2). Note in particular that these properties show, on one hand, that the nonzero singular points of $\widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k]}}^{\prime[k;u]}(\tau)$ are the singular points $\omega \in \Omega_{\rho_k}^*$ of $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]}}^{\prime[k;u]}(\tau)$ (see also proposition 3.13) and, on the other hand, that the fronts of singularities of functions $\widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k]}}^{\prime[k;u]}(\tau)$ and $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]}}^{\prime[k;u]}(\tau)$ at any of these points coincide.

Proposition 3.14 (Description of $\widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k];\omega,\gamma}}^{\nabla[k;u]}$, [21, Thm. 4.24])

Let $k \in \{1, \dots, p_1\}$ and $u \in \{0, \dots, \rho_k - 1\}$.

Let $\omega \in \Omega_{\rho_k}^*$ and γ a path on $\mathbb{C} \setminus \Omega_{\rho_k}$ starting from a point of $V_0(\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]}}^{\prime[k;u]})$ and ending in a neighborhood of ω .

1. Suppose that ω has a good front. Let

$$\mathcal{Q}_\omega = \left\{ q_{\omega,k} \left(\frac{1}{\mu_k^v t^{1/\rho_k}} \right) ; v = 0, \dots, \rho_k - 1 \right\}$$

with $\mu_k := e^{-2i\pi/\rho_k}$. Then,

$$\widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k];\omega,\gamma}}^{\nabla[k;u]} \in \sum_{q \in \mathcal{Q}_\omega} \mathcal{N}il_{\Omega_{\rho_k}^* - \omega, 0}^{\nabla} \otimes \tilde{e}^q|_\omega.$$

In particular, if ω has besides a monomial front, then

$$\boxed{\mathbf{f}_{\rho_k^-; \boldsymbol{\theta}^{[k]}; \omega, \gamma}^{\nabla [k; u]} \in \mathcal{N}il_{\Omega_{\rho_k^- \omega, 0}^{\nabla}}^{s-res} | \omega.}$$

2. Suppose that ω has a bad front. Let

$$\mathcal{Q}_\omega = \left\{ q_{\omega, k, \ell} \left(\frac{1}{\mu_k^v t^{1/\rho_k}} \right) ; \ell = 1, \dots, s_k \text{ and } v = 0, \dots, \rho_k - 1 \right\}$$

with $\mu_k := e^{-2i\pi/\rho_k}$. Then,

$$\boxed{\mathbf{f}_{\rho_k^-; \boldsymbol{\theta}^{[k]}; \omega, \gamma}^{\nabla [k; u]} \in \sum_{q \in \mathcal{Q}_\omega} \mathcal{D}et_{\Omega_{\rho_k^- \omega, \tilde{0}}^{\nabla}}^{s-res} \otimes e^q | \omega.}$$

Notation e^q stands for the singularity of $\mathcal{C}^{\leq 1}$ defines by e^q (section 3.4.1).

The fundamental identity

$$\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]}}^{[k; u]} = \begin{bmatrix} \widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k]}}^{\nabla [k; u]} \\ 0_{N_{>k} \times n_1} \end{bmatrix} + \mathbf{E}_{k, u} * \begin{bmatrix} \widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k]}}^{\nabla [k; u]} \\ 0_{N_{>k} \times n_1} \end{bmatrix} \quad (3.8)$$

leads then us to the result in view in this section:

Theorem 3.15 (Description of $\mathbf{f}_{\boldsymbol{\theta}^{[k]}; \omega, \gamma}^{\nabla [k; u]}$)

Let $k \in \{1, \dots, p_1\}$ and $u \in \{0, \dots, \rho_k - 1\}$.

Let $\omega \in \Omega_{\rho_k}^*$ be a Stokes value of level ρ_k of $\tilde{f}(x)$.

Let γ a path on $\mathbb{C} \setminus \Omega_{\rho_k}$ starting from a point of $V_0(\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]}}^{[k; u]})$ and ending in a neighborhood of ω .

1. Suppose that ω has a good front. Let

$$\mathcal{Q}_\omega = \left\{ q_{\omega, k} \left(\frac{1}{\mu_k^v t^{1/\rho_k}} \right) ; v = 0, \dots, \rho_k - 1 \right\}$$

with $\mu_k := e^{-2i\pi/\rho_k}$. Then,

$$\boxed{\mathbf{f}_{\boldsymbol{\theta}^{[k]}; \omega, \gamma}^{\nabla [k; u]} \in \sum_{q \in \mathcal{Q}_\omega} \mathcal{N}il_{\Omega_{\rho_k^- \omega, 0}^{\nabla}}^{res; \leq \kappa_k} \otimes e^q | \omega.}$$

In particular, if ω has besides a monomial front, then

$$\boxed{\mathbf{f}_{\boldsymbol{\theta}^{[k]}; \omega, \gamma}^{\nabla [k; u]} \in \mathcal{N}il_{\Omega_{\rho_k^- \omega, 0}^{\nabla}}^{res; \leq \kappa_k} | \omega.}$$

2. Suppose that ω has a bad front. Let

$$\mathcal{Q}_\omega = \left\{ q_{\omega,k,\ell} \left(\frac{1}{\mu_k^v t^{1/\rho_k}} \right) ; \ell = 1, \dots, s_k \text{ and } v = 0, \dots, \rho_k - 1 \right\}$$

with $\mu_k := e^{-2i\pi/\rho_k}$. Then,

$$\boxed{\mathbf{f}_{\boldsymbol{\theta}^{[k]};\omega,\gamma}^{\nabla[k;u]} \in \sum_{q \in \mathcal{Q}_\omega} \mathcal{D}et_{\Omega_{\rho_k}^{-\omega, \tilde{0}}}^{\nabla, res; \leq \kappa_k} \otimes e^q|_\omega.}$$

Note that conditions $\leq \kappa_k$ which occur in singularities $\mathbf{f}_{\boldsymbol{\theta}^{[k]};\omega,\gamma}^{\nabla[k;u]}$ are due to the exponential growth of entire functions $E_{k,u}$ at infinity (see lemma 3.12). Note also that a more precise description of singularities with good monomial front will be given in next section 4 in the case of some special geometric configurations of singular points of $\Omega_{\rho_k}^*$.

4 Effective calculation of Stokes multipliers

In this section, we are interested in the effective calculation of Stokes multipliers of $f(x)$. We shall prove in particular that, for some special geometric configurations of Stokes values of $\tilde{f}(x)$, these calculations can be reduced, by means of *explicit* and *theoretical* formulæ, to the effective calculations of some *connection constants* given by the singularities of $\hat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]}}^{[k;u]}(\tau)$ in the Borel plane. These *connection-to-Stokes formulæ*, which we shall display in theorem 4.11 below, generalize thus to any level of system (A) those already stated by M. Loday-Richaud and the author for systems with a single level [12, 23] and by the author for lowest and highest levels [21, 22].

Before starting the calculations, let us recall some definitions and notations about the Stokes phenomenon and Stokes-Ramis matrices.

4.1 Stokes phenomenon and Stokes-Ramis matrices

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be an anti-Stokes direction of $\tilde{F}(x)$ (see definition 2.3).

◁ **Stokes phenomenon.** For any $\eta > 0$ small enough, directions $\theta \pm \eta$ are not anti-Stokes directions of $\tilde{F}(x)$. Then, all the sums $s_{r;\theta \pm \eta}(\tilde{F})$ exist (see theorem 2.7) and we define the *lateral sums of $\tilde{F}(x)$ in direction θ* as the respective analytic continuations of the $s_{r;\theta \pm \eta}(\tilde{F})$'s to a sector with vertex 0,

bisected by θ and opening π/r (recall that r is the highest level of $\tilde{F}(x)$). We denote these sums by $s_{r;\theta^\pm}(\tilde{F})$.

We also define the *lateral sums of $\tilde{Y}(x)$ in direction θ* by $Y_{\theta^\pm}(x) := s_{r;\theta^\pm}(\tilde{F})(x)Y_{0;\theta^*}(x)$, where $Y_{0;\theta^*}(x)$ is the actual analytic function $Y_{0;\theta^*}(x) := x^L e^{Q(1/x)}$ defined by the choice $\arg(x)$ close to θ^* (denoted below by $\arg(x) \simeq \theta^*$) with θ^* an argument of θ , say its principal determination in $] - 2\pi, 0]$ ⁷.

The *Stokes phenomenon* of system (A) stems from the fact that the two lateral sums $s_{r;\theta^-}(\tilde{F})$ and $s_{r;\theta^+}(\tilde{F})$ are not analytic continuations from each other in general. This defect of analyticity is quantified by the collection of *Stokes-Ramis automorphisms* $St_{\theta^*} : Y_{\theta^+} \mapsto Y_{\theta^-}$ for all the anti-Stokes directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of $\tilde{F}(x)$.

◁ **Stokes-Ramis matrices.** The *Stokes-Ramis matrices*⁸ are then defined as the matrix representations of the St_{θ^*} 's in $GL_n(\mathbb{C})$:

Definition 4.1 (Stokes-Ramis matrix) One calls *Stokes-Ramis matrix associated with $\tilde{Y}(x)$ in direction θ* the matrix of St_{θ^*} in the basis Y_{θ^+} . We still denote it by St_{θ^*} ; it is uniquely determined by the relation

$$Y_{\theta^-}(x) = Y_{\theta^+}(x)St_{\theta^*} \quad \text{for } \arg(x) \simeq \theta^*.$$

Let us now split $St_{\theta^*} = [St_{\theta^*}^{j;\ell}]$ into blocks fitting to the Jordan block-structure of L (for $j, \ell = 1, \dots, J$, the matrix $St_{\theta^*}^{j;\ell}$ has size $n_j \times n_\ell$). Then, $St_{\theta^*}^{j;j} = I_{n_j}$ and $St_{\theta^*}^{j;\ell} = 0$ if θ is not a direction of maximal decay of polynomial $q_j - q_\ell$; otherwise, the entries of $St_{\theta^*}^{j;\ell}$ are called *Stokes multipliers of $\tilde{F}^{\bullet;\ell}(x)$ in direction θ* .

◁ **Factorization of Stokes-Ramis matrices.** Like $\tilde{F}(x)$, Stokes-Ramis matrix St_{θ^*} can be factored by levels. This result was first proved by J.-P. Ramis in [19, 20] by using the factorization theorem of $\tilde{F}(x)$ (see theorem 2.7); a quite different proof based on Stokes cocycles and mainly algebraic was given later by M. Loday-Richaud in [9].

⁷Any choice of the argument is convenient. However, to be compatible, on the Riemann sphere, with the usual choice $0 \leq \arg(z = 1/x) < 2\pi$ of the principal determination at infinity, we suggest to choose $-2\pi < \arg(x) \leq 0$ as principal determination about 0.

⁸In the literature, a Stokes matrix has a more general meaning where one allows to compare any two asymptotic solutions whose domains of definition overlap. According to the custom initiated by J.-P. Ramis [20] in the spirit of Stokes' work, we exclude this case here. We consider only matrices providing the transition between the sums on each side of a same anti-Stokes direction.

Recall that the levels of $\tilde{F}(x)$ are $r_1 < \dots < r_p = r$ and are given by the degrees $r_{j,\ell}$ of nonzero polynomials $q_j - q_\ell$ (see section 2.1).

Theorem 4.2 (Factorization of St_{θ^*} , [9, 19, 20]) *With notations as above, the Stokes-Ramis matrix St_{θ^*} can be written as*

$$St_{\theta^*} = St_{r_1;\theta^*} \dots St_{r_p;\theta^*} \quad , \quad St_{r_k;\theta^*} = [St_{r_k;\theta^*}^{j;\ell}] \in GL_n(\mathbb{C})$$

where, for all $k = 1, \dots, p$,

- $St_{r_k;\theta^*}^{j;j} = I_{n_j}$,
- $St_{r_k;\theta^*}^{j;\ell} = 0$ if θ is not a direction of maximal decay of $q_j - q_\ell$ or $r_{j,\ell} \neq r_k$.

Moreover, for any $\rho \in \{r_1, \dots, r_p\}$, the product $St_{\rho^-;\theta^*} := St_{r_1;\theta^*} \dots St_{\rho;\theta^*}$ is the Stokes-Ramis matrix of system (A_ρ) associated with $\tilde{Y}_{\rho^-}(x)$ in direction θ (see page 7).

Definition 4.3 (Stokes multipliers of level r_k) Let $k \in \{1, \dots, p\}$.

1. The matrix $St_{r_k;\theta^*}$ is called *Stokes-Ramis matrix of level r_k associated with $\tilde{Y}(x)$ in direction θ* .
2. When θ is a direction of maximal decay of $q_j - q_\ell$ and $r_{j,\ell} = r_k$, the entries of $St_{r_k;\theta^*}^{j;\ell}$ are called *Stokes multipliers of level r_k of $\tilde{F}^{\bullet;\ell}(x)$ in direction θ* .

For some special geometric configurations of the Stokes values of highest level $\rho_{p_1} = r_p = r$ of $\tilde{f}(x)$, it was proved in [21] that the Stokes multipliers $st_{r;\theta^*}^{j;\bullet} := St_{r;\theta^*}^{j;1}$ of $\tilde{f}(x)$ can be expressed in terms of connection constants given by the *principal singularities* (see definition 4.6 below) of functions $\hat{\mathbf{f}}_{\theta^{[p_1]^-}}^{[p_1;u]}(\tau)$, $u = 0, \dots, r - 1$ (these functions are defined in the same way as the lateral sum $s_{\underline{r};\theta^-}(\tilde{F})$). The *highest level's connection-to-Stokes formulæ* thus obtained provide then an efficient tool for the effective calculation of the $st_{r;\theta^*}^{j;\bullet}$'s.

In section 4.2 below, we propose to extend this result to any level ρ_k of $\tilde{f}(x)$, $k \in \{1, \dots, p_1\}$.

4.2 Connection-to-Stokes formulæ of level ρ_k in the case of a SG-Configuration

In this section, we fix $k \in \{1, \dots, p_1\}$ and, as before, we denote by $\Omega_{\rho_k}^*$ the set of all the Stokes values of level ρ_k of $\tilde{f}(x)$. Recall (cf. definition 2.1) that the elements of $\Omega_{\rho_k}^*$ determine all the anti-Stokes directions of level ρ_k of $\tilde{f}(x)$; precisely, any $\omega \in \Omega_{\rho_k}^*$ generates a collection $(\theta_\ell)_{\ell=0, \dots, \rho_k-1}$ of ρ_k anti-Stokes directions of level ρ_k of $\tilde{f}(x)$ regularly distributed around the origin and defined by its ρ_k -th roots. In the sequel, we choose such a collection (θ_ℓ) and we suppose, to fix ideas, that their principal determinations $\theta_\ell^* \in]-2\pi, 0]$ (see note 7) satisfy $-2\pi < \theta_{\rho_k-1}^* < \dots < \theta_0^* \leq 0$.

Let us now denote $\theta^{[k]} := \rho_k \theta_0$ and $\Omega_{\rho_k; \theta^{[k]}}^*$ the set of all the Stokes values of $\Omega_{\rho_k}^*$ with argument $\theta^{[k]}$. Note that $\omega \in \Omega_{\rho_k; \theta^{[k]}}^*$ if and only if $\omega \in \Omega_{\rho_k}^*$ and ω generates the collection (θ_ℓ) . Note also that, for any $\ell \in \{0, \dots, \rho_k - 1\}$, the Stokes multipliers of level ρ_k of $\tilde{f}(x)$ in direction θ_ℓ are all the entries of all matrices $st_{\rho_k; \theta_\ell^*}^{j; \bullet}$ with j such that

$$q_j \left(\frac{1}{x} \right) = -\frac{a_{j, \rho_k}}{x^{\rho_k}} + o \left(\frac{1}{x^{\rho_k}} \right) \quad \text{and} \quad a_{j, \rho_k} \in \Omega_{\rho_k; \theta^{[k]}}^*.$$

◁ **The SG-Configuration.** The connection-to-Stokes formulæ of level ρ_k strongly depend on the nature and the geometric configuration of the Stokes values of $\Omega_{\rho_k; \theta^{[k]}}^*$. Henceforth, in the rest of the article, we restrict ourselves to the following Special Geometric Configuration (in short, SG-Configuration):

Definition 4.4 (SG-Configuration) The set $\Omega_{\rho_k; \theta^{[k]}}^*$ is said to have a *SG-Configuration* when *all* its elements have a *good front*.

Let us now consider $\omega \in \Omega_{\rho_k; \theta^{[k]}}^*$. According to the following technical lemma due to M. Loday-Richaud, we can always suppose that ω has a *good monomial front*.

Lemma 4.5 (M. Loday-Richaud, [8]) *Let $\omega \in \Omega_{\rho_k; \theta^{[k]}}^*$ with a good front and $q_\omega(1/x)$ the unique element of $Fr_{\rho_k}(\omega)$.*

1. *There exists a change of the variable x of the form*

$$x = \frac{y}{1 + \alpha_1 y + \dots + \alpha_{r-1} y^{r-1}} \quad , \quad \alpha_1, \dots, \alpha_{r-1} \in \mathbb{C} \quad (4.1)$$

such that the polar part of $q_\omega(1/x(y))$ reads $-\omega/y^r$.

2. *The Stokes-Ramis matrices of system (A) are preserved by the change of variable (4.1).*

Note that, although lemma 4.5 be proved in [8] in the case of systems of dimension 2 (hence, with a single level), it can be extended to any system of dimension ≥ 3 . Indeed, since the change of variable (4.1) is tangent to identity, it “preserves” levels, Stokes values and summation operators.

Under the hypotheses above, we shall now prove (see theorem 4.11) that the Stokes multipliers $(st_{\rho_k; \theta_\ell}^{j; \bullet})_{\ell=0, \dots, \rho_k-1}$ for j such that $q_j(1/x) \in Fr_{\rho_k}(\omega)$ are expressed in terms of the connection constants given by the *principal singularities* of functions $\widehat{\mathbf{f}}_{\theta^{[k]}^-}^{[k; u]}(\tau)$ at ω . To do that, let us first give some precisions about the structure of these singularities.

◁ **Principal singularities with good monomial front, principal majors and connection constants.** As we said at the beginning of section 3.4.2, the singularities $\widehat{\mathbf{f}}_{\theta^{[k]}^-; \omega, \gamma}^{[k; u]}$ depend on the chosen path γ for the analytic continuations $\widehat{\mathbf{f}}_{\theta^{[k]}^-; \omega, \gamma}^{[k; u]}$ of functions $\widehat{\mathbf{f}}_{\theta^{[k]}^-}^{[k; u]}$ and meanwhile, on the chosen determination of the argument around ω . Recall that such a path γ starts from a point of $V_0(\widehat{\mathbf{f}}_{\theta^{[k]}^-}^{[k; u]})$ and ends in a neighborhood of ω . Here below, we consider a path $\gamma_{\tau_0, \omega}^+$ defined as follows:

- τ_0 is a point of $V_0(\widehat{\mathbf{f}}_{\theta^{[k]}^-}^{[k; u]}) \cap]0, \infty e^{i\theta^{[k]}}[$ (lied in the first sheet of Riemann surface $\mathcal{R}_{\Omega_{\rho_k}}$ or $\widetilde{\mathcal{R}}_{\Omega_{\rho_k}}$ (see definition 3.8)⁹,
- $\gamma_{\tau_0, \omega}^+$ is a path starting from τ_0 , going along the straight line $[0, \omega]$ to a point τ close to ω and avoiding all singular points of $\Omega_{\rho_k}^* \cap [0, \omega]$ to the right as shown on figure 4.1 below,
- we choose the principal determination of the variable τ around ω , say $\arg(\tau) \in]-2\pi, 0]$ as in section 4.1 (*cf.* note 7).

⁹Note that this last condition is, of course, always fulfilled when $V_0(\widehat{\mathbf{f}}_{\theta^{[k]}^-}^{[k; u]})$ is a disc or a sector with opening $< 2\pi$ (*cf.* note 5).

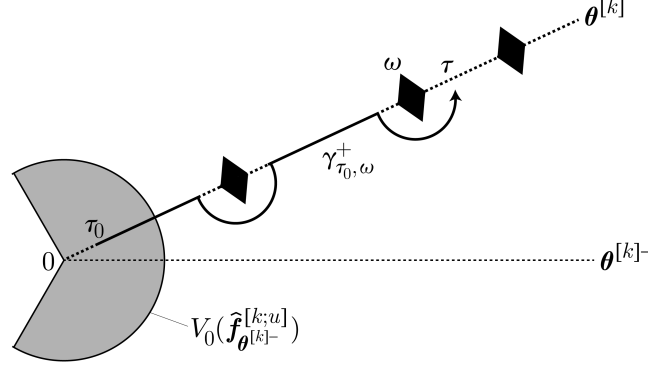


Figure 4.1 – A path $\gamma_{\tau_0, \omega}^+$ in the case of a sector $V_0(\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]})$ with opening $< 2\pi$

The analytic continuation $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}; \omega, +}^{[k;u]} := \widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}; \omega, \gamma_{\tau_0, \omega}^+}^{[k;u]}$ is called *right analytic continuation* of $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]}$ at ω . Note that it does not depend on the choice of τ_0 . The principal singularity of $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]}$ at ω is then defined as follows:

Definition 4.6 (Principal singularity) Given $u \in \{0, \dots, \rho_k - 1\}$, we call *principal singularity* of $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]}$ at ω the singularity $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}; \omega, +}^{\nabla [k;u]}$ defined by the right analytic continuation $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}; \omega, +}^{[k;u]}$ of $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]}$ at ω . A major $\widetilde{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}; \omega, +}^{[k;u]}$ is then called a *principal major*.

Under our two hypotheses “ $\Omega_{\rho_k; \boldsymbol{\theta}^{[k]}}^*$ has a SG-Configuration” and “ $\omega \in \Omega_{\rho_k; \boldsymbol{\theta}^{[k]}}^*$ has a good monomial front”, the structure of the principal singularity $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}; \omega, +}^{\nabla [k;u]}$ given in theorem 3.15 can be improved and a much more precise description can be displayed. As in section 3.4.2, this description stems from the study of the associated principal singularity of $\widehat{\mathbf{f}}_{\rho_k^-; \boldsymbol{\theta}^{[k]-}}^{[k;u]}$ at ω . The latter (see proposition 4.8 below) is obtained by applying once again the results of [21] to system (A'_{ρ_k}) .

Before stating it, let us introduce some notations. According to the block-structures

$$Q \begin{pmatrix} 1 \\ x \end{pmatrix} = \bigoplus_{j=1}^J q_j \begin{pmatrix} 1 \\ x \end{pmatrix} I_{n_j} \quad \text{and} \quad L = \bigoplus_{j=1}^J (\lambda_j I_{n_j} + J_{n_j})$$

of matrices $Q(1/x)$ and L (section 1) and the definition of matrices $Q_{\leq k}$ and $L_{\leq k}$ (see notation 3.4), we can write

$$Q_{\leq k} \left(\frac{1}{x} \right) = \bigoplus_{j=1}^{J_k} q_j \left(\frac{1}{x} \right) I_{n_j} \quad \text{and} \quad L_{\leq k} = \bigoplus_{j=1}^{J_k} (\lambda_j I_{n_j} + J_{n_j})$$

with a convenient $J_k \in \{1, \dots, J\}$. Note that $J_{p_1} = J$. Note also that $n_1 + \dots + n_{J_k} = N_{\leq k}$ the dimension of $Q_{\leq k}$ and $L_{\leq k}$.

Notation 4.7 With notations as above, we split below

- $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]}$ into J row-blocks $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]j;\bullet}$ fitting to the Jordan block-structure of L (for $j = 1, \dots, J$, the matrix $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]j;\bullet}$ has size $n_j \times n_1$),
- $\widehat{\mathbf{f}}'_{\rho_k^-; \boldsymbol{\theta}^{[k]-}}^{[k;u]}$ into J_k row-blocks $\widehat{\mathbf{f}}'_{\rho_k^-; \boldsymbol{\theta}^{[k]-}}^{[k;u]j;\bullet}$ fitting to the Jordan block-structure of $L_{\leq k}$ (for $j = 1, \dots, J_k$, the matrix $\widehat{\mathbf{f}}'_{\rho_k^-; \boldsymbol{\theta}^{[k]-}}^{[k;u]j;\bullet}$ has size $n_j \times n_1$).

In the same way, any matrix of size $n \times p$ (resp. $N_{\leq k} \times p$) with $p \geq 1$ is split like $\widehat{\mathbf{f}}_{\boldsymbol{\theta}^{[k]-}}^{[k;u]}$ (resp. $\widehat{\mathbf{f}}'_{\rho_k^-; \boldsymbol{\theta}^{[k]-}}^{[k;u]}$).

Proposition 4.8 (Description of $\widehat{\mathbf{f}}'_{\rho_k^-; \boldsymbol{\theta}^{[k]-}; \omega, +}^{\nabla [k;u]}$, [21, Prop. 5.4])

Let $k \in \{1, \dots, p_1\}$ and $u \in \{0, \dots, \rho_k - 1\}$.

Let $\omega \in \Omega_{\rho_k; \boldsymbol{\theta}^{[k]}}^*$ with good monomial front. Then, the principal singularity

$\widehat{\mathbf{f}}'_{\rho_k^-; \boldsymbol{\theta}^{[k]-}; \omega, +}^{\nabla [k;u]}$ admits a major $\widetilde{\mathbf{f}}'_{\rho_k^-; \boldsymbol{\theta}^{[k]-}; \omega, +}^{\nabla [k;u]}$ of the form

$$\widetilde{\mathbf{f}}'_{\rho_k^-; \boldsymbol{\theta}^{[k]-}; \omega, +}^{\nabla [k;u]j;\bullet}(\omega + \tau) = \tau^{\frac{\lambda_j - u}{\rho_k} - 1} \tau^{\frac{J_{n_j}}{\rho_k}} \mathbf{K}_{\omega^*, +}^{\nabla [k;u]j;\bullet} \tau^{-\frac{J_{n_1}}{\rho_k}} + \text{rem}_{\omega^*, +}^{\nabla [k;u]j;\bullet}(\tau)$$

for all $j = 1, \dots, J_k$ with a remainder

$$\text{rem}_{\omega^*, +}^{\nabla [k;u]j;\bullet}(\tau) := \sum_{\lambda_\ell; q_\ell(1/x) \in \text{Fr}_{\rho_k}(\omega)} \sum_{v=0}^{\rho_k-1} \tau^{\frac{\lambda_\ell - v}{r}} \mathbf{R}_{\lambda_\ell, v; \omega^*, +}^{\nabla [k;u]j;\bullet}(\ln \tau)$$

where

- $\mathbf{K}_{\omega^*, +}^{\nabla [k;u]j;\bullet}$ denotes a $n_j \times n_1$ -constant matrix such that $\mathbf{K}_{\omega^*, +}^{\nabla [k;u]j;\bullet} = 0$ as soon as $q_j(1/x) \notin \text{Fr}_{\rho_k}(\omega)$,

- $\mathbf{R}_{\lambda_\ell, v; \omega^*, +}^{[k; u]j; \bullet}(X)$ denotes a $n_j \times n_1$ -polynomial matrix with summable-resurgent coefficients in $\mathcal{R}es_{\Omega_{\rho_k}^{sum} - \omega, 0}$ whose the columns are of log-degree

$$N[\ell] = \begin{cases} [(n_\ell - 1) & (n_\ell - 1) + 1 & \cdots & (n_\ell - 1) + (n_1 - 1)] & \text{if } \lambda_\ell \neq 0 \\ [n_\ell & n_\ell + 1 & \cdots & n_\ell + (n_1 - 1)] & \text{if } \lambda_\ell = 0. \end{cases}$$

The entries of nontrivial matrices $\mathbf{K}_{\omega^*, +}^{[k; u]j; \bullet}$ are called the connection constants of $\widehat{\mathbf{f}}_{\rho_k^-; \theta^{[k]'}}^{[k; u]}$ at ω .

The structure of the principal singularities $\mathbf{f}_{\theta^{[k]^-}; \omega, +}^{\nabla [k; u]}$ stems then from proposition 4.8 and from the fundamental identity (3.8) stated in section 3.3.2. More precisely, we have the following.

Theorem 4.9 (Description of $\mathbf{f}_{\theta^{[k]^-}; \omega, +}^{\nabla [k; u]}$)

Let $k \in \{1, \dots, p_1\}$ and $u \in \{0, \dots, \rho_k - 1\}$.

Let $\omega \in \Omega_{\rho_k; \theta^{[k]}}^*$ with good monomial front. Then, the principal singularity

$\mathbf{f}_{\theta^{[k]^-}; \omega, +}^{\nabla [k; u]}$ admits a major $\widetilde{\mathbf{f}}_{\theta^{[k]^-}; \omega, +}^{[k; u]}$ of the form

$$\widetilde{\mathbf{f}}_{\theta^{[k]^-}; \omega, +}^{[k; u]}(\omega + \tau) = \tau^{\frac{\lambda_j - u}{\rho_k} - 1} \tau^{\frac{J_{n_j}}{\rho_k}} \mathbf{K}_{\omega^*, +}^{[k; u]j; \bullet} \tau^{-\frac{J_{n_1}}{\rho_k}} + \text{rem}_{\omega^*, +}^{[k; u]j; \bullet}(\tau)$$

for all $j = 1, \dots, J$ with a remainder

$$\text{rem}_{\omega^*, +}^{[k; u]j; \bullet}(\tau) := \sum_{\lambda_\ell; q_\ell(1/x) \in Fr_{\rho_k}(\omega)} \sum_{v=0}^{\rho_k - 1} \tau^{\frac{\lambda_\ell - v}{r}} \mathbf{R}_{\lambda_\ell, v; \omega^*, +}^{[k; u]j; \bullet}(\ln \tau)$$

where

- $\mathbf{K}_{\omega^*, +}^{[k; u]j; \bullet}$ denotes a $n_j \times n_1$ -constant matrix such that

$$\mathbf{K}_{\omega^*, +}^{[k; u]j; \bullet} = \begin{cases} \mathbf{K}_{\omega^*, +}^{[k; u]j; \bullet} & \text{if } q_j(1/x) \in Fr_{\rho_k}(\omega) \\ 0_{n_j \times n_1} & \text{otherwise} \end{cases},$$

- $\mathbf{R}_{\lambda_\ell, v; \omega^*, +}^{[k; u]j; \bullet}(X)$ denotes a $n_j \times n_1$ -polynomial matrix with resurgent coefficients in $\mathcal{R}es_{\Omega_{\rho_k}^{\leq \kappa_k} - \omega, 0}$ whose the columns are of log-degree $N[\ell]$ (cf. notation just above).

Note that the nontrivial constant matrices $\mathbf{K}_{\omega^*,+}^{l[k;u]j;\bullet}$ and $\mathbf{K}_{\omega^*,+}^{[k;u]j;\bullet}$ coincide. This shows in particular that the connection constants of $\widehat{\mathbf{f}}_{\rho_k^-; \theta^{[k]-}}^{l[k;u]}$ at ω can actually be directly calculated by considering the principal singularity of $\widehat{\mathbf{f}}_{\theta^{[k]-}}^{[k;u]}$ at ω . Hence, the following definition:

Definition 4.10 (Connection constants of $\widehat{\mathbf{f}}_{\theta^{[k]-}}^{[k;u]}$ at ω) Given $k \in \{1, \dots, p_1\}$ and $u \in \{0, \dots, \rho_k - 1\}$, we call *connection constants of $\widehat{\mathbf{f}}_{\theta^{[k]-}}^{[k;u]}$ at ω* the entries of the nontrivial constant matrices $\mathbf{K}_{\omega^*,+}^{[k;u]j;\bullet}$.

Note that, in practice, the matrices $\mathbf{K}_{\omega^*,+}^{[k;u]j;\bullet}$ for $j \in \{1, \dots, J\}$ such that $q_j(1/x) \in Fr_{\rho_k}(\omega)$ can be determined as the coefficients of the monomial $\tau^{(\lambda_j - u)/\rho_k - 1}$ in the principal major $\widetilde{\mathbf{f}}_{\theta^{[k]-}; \omega, +}^{[k;u]}(\omega + \tau)$.

We are now able to state the connection-to-Stokes formulæ of level ρ_k .

◁ **Connection-to-Stokes formulæ of level ρ_k .** As said in factorization theorem 4.2, the matrix $St_{\rho_k^-; \theta_\ell^*}$ is the Stokes-Ramis matrix of system (A_{ρ_k}) associated with $\widetilde{Y}_{\rho_k^-}(x)$ in direction θ_ℓ . According to block-diagonalisation theorem 3.6, this matrix has a block-decomposition $St_{\rho_k^-; \theta_\ell^*} = St_{\rho_k^-; \theta_\ell^*}' \oplus St_{\rho_k^-; \theta_\ell^*}''$, where $St_{\rho_k^-; \theta_\ell^*}'$ is the Stokes-Ramis matrix of system $(A_{\rho_k}^\square)$ associated with $\widetilde{Y}_{\rho_k^-}^\square(x)$ in direction θ_ℓ . Moreover, $St_{\rho_k^-; \theta_\ell^*}'$ and $St_{\rho_k^-; \theta_\ell^*}''$ can be factored like $St_{\rho_k^-; \theta_\ell^*}$ by levels $\leq \rho_k$:

$$St_{\rho_k^-; \theta_\ell^*}' = St_{r_1; \theta_\ell^*}' \dots St_{\rho_k; \theta_\ell^*}' \quad \text{and} \quad St_{\rho_k^-; \theta_\ell^*}'' = St_{r_1; \theta_\ell^*}'' \dots St_{\rho_k; \theta_\ell^*}''$$

(recall that r_1 is the lowest level) and we have $St_{\rho; \theta_\ell^*} = St_{\rho; \theta_\ell^*}' \oplus St_{\rho; \theta_\ell^*}''$ for any level $\rho \leq \rho_k$. In particular, the Stokes multipliers $st_{\rho_k; \theta_\ell^*}^{l[j]; \bullet}$ of level ρ_k of $\widetilde{f}'(x)$ coincide with the Stokes multipliers $st_{\rho_k; \theta_\ell^*}^{j; \bullet}$ of level ρ_k of $\widetilde{f}(x)$.

Besides, since ρ_k is the highest level of system (A'_{ρ_k}) , [21, Thm. 5.7] applies and tells us that, for any $\omega \in \Omega_{\rho_k; \theta^{[k]}}^*$ with good monomial front and any j such that $q_j(1/x) \in Fr_{\rho_k}(\omega)$, the $st_{\rho_k; \theta_\ell^*}^{l[j]; \bullet}$'s are expressed in terms of the connection constants $\mathbf{K}_{\omega^*,+}^{l[k;u]j;\bullet}$ of $\widehat{\mathbf{f}}_{\rho_k^-; \theta^{[k]-}}^{l[k;u]}$ at ω . On the other hand, we proved in theorem 4.9 above that the connection constants $\mathbf{K}_{\omega^*,+}^{l[k;u]j;\bullet}$ are also the connection constants $\mathbf{K}_{\omega^*,+}^{[k;u]j;\bullet}$ of $\widehat{\mathbf{f}}_{\theta^{[k]-}}^{[k;u]}$ at ω .

Thereby, the connection-to-Stokes formulæ between $st_{\rho_k; \theta_\ell^*}^{l[j]; \bullet}$ and $\mathbf{K}_{\omega^*,+}^{l[k;u]j;\bullet}$ already stated in [21, Thm. 5.7] coincide with the connection-to-Stokes formulæ between $st_{\rho_k; \theta_\ell^*}^{j; \bullet}$ and $\mathbf{K}_{\omega^*,+}^{[k;u]j;\bullet}$ in view in this section. Hence the following.

Theorem 4.11 (Connection-to-Stokes formulæ of level ρ_k)

Let $k \in \{1, \dots, p_1\}$ and $\omega \in \Omega_{\rho_k; \theta^{[k]}}^*$ with a good monomial front.

Let $j \in \{1, \dots, J\}$ such that $q_j(1/x) \in Fr_{\rho_k}(\omega)$.

Then, the data of the Stokes multipliers $(st_{\rho_k; \theta_\ell^*}^{j; \bullet})_{\ell=0, \dots, \rho_k-1}$ of level ρ_k of $\tilde{f}(x)$ and the data of the connection constants $(\mathbf{K}_{\omega^*, +}^{[k; u]j; \bullet})_{u=0, \dots, \rho_k-1}$ of $\hat{\mathbf{f}}_{\theta^{[k]-}}^{[k; u]}(\tau)$ at ω are equivalent and are related, for all $\ell = 0, \dots, \rho_k - 1$, by relations

$$\boxed{st_{\rho_k; \theta_\ell^*}^{j; \bullet} = \sum_{u=0}^{\rho_k-1} \mu_k^{\ell(uI_{n_j} - L_j)} \mathbf{I}_{\omega^*}^{[k; u]j; \bullet} \mu_k^{\ell J_{n_1}} \quad (4.2)}$$

where $\mu_k := e^{-2i\pi/\rho_k}$ and where

$$\boxed{\mathbf{I}_{\omega^*}^{[k; u]j; \bullet} := \int_{\gamma_0} \tau^{\frac{\lambda_j - u}{\rho_k} - 1} \tau^{\frac{J_{n_j}}{\rho_k}} \mathbf{K}_{\omega^*, +}^{[k; u]j; \bullet} \tau^{-\frac{J_{n_1}}{\rho_k}} e^{-\tau} d\tau \quad (4.3)}$$

with γ_0 a Hankel type path around the nonnegative real axis \mathbb{R}^+ with argument from -2π to 0.

Note that relation (4.2) is similar to the ones obtained by M. Loday-Richaud and the author in [12, 21–23] for systems with a unique level and for the lowest and highest levels of systems with multi-levels. In particular, an expanded form providing each entry of formula (4.2) can be found in [23, Cor. 4.6]. This can be useful for effective numerical calculations.

Here below, we recall this expanded form in the special case where the matrix L of exponents of formal monodromy is diagonal: $L = \text{diag}(\lambda_1, \dots, \lambda_n)$. In this case, the matrices $st_{\rho_k; \theta_\ell^*}^{j; \bullet}$ and $\mathbf{K}_{\omega^*, +}^{[k; u]j; \bullet}$ are reduced to just one entry which we respectively denote $st_{\rho_k; \theta_\ell^*}^j$ and $\mathbf{K}_{\omega^*, +}^{[k; u]j}$. Then, identity (4.3) becomes

$$\int_{\gamma_0} \tau^{\frac{\lambda_j - u}{\rho_k} - 1} \mathbf{K}_{\omega^*, +}^{[k; u]j} e^{-\tau} d\tau = 2i\pi \frac{e^{-i\pi \frac{\lambda_j - u}{\rho_k}}}{\Gamma\left(1 - \frac{\lambda_j - u}{\rho_k}\right)} \mathbf{K}_{\omega^*, +}^{[k; u]j}$$

and the connection-to-Stokes formulæ (4.2) become

$$\boxed{st_{\rho_k; \theta_\ell^*}^j = 2i\pi \sum_{u=0}^{\rho_k-1} \mu_k^{\ell(u-\lambda_j)} \frac{e^{-i\pi \frac{\lambda_j - u}{\rho_k}}}{\Gamma\left(1 - \frac{\lambda_j - u}{\rho_k}\right)} \mathbf{K}_{\omega^*, +}^{[k; u]j}, \quad \ell = 0, \dots, \rho_k - 1. \quad (4.4)}$$

◁ **Effective calculation.** When $\Omega_{\rho_k; \theta^{[k]}}^*$ has the SG-Configuration, theorem 4.11 tells us that the effective calculation of the Stokes multipliers of level ρ_k associated with $\omega \in \Omega_{\rho_k; \theta^{[k]}}^*$ is reduced, after applying lemma 4.5 if needed, to the effective calculation of the connection constants at ω .

In section 4.3 below, we treat in detail one typical example to illustrate the connection-to-Stokes formulæ (4.2).

For the convenience of the reader, we briefly recall here below how to characterize the formal series $\tilde{\mathbf{f}}^{[k;u]}(t)$'s and their Borel transforms $\hat{\mathbf{f}}_{\theta^{[k]}}^{[k;u]}(\tau)$'s.

• *Case $\rho_k = \rho_1 = 1$.* In this case, the $\tilde{\mathbf{f}}^{[k;u]}(t)$'s are reduced to just one series $\tilde{\mathbf{f}}^{[k;0]}(t) = \tilde{f}(x)$ and we keep denoting the variable x for t . According to normalizations $(N_1) - (N_4)$ of the formal fundamental solution $\tilde{Y}(x)$ of system (A) , the formal series $\tilde{F}(x)$ is uniquely determined by the homological system

$$x^{r+1} \frac{dF}{dx} = A(x)F - FA_0(x) \quad , \quad A_0(x) := x^{r+1} \frac{dQ}{dx} + x^r L \quad (A_H)$$

associated with system (A) jointly with the initial condition $\tilde{F}(0) = I_n$ [5]. Hence, by considering its first n_1 columns, we deduce that $\tilde{f}(x)$ is uniquely determined by the system

$$\boxed{x^2 \frac{df}{dx} = x^{1-r} A(x)f - xfJ_{n_1}} \quad (A_H^{[1]})$$

jointly with the initial condition $\tilde{f}(0) = I_{n, n_1}$ (first n_1 columns of the identity matrix of size n). Recall that $q_1 \equiv 0$ and $\lambda_1 = 0$ (*cf.* normalization (N_4)).

• *Case $\rho_k \geq 2$.* In this case, a system characterizing the formal series $\tilde{\mathbf{f}}^{[k;u]}(t)$'s, $u = 0, \dots, \rho_k - 1$, is provided by the classical method of rank reduction [10] by considering the homological system of the ρ_k -reduced system associated with system (A) . More precisely, writing system (A) in the form

$$x^{\rho_k+1} \frac{dY}{dx} = \mathcal{A}(x)Y \quad , \quad \mathcal{A}(x) := x^{\rho_k-r} A(x) \in M_n(\mathbb{C}\{x\}[x^{-1}])$$

one can prove, similarly as in the case $\rho_k = 1$, that the formal series

$$\tilde{\mathbf{f}}^{[k]}(t) = \begin{bmatrix} \tilde{\mathbf{f}}^{[k;0]}(t) \\ \vdots \\ \tilde{\mathbf{f}}^{[k;\rho_k-1]}(t) \end{bmatrix} \in M_{\rho_k n, n_1}(\mathbb{C}[[t]])$$

is uniquely determined by the system

$$\boxed{\rho_k t^2 \frac{d\mathbf{f}}{dt} = \mathbf{A}^{[k]}(t)\mathbf{f} - t\mathbf{f}J_{n_1}} \quad (\mathbf{A}_H^{[k]})$$

jointly with the initial condition $\tilde{\mathbf{f}}^{[k]}(0) = I_{\rho_k n, n_1}$ (first n_1 columns of the identity matrix of size $\rho_k n$); the matrix $\mathbf{A}^{[k]}(t) \in M_{\rho_k n}(\mathbb{C}\{t\}[t^{-1}])$ is defined by

$$\mathbf{A}^{[k]}(t) = \begin{bmatrix} \mathcal{A}^{[k;0]}(t) & t\mathcal{A}^{[k;\rho_k-1]}(t) & \cdots & \cdots & t\mathcal{A}^{[k;1]}(t) \\ \mathcal{A}^{[k;1]}(t) & \mathcal{A}^{[k;0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathcal{A}^{[k;0]}(t) & t\mathcal{A}^{[k;\rho_k-1]}(t) \\ \mathcal{A}^{[k;\rho_k-1]}(t) & \cdots & \cdots & \mathcal{A}^{[k;1]}(t) & \mathcal{A}^{[k;0]}(t) \end{bmatrix} - \bigoplus_{u=0}^{\rho_k-1} utI_n$$

where the $\mathcal{A}^{[k;u]}(t)$'s denote, as in section 3.3.2, the ρ_k -reduced series of $\mathcal{A}(x)$.

• Then, applying a formal Borel transform to systems $(A_H^{[1]})$ and $(\mathbf{A}_H^{[k]})$, we obtain convolution systems $(A_H^{[1]*})$ and $(\mathbf{A}_H^{[k]*})$ satisfied by the functions $\hat{\mathbf{f}}_{\theta^{[k]}-}^{[k;u]}(\tau)$, $u = 0, \dots, \rho_k - 1$. In the special case where $\rho_k = \rho_1 = 1$, we simply denote $\hat{f}(\xi)$ for $\hat{\mathbf{f}}_{\theta^{[k]}-}^{[k;u]}(\tau)$ (in this case, the formal Borel transform of $\tilde{f}(x)$ defines an analytic function at 0 (see proposition 3.3)).

Recall that the formal Borel transformation is an isomorphism from the \mathbb{C} -differential algebra $(\mathbb{C}[[t]], +, \cdot, t^2 \frac{d}{dt})$ to the \mathbb{C} -differential algebra $(\delta\mathbb{C} \oplus \mathbb{C}[[\tau]], +, *, \tau \cdot)$ that changes ordinary product \cdot into convolution product $*$ and changes derivation $t^2 \frac{d}{dt}$ into multiplication by τ . It also changes multiplication by $\frac{1}{t}$ into derivation $\frac{d}{d\tau}$ allowing thus to extend the isomorphism from the meromorphic series $\mathbb{C}[[t]][t^{-1}]$ to $\mathbb{C}[\delta^{(k)}, k \in \mathbb{N}] \oplus \mathbb{C}[[\tau]]$.

Recall also that, in the special case where matrix $A(x)$ of initial system (A) has rational coefficients, convolution systems $(A_H^{[1]*})$ and $(\mathbf{A}_H^{[k]*})$ can actually be always replaced by a convenient linear differential system.

4.3 Example

In this section, we consider the system

$$x^4 \frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2x^5 & x^2 & 0 & 0 \\ -x^4 & 0 & 2x + \frac{x^3}{2} & 0 \\ x^3 & x^3 & -2x^3 & 6 + \frac{x^3}{4} \end{bmatrix} Y \quad (4.5)$$

of dimension $n = 4$ and rank $r = 3$ together with its formal fundamental solution $\tilde{Y}(x) = \tilde{F}(x)x^L e^{Q(1/x)}$ at 0 where

$$\bullet Q\left(\frac{1}{x}\right) = \text{diag}\left(0, -\frac{1}{x}, -\frac{1}{x^2}, -\frac{2}{x^3}\right), \quad L = \text{diag}\left(0, 0, \frac{1}{2}, \frac{1}{4}\right),$$

$$\bullet \tilde{F}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \tilde{f}^2(x) & 1 & 0 & 0 \\ \tilde{f}^3(x) & 0 & 1 & 0 \\ \tilde{f}^4(x) & * & * & 1 \end{bmatrix} \text{ satisfies } \tilde{F}(x) = I_4 + O(x^3). \text{ More precisely,}$$

$$\begin{cases} \tilde{f}^2(x) = -2x^3 - 6x^4 - 24x^5 - 120x^6 + O(x^7) & \in x^3\mathbb{C}[[x]] \\ \tilde{f}^3(x) = \frac{1}{2}x^3 + \frac{5}{8}x^5 + O(x^7) & \in x^3\mathbb{C}[[x^2]] \\ \tilde{f}^4(x) = -\frac{1}{6}x^3 + \frac{61}{144}x^6 + O(x^7) & \in x^3\mathbb{C}[[x]] \end{cases} . \quad (4.6)$$

As before, we denote by $\tilde{f}(x)$ the first column of $\tilde{F}(x)$. According to calculations above, $\tilde{f}(x)$ has, like $\tilde{F}(x)$, three levels: $\rho_1 = 1$, $\rho_2 = 2$ and $\rho_3 = r = 3$ the rank of system (4.5).

The aim of this example is to illustrate the connection-to-Stokes formulæ given in theorem 4.11 by making explicit *all* the Stokes multipliers of $\tilde{f}(x)$.

Note that, although system (4.5) may seem a little bit involved, it is actually simple enough to allow *exact* calculations. This “simplicity” is due to the fact that its matrix is triangular. Of course, such a case is anecdotal and, in a more general situation, *i.e.*, for systems for which the matrices are not triangular, such exact calculations are not possible anymore. Nevertheless, it is worth to be treated since it allows to easily illustrate formulæ (4.2).

◁ **Stokes multipliers of level $\rho_1 = 1$.** According to calculations above, the direction $\theta = 0$ of maximal decay of exponential $e^{-1/x}$ is the unique anti-Stokes direction of level 1 of $\tilde{f}(x)$ and its corresponding Stokes-Ramis matrix $St_{1,0}$ of level 1 reads as

$$St_{1,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ st_{1,0}^2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Furthermore, using notations of section 4.2, we have $\theta^{[1]} = \theta = 0$, $\Omega_{1,0}^* = \{1\}$ and $Fr_1(1) = \{-1/x\}$. Thereby, $\Omega_{1,0}^*$ has a SG-Configuration, $\omega = 1$ has a good monomial front and, consequently, the connection-to-Stokes formulæ of level 1 allow us to express the Stokes multiplier $st_{1,0}^2$ in terms of the connection constant $K_{1,+}^2$ of $\hat{f}(\xi) = \hat{f}_{0-}(\xi)$ at $\xi = 1$ (recall indeed that, since 1 is the lowest level of $\tilde{f}(x)$, the Borel transform $\hat{f}(\xi)$ defines an analytic function at 0 –see proposition 3.3). More precisely, since matrix L is diagonal, identity (4.4) applies and implies relation

$$\boxed{st_{1,0}^2 = 2i\pi K_{1,+}^2} \quad (4.7)$$

We are left to calculate the connection constant $K_{1,+}^2$. To do that, we proceed as follows: according to relation $(A_H^{[1]})$, the formal series $\tilde{f}(x)$ is solution of system

$$x^2 \frac{df}{dx} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2x^3 & 1 & 0 & 0 \\ -x^2 & 0 & \frac{2}{x} + \frac{x}{2} & 0 \\ x & x & -2x & \frac{6}{x^2} + \frac{x}{4} \end{bmatrix} f.$$

Thereby, its components $\tilde{f}^j(x)$, $j \in \{2, 3, 4\}$, satisfy identities

$$\begin{cases} x^2 \frac{d\tilde{f}^2}{dx} - \tilde{f}^2 = 2x^3 \\ x^2 \frac{d\tilde{f}^3}{dx} - \frac{2}{x} \tilde{f}^3 - \frac{x}{2} \tilde{f}^3 = -x^2 \\ x^2 \frac{d\tilde{f}^4}{dx} - \frac{6}{x^2} \tilde{f}^4 - \frac{x}{4} \tilde{f}^4 = x + x\tilde{f}^2 - 2x\tilde{f}^3 \end{cases}$$

and, consequently, the Borel transforms $\hat{f}^j(\xi)$ are the unique analytic solutions at 0 of the differential system

$$\begin{cases} (\xi - 1)\hat{f}^2 = \xi^2 \\ -2 \frac{d^2 \hat{f}^3}{d\xi^2} + \xi \frac{d\hat{f}^3}{d\xi} + \frac{1}{2} \hat{f}^3 = -1, \hat{f}^3(0) = 0, \frac{d\hat{f}^3}{d\xi}(0) = 0 \\ -6 \frac{d^3 \hat{f}^4}{d\xi^3} + \xi \frac{d\hat{f}^4}{d\xi} + \frac{3}{4} \hat{f}^4 = \hat{f}^2 - 2\hat{f}^3, \hat{f}^4(0) = 0, \frac{d\hat{f}^4}{d\xi}(0) = 0, \frac{d^2 \hat{f}^4}{d\xi^2}(0) = -\frac{1}{6} \end{cases}$$

In particular, $\hat{f}^3(\xi)$ is entire on all \mathbb{C} . Moreover, choosing a determination of the logarithm such that $\ln(\xi) > 0$ for $\xi > 0$ and integrating system above with Lagrange method (variation of constants), we have, for all $|\xi| < 1$,

$$\hat{f}^2(\xi) = -\frac{\xi^2}{1-\xi} \quad \text{and} \quad \hat{f}^4(\xi) = h_1(\xi) + h_2(\xi) \ln(1-\xi)$$

with $h_1(\xi)$ and $h_2(\xi)$ analytic at 0 (in fact, on all \mathbb{C}). Then (see definition 4.10), the connection constant $K_{1,+}^2$ is equal to

$$K_{1,+}^2 = 1$$

and, applying identity (4.7), we finally obtain

$$st_{1,0}^2 = 2i\pi.$$

◁ **Stokes multipliers of level $\rho_2 = 2$.** Let us now turn to the calculation of the Stokes multipliers of level 2. According to the form of $\tilde{Y}(x)$, the anti-Stokes directions of level 2 of $\tilde{f}(x)$ are the two directions $\theta_0 = 0$ and $\theta_1 = -\pi$ of maximal decay of exponential e^{-1/x^2} . Moreover, the corresponding Stokes-Ramis matrices $St_{2;0}$ and $St_{2;-\pi}$ of level 2 read as

$$St_{2;0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ st_{2;0}^3 & * & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad St_{2;-\pi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ st_{2;-\pi}^3 & * & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, since $\theta^{[2]} = 2\theta_0 = 0$, $\Omega_{2;0}^* = \{1\}$ and $Fr_2(1) = \{-1/x^2\}$, the connection-to-Stokes formulæ of level 2 apply (indeed, $\Omega_{2;0}^*$ has a SG-Configuration and the Stokes value $\omega = 1$ has a good monomial front) and tell us that the Stokes multipliers $st_{2;0}^3$ and $st_{2;-\pi}^3$ are expressed in terms of the connection constants $\mathbf{K}_{1,+}^{[2;0]3}$ and $\mathbf{K}_{1,+}^{[2;1]3}$ of $\hat{\mathbf{f}}_{0^-}^{[2;0]}(\tau)$ and $\hat{\mathbf{f}}_{0^-}^{[2;1]}(\tau)$ at $\tau = 1$. More precisely, since matrix L is diagonal, formulæ (4.4) give us

$$\left\{ \begin{array}{l} st_{2;0}^3 = \frac{(1+i)\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)} \mathbf{K}_{1,+}^{[2;0]3} + (-4+4i)\Gamma\left(\frac{3}{4}\right) \mathbf{K}_{1,+}^{[2;1]3} \\ st_{2;-\pi}^3 = \frac{(-1+i)\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)} \mathbf{K}_{1,+}^{[2;0]3} + (4+4i)\Gamma\left(\frac{3}{4}\right) \mathbf{K}_{1,+}^{[2;1]3} \end{array} \right. \quad (4.8)$$

To calculate the connection constants $\mathbf{K}_{1,+}^{[2;0]3}$ and $\mathbf{K}_{1,+}^{[2;1]3}$, we proceed similarly as the previous case $\rho_1 = 1$: using relation $(\mathbf{A}_{\mathbf{H}}^{[2]})$, the formal series $\tilde{\mathbf{f}}^{[2]}(t)$ (see page 34 for notation) is solution of system

$$2t^2 \frac{d\mathbf{f}}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2t^2 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 2 + \frac{t}{2} & 0 & -t^2 & 0 & 0 & 0 \\ t & t & -2t & \frac{t}{4} & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & -t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2t^2 & -t & 0 & 0 \\ -t & 0 & 0 & 0 & 0 & 0 & 2 - \frac{t}{2} & 0 \\ 0 & 0 & 0 & -\frac{6}{t} & t & t & -2t & -\frac{3t}{4} \end{bmatrix} \mathbf{f}.$$

Thereby, since $\tilde{f}^1(x) = 1$ implies $\tilde{\mathbf{f}}^{[2;0]1}(t) = 1$ and $\tilde{\mathbf{f}}^{[2;1]1}(t) = 0$, its components $\tilde{\mathbf{f}}^{[2;u]j}(t)$, $u \in \{0, 1\}$ and $j \in \{2, 3, 4\}$, are uniquely determined by

relations

$$\left\{ \begin{array}{l} 2t^2 \frac{d\tilde{\mathbf{f}}^{[2;0]2}}{dt} = 2t^2 + t\tilde{\mathbf{f}}^{[2;1]2}, \quad 2t^2 \frac{d\tilde{\mathbf{f}}^{[2;1]2}}{dt} + t\tilde{\mathbf{f}}^{[2;1]2} = \tilde{\mathbf{f}}^{[2;0]2} \\ \tilde{\mathbf{f}}^{[2;0]3} = 0, \quad 2t^2 \frac{d\tilde{\mathbf{f}}^{[2;1]3}}{dt} - \left(2 - \frac{t}{2}\right) \tilde{\mathbf{f}}^{[2;1]3} = -t \\ 2t^2 \frac{d\tilde{\mathbf{f}}^{[2;0]4}}{dt} - \frac{t}{4} \tilde{\mathbf{f}}^{[2;0]4} + 6\tilde{\mathbf{f}}^{[2;1]4} = t + t\tilde{\mathbf{f}}^{[2;0]2} \\ 2t^2 \frac{d\tilde{\mathbf{f}}^{[2;1]4}}{dt} + \frac{3t}{4} \tilde{\mathbf{f}}^{[2;1]4} + \frac{6}{t} \tilde{\mathbf{f}}^{[2;0]4} = t\tilde{\mathbf{f}}^{[2;1]2} - 2t\tilde{\mathbf{f}}^{[2;1]3} \end{array} \right.$$

jointly with initial conditions $\tilde{\mathbf{f}}^{[2;u]j}(0) = 0$. In particular,

$$\left\{ \begin{array}{l} \tilde{\mathbf{f}}^{[2;0]2}(t) = -6t^2 + O(t^3), \quad \tilde{\mathbf{f}}^{[2;1]2}(t) = -2t - 24t^2 + O(t^3) \\ \tilde{\mathbf{f}}^{[2;0]3}(t) = 0, \quad \tilde{\mathbf{f}}^{[2;1]3}(t) = \frac{1}{2}t + \frac{5}{8}t^2 + O(t^3) \\ \tilde{\mathbf{f}}^{[2;0]4}(t) = -\frac{1}{6}t + O(t^3), \quad \tilde{\mathbf{f}}^{[2;1]4}(t) = O(t^3) \end{array} \right. .$$

This leads then us, after a Borel transformation, to the following properties.

- The formal Borel transforms $\hat{\mathbf{f}}^{[2;0]2}$ and $\hat{\mathbf{f}}^{[2;1]2}$ satisfy relations

$$\left\{ \begin{array}{l} \hat{\mathbf{f}}^{[2;1]2} = 2 \frac{d}{d\tau} \left(\tau \hat{\mathbf{f}}^{[2;0]2} \right) - 2 \\ 4\tau^2 \frac{d^2 \hat{\mathbf{f}}^{[2;0]2}}{d\tau^2} + (14\tau - 1) \frac{d\hat{\mathbf{f}}^{[2;0]2}}{d\tau} + 6\hat{\mathbf{f}}^{[2;0]2} = 6 \quad (*) \end{array} \right.$$

Therefore, due to the Newton polygon at 0 of (*), $\hat{\mathbf{f}}^{[2;0]2}$ (hence, $\hat{\mathbf{f}}^{[2;1]2}$) is 1-summable in any direction $\theta \neq 0$. In particular, functions $\hat{\mathbf{f}}_{0^-}^{[2;u]2}$'s are given, for instance, by the 1-sums $s_{1; -\frac{\pi}{2}}(\hat{\mathbf{f}}^{[2;u]2})$ in direction $-\frac{\pi}{2}$ (see notation 2.5). Moreover, since 0 is the only singular point of (*), these functions can be analytically continued on all the Riemann surface $\tilde{\mathbb{C}}$ of the logarithm.

- $\hat{\mathbf{f}}^{[2;0]3} = \hat{\mathbf{f}}_{0^-}^{[2;0]3} = 0$ and $\hat{\mathbf{f}}^{[2;1]3}$ defines an analytic function at 0 which is the unique solution of the differential equation

$$2(\tau - 1) \frac{d\hat{\mathbf{f}}^{[2;1]3}}{d\tau} + \frac{5}{2} \hat{\mathbf{f}}^{[2;1]3} = 0 \quad , \quad \hat{\mathbf{f}}^{[2;1]3}(0) = \frac{1}{2}.$$

In particular, we have $\widehat{\mathbf{f}}^{[2;1]3} = \widehat{\mathbf{f}}_{0^-}^{[2;1]3}$ and, for all $|\tau| < 1$,

$$\widehat{\mathbf{f}}^{[2;1]3}(\tau) = \frac{1}{2}(1 - \tau)^{-5/4}.$$

- The function $\varphi := \begin{bmatrix} \widehat{\mathbf{f}}_{0^-}^{[2;0]4} \\ \widehat{\mathbf{f}}_{0^-}^{[2;1]4} \end{bmatrix}$ is an analytic solution at $0 \in \widetilde{\mathbb{C}}$ of the differential system

$$\begin{bmatrix} 6 & 0 \\ 2\tau & 6 \end{bmatrix} \frac{d^2\varphi}{d\tau^2} + \begin{bmatrix} 0 & 2 \\ \frac{15}{4} & 0 \end{bmatrix} \frac{d\varphi}{d\tau} + \begin{bmatrix} 0 & \frac{11}{4} \\ 0 & 0 \end{bmatrix} \varphi = \begin{bmatrix} \widehat{\mathbf{f}}_{0^-}^{[2;1]2} - 2\widehat{\mathbf{f}}_{0^-}^{[2;1]3} \\ \frac{d\widehat{\mathbf{f}}_{0^-}^{[2;0]2}}{d\tau} \end{bmatrix}.$$

In particular, since all the solutions of the homogeneous system are entire on all \mathbb{C} , we have, for all $|\tau| < 1$ and $u \in \{0, 1\}$,

$$\widehat{\mathbf{f}}_{0^-}^{[2;u]4}(\tau) = h_{1,u}(\tau) + h_{2,u}(\tau)(1 - \tau)^{-1/4}$$

with $h_{1,u}(\tau)$ analytic at $0 \in \widetilde{\mathbb{C}}$ (in fact, on all $\widetilde{\mathbb{C}}$) and $h_{2,u}(\tau)$ analytic at $0 \in \mathbb{C}$ (in fact, on all \mathbb{C}).

Hence, applying definition 4.10, the connection constants $\mathbf{K}_{1,+}^{[2;0]3}$ and $\mathbf{K}_{1,+}^{[2;1]3}$ are given by

$$\mathbf{K}_{1,+}^{[2;0]3} = 0, \quad \mathbf{K}_{1,+}^{[2;1]3} = \frac{-\sqrt{2} + i\sqrt{2}}{4}$$

and identities (4.8) then imply

$$st_{2;0}^3 = -2i\sqrt{2}\Gamma\left(\frac{3}{4}\right), \quad st_{2;-\pi}^3 = -2\sqrt{2}\Gamma\left(\frac{3}{4}\right).$$

◁ **Stokes multipliers of level $\rho_3 = 3$.** We are left to calculate the Stokes multipliers of level 3 of $\widetilde{f}(x)$. According to the form of $\widetilde{Y}(x)$, the anti-Stokes directions of level 3 of $\widetilde{f}(x)$ are the directions $\theta_0 = 0$, $\theta_1 = -\frac{2\pi}{3}$ and $\theta_2 = -\frac{4\pi}{3}$ of maximal decay of exponential e^{-2/x^3} and the corresponding Stokes-Ramis matrices read as

$$St_{3;\theta_\ell} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ st_{3;\theta_\ell}^4 & * & * & 1 \end{bmatrix} \quad \text{for } \ell = 0, 1, 2.$$

As before, the calculation of the Stokes multipliers $st_{3;\theta_\ell}^4$'s can be reduced to a calculation of connection constants in the Borel plane by means of connection-to-Stokes formulæ of level 3 (indeed, we have $\theta^{[3]} = 3\theta_0 = 0$, $\Omega_{3;\mathbf{0}}^* = \{2\}$ and $Fr_3(2) = \{-2/x^3\}$; hence, $\Omega_{3;\mathbf{0}}^*$ has once more a SG-Configuration and the Stokes value $\omega = 2$ has a good monomial front). More precisely, applying (4.4) since matrix L is diagonal, we have relations

$$\left\{ \begin{array}{l} st_{3;0}^4 = \frac{2i\pi e^{-\frac{i\pi}{12}}}{\Gamma(\frac{11}{12})} \mathbf{K}_{2,+}^{[3;0]4} + (-4+4i)\Gamma(\frac{3}{4}) \mathbf{K}_{2,+}^{[3;1]4} + \frac{24i\pi e^{\frac{7i\pi}{12}}}{7\Gamma(\frac{7}{12})} \mathbf{K}_{2,+}^{[3;2]4} \\ st_{3;-\frac{2\pi}{3}}^4 = \frac{i\pi(\sqrt{3}+i)e^{-\frac{i\pi}{12}}}{\Gamma(\frac{11}{12})} \mathbf{K}_{2,+}^{[3;0]4} + (4+4i)\Gamma(\frac{3}{4}) \mathbf{K}_{2,+}^{[3;1]4} + \frac{12i\pi(-\sqrt{3}+i)e^{\frac{7i\pi}{12}}}{7\Gamma(\frac{7}{12})} \mathbf{K}_{2,+}^{[3;2]4} \\ st_{3;-\frac{4\pi}{3}}^4 = \frac{i\pi(1+i\sqrt{3})e^{-\frac{i\pi}{12}}}{\Gamma(\frac{11}{12})} \mathbf{K}_{2,+}^{[3;0]4} + (4-4i)\Gamma(\frac{3}{4}) \mathbf{K}_{2,+}^{[3;1]4} + \frac{12i\pi(1-i\sqrt{3})e^{\frac{7i\pi}{12}}}{7\Gamma(\frac{7}{12})} \mathbf{K}_{2,+}^{[3;2]4} \end{array} \right. \quad (4.9)$$

where the $\mathbf{K}_{2,+}^{[3;u]4}$'s denote the connection constants of the $\widehat{\mathbf{f}}_{0^-}^{[3;u]}(\tau)$'s at $\tau = 2$. To evaluate these constants, we proceed in the same way as the two previous cases $\rho_1 = 1$ and $\rho_2 = 2$: using relation $(\mathbf{A}_{\mathbf{H}}^{[3]})$ page 34, the formal series $\widetilde{\mathbf{f}}^{[3]}(t)$ is solution of system

$$3t^2 \frac{d\mathbf{f}}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2t^2 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{t}{2} & 0 & 0 & 0 & 0 & 0 & -t^2 & 0 & 2t & 0 \\ t & t & -2t & 6 + \frac{t}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t & 0 & 0 & 2t^2 & t & 0 & 0 \\ -t & 0 & 2 & 0 & 0 & 0 & -\frac{t}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & t & -2t & 6 - \frac{3t}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2t & 0 & 0 & 0 \\ 2t & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2t & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 & 2 & 0 & 0 & 0 & -\frac{3t}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & -2t & 6 - \frac{7t}{4} \end{bmatrix} \mathbf{f}.$$

Thereby, since $\widetilde{f}^1(x) = 1$ implies $\widetilde{\mathbf{f}}^{[3;0]1}(t) = 1$ and $\widetilde{\mathbf{f}}^{[3;1]1}(t) = \widetilde{\mathbf{f}}^{[3;2]1}(t) = 0$, its components $\widetilde{\mathbf{f}}^{[3;u]j}(t)$, $u \in \{0, 1, 2\}$ and $j \in \{2, 3, 4\}$, are uniquely

determined by relations

$$\left\{ \begin{array}{l} 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;0]2}}{dt} = t\tilde{\mathbf{f}}^{[3;1]2}, \quad 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;1]2}}{dt} + t\tilde{\mathbf{f}}^{[3;1]2} = t\tilde{\mathbf{f}}^{[3;2]2} \\ 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;2]2}}{dt} + 2t\tilde{\mathbf{f}}^{[3;2]2} = 2t + \tilde{\mathbf{f}}^{[3;0]2} \\ 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;0]3}}{dt} - \frac{t}{2}\tilde{\mathbf{f}}^{[3;0]3} = 2t\tilde{\mathbf{f}}^{[3;2]3}, \quad 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;1]3}}{dt} + \frac{t}{2}\tilde{\mathbf{f}}^{[3;1]3} = -t + 2\tilde{\mathbf{f}}^{[3;0]3} \\ 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;2]3}}{dt} + \frac{3t}{2}\tilde{\mathbf{f}}^{[3;2]3} = 2\tilde{\mathbf{f}}^{[3;1]3} \\ 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;0]4}}{dt} - \left(6 + \frac{t}{4}\right)\tilde{\mathbf{f}}^{[3;0]4} = t + t\tilde{\mathbf{f}}^{[3;0]2} - 2t\tilde{\mathbf{f}}^{[3;0]3} \\ 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;1]4}}{dt} - \left(6 - \frac{3t}{4}\right)\tilde{\mathbf{f}}^{[3;1]4} = t\tilde{\mathbf{f}}^{[3;1]2} - 2t\tilde{\mathbf{f}}^{[3;1]3} \\ 3t^2 \frac{d\tilde{\mathbf{f}}^{[3;2]4}}{dt} - \left(6 - \frac{7t}{4}\right)\tilde{\mathbf{f}}^{[3;2]4} = t\tilde{\mathbf{f}}^{[3;2]2} - 2t\tilde{\mathbf{f}}^{[3;2]3} \end{array} \right. \quad (4.10)$$

jointly with initial conditions $\tilde{\mathbf{f}}^{[2;u]j}(0) = 0$. In particular,

$$\left\{ \begin{array}{l} \tilde{\mathbf{f}}^{[3;0]2}(t) = -2t + O(t^2), \quad \tilde{\mathbf{f}}^{[3;1]2}(t) = -6t + O(t^2), \quad \tilde{\mathbf{f}}^{[3;2]2}(t) = -24t + O(t^2) \\ \tilde{\mathbf{f}}^{[3;0]3}(t) = \frac{1}{2}t + O(t^2), \quad \tilde{\mathbf{f}}^{[3;1]3}(t) = O(t^2), \quad \tilde{\mathbf{f}}^{[3;2]3}(t) = \frac{5}{8}t + O(t^2) \\ \tilde{\mathbf{f}}^{[3;0]4}(t) = -\frac{1}{6}t + O(t^2), \quad \tilde{\mathbf{f}}^{[3;1]4}(t) = O(t^2), \quad \tilde{\mathbf{f}}^{[3;2]4}(t) = O(t^2) \end{array} \right. .$$

As before, a Borel transformation of relations (4.10) above provides us some properties about the formal Borel transforms $\hat{\mathbf{f}}^{[3;u]j}$, hence about the functions $\hat{\mathbf{f}}_{0^-}^{[3;u]}$. More precisely, we have the following considerations.

- According to the first six identities of (4.10), one easily checks that the $\hat{\mathbf{f}}^{[3;u]j}$'s with $u \in \{0, 1, 2\}$ and $j \in \{2, 3\}$ satisfy relations

$$\left\{ \begin{array}{l} \hat{\mathbf{f}}^{[3;1]2} = 3\tau \frac{d\hat{\mathbf{f}}^{[3;0]2}}{d\tau} + 3\hat{\mathbf{f}}^{[3;0]2}, \quad \hat{\mathbf{f}}^{[3;2]2} = 9\tau^2 \frac{d^2\hat{\mathbf{f}}^{[3;0]2}}{d\tau^2} + 30\tau \frac{d\hat{\mathbf{f}}^{[3;0]2}}{d\tau} + 12\hat{\mathbf{f}}^{[3;0]2} \\ 27\tau^3 \frac{d^3\hat{\mathbf{f}}^{[3;0]2}}{d\tau^3} + 189\tau^2 \frac{d^2\hat{\mathbf{f}}^{[3;0]2}}{d\tau^2} + (276\tau - 1) \frac{d\hat{\mathbf{f}}^{[3;0]2}}{d\tau} + 60\hat{\mathbf{f}}^{[3;0]2} = 0 \quad (*) \\ \hat{\mathbf{f}}^{[3;2]3} = \frac{3\tau}{2} \frac{d\hat{\mathbf{f}}^{[3;0]3}}{d\tau} + \frac{5}{4}\hat{\mathbf{f}}^{[3;0]3}, \quad \hat{\mathbf{f}}^{[3;1]3} = \frac{9\tau^2}{4} \frac{d\hat{\mathbf{f}}^{[3;0]3}}{d\tau} + 3\tau\hat{\mathbf{f}}^{[3;0]3} - \frac{3}{16} * \hat{\mathbf{f}}^{[3;0]3} \\ 216\tau^3 \frac{d^3\hat{\mathbf{f}}^{[3;0]3}}{d\tau^3} + (1620\tau^2 - 64) \frac{d^2\hat{\mathbf{f}}^{[3;0]3}}{d\tau^2} + 2550\tau \frac{d\hat{\mathbf{f}}^{[3;0]3}}{d\tau} + 585\hat{\mathbf{f}}^{[3;0]3} = 0 \quad (**) \end{array} \right.$$

Then, due to the Newton polygons at 0 of (*) and (**), the $\widehat{\mathbf{f}}^{[3;u]2}$'s (resp. $\widehat{\mathbf{f}}^{[3;u]3}$'s) is $\frac{1}{2}$ -summable in any direction $\theta \neq 0$ (resp. 2-summable in any direction $\theta \notin \{-\pi, 0\}$). In particular, functions $\widehat{\mathbf{f}}_{0^-}^{[3;u]2}$'s and $\widehat{\mathbf{f}}_{0^-}^{[3;u]3}$'s are respectively given, for instance, by the sums $s_{\frac{1}{2};-\frac{\pi}{4}}(\widehat{\mathbf{f}}^{[3;u]2})$ and $s_{2;-\frac{\pi}{4}}(\widehat{\mathbf{f}}^{[3;u]3})$ in direction $-\frac{\pi}{4}$ (see notation 2.5). Besides, since 0 is the only singular point of (*) and (**), these functions can be analytically continued on all the Riemann surface $\widetilde{\mathbb{C}}$ of the logarithm.

- Let us now consider the last three identities of (4.10). Then, according to calculations just above, the functions $\widehat{\mathbf{f}}_{0^-}^{[3;u]4}$ are uniquely determined by the differential equations

$$\left\{ \begin{array}{l} 3(\tau - 2) \frac{d\widehat{\mathbf{f}}_{0^-}^{[3;0]4}}{d\tau} + \frac{11}{4} \widehat{\mathbf{f}}_{0^-}^{[3;0]4} = g_0 \quad , \quad \widehat{\mathbf{f}}_{0^-}^{[3;0]4}(0) = -\frac{1}{6} \\ 3(\tau - 2) \frac{d\widehat{\mathbf{f}}_{0^-}^{[3;1]4}}{d\tau} + \frac{15}{4} \widehat{\mathbf{f}}_{0^-}^{[3;1]4} = g_1 \quad , \quad \widehat{\mathbf{f}}_{0^-}^{[3;1]4}(0) = 0 \\ 3(\tau - 2) \frac{d\widehat{\mathbf{f}}_{0^-}^{[3;2]4}}{d\tau} + \frac{17}{4} \widehat{\mathbf{f}}_{0^-}^{[3;2]4} = g_2 \quad , \quad \widehat{\mathbf{f}}_{0^-}^{[3;2]4}(0) = 0 \end{array} \right.$$

(recall indeed that the $\widehat{\mathbf{f}}_{0^-}^{[3;u]4}$'s are continuous at 0 with $\widehat{\mathbf{f}}_{0^-}^{[3;u]4}(0) = \widehat{\mathbf{f}}^{[3;u]4}(0)$), where the g_u 's denote, for all $u \in \{0, 1, 2\}$, the functions $g_u = \widehat{\mathbf{f}}_{0^-}^{[3;u]2} - 2\widehat{\mathbf{f}}_{0^-}^{[3;u]3}$ which are integrable at 0. In particular, Lagrange method (= variation of constants) tells us that, for all $|\tau| < 2$,

$$\left\{ \begin{array}{l} \widehat{\mathbf{f}}_{0^-}^{[3;0]4}(\tau) = -\frac{2^{-\frac{1}{12}}}{3} (2 - \tau)^{-\frac{11}{12}} - \frac{1}{3} (2 - \tau)^{-\frac{11}{12}} \int_0^\tau (2 - \eta)^{-\frac{1}{12}} g_0(\eta) d\eta \\ \widehat{\mathbf{f}}_{0^-}^{[3;1]4}(\tau) = -\frac{1}{3} (2 - \tau)^{-\frac{5}{4}} \int_0^\tau (2 - \eta)^{\frac{1}{4}} g_1(\eta) d\eta \\ \widehat{\mathbf{f}}_{0^-}^{[3;2]4}(\tau) = -\frac{1}{3} (2 - \tau)^{-\frac{19}{12}} \int_0^\tau (2 - \eta)^{\frac{7}{12}} g_2(\eta) d\eta \end{array} \right.$$

where the integrals can be written on the form

$$\left\{ \begin{array}{l} \int_0^\tau (2-\eta)^{-\frac{1}{12}} g_0(\eta) d\eta = \alpha_0 + (2-\tau)^{\frac{11}{12}} h_0(\tau) \\ \int_0^\tau (2-\eta)^{\frac{1}{4}} g_1(\eta) d\eta = \alpha_1 + (2-\tau)^{\frac{5}{4}} h_1(\tau) \\ \int_0^\tau (2-\eta)^{\frac{7}{12}} g_2(\eta) d\eta = \alpha_2 + (2-\tau)^{\frac{19}{12}} h_2(\tau) \end{array} \right.$$

with $\alpha_u \in \mathbb{C}$ and $h_u(\tau)$ analytic on the disc $D(2, 2)$ with center 2 and radius 2 for all u .

Then, applying definition 4.10, the connection constants $\mathbf{K}_{2,+}^{[3;u]4}$ are given by

$$\mathbf{K}_{2,+}^{[3;0]4} = -\frac{\alpha_0 + 2^{-\frac{1}{12}}}{3} e^{-\frac{11i\pi}{12}}, \quad \mathbf{K}_{2,+}^{[3;1]4} = -\frac{\alpha_1}{3} e^{-\frac{5i\pi}{4}}, \quad \mathbf{K}_{2,+}^{[3;2]4} = -\frac{\alpha_2}{3} e^{-\frac{19i\pi}{12}}$$

and, consequently, identities (4.9) give us

$$\left\{ \begin{array}{l} st_{3,0}^4 = i\pi \left(\frac{2^{\frac{11}{12}}}{3\Gamma(\frac{11}{12})} + \frac{2\alpha_0}{3\Gamma(\frac{11}{12})} + \frac{4\sqrt{2}\Gamma(\frac{3}{4})\alpha_1}{3\pi} + \frac{8\alpha_2}{7\Gamma(\frac{7}{12})} \right) \\ st_{3,-\frac{2\pi}{3}}^4 = \pi \left(\frac{(i\sqrt{3}-1)2^{\frac{11}{12}}}{6\Gamma(\frac{11}{12})} + \frac{(i\sqrt{3}-1)\alpha_0}{3\Gamma(\frac{11}{12})} + \frac{4\sqrt{2}\Gamma(\frac{3}{4})\alpha_1}{3\pi} - \frac{4(i\sqrt{3}+1)\alpha_2}{7\Gamma(\frac{7}{12})} \right) \\ st_{3,-\frac{4\pi}{3}}^4 = \pi \left(\frac{(i-\sqrt{3})2^{\frac{11}{12}}}{6\Gamma(\frac{11}{12})} + \frac{(i-\sqrt{3})\alpha_0}{3\Gamma(\frac{11}{12})} - \frac{4i\sqrt{2}\Gamma(\frac{3}{4})\alpha_1}{3\pi} + \frac{4(i+\sqrt{3})\alpha_2}{7\Gamma(\frac{7}{12})} \right) \end{array} \right.$$

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