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| Daniel Barlet. Strongly quasi-proper maps and the f-flattning theorem. 2015. hal-01139915

HAL Id: hal-01139915

<https://hal.science/hal-01139915>

Preprint submitted on 7 Apr 2015

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Strongly quasi-proper maps and the f-flattning theorem.

Daniel Barlet*

07/04/15

ABSTRACT. We complete and precise the results of [B.13] and we prove a strong version of the semi-proper direct image theorem with values in the space $\mathcal{C}_n^f(M)$ of finite type closed n -cycles in a complex space M . We describe the strongly quasi-proper maps as the class of holomorphic surjective maps which admit a meromorphic family of fibers and we prove stability properties of this class. In the Appendix we give a direct and short proof of D. Mathieu's flattning theorem (see [M.00]) for a strongly quasi-proper map which is easier and more accessible.

AMS CLASSIFICATION 2010. 32-H-02, 32-H-04, 32-C-25, 32-H-quasi-35, 32-15, 32-K-05, 32-L-05.

KEY WORDS. Finite Type Cycles, Geometrically f-Flat map, Strongly Quasi-Proper map, Stein Factorization, Semi-Proper Direct Image.

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1 Introduction.

It is now a classical result that there exists a geometric flattening theorem for a proper surjective holomorphic map between irreducible complex spaces. This is a rather simple consequence of the existence of the cycle's space of a complex space (see [B.75] and [B-M] ch.IV section 9). This fact can be seen as the meromorphic along the subset of "big fibers" of the "fiber map" for such a morphism. Of course this is a geometric version of the deep flattening theorem of H. Hironaka (see [H.75]). We consider an analogous statement for a large class of surjective morphisms between irreducible complex spaces with non compact fibers: the strongly quasi-proper maps. The main problem here is the fact that the standard notion of a quasi-proper map is not strong enough for our purpose : the strict transform of a quasi-proper map by a (proper) modification of the target space is no longer quasi-proper in general (see the example following the proposition 3.2.7). This comes from the fact that the quasi-properness is not sufficient to control the limits of generic fibers near a " big fiber". This motivate to work with the notion of strongly proper maps introduced in [M.00], [B.08] in an implicit way and in [B.13] explicitly.

We complete and precise the study of the class "strongly quasi-proper maps" introduced in *loc. cit.* which enjoys several interesting stability properties and which is an useful tool to prove the existence of meromorphic quotients in the category of reduced complex spaces (see [B.13]).

The main results obtained here are :

1. A stronger criterium for proving the SQP property and the fact that this property is equivalent to the existence of a meromorphic fiber map, see the proposition 3.2.2 and the theorem 4.1.1.

2. A more precise version of a the semi-proper direct image with value in the space $\mathcal{C}_n^f(M)$, see the theorem 2.3.2 which uses a new analytic continuation result given in the theorem 2.2.1.
3. The strong stability theorem for SQP maps, see the theorem 3.2.12.

2 Analytic structure on $\mathcal{C}_n^f(M)$.

2.1 Definitions and examples of analytic subsets.

We suppose that the notion of analytic family of n -cycles is known and also the definition of the topology of the space $\mathcal{C}_n^{loc}(M)$ for a complex space M (see [B.M] ch. IV section 2).

Recall that a n -cycle has finite type if it has only finitely many irreducible components. The subset of finite type n -cycles in M is denoted $\mathcal{C}_n^f(M)$. We define the topology on it as follows :

For $W := (W_1, \dots, W_m)$ a finite set of relatively compact open sets in M let $\Omega(W)$ the subset of n -cycles C in M such that each irreducible component of $|C|$ meets each W_i for $i = 1, \dots, m$. Remark that finite intersection of some $\Omega(W)$ is again of the form $\Omega(W)$. Now define open sets in $\mathcal{C}_n^f(M)$ as union of subsets of the form $\mathcal{U} \cap \Omega(W)$ where \mathcal{U} is an open set in $\mathcal{C}_n^{loc}(M)$.

So the inclusion $\mathcal{C}_n^f(M) \rightarrow \mathcal{C}_n^{loc}(M)$ is continuous, but it is not a homeomorphism on its image (with the induced topology). For instance, it is easy to see that a continuous family of n -cycles $(X_s)_{s \in S}$ in M parametrized by a Hausdorff topological space S , such that each cycle is of finite type is f-continuous (so corresponds to a continuous map $S \rightarrow \mathcal{C}_n^f(M)$ with the topology defined above) if and only if its set-theoretic graph $|G| \subset S \times M$ is quasi-proper over S ; here we use the following definition of quasi-proper, valid as long as we know that the fibers of the continuous map $\pi : G \rightarrow S$ are complex analytic subsets (here $G \subset S \times M$ is closed and the fibers are cycles in the complex space M) :

For any point $s_0 \in S$ there exists a neighborhood S_0 of s_0 in S and a closed S_0 -proper subset K in $\pi^{-1}(S_0)$ such that for any $s \in S_0$ any irreducible component of the fiber $\pi^{-1}(s)$ meets K .

Lemma 2.1.1 *For any complex space M and any integer n the topology of the space $\mathcal{C}_n^f(M)$ has a countable basis¹.*

So this space is metrizable and then any point has a countable basis of closed neighborhoods.

PROOF. This an easy consequence of the analogous result for the topology of $\mathcal{C}_n^{loc}(M)$ which is proved in [B-M] IV section 2.4:

Let D be a countable dense subset in M and consider also a countable covering of M

¹By definition a complex space is countable at infinity.

by domains of charts $(U_p, j_p)_{p \in \mathbb{N}}$. Now for each point $d \in D$ let $p(d)$ be the smallest integer p such that $d \in U_p$. Define for $r \in \mathbb{Q}^{+*}$ small enough the pull-back $P_{d,r}$ in $U_{p(d)}$ of the polydisc with center $j_{p(d)}(d)$ and radius r . Then take a countable basis $\mathcal{U}_\nu, \nu \in \mathbb{N}$, for the topology of $\mathcal{C}_n^{\text{loc}}(M)$ and consider the family $(\Omega(W_j) \cap \mathcal{U}_\nu)_{j,\nu}$ where the countable family of relatively compact open sets $W_j =: (W_{j_1} \dots W_{j_\mu})$ runs over the finite subsets of the countable family of the $P_{d,r} \subset\subset M$, gives a countable basis for the topology of $\mathcal{C}_n^f(M)$. \blacksquare

Proposition 2.1.2 *Let M be a reduced complex space and consider a sequence of finite type n -cycles $(C_m)_{m \geq 0}$ in M with the following properties :*

- i) there exists a compact set K in M such that each irreducible component of each C_m meets K .*
- ii) The sequence $(C_m)_{m \geq 0}$ converges for the topology of $\mathcal{C}_n^{\text{loc}}(M)$ to a n -cycle C .*
- iii) The cycle C is in $\mathcal{C}_n^f(M)$.*

Then the sequence $(C_m)_{m \geq 0}$ converges to C in the topology of $\mathcal{C}_n^f(M)$.

PROOF. We begin by proving the case where each C_m is irreducible. Note first that C is not the empty n -cycle because, up to pass to a subsequence, we may assume that for each $m \geq 0$ we can choose a point in $x_m \in C_m \cap K$ in order that the sequence $(x_m)_{m \geq 0}$ converges to a point $x \in K$. Then we have $x \in |C|$.

Let $W := (W_1, \dots, W_\mu)$ be a finite collection of relatively compact open sets open M such that each irreducible component of $|C|$ meets each W_i for $i = 1, \dots, \mu$. Choose also $F_j, j \in J$, a finite number of n -scales on M adapted to C and put $l_j := \deg_{F_j}(C)$. Define the open set of $\mathcal{C}_n^f(M)$ containing C

$$\mathcal{V} := \Omega(W_1, \dots, W_\mu) \cap_{j \in J} \Omega_{l_j}(F_j).$$

Choose also for each $i \in [1, \mu]$ a n -scales $E_i := (U_i, B_i, j_i)$ on W_i adapted to C and such that $\deg_{E_i}(C) = k_i \geq 1$. This is possible because C has at least one irreducible component and such an irreducible component meets each W_i . So for $m \geq m_0$ the scales F_j and E_i will be adapted to C_m and we shall have for all $j \in J$ and all $i \in [1, \mu]$

$$l_j = \deg_{F_j}(C) = \deg_{F_j}(C_m) \quad \text{and} \quad \deg_{E_i}(C_m) = \deg_{E_i}(C) = k_i \geq 1.$$

This shows that for $m \geq m_0$ the unique irreducible component of C_m meets each W_i . So, for $m \geq m_0$ each C_m for $m \geq m_0$ is in the given open set \mathcal{V} of $\mathcal{C}_n^f(M)$. So the sequence $(C_m)_{m \geq 0}$ converges to C in the sense of the topology of $\mathcal{C}_n^f(M)$. This is enough to conclude the proof in this case.

Consider now the general case. We shall prove first that it is enough to prove the following claim :

CLAIM. For any relatively compact open set W meeting each irreducible component of C , there exists m_1 such for $m \geq m_1$ any irreducible component of any C_m meets W .

THE CLAIM IS ENOUGH. Consider as before an open set

$$\mathcal{V} := \Omega(W_1, \dots, W_\mu) \cap_{j \in J} \Omega_{l_j}(F_j)$$

in $\mathcal{C}_n^f(M)$ containing C . If the claim is true we find an integer m_2 such that for each $m \geq m_2$ we have $C_m \in \Omega(W_1, \dots, W_\mu)$. Now the convergence of the sequence for the topology of $\mathcal{C}_n^{loc}(M)$ gives an integer m_3 such that for each $m \geq m_3$ we have $C_m \in \cap_{j \in J} \Omega_{l_j}(F_j)$. So the convergence for the topology of $\mathcal{C}_n^f(M)$ is proved.

PROOF OF THE CLAIM. Take any open set $W \subset M$ such that any irreducible component of C meets W . If there is infinitely many $m \geq 0$ for which an irreducible component Γ_m of C_m does not meet W then extract a sub-sequence (Γ_p) of irreducible components in this sequence which converges in the topology of $\mathcal{C}_n^{loc}(M)$ to a **non empty**² cycle $\Gamma \leq C$. But any irreducible component of Γ has to meet W giving a contradiction as $\Gamma_p \cap W = \emptyset$ by construction. So the claim is proved and the proof is complete. \blacksquare

Corollaire 2.1.3 *Let M be a reduced complex space and let $A \subset \mathcal{C}_n^f(M) \setminus \{\emptyset\}$ be a compact subset in $\mathcal{C}_n^{loc}(M)$. Assume that the following condition is fulfilled :*

- *There exists a compact set $K \subset M$ such for any irreducible component Γ of each $C \in A$ the intersection $\Gamma \cap K$ is not empty.* (Ⓐ)

Then A is compact in $\mathcal{C}_n^f(M)$.

Note that conversely any compact subset in $\mathcal{C}_n^f(M) \setminus \{\emptyset\}$ is compact in $\mathcal{C}_n^{loc}(M)$ and satisfies (Ⓐ) : choose for each $C \in A$ an open set $\mathcal{U}_C \cap \Omega(W_C)$ containing C . As A is compact we can extract a finite sub-covering given by C_1, \dots, C_N and take $K := \cup_{j=1}^N W_{C_j}$.

PROOF. Note that any limit of a sequence in A for the topology of $\mathcal{C}_n^{loc}(M)$ is not the empty cycle, as it contains a point in K . Now, by assumption, for any sequence in A we may find a sub-sequence converging for the topology of $\mathcal{C}_n^{loc}(M)$ to a cycle in $\mathcal{C}_n^f(M)$. So, applying the proposition 2.1.2, we conclude that such a sub-sequence converges for the topology of $\mathcal{C}_n^f(M)$. \blacksquare

Definition 2.1.4 *Let S be a reduced complex space. A map $\varphi : S \rightarrow \mathcal{C}_n^f(M)$ will be called **holomorphic** when it classifies a f -analytic family of n -cycles in M .*

²remember that each Γ_m meets the compact set K .

Recall that an analytic family of n -cycles in M is f -analytic if and only if its set theoretic graph $|G| \subset S \times M$ is **quasi-proper** on S . Of course this condition implies that each cycle in the family is a finite type cycle. But the quasi-properness of the graph asks more : on any compact set L in S we can find a compact set K in M such any irreducible component of any cycle parametrized by a point $s \in L$ has to meet K .

Definition 2.1.5 *Let \mathcal{U} be an open set in $\mathcal{C}_n^f(M)$. A continuous map $f : \mathcal{U} \rightarrow T$ to a Banach analytic set T is called **holomorphic** if and only if for any reduced complex space S and any holomorphic map $\varphi : S \rightarrow \mathcal{U}$ the composed map*

$$f \circ \varphi : S \rightarrow T$$

*is holomorphic. We shall say that a closed subset \mathcal{X} in an open set \mathcal{U} of $\mathcal{C}_n^f(M)$ is **analytic** when there exists locally on \mathcal{U} an holomorphic map f with values in a Banach space E such that $\mathcal{X} = f^{-1}(0)$.*

*A continuous map of an analytic subset \mathcal{X} of \mathcal{U} to a Banach space E will be called **holomorphic** if it is locally induced on \mathcal{X} by a holomorphic map of $\mathcal{C}_n^f(M)$ with values in E .*

The next lemma gives a first example of a closed analytic subset in $\mathcal{C}_n^f(M)$.

Lemma 2.1.6 *Let $NR := \{C \in \mathcal{C}_n^f(M) \mid C \neq |C|\}$ be the subset of non reduced cycles. Then NR is a closed analytic subset in $\mathcal{C}_n^f(M)$.*

PROOF. As the empty n -cycle is open and closed in $\mathcal{C}_n^f(M)$ it is enough to consider non empty cycles. Let C_0 be any non empty cycle in $\mathcal{C}_n^f(M)$. Choose for each irreducible component Γ_i of C_0 an n -scale $E_i := (U_i, B_i, j_i)$ on M adapted to C_0 such that the degree of $|C_0|$ and Γ_i in E_i are equal to 1, and note $k_i := \deg_{E_i}(C_0)$. Remark that C_0 is reduced if and only if we have $k_i = 1$ for each $i \in I$. Let $W := \cup_{i \in I} j_i^{-1}(U_i \times B_i)$ and $\mathcal{V} := \Omega(W) \cap (\cap_{i \in I} \Omega_{k_i}(E_i))$. Then a cycle $C \in \mathcal{V}$ is not reduced if and only if there exists at least one $i \in I$ such that $C \cap j_i^{-1}(U_i \times B_i)$ is not reduced. As each map $\mathcal{V} \rightarrow H(\bar{U}_i, \text{Sym}^{k_i}(B_i))$ is holomorphic, the proof is consequence of the following claim :

CLAIM.

- The subset of $H(\bar{U}, \text{Sym}^k(B))$ corresponding to non reduced cycles in $U \times B$ is a closed analytic subset.

Consider the discriminant map $\Delta_0 : \text{Sym}^k(\mathbb{C}^p) \rightarrow S^{k \cdot (k-1)}(\mathbb{C}^p)$ defined by

$$(x_1, \dots, x_k) \mapsto \prod_{1 \leq i < j \leq k} (x_i - x_j)^2.$$

It is induced by a polynomial map $\oplus_{i=1}^k S^i(\mathbb{C}^p) \rightarrow S^{k \cdot (k-1)}(\mathbb{C}^p)$ thanks to the symmetric function theorem (for this vector case see [B-M] theorem II 4.2.7), and so

we have a holomorphic map $\Delta : H(\bar{U}, \text{Sym}^k(B)) \rightarrow H(\bar{U}, S^{k \cdot (k-1)}(\mathbb{C}^p))$ given by $f \mapsto \Delta_0 \circ f$. Of course, if $f \in H(\bar{U}, \text{Sym}^k(B))$ defines a non reduced cycle in $U \times B$ we have $\Delta_0 \circ f = 0$ in $H(\bar{U}, S^{k \cdot (k-1)}(\mathbb{C}^p))$. Conversely, if $\Delta_0 \circ f = 0$ in $H(\bar{U}, S^{k \cdot (k-1)}(\mathbb{C}^p))$, then choose a point t_0 which is not a ramification point for the reduced cycle which is the support of the cycle X associated to f . Then locally $|X|$ is the disjoint union of $h \leq k$ graphs of holomorphic functions $f_1, \dots, f_h : V(t_0) \rightarrow B$ which are two by two disjoint. If X is reduced, we have $h = k$ and on the open neighborhood $V(t_0)$ we have

$$\Delta_0 \circ f = \prod_{1 \leq i < j \leq k} (f_i - f_j)^2$$

which nowhere vanishes on $V(t_0)$. Contradiction. So the claim is proved. \blacksquare

The second interesting example of an analytic subset of $\mathcal{C}_n^f(M)$ and of holomorphic map on it is given by the following proposition.

Proposition 2.1.7 *Let $\pi : M \rightarrow S$ be a holomorphic map between two reduced complex spaces. Define $\mathcal{C}_n^f(\pi)$ as the subset of $\mathcal{C}_n^f(M)$ of the n -cycles contained in a fiber of π . Then $\mathcal{C}_n^f(\pi)$ is an analytic (closed) subset in $\mathcal{C}_n^f(M)$ and the obvious map $p : \mathcal{C}_n^f(\pi) \rightarrow S$ is holomorphic.*

PROOF. First we shall show that the complement of $\mathcal{C}_n^f(\pi)$ is open: take $C_0 \notin \mathcal{C}_n^f(\pi)$. Then it contains two points x and y such that $\pi(x) \neq \pi(y)$. Take two scales $E := (U, B, j)$ and $E' := (U', B', j')$ adapted to C_0 such that x is in the center $j^{-1}(U \times B)$ of E and y in the center $(j')^{-1}(U' \times B')$ of E' , small enough such that $\pi(j^{-1}(\bar{U} \times \bar{B}))$ and $\pi((j')^{-1}(\bar{U}' \times \bar{B}'))$ are disjoint. Note that the degrees of C_0 in E and in E' are positive. Now a cycle C such that E and E' are adapted to C with the same degrees as C_0 in these scales cannot be in $\mathcal{C}_n^f(\pi)$ because it has two points with different images by π . This defines an open set in the complement of $\mathcal{C}_n^f(\pi)$ containing C_0 .

To obtain a local holomorphic equation for $\mathcal{C}_n^f(\pi)$, recall the following facts :

- i) For any n -scale $E := (U, B, j)$ on M let $\Omega_k(E)$ be the open set in $\mathcal{C}_n^{\text{loc}}(M)$ of cycles for which E is adapted and the degree in E is k . Then the map

$$r_{E,k} : \Omega_k(E) \rightarrow H(\bar{U}, \text{Sym}^k(B))$$

is holomorphic. This is a consequence of the definition of an analytic family of cycles !

- ii) If for a given cycle C_0 we have adapted scales E_1, \dots, E_m such that any irreducible component of C_0 meets the union W of the centers of the $E_i, i \in [1, m]$; then the subset of $[\bigcap_{i \in [1, m]} \Omega_{k_i}(E_i)] \cap \Omega(W)$, where $k_i := \text{deg}_{E_i}(C_0)$, is an open set in $\mathcal{C}_n^f(M)$ and the holomorphic map $\prod_{i \in [1, m]} r_{E_i, k_i}$ is injective on this open set.

Then the following lemma allows to conclude. ■

Lemma 2.1.8 *Let U and B be relatively compact polydiscs respectively in \mathbb{C}^n and \mathbb{C}^p , and let $\pi : W \rightarrow F$ be a holomorphic map of an open neighborhood W of $\bar{U} \times \bar{B}$ to a Banach space F . Then the subset \mathcal{X} of $H(\bar{U}, \text{Sym}^k(B))$ of multiform graphs contained in a fiber of π is a closed Banach analytic subset of $H(\bar{U}, \text{Sym}^k(B))$. Moreover, the map $\varphi : \mathcal{X} \rightarrow F$ defined by sending $X \in \mathcal{X}$ to the point $\pi(X) \in F$ such that $|X| \subset \pi^{-1}(\pi(X))$ is holomorphic.*

PROOF. Consider for each $h \in [1, k]$ the holomorphic map

$$N_h(\pi) : \text{Sym}^k(W) \rightarrow S^h(F)$$

given by the h -th Newton symmetric function $(z_1, \dots, z_k) \mapsto \sum_{j=1}^k \pi(z_j)^h$, where $S^h(F)$ is the h -th symmetric power of F^3 . These maps are holomorphic and for $f \in H(\bar{U}, \text{Sym}^k(B))$ we can compose the associated map $\tilde{f} \in H(\bar{U}, \text{Sym}^k(W))$, sending t to $((t, x_1), \dots, (t, x_k))$ if $f(t) := (x_1, \dots, x_k)$, with $\bigoplus_{h=1}^k N_h(\pi)$ to obtain a holomorphic map

$$\Phi : H(\bar{U}, \text{Sym}^k(B)) \longrightarrow H(\bar{U}, \bigoplus_{h=1}^k S^h(F)).$$

Now fix a point $t_0 \in U$ and a non empty open polydisc $U' \subset\subset U$ and consider the holomorphic maps

$$\Psi : H(\bar{U}, \bigoplus_{h=1}^k S^h(F)) \rightarrow H(\bar{U}', L(\mathbb{C}^n, \bigoplus_{h=1}^k S^h(F)))$$

and

$$\chi : H(\bar{U}, \bigoplus_{h=1}^k S^h(F)) \rightarrow \bigoplus_{h=2}^k S^h(F)$$

defined as follows : $\Psi(f)$ is given by the derivative of f on \bar{U}' and $\chi(f)$ is given by the collection of the $k^{h-1} \cdot N_h(f(t_0)) - (N_1(f(t_0)))^h \in S^h(F)$ for $h \in [2, k]$. These maps are holomorphic and the Banach analytic subset $Z := \Psi^{-1}(0) \cap \chi^{-1}(0)$ is the subset in $H(\bar{U}, \bigoplus_{h=1}^k S^h(F))$ corresponding to constant maps $\bar{U} \rightarrow \bigoplus_{j=1}^k S^h(F)$ such that the value is of the form $k \cdot a \oplus k \cdot a^2 \oplus \dots \oplus k \cdot a^k$ for some $a \in F$. So $\Phi^{-1}(Z)$ is exactly the subset \mathcal{X} of $H(\bar{U}, \text{Sym}^k(B))$.

To conclude the proof, it is enough to remark that the holomorphic map $\frac{1}{k} \cdot (ev_1 \circ \Phi)$ induces on \mathcal{X} the desired map, where $ev_1 : H(\bar{U}, \bigoplus_{h=1}^k S^h(F)) \rightarrow F$ is given by evaluation at t_0 of the component on $S^1(F) = F$. ■

Note that an obvious consequence of the previous proposition is the fact that if N is a closed analytic subset in M , then $\mathcal{C}_n^f(N)$ is a closed analytic subset of $\mathcal{C}_n^f(M)$.

An useful variant of this result is given by the following lemma.

³That is to say the Banach space generated by $x^h, x \in F$, in the Banach space of continuous homogeneous polynomials of degree h on the dual F' of F .

Lemma 2.1.9 *Let M and S be reduced complex spaces. Let $(X_s)_{s \in S}$ be an f -analytic family of n -cycles of M parametrized by S . Let T be an analytic set in M . Then the subset*

$$Z := \{s \in S \mid |X_s| \text{ has an irreducible component contained in } T\}$$

is a closed analytic subset in S .

PROOF. It is enough to consider the case of an analytic family of multiform graphs in a product $U \times B$ of polydiscs, classified by an holomorphic map

$$f : S \times U \rightarrow \text{Sym}^k(B),$$

where $T \subset U \times B$ is the subset given by $T := g^{-1}(0)$ for a holomorphic function $g : U \times B \rightarrow \mathbb{C}$. If $\tilde{f} : S \times U \rightarrow \text{Sym}^k(U \times B)$ is the holomorphic map associated to f keeping the U -component, then consider, for $U' \subset\subset U$, the holomorphic map

$$\Phi : S \rightarrow H(\bar{U}', \mathbb{C}) \quad \text{given by} \quad s \mapsto (t \mapsto \text{Nr}(g)[\tilde{f}(s, t)])$$

where $\text{Nr}(g) : \text{Sym}^k(U \times B) \rightarrow \mathbb{C}$ is the norm⁴ of g .

Then for U' non empty, the set $\Phi^{-1}(0)$ in S is the corresponding Z . As Φ is holomorphic, this conclude the proof. ■

This lemma allows to give the following corollary to the proposition 2.1.7

Corollaire 2.1.10 *Let T be a closed analytic subset of the complex space M . Then the subset $\mathcal{T} \subset \mathcal{C}_n^f(M)$ of cycles having an irreducible component contained in T is a closed analytic subset in $\mathcal{C}_n^f(M)$.*

PROOF. First we want to prove that the complement of \mathcal{T} is open. Let $C_0 \notin \mathcal{T}$ and choose a point $x_j, j \in [1, m]$, which is not in T , and choose in each irreducible component of the cycle C_0 a n -scales $E_j, j \in [1, m]$, on $M \setminus T$, such that x_j is in the center of E_j for each j . Then $k_j := \deg_{E_j}(C_0)$ is positive for each j . Let W be the union of the centers of the scales $E_j, j \in [1, m]$ and defined the open set \mathcal{U} in $\mathcal{C}_n^f(M)$ which contains C_0 by

$$\mathcal{U} := \Omega(W) \cap \left[\bigcap_{j \in [1, m]} \Omega_{k_j}(E_j) \right]$$

where $\Omega_k(E)$ denotes the open set of cycles C such E is adapted to C with $\deg_E(C) = k$ and $\Omega(W)$ is the set of cycles such that any irreducible component meets W . Then \mathcal{U} does not meet \mathcal{T} and so \mathcal{T} is closed.

To prove the analyticity of \mathcal{T} consider a cycle C_0 in \mathcal{T} and again choose a point x_j in each irreducible component of C_0 and for each $j \in [1, m]$ a n -scale E_j adapted to C_0 with center containing x_j . Construct the open set \mathcal{U} as above and write in this

⁴So $\text{Nr}(g)[z_1, \dots, z_k] = \prod_{j=1}^k g(z_j)$ for $[z_1, \dots, z_k] \in \text{Sym}^k(U \times B)$.

open set the Banach analytic equations given by the previous lemma in each scale E_j . The (finite) union of the analytic sets so defined for each $j \in [1, m]$ is equal to $\mathcal{T} \cap \mathcal{U}$, concluding the proof. ■

Note that this result may not be true in the case of an analytic family of cycles which is not f-analytic as it is shown by the following example.

EXAMPLE. Let $M := D = \{z \in \mathbb{C} / |z| < 1\}$, $T := \{1 - \frac{1}{n}, n \in \mathbb{N}, n \geq 3\}$ and consider the family of 0-cycles in D parametrized by D :

$$X_s := \{1 - \frac{s+1}{s+m}, m \in \mathbb{N}, m \geq 3\} \cap D \quad \text{for } s \in D.$$

We have $X_0 = T$ and a necessary and sufficient condition on $s \in D$ in order that X_s meets T is that there exists $m, n \in \mathbb{N} \setminus \{0, 1, 2\}$ with $\frac{1}{n} = \frac{s+1}{s+m}$. This gives that X_s meets T if and only iff $s = \frac{p}{q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0, 1\}$ and $|\frac{p}{q}| < 1$. This is a dense set in $] - 1, +1[$! □

We shall conclude this paragraph by an elementary but useful result on the number of irreducible components of cycles in a f-continuous family of cycles.

Definition 2.1.11 Let $X := \sum_{i \in I} n_i X_i$ be a finite type n -cycle in a complex space M , where X_i are the irreducible components of $|X|$, n_i are positive integers and I is a finite set. We define the **weight of X** as the integer $w(X) := \sum_{i \in I} n_i$.

Of course when X is a reduced cycle, the weight of X is simply the number of irreducible components of X .

Lemma 2.1.12 Let M be a complex space and n an integer. The weight function $w : \mathcal{C}_n^f(M) \rightarrow \mathbb{N}$ is lower semi-continuous on $\mathcal{C}_n^f(M)$.

PROOF. Let C_0 be a non empty cycle in $\mathcal{C}_n^f(M)$ and choose on M , for each irreducible component Γ_i of $|C_0|$, an adapted scale $E_i := (U_i, B_i, j_i)$ such that we have $\deg_{E_i}(\Gamma_i) = \deg_{E_i}(|C_0|) = 1$ and $\deg_{E_i}(C_0) = k_i$. Then we have $w(C_0) = \sum_{i \in I} k_i$. Let $W := \cup_{i \in I} j_i^{-1}(U_i \times B_i)$ and define the open neighbourhood \mathcal{V} of C_0 in $\mathcal{C}_n^f(M)$ as :

$$\mathcal{V} := \Omega(W) \cap (\cap_{i \in I} \Omega_{k_i}(E_i)).$$

Then for any $C \in \mathcal{V}$ we have the inequality $1 \leq w(C) \leq \sum_{i \in I} k_i = w(C_0)$. This is consequence of the fact that each irreducible component of C has to meet W , and that the degree of C in the scale E_i is equal to k_i . ■

Then, using the existence of a meromorphic geometrically flat Stein reduction for a meromorphic geometrically f-flat map proved in [B.13] theorem 4.2.4, we obtain the following result.

Proposition 2.1.13 *Let M be a complex space, n an integer and $\varphi : N \rightarrow \mathcal{C}_n^f(M)$ be a holomorphic map where N is an irreducible complex space such that for y in a dense Zariski open set the cycle $\varphi(y)$ is reduced. Then the number of irreducible components of $\varphi(y)$ is constant on a dense open set in N . ■*

Note that, thanks to the lemma 2.1.6, if there exists one $y \in N$ such that $\varphi(y)$ is a reduced cycle, then there exists a dense Zariski open set N' in N such for all $y \in N'$ the cycle $\varphi(y)$ is reduced.

PROOF. Let $G \subset N \times M$ be the graph of the f -analytic family of n -cycles of M classified by φ . As the projection $\pi : G \rightarrow N$ is geometrically f -flat, we can apply the theorem 4.2.4 of [B.13], see also the theorem 4.1.1 in section 4. Let $\nu : \tilde{G} \rightarrow G$ the normalization of G , $q : \tilde{G} \rightarrow Q$ and $p : Q \rightarrow N$ the maps given by the theorem. Then q is geometrically f -flat with irreducible general fibers and p is proper finite and surjective. If k is the generic degree of p , then it is easy to see, using the lemma 2.1.9 with the subset $\nu^{-1}(T) \subset \tilde{G}$, where $T \subset G$ is the subset of non normal points in G , that there is a dense open subset in N on which each $\varphi(y)$ has exactly k irreducible components. ■

The next lemma shows that the graph of a f -analytic family of cycles in M parametrized by an irreducible complex space is again a finite type cycle, assuming that the generic cycle is reduced. It is an easy exercise left to the reader to show that the existence of a reduced cycle in such a family is not necessary for this result.

Lemma 2.1.14 *Let M be a complex space and N an irreducible complex space. Let $\varphi : N \rightarrow \mathcal{C}_n^f(M)$ be a holomorphic map such that for some y in N the cycle $\varphi(y)$ is reduced. Let $G \subset N \times M$ be the graph of the f -analytic family of n -cycles in M classified by the map φ . Then G has finitely many irreducible components.*

PROOF. Let N' be the set of points in N such that $\varphi(y)$ is reduced. Then the irreducibility of N and the lemma 2.1.6 imply that N' is a dense Zariski open set in N . Denote $p : G \rightarrow N$ the projection. Then $G' := p^{-1}(N')$ is a dense Zariski open set in G . Then the proposition 2.1.13 gives that the number of irreducible components of $\varphi(p(z))$ for any z in a dense open set $G'' := p^{-1}(N'')$ is constant equal to a positive integer k , where $N'' \subset N'$ is a dense open set.

Note I the set of irreducible components of G . For $i \in I$ we shall denote G_i the corresponding irreducible component. Let Z be the analytic subset in G corresponding to points which are in two distinct irreducible components of G . It is a closed analytic subset with no interior point in G and also in any irreducible component G_i of G . Now for each i the image of the restriction p_i of the projection p to G_i contains an open dense subset N_i ⁵, as this restriction is equidimensional between irreducible complex spaces. Now the set I is countable, so the intersection $N_0 := N'' \cap (\bigcap_{i \in I} N_i)$ is dense by Baire's theorem. Let N_1 be the dense Zariski open

⁵By the remark 2 following the definition 3.1.4 each p_i is in fact surjective.

set in N such that $\varphi(y)$ has no irreducible component contained in Z . Consider now a point y in $N_0 \cap N_1$. Then $\varphi(p(y))$ is reduced, have k irreducible components, and each irreducible component G_i contains at least one of these components; but any irreducible component of $\varphi(p(y))$ cannot be in two different G_i because y is in N_1 . So the set I has at most k elements. ■

2.2 An analytic extension theorem.

The main goal of this paragraph is to prove the following analytic continuation result.

Theorem 2.2.1 *Let M be a complex space and n an integer. Consider a f -continuous family $(X_s)_{s \in S}$ of finite type n -cycles in M parametrized by a reduced complex space S . Fix a point s_0 in S and assume that there exists an open set M' in M meeting any irreducible component of $|X_{s_0}|$ and such that the family $(X_s \cap M')_{s \in S}$ is analytic at s_0 . Then there exists an open neighborhood S_0 of s_0 in S such that the family $(X_s)_{s \in S_0}$ is f -analytic.*

Let us make explicit the situation of the previous theorem in term of classifying maps : we have a continuous map $\varphi : S \rightarrow \mathcal{C}_n^f(M)$ such that the composed map $r \circ \varphi$ is holomorphic at s_0 , where $r : \mathcal{C}_n^f(M) \rightarrow \mathcal{C}_n^{loc}(M')$ is the restriction map. Then the statement is that φ is holomorphic on an open neighborhood S_0 of s_0 in S , assuming that M' meets each irreducible component of $|X_{s_0}|$.

Remark that the map r is holomorphic⁶ so that the holomorphy at s_0 of $r \circ \varphi$ is a necessary condition for the holomorphy of φ on an open neighborhood of s_0 . The theorem says that this condition is sufficient.

One key point in the proof of the previous theorem is given by the following analytic continuation result.

Proposition 2.2.2 *Let S be a reduced complex space and let $U_1 \subset U_2$ be two open polydiscs in \mathbb{C}^n . Let $f : S \times U_2 \rightarrow \mathbb{C}$ be a continuous function, holomorphic on $\{s\} \times U_2$ for each fixed $s \in S$ and assume also that the restriction of f to $S \times U_1$ is holomorphic. Then f is holomorphic on $S \times U_2$.*

PROOF OF THE PROPOSITION. Consider first the case where S is smooth. As the problem is local on S it is enough to treat the case where S is an open set in some \mathbb{C}^m . Fix then a relatively compact open polydisc P in S . The function f defines a map $F : U_2 \rightarrow \mathcal{C}^0(\bar{P}, \mathbb{C})$ where $\mathcal{C}^0(\bar{P}, \mathbb{C})$ is the Banach space of continuous functions on \bar{P} , via the formula $F(t)[s] = f(s, t)$ for $t \in U_2$ et $s \in \bar{P}$. The map F is holomorphic:

⁶meaning that for any holomorphic map $\psi : T \rightarrow \mathcal{C}_n^f(M)$ of a reduced complex space T the composed map $r \circ \psi$ is holomorphic.

this is an easy consequence of Cauchy's formula on a polydisc $U \subset\subset U_2$ with fixed $s \in S$ which computes the partial derivatives in $t := (t_1, \dots, t_n)$:

$$\frac{\partial f}{\partial t_i}(s, t) = \frac{1}{(2i\pi)^n} \int_{\partial\partial U} f(s, \tau) \cdot \frac{d\tau_1 \wedge \dots \wedge d\tau_n}{(\tau_1 - t_1) \dots (\tau_i - t_i)^2 \dots (\tau_n - t_n)} \quad \forall t \in U \quad \forall i \in [1, n].$$

This shows that F is \mathbb{C} -differentiable and its differential in $t \in U$ is given by $h \mapsto \sum_{i=1}^n F_i(t) \cdot h_i$, $h \in \mathbb{C}^n$, where F_i is the map associated to the function

$$(s, t) \mapsto \frac{\partial f}{\partial t_i}(s, t) \quad i \in [1, n]$$

which is holomorphic for any fixed $s \in S$ thanks to the Cauchy formula above.

Let $H(\bar{P}, \mathbb{C})$ be the (closed) subspace of $\mathcal{C}^0(\bar{P}, \mathbb{C})$ of continuous functions which are holomorphic on P . Our assumption implies that the restriction of F to U_1 takes its values in this subspace. Let us show that for each point $t \in U_2$, $F(t)$ is still in $H(\bar{P}, \mathbb{C})$: assume this is not true. Then there exists $t_0 \in U_2$ with $F(t_0) \notin H(\bar{P}, \mathbb{C})$, and so, by the Hahn-Banach theorem, there exists a continuous linear form λ on $\mathcal{C}^0(\bar{P}, \mathbb{C})$, vanishing on $H(\bar{P}, \mathbb{C})$ and such that $\lambda(F(t_0)) \neq 0$. But the function $t \mapsto \lambda(F(t))$ is holomorphic on U_2 and vanishes on U_1 . So it vanishes identically contradicting the fact that $\lambda(F(t_0)) \neq 0$. So F is an holomorphic map with values in $H(\bar{P}, \mathbb{C})$ and f is holomorphic on $S \times U_2$ when S is smooth.

The case where S is a weakly normal complex space is then an immediate consequence, as the continuity of f on $S \times U_2$ and the holomorphicity of f on $S_{reg} \times U_2$, obtained above, are enough to conclude.

When S is a general reduced complex space the function f is then a continuous meromorphic function on $S \times U_2$ which is holomorphic on $S \times U_1$. So the closed analytic subset $Y \subset S \times U_2$ of points at which f is not holomorphic has empty interior in each $\{s\} \times U_2$. So the criterium 3.1.7 of analytic continuation of chapter IV in [B-M] allows to conclude. \blacksquare

REMARK. It is an easy exercise to weaken the hypothesis of the previous proposition replacing the continuity of f by the hypothesis “ f is measurable and locally bounded on $S \times U_2$ ”. Then replace the Banach space $\mathcal{C}^0(\bar{P}, \mathbb{C})$ by the Banach space of bounded measurable functions on \bar{P} and in the second step consider the case where S is normal; conclude following the same lines. \square

PROOF OF THE THEOREM 2.2.1. Consider the graph $|G| \subset S \times M$ of the f -continuous family $(X_s)_{s \in S}$ and let A be the open set of points in $|G|$ such that the family is analytic in a neighborhood. Precisely, the point $(\sigma, \zeta) \in |G|$ is in A if there exist open neighborhoods S_σ and U_ζ respectively of σ in S and of ζ in M such that the family $(X_s \cap M_\zeta)_{s \in S_\sigma}$ is analytic. Remark that, because of our assumption, A meets each irreducible component of $\{s_0\} \times |X_{s_0}|$.

First, assume that there exists a smooth point of $|X_{s_0}|$ in the boundary of the set

$A \cap (\{s_0\} \times |X_{s_0}|)$. Consider now such a point (s_0, z_0) and choose also a n -scale $E := (U, B, j)$ adapted to X_{s_0} satisfying the following conditions:

$$\deg_E(|X_{s_0}|) = 1 \quad j_*(X_{s_0}) = k.(U \times \{0\}) \quad z_0 \in j^{-1}(U \times B) \quad j(z_0) := (t_0, 0).$$

Then we have a continuous classifying map $f : S_1 \times U \rightarrow \text{Sym}^k(B)$ where S_1 is an open neighborhood of s_0 in S . The map f is holomorphic for each fixed $s \in S_1$. As the point (s_0, z_0) is in the boundary of the open set $A \cap (\{s_0\} \times |X_{s_0}|)$ of $\{s_0\} \times |X_{s_0}|$, there exists a (non empty) polydisc $U_1 \subset U$ such that the restriction of f to $S_1 \times U_1$ is holomorphic near s_0 . So, up to shrink S_1 , we can assume that f is holomorphic on $S_1 \times U_1$. Applying the proposition 2.2.2 to each scalar component of f , we conclude that f is holomorphic on $S_1 \times U$. As we can apply the previous argument to any linear projection of $U \times B$ on U near the vertical one, we obtain also the fact that f is an isotropic map. This contradicts the fact that (s_0, z_0) is in the boundary of $A \cap (\{s_0\} \times |X_{s_0}|)$.

If the boundary of $A \cap (\{s_0\} \times |X_{s_0}|)$ is contained in the singular set of $|X_{s_0}|$, then we can apply the criterium ([B-M] ch.IV critère 3.9.1) to obtain directly that A contains $|X_{s_0}|$.

So in all cases, the family $(X_s)_{s \in S}$ is analytic at s_0 . As the graph $|G|$ is quasi-proper on S by assumption, it is enough to apply the proposition 2.2.3 to conclude the proof. \blacksquare

Proposition 2.2.3 *Let M and S be a reduced complex spaces and let $(X_s)_{s \in S}$ be a f -continuous family of n -cycles in M . Assume that this family is analytic at s_0 . Then there exists an open neighbourhood S' of s_0 in S such the family $(X_s)_{s \in S'}$ is a f -analytic family of n -cycles in M .*

PROOF. The quasi-properness hypothesis on the graph of the family gives an open neighbourhood S_1 of s_0 in S and a relatively compact open set W in M such that any irreducible component of any $X_s, s \in S_1$, meets W . Now for each irreducible component Γ of $|X_{s_0}|$ choose a point $z_\Gamma \in \Gamma \cap W$, an open neighbourhood S_Γ of s_0 in S_1 and an open neighbourhood W_Γ of z_Γ in W such that the family $(X_s \cap W_\Gamma)_{s \in S_\Gamma}$ is analytic. Then, as there is a finite number of Γ , the subset $S' := \bigcap_\Gamma S_\Gamma$ is an open neighbourhood of s_0 in S_1 , and the family $(X_s \cap V)_{s \in S'}$ is analytic where $V := \bigcup_\Gamma W_\Gamma$. Then, for each point $s_1 \in S'$, we may apply the first part of the proof of the theorem 2.2.1 which is given above (without the present proposition) and conclude that our family is holomorphic at any point $s_1 \in S'$. \blacksquare

Remark that the previous result is not true in general for a family of cycles which is analytic at s_0 and is not f -continuous in a neighbourhood of s_0 .

2.3 The semi-proper direct image theorem.

The aim of this paragraph is to give an improvement to the following important result given by the theorem 5.0.5 of [B.13]. First recall this result from *loc. cit.*

Theorem 2.3.1 *Let M and S be a reduced complex spaces and fix an integer n . Assume that we have a holomorphic map $\varphi : S \rightarrow \mathcal{C}_n^f(M)$ which is semi-proper⁷. Then the closed subset $\varphi(S)$ has a natural structure of weakly normal complex space such that the tautological family of n -cycles parametrized by $\varphi(S)$ is an f -analytic family of cycles in M .*

Here is our improvement.

Theorem 2.3.2 *Let M and S be a reduced complex spaces and fix an integer n . Assume that we have a holomorphic map $\varphi : S \rightarrow \mathcal{C}_n^f(M)$ which is semi-proper. Then the closed subset $\varphi(S)$ in $\mathcal{C}_n^f(M)$ is analytic and the sheaf induced by the sheaf of holomorphic functions on $\mathcal{C}_n^f(M)$ defines a structure of reduced (locally finite dimensionnal) complex space on it.*

This improvement of the theorem 5.0.5 of [B.13] will allow us to avoid to restrict several statements to weakly normal complex spaces. It is also an opportunity to precise the proof of this delicate result.

The main tool for the proof of the theorem 2.3.2 is the semi-proper direct image theorem of [M.00] and the theorem 2.2.1.

Before going to the proof of the theorem 2.3.2 we have to give some details here on a crucial point for the use of the semi-proper direct image theorem with value in a Banach space in order to have a semi-proper direct image theorem with values in $\mathcal{C}_n^f(M)$ for any complex space M . This is to be compared with condition (H) in [M.00].

Proposition 2.3.3 *Let M and S be reduced complex spaces and $\varphi : S \rightarrow \mathcal{C}_n^f(M)$ be a holomorphic semi-proper map. Then for each $C \in \varphi(S)$ there exists an open neighborhood \mathcal{V} of C in $\mathcal{C}_n^f(M)$ and a holomorphic map $f : \mathcal{V} \rightarrow \mathcal{U}$ in an open set \mathcal{U} of a Banach space, such that the map $f \circ \varphi : \varphi^{-1}(\mathcal{V}) \rightarrow \mathcal{U}$ is semi-proper.*

PROOF. First recall that a basis of the topology of $\mathcal{C}_n^f(M)$ is given by finite intersections of open sets of the type $\Omega_k(E)$ and $\Omega(W)$; here $\Omega_k(E)$, associated to a natural integer $k \geq 0$ and a n -scale E on M , is defined as the subset of $C \in \mathcal{C}_n^f(M)$ such that E is adapted to C and $\deg_E(C) = k$ and the open set $\Omega(W)$, associated to a relatively compact open set W in M , is defined as the subset of $C \in \mathcal{C}_n^f(M)$ such that each irreducible component of C meets W .

⁷This means that for any cycle $C \in \mathcal{C}_n^f(M)$ we can find an open neighborhood \mathcal{V} of C and a compact set K in S such that $\varphi(S) \cap \mathcal{V} = \varphi(K) \cap \mathcal{V}$.

Fix $C_0 \in \mathcal{C}_n^f(M)$ and, using the semi-properness of φ , let

$$\mathcal{V} := \left(\bigcap_{j=1}^J \Omega(W_j) \right) \cap \left(\bigcap_{i=1}^N \Omega_{k_i}(E_i) \right)$$

an open neighborhood of C_0 in $\mathcal{C}_n^f(M)$ and $K \subset S$ a compact set in S such that

$$\varphi(S) \cap \mathcal{V} = \varphi(K) \cap \mathcal{V}. \quad (*)$$

As $\varphi(K)$ is a compact subset in $\mathcal{C}_n^f(M)$, there exists a relatively compact open set W_0 in M such that for any $s \in K$ any irreducible component of $\varphi(s)$ meets W_0 . As we can make \mathcal{V} smaller around C_0 keeping the condition (*) above, we shall assume that W_0 is one of the $W_i, i \in [1, M]$ and that the union of the centers of the scales $E_i, i \in [1, N]$ contains the compact set \bar{W}_0 .

We consider now the holomorphic map

$$f : \mathcal{V} \cap \varphi(S) \longrightarrow \prod_{i=1}^N H(\bar{U}_i, \text{Sym}^{k_i}(B_i)) \subset \mathcal{U}$$

given by $f(C) = (f_i)_{i \in [1, N]}$ where $f_i \in H(\bar{U}_i, \text{Sym}^{k_i}(B_i))$ is the classifying map of C in the scale $E_i := (U_i, B_i, j_i)$ and where \mathcal{U} is an open set in the Banach space $\prod_{i=1}^N H(\bar{U}_i, E_i(k_i))$ ⁸ such that $\prod_{i=1}^N H(\bar{U}_i, \text{Sym}^{k_i}(B_i))$ is a closed analytic subset of \mathcal{U} . Then $f : \mathcal{V} \rightarrow \mathcal{U}$ is holomorphic and injective. We shall show that f induces an homeomorphism of $\mathcal{V} \cap \varphi(S)$ onto $f(\mathcal{V} \cap \varphi(S))$.

As this map is bijective and continuous, it is enough to prove that this map is closed; if the sequence $(f(C_\nu))$ converges to $f(C)$ in \mathcal{U} where C_ν and C are in $\mathcal{V} \cap \varphi(S)$, we shall prove that the sequence (C_ν) converges to C in \mathcal{V} (that is to say in $\mathcal{C}_n^f(M)$).

The convergence of $(f(C_\nu))$ to $f(C)$ implies the convergence of $(C_\nu \cap W_0)$ to $C \cap W_0$ in $\mathcal{C}_n^{loc}(W_0)$, as we assume that the union of the centers of the scales E_i covers \bar{W}_0 (see [B-M] ch. IV). Write $C_\nu = \varphi(s_\nu)$ where s_ν is in K . Up to pass to a subsequence, we can assume that the sequence (s_ν) converges to $t \in K$. Then (C_ν) converges to $D := \varphi(t)$ in $\mathcal{C}_n^f(M)$ as φ is continuous. Then D is in $\varphi(K)$ but it is not, a priori, clear that D is in \mathcal{V} and then equal to C . Nevertheless we have $C \cap W_0 = D \cap W_0$ by the uniqueness of a limit in $\mathcal{C}_n^{loc}(W_0)$. As C and D are in $\varphi(K)$, each irreducible components of C and D meets W_0 , and this allows to conclude that $D = C$. As each converging subsequence of the sequence (C_ν) converges to C and as we are in the compact set $\varphi(K)$ of $\mathcal{C}_n^f(M)$, we obtain that the sequence (C_ν) converges to C in $\mathcal{C}_n^f(M)$.

Now the composed map $f \circ \varphi : \varphi^{-1}(\mathcal{V}) \rightarrow f(\mathcal{V} \cap \varphi(S))$ is semi-proper. Then, up to shrink the open set \mathcal{U} around $f(\mathcal{V} \cap \varphi(S))$, the map $f \circ \varphi : \varphi^{-1}(\mathcal{V}) \rightarrow \mathcal{U}$ is also semi-proper⁹.

⁸We denote by $E_i(k_i)$ the Zariski tangent space at $k_i \cdot \{0\}$ of $\text{Sym}^{k_i}(\mathbb{C}^{p_i})$. So $\text{Sym}^{k_i}(\mathbb{C}^{p_i})$ is an algebraic subset of $E_i(k_i)$; see [B-M] ch.I.

⁹The image of a semi-proper map is locally compact; so it is locally closed in \mathcal{U} .

PROOF OF THE THEOREM 2.3.2. Assume now that in the previous proof we choose for each scale $E_i, i \in [1, N]$ an open polydisc $U'_i \subset\subset U_i$ such that the union of the $j_i^{-1}(U'_i \times B_i)$ covers the compact set \bar{W}_0 and that we replace the Banach analytic set $H(\bar{U}_i, \text{Sym}^k(B_i))$ by the Banach analytic set $\Sigma_{U_i, U'_i}(k_i)$ which classifies the $f \in H(\bar{U}_i, \text{Sym}^k(B_i))$ which are isotropic on \bar{U}'_i . As the natural projection $\Sigma_{U_i, U'_i}(k_i) \rightarrow H(\bar{U}_i, \text{Sym}^k(B_i))$ is an holomorphic homeomorphism (see [B.75] chapter III or [B-M 2] chapter V), and as the map

$$\tilde{f} : \mathcal{V} \rightarrow \prod_{i=1}^N \Sigma_{U_i, U'_i}(k_i)$$

is holomorphic, we obtain that the map $\tilde{f} \circ \varphi : \varphi^{-1}(\mathcal{V}) \rightarrow \prod_{i=1}^N \Sigma_{U_i, U'_i}(k_i)$ is semi-proper on its image and so there exists an open set $\tilde{\mathcal{U}}$ in the ambient Banach space of $\prod_{i=1}^N \Sigma_{U_i, U'_i}(k_i)$ such the map $\tilde{f} : \mathcal{V} \rightarrow \tilde{\mathcal{U}}$ is semi-proper. So the semi-proper direct image theorem with values in a Banach open set gives that $\tilde{f}(\mathcal{V} \cap \varphi(S))$ is a locally finite dimensional analytic subset in $\tilde{\mathcal{U}}$.

Consider now the f-continuous family of n -cycles in M parametrized by $\tilde{f}(\mathcal{V} \cap \varphi(S))$ induced via the homeomorphism \tilde{f} by the tautological family on $\mathcal{V} \cap \varphi(S)$. The universal properties of the Banach analytic spaces $\Sigma_{U_i, U'_i}(k_i)$ imply that this family of n -cycles in M is analytic on the open set $\cup_{i=1}^N j_i^{-1}(U'_i \times B_i)$ which contains the open set W_0 . As W_0 meets each irreducible component of each cycle in $\mathcal{V} \cap \varphi(S)$, we can apply the theorem 2.2.1 and conclude that this family is f-analytic. By the universal property¹⁰ of $\mathcal{C}_n^f(M)$, the corresponding classifying map

$$\tilde{f}^{-1} : \tilde{f}(\mathcal{V} \cap \varphi(S)) \longrightarrow \mathcal{V} \cap \varphi(S) \subset \mathcal{C}_n^f(M)$$

is holomorphic. To conclude, it is enough to use the fact that \tilde{f} and \tilde{f}^{-1} are holomorphic between $\mathcal{V} \cap \varphi(S)$ and $\tilde{f}(\mathcal{V} \cap \varphi(S))$. \blacksquare

REMARK. We can avoid to use the full strength of the theorem 2.2.1 in the proof above, because the semi-properness of the map $\tilde{f} \circ \varphi : \varphi^{-1}(\mathcal{V}) \rightarrow \tilde{f}(\mathcal{V} \cap \varphi(S))$ implies the semi-properness of the holomorphic map

$$(\tilde{f} \circ \varphi) \times \text{id}_M : |G|_{\varphi^{-1}(\mathcal{V})} \rightarrow \tilde{f}(\mathcal{V} \cap \varphi(S)) \times M$$

where $|G| \subset S \times M$ is the set-theoretic graph of the f-analytic family classified by φ . Then the (finite dimensional) semi-proper direct image theorem of Kuhlmann gives the analyticity of the set-theoretic graph of the f-continuous family parametrized by $\tilde{f}(\mathcal{V} \cap \varphi(S))$, so the fact that this family is weakly holomorphic. \square

¹⁰Recall that we have already proved that $\tilde{f}(\mathcal{V} \cap \varphi(S))$ is a reduced (locally finite dimensional) complex space.

2.4 Meromorphic maps to $\mathcal{C}_n^f(M)$.

TERMINOLOGY.

- A **modification** between two reduced complex spaces will be always a proper holomorphic map which induces an isomorphism between two dense Zariski open sets.
- We shall say that a holomorphic map $p : P \rightarrow N$ between two irreducible complex spaces is **dominant** if there exists an open dense subset $\Lambda \subset P$ such that the restriction of p to Λ is an open map with dense image in N . When P is not irreducible, we shall say that p is dominant when its restriction to each irreducible component of P is dominant.

Remark that this implies that the restriction of p to the smooth parts of P has generically maximal rank equal to $\dim N$.

Definition 2.4.1 *Let $\pi : M \rightarrow N$ and $p : P \rightarrow N$ be two holomorphic maps where M, N, P are complex spaces. Assume that N is irreducible and that π and p are dominant. Then we shall call the **strict fiber product** of π and p , the map*

$$\tilde{\pi} : \tilde{M} := M \times_{N, str} P \rightarrow P$$

which is the restriction of the projection of the usual fiber product to the union of the irreducible components of $M \times_N P$ which are dominant over P .

Of course, the strict fiber product has two holomorphic projections on M and P which factorize its natural projection on N . They are both dominant.

The reader will find easily examples of fiber products which are not equal to the corresponding strict fiber product. For instance, if $\tau : M \rightarrow N$ is a modification which is not injective, the fiber product $M \times_N M$ has at least one irreducible component which is not contained in the corresponding strict fiber product which coincides with the map τ itself.

It is an easy exercise left to the reader to prove that if we assume that $p : P \rightarrow N$ is a modification and that $\pi : M \rightarrow N$ is dominant the projection $M \times_{N, str} P \rightarrow M$ is a modification.

The definition of a meromorphic map from a reduced complex space N to $\mathcal{C}_n^f(M)$ is the usual one.

Definition 2.4.2 *Fix a complex space M and an integer n . Let N be a reduced complex space and $\Sigma \subset N$ a nowhere dense closed analytic subset in N . We shall say that a holomorphic map $\varphi : N \setminus \Sigma \rightarrow \mathcal{C}_n^f(M)$ is **meromorphic along Σ** (or more simply that $\varphi : N \dashrightarrow \mathcal{C}_n^f(M)$ is meromorphic) when there exists a modification $\sigma : N_1 \rightarrow N$ with center in Σ and a holomorphic map $\varphi_1 : N_1 \rightarrow \mathcal{C}_n^f(M)$ extending $\varphi \circ \sigma$.*

To define the “graph” of such a meromorphic map we shall need the following corollary of the theorem 2.3.2.

Corollaire 2.4.3 *Fix a complex space M and an integer n . Let N and P be reduced complex spaces and let $\varphi : N \rightarrow P \times \mathcal{C}_n^f(M)$ be a semi-proper holomorphic map. Then $\varphi(N)$ is a closed analytic subset in $P \times \mathcal{C}_n^f(M)$ which is locally of finite dimension.*

The proof will be an easy consequence of the theorem 2.3.2 using the following lemma.

Lemma 2.4.4 *Fix a complex space M and an integer n . Let P be a reduced complex space. Denote $p : P \times M \rightarrow P$ and $q : P \times M \rightarrow M$ the projections. Then the closed analytic subset $\mathcal{C}_n^f(p) \subset \mathcal{C}_n^f(P \times M)$ is bi-holomorphic to the product $P \times \mathcal{C}_n^f(M)$.*

PROOF. Denote $\alpha : \mathcal{C}_n^f(p) \rightarrow P$ the natural holomorphic projection (see the proposition 2.1.7) and $\beta : \mathcal{C}_n^f(p) \rightarrow \mathcal{C}_n^f(M)$ be the holomorphic map induced by the direct image by q . Remark that for any p -relative n -cycle the restriction of q is a holomorphic homeomorphism on its image which is closed (so the restriction of q to such a cycle is a homeomorphism), then the direct image theorem with parameter applies (see [B.75] or [B-M] ch.IV). So the map $(\alpha, \beta) : \mathcal{C}_n^f(p) \rightarrow P \times \mathcal{C}_n^f(M)$ is holomorphic and bijective. The inverse map γ , given by $\gamma(p, C) := \{p\} \times C \in \mathcal{C}_n^f(p) \subset \mathcal{C}_n^f(P \times M)$, is also holomorphic thanks to the product theorem¹¹ for analytic families of cycles (see also [B.75] or [B-M] ch.IV). ■

PROOF OF THE COROLLARY. The holomorphic map φ defines a holomorphic map $\psi : N \rightarrow \mathcal{C}_n^f(P \times M)$ with values in $\mathcal{C}_n^f(p)$. The map ψ is holomorphic and semi-proper and we conclude by applying the theorem 2.3.2. ■

Lemma 2.4.5 *Let $\varphi : N \dashrightarrow \mathcal{C}_n^f(M)$ be a meromorphic map and consider a modification $\sigma : N_1 \rightarrow N$ such that there exists a holomorphic map $\varphi_1 : N_1 \rightarrow \mathcal{C}_n^f(M)$ extending $\varphi \circ \sigma$. Then the holomorphic map $(\sigma, \varphi_1) : N_1 \rightarrow N \times \mathcal{C}_n^f(M)$ is proper and its image is the closure in $N \times \mathcal{C}_n^f(M)$ of the graph of $\varphi|_{N \setminus \Sigma}$.*

PROOF. To prove the properness of the map (σ, φ_1) we have to prove that it is a closed map and that all fibers are compact. The second point is obvious as σ is proper. So consider a closed set F in N and a sequence (y_ν) in F such that $(\sigma(y_\nu), \varphi_1(y_\nu))$ converges to $(x, C) \in N \times \mathcal{C}_n^f(M)$. Take a compact neighbourhood V of x in N . As σ is proper, $\sigma^{-1}(V)$ is compact and contains all y_ν for ν large enough. So, up to pass to a subsequence, we can assume that (y_ν) converges to y in N_1 . Then the sequence $(\varphi_1(y_\nu))$ converges to $\varphi_1(y)$ in $\mathcal{C}_n^f(M)$. So we have $y \in F$ and $(x, C) = (\sigma, \varphi_1)(y)$. The last assertion is obvious. ■

¹¹Note that even in this case the product theorem is not obvious because a n -scale on $P \times M$ adapted to a cycle like $\{p\} \times C$ does not necessary comes from a n -scale on M .

Note that the projection τ of the reduced complex space $\tilde{N} := (\sigma, \varphi_1)(N_1)$ on N is a proper modification and that the map $\tilde{\varphi} : \tilde{N} \rightarrow \mathcal{C}_n^f(M)$ is holomorphic and extends $\varphi \circ \tau$. Moreover, for any modification $\sigma_2 : N_2 \rightarrow N$ such that there exists a holomorphic map $\varphi_2 : N_2 \rightarrow \mathcal{C}_n^f(M)$ extending $\varphi \circ \sigma_2$ the holomorphic map (σ_2, φ_2) factorizes through \tilde{N} , meaning that there exists a holomorphic map $h : N_2 \rightarrow \tilde{N}$ such that $\sigma_2 = \tau \circ h$ and $\varphi_2 = \tilde{\varphi} \circ h$.

Definition 2.4.6 *In the situation of the lemma above the reduced complex space $\tilde{N} \subset N \times \mathcal{C}_n^f(M)$ will be called the **graph** of the meromorphic map φ .*

3 Strongly quasi-proper maps.

We first explain that geometrically f-flat maps (resp. strongly quasi-proper maps) is the class of holomorphic maps admitting a holomorphic fiber map (resp. a meromorphic fiber map) defined on the target space with values in the space of finite type cycles of the source space (of the appropriate dimension). Then we give some basic results for these notions including stability results in order to dispose of easy criteria ensuring that a holomorphic map is strongly quasi-proper, condition which is not so simple to verify directly on the definition.

Of course these tools are essential for the applications given in [B.13] mainly because they allow to use the holomorphic semi-proper direct image theorem 2.3.2 with values in the space $\mathcal{C}_n^f(M)$ where M is any complex space and n any integer. for an example of how these tools allow to give some results on meromorphic quotients for some holomorphic action of a complex Lie group on a reduced complex space see [B.15].

3.1 Geometrically f-flat maps.

We recall first the notion of geometrically f-flat holomorphic map.

Definition 3.1.1 *Let M be a pure dimensional complex space and let N be an irreducible complex space. Put $n := \dim M - \dim N$. Let $\pi : M \rightarrow N$ be a holomorphic map. We shall say that π is a **geometrically f-flat map** (**GF map** for short) if there exists a holomorphic map $\varphi : N \rightarrow \mathcal{C}_n^f(M)$ such that for y generic¹² in N the cycle $\varphi(y)$ is reduced and such that the projection on M of the graph G of the f-analytic family of n -cycles in M classified by φ , is an isomorphism.*

$$\begin{array}{ccc} M & \xrightarrow{\cong} & G \subset N \times M \\ & \searrow \pi & \swarrow pr \\ & & N \end{array}$$

¹²One y such $\varphi(y)$ is reduced is enough as N is irreducible ; see lemma 2.1.6.

Note that the lemma 2.1.14 implies that in this situation M has finitely many irreducible components. The restriction of π to each of these irreducible components is quasi-proper and equidimensional (so surjective), but it is not true that these restrictions are GF maps ; see the example below. But it is not far to be true as we shall see thanks to the lemma 3.1.3 and the remark 2 following the definition 3.1.4

EXAMPLE. Let $X := \{(x, y) \in \mathbb{C}^2 / x^2 = y^3\}$ and let $f : X \rightarrow \mathbb{C}$ the continuous meromorphic function defined by $f(x, y) = x/y$. Let $G_{\pm} \subset X \times \mathbb{C}$ the graph of $\pm f$ and define $Y := G_+ \cup G_- \subset X \times \mathbb{C}$. Then the projection $\pi : Y \rightarrow X$ is a GF map, as the map $F : X \rightarrow \text{Sym}^2(\mathbb{C}) \simeq \mathbb{C}^2$ given by $F(x, y) = (0, y)$ is holomorphic and classifies the fibers of π (note that $f^2(x, y) = y$). But the restriction of the projection to the irreducible component G_+ of Y is not a GF map as f is not holomorphic on X .

Lemma 3.1.2 *Let $\pi : M \rightarrow N$ be a holomorphic n -equidimensional map between a pure dimensional complex space M to an irreducible complex space N , where the integer n is equal to $\dim M - \dim N$. Assume that there exists a holomorphic map*

$$\varphi : N \rightarrow \mathcal{C}_n^f(M)$$

such that for any y in a dense set in N the cycle $\varphi(y)$ is reduced and equal to the fiber of π . Then π is a GF map.

PROOF. As for y in a dense set we have $\varphi(y) \in \mathcal{C}_n^f(\pi)$ which is a closed (analytic) subset in $\mathcal{C}_n^f(M)$ (see the proposition 2.1.7), we have $|\varphi(y)| \subset \pi^{-1}(y)$ for any $y \in N$. Consider the second projection $pr : G \rightarrow M$ of the graph $G \subset N \times M$ of the f -analytic family of n -cycles classified by φ . As π is an open map, there exists a dense subset of x in M such that $(\pi(x), x)$ is in G . So the holomorphic map $(\pi, \text{id}_M) : M \rightarrow N \times M$ takes its values in G and it gives a holomorphic inverse to pr . So π is a GF map. ■

REMARK. A GF-map is quasi-proper, equidimensional and surjective. But a holomorphic quasi-proper equidimensional (and then surjective) map between irreducible complex spaces is not always a GF map : for instance the (weak) normalization of a reduced complex space N is not a GF-map if N is not (weakly) normal. Nevertheless the next lemma explains that such a map it is not so far to be a GF map.

Lemma 3.1.3 *Let $\pi : M \rightarrow N$ be a holomorphic map between a pure dimensional complex space M and an irreducible complex space N . Let $n := \dim M - \dim N$ and let $\nu : \tilde{N} \rightarrow N$ be the normalization of N . Then the following properties are equivalent :*

- i) π is quasi-proper and n -equidimensional.*

ii) The projection¹³ $\tilde{\pi} : \tilde{M} := M \times_{N, str} \tilde{N} \rightarrow \tilde{N}$ is a GF-map.

PROOF. Assume i) ; first let us prove that $\tilde{\pi}$ is quasi-proper. Let \tilde{K} be a compact set in \tilde{N} ; as $K := \nu(\tilde{K})$ is a compact set in N there exists a compact set L in M such that each irreducible component of a $\pi^{-1}(y)$ for $y \in K$ meets L . Then the compact set $(L \times \nu^{-1}(K)) \cap (M \times_N \tilde{N})$ meets any irreducible component of a $\tilde{\pi}^{-1}(\tilde{y})$ for each $\tilde{y} \in \nu^{-1}(K)$. So $\tilde{\pi}$ is quasi-proper as any irreducible component of a fiber of $\tilde{\pi}$ is an irreducible component of a fiber of π . The n -equidimensionality of $\tilde{\pi}$ is obvious. Now $\tilde{\pi} : M \times_{N, str} \tilde{N} \rightarrow \tilde{N}$ is quasi-proper and equidimensional on a normal complex space, so there exists a holomorphic map (see [B.M] ch.IV) $\varphi : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$ such that $|\varphi(\tilde{y})| = \tilde{\pi}^{-1}(s)$ for \tilde{y} general in \tilde{N} , where we put $\tilde{M} := M \times_{N, str} \tilde{N}$. Using the analyticity of the direct image of cycles by ν the lemma 3.1.2 gives that $\tilde{\pi}$ is a GF map.

Conversely, assume ii). First we prove the n -equidimensionality of π . Assume there exists a non normal point $y \in N$ such that the fiber $\pi^{-1}(y)$ has an irreducible component Γ of dimension strictly bigger than n . This irreducible component does not appears in a fiber of $\tilde{\pi}$. We can find a sequence of points in $\pi^{-1}(N_{norm})$ converging to the generic point of Γ , where N_{norm} is the open dense set of normal points in N . As \tilde{M} is a finite modification of M we can assume, up to pass to a sub-sequence, that this sequence converges in \tilde{M} . We then find a point y in N such that the fiber of $\tilde{\pi}$ contains the generic point of Γ . This contradicts our assumption.

Let us prove the quasi-properness of π . Let K be a compact in N ; then $\nu^{-1}(K)$ is a compact in \tilde{N} and so there exists a compact L in M such that any irreducible component of $\tilde{\pi}^{-1}(\tilde{y})$ for $\tilde{y} \in \nu^{-1}(K)$ meets $(L \times \nu^{-1}(K)) \cap \tilde{M}$. Then any irreducible component of $\pi^{-1}(y)$ for $y \in K$ meets L . ■

Definition 3.1.4 A holomorphic map $\pi : M \rightarrow N$ between a pure dimensional complex space M to an irreducible complex space N which satisfies the conditions of the lemma above will be called a **weakly geometrically f-flat map** (in short a **wGF map**).

REMARKS.

1. As in the situation of the previous lemma the projection $M \times_{N, str} \tilde{N} \rightarrow M$ is a modification, this implies that M has finitely many irreducible components because the lemma 2.1.14 implies that $\tilde{M} := M \times_{N, str} \tilde{N}$ has only finitely many irreducible components.
2. If the holomorphic map $\pi : M \rightarrow N$ is wGF map its restriction to any irreducible component of M is again wGF because this restriction is equidimensional and quasi-proper.
3. Conversely, if $\pi : M \rightarrow N$ is holomorphic, if M is pure dimensional with finitely many irreducible components and if the restriction of π on each irreducible

¹³See the definition 2.4.1.

component of M is a wGF map, then π is a wGF map, again thanks to the lemma 3.1.3.

4. If $\pi : M \rightarrow N$ is a wGF map then for any closed irreducible analytic subset $P \subset N$ the restriction $\pi_{\pi^{-1}(P)} : \pi^{-1}(P) \rightarrow P$ is a wGF map.

The first stability results for this notion of wGF map are given by the following lemma.

Lemma 3.1.5 *Let $\alpha : M \rightarrow N$ be a wGF map and $\sigma : P \rightarrow N$ be a dominant holomorphic map of an irreducible complex space P to N . The the natural projection of the fiber product $\tilde{\alpha} : M \times_{N, str} P \rightarrow P$ is a wGF map. If α is a GF map, so is $\tilde{\alpha}$.*

PROOF. Assume first that α is geometrically f-flat; its fibers are classified by a holomorphic map

$$\varphi : N \rightarrow \mathcal{C}_n^f(M)$$

where $n := \dim M - \dim N$. Compose this map with σ ; this gives a holomorphic map $\varphi \circ \sigma$ which classifies a f-analytic family of n -cycles in M parametrized by P . Let $G \subset P \times M$ the graph of this family. As for y generic in N the cycle $\varphi(y)$ is reduced and the map σ is dominant, the graph G of the family parametrized by P is reduced and given by

$$G := \{(x, z) \in M \times P / x \in |\varphi(\sigma(z))|\}.$$

But, as $|\varphi(y)| = \alpha^{-1}(y)$ for each $y \in N$, we see that $G = M \times_{N, str} P$ and that for z in a dense set in P the fiber of the projection of G on P is the fiber of the map $\tilde{\alpha}$. So this map is geometrically f-flat, thanks to the lemma 3.1.2.

Consider now the case where α is only a wGF map. So, thanks to the lemma 3.1.3, if $\nu : \tilde{N} \rightarrow N$ is the normalization of N , the fiber product $\tilde{M} := M \times_{N, str} \tilde{N}$ has a projection $\tilde{\alpha}$ on \tilde{N} which is a GF map. Consider now the normalization $\mu : \tilde{P} \rightarrow P$. We have the following commutative diagram, where $\tilde{\sigma}$ is a lifting of $\sigma \circ \mu$ to \tilde{N} which exists by normality of \tilde{P} as σ is dominant.

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\tilde{\alpha}} & \tilde{N} & \xleftarrow{\tilde{\sigma}} & \tilde{P} \\ \tilde{\nu} \downarrow & & \nu \downarrow & & \mu \downarrow \\ M & \xrightarrow{\alpha} & N & \xleftarrow{\sigma} & P \end{array}$$

Let $\varphi : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$ the holomorphic map classifying the fibers of the GF map $\tilde{\alpha}$. Composed with $\tilde{\sigma}$ and with the direct image map $\tilde{\nu}_*$ for n -cycles (see [B.M] ch. IV case of a proper map) it gives a holomorphic map

$$\psi : \tilde{P} \xrightarrow{\tilde{\sigma}} \mathcal{C}_n^f(\tilde{M}) \xrightarrow{\tilde{\nu}_*} \mathcal{C}_n^f(M)$$

and it is easy to see that for z in a general subset in \tilde{P} its value is the reduced cycle C in M where the cycle $C \times \{z\}$ in $M \times_{N, str} \tilde{P}$ which is the fiber of the natural projection $p : M \times_{N, str} \tilde{P} \rightarrow \tilde{P}$, as we have $(M \times_{N, str} P) \times_{P, str} \tilde{P} \simeq M \times_{N, str} \tilde{P}$. So p is a GF map from the first case proved above, and we conclude the proof thanks to the lemma 3.1.3. \blacksquare

REMARK. Let $\alpha : M \rightarrow N$ be a wGF map and $\sigma : P \rightarrow N$ any holomorphic map, it is easy to see that the projection $M \times_{N, str} P \rightarrow P$ is again a wGF map using the lemma 3.1.3 ; this generalizes the previous lemma and the remark 4 following the definition 3.1.4. Note that the case of a GF map is not clear because the image of σ may be inside the locus of non reduced fibers of α .

Lemma 3.1.6 *Let $\alpha : M \rightarrow N$ and $\beta : N \rightarrow P$ two wGF maps between a pure dimensional complex space M and irreducible complex spaces N and P . Then $\beta \circ \alpha$ is a wGF map.*

PROOF. Put $m := \dim M - \dim N$ and also $n := \dim N - \dim P$. Fix an irreducible component Δ of $(\beta \circ \alpha)^{-1}(z)$ for some z in P . Then α induces a map $\alpha' : \Delta \rightarrow \beta^{-1}(z)$ where $\beta^{-1}(z)$ has pure dimension n and α' has pure dimension m fibers. So Δ has dimension at most $m+n$, and, as the map $\beta \circ \alpha$ is surjective between each irreducible component of M and the irreducible complex space P with $\dim M - \dim P = m+n$, we obtain that Δ has dimension $m+n$ and also that $\alpha(\Delta)$ is dense in an irreducible component Γ of $\beta^{-1}(z)$. So $\beta \circ \alpha$ is equidimensional (and surjective).

We shall show that it is also quasi-proper. Let z_0 be a point in P . The quasi-properness of β implies that there exists an open neighborhood V of z_0 in P and a relatively compact open set W in N such that any irreducible component Γ of any fiber $\beta^{-1}(z)$ for $z \in V$ meets W . As W is relatively compact and α quasi-proper, there exists a relatively compact open set U in M such any irreducible component Γ of a fiber $\alpha^{-1}(y)$ with $y \in W$ meets U . Take now $z \in V$ and Δ an irreducible component of $(\beta \circ \alpha)^{-1}(z)$. Let Γ be the irreducible component of $\beta^{-1}(z)$ in which $\alpha(\Delta)$ is dense (see above). As Γ meets W the dense set $\alpha(\Delta)$ in Γ meets the non empty open set $\Gamma \cap W$ of Γ . Let y be a point in $\alpha(\Delta) \cap W$ such that there exists a point x in Δ with $\alpha(x) = y$ and such that x is a smooth point in $(\beta \circ \alpha)^{-1}(z)$. Such a point x exists because we may choose y to be a smooth point in $\beta^{-1}(z)$ and also in the image of the open dense set in Δ of points which are smooth in $(\beta \circ \alpha)^{-1}(z)$. Now an irreducible component Γ of $\alpha^{-1}(y)$ containing x has to be contained in Δ as it is contained in $(\beta \circ \alpha)^{-1}(z)$ and contains x . But such a Γ meets U as y is in W . So Δ meets U . This gives the quasi-properness of $\beta \circ \alpha$. \blacksquare

The next lemma will be used later on.

Lemma 3.1.7 *Let $\tau : M_1 \rightarrow M$ be a modification and $\alpha : M \rightarrow N$ a holomorphic map between irreducible complex spaces. Assume that $\alpha \circ \tau$ is a GF map. Then α is a GF map.*

PROOF. Our assumption gives a holomorphic map $\varphi : N \rightarrow \mathcal{C}_n(M_1)$ which classifies the fibers of $\tau \circ \alpha$. Denote by Σ the center of τ . Then the corollary 2.1.10 implies that for generic $y \in N$ no irreducible component of $|\varphi(y)|$ is contained in $\tau^{-1}(\Sigma)$ (see the corollary 2.1.10); then the direct image $\tau_*(\varphi(y))$ has its support equal to $\alpha^{-1}(y)$. This means that the composition of φ with the direct image of n -cycle by τ gives a holomorphic map $\psi : N \rightarrow \mathcal{C}_n(M)$ such that for generic $y \in N$ we have $\psi(y) = |\psi(y)| = \alpha^{-1}(y)$. As α is n -equidimensional because $\dim \alpha^{-1}(z) \leq \dim(\alpha \circ \tau)^{-1}(z)$ for each $z \in N$, the lemma 3.1.2 allows to conclude that α is a GF map. ■

Lemma 3.1.8 *Let $\alpha : M \rightarrow T$ and $p : T \rightarrow N$ holomorphic maps between normal complex spaces such that p is proper finite and surjective and $p \circ \alpha$ is GF. Then α is a GF map.*

PROOF. As N is normal, it is enough to show that α is quasi-proper and equidimensional. Let δ be an irreducible component of $\alpha^{-1}(t)$ for some $t \in T$. As p has finite fibers, δ is an irreducible component of $(p \circ \alpha)^{-1}(p(t))$ and so has pure dimension $n := \dim M - \dim N$. So α is n -equidimensional and any irreducible component of the fiber of α at a point $t \in T$ is an irreducible component of the fiber of $p \circ \alpha$ at the point $p(t)$.

Let K be a compact subset in T ; then $p(K)$ is compact and there exists a compact set L in M such that for any $y \in p(K)$ any irreducible component γ of $(p \circ \alpha)^{-1}(y)$ meets L . Now an irreducible component of a fiber at a point $t \in K$ is an irreducible component of a fiber of $p \circ \alpha$ at the point $p(t) \in p(K)$. So it meets the compact L . ■

It is a simple exercise to show an analogous result for wGF maps instead of GF maps (and the normality assumptions can be omitted).

We conclude this paragraph on GF maps by the following “embedding theorem”.

Theorem 3.1.9 *Let $\pi : M \rightarrow N$ a holomorphic GF map between irreducible complex spaces. Then the fiber map $\varphi : N \rightarrow \mathcal{C}_n^f(M)$ is a proper holomorphic embedding.*

PROOF. The first step of the proof is to show that the map φ is proper, that is to say that φ is closed with compact fibers. But as this map is clearly injective we only have to prove that for any closed subset $F \subset N$ its image $\varphi(F)$ is closed in $\mathcal{C}_n^f(M)$. So consider a sequence (y_ν) of points in F such that the sequence $(\varphi(y_\nu))$ converges to an element $C_0 \in \mathcal{C}_n^f(M)$. Let $W \subset\subset M$ a relatively compact open set in M such that each irreducible component of $|C_0|$ meets W . As $\Omega(W)$ is an open set in $\mathcal{C}_n^f(M)$ containing C_0 the cycle $\varphi(y_\nu)$ is in $\Omega(W)$ for ν large enough. So choose for each such ν a point x_ν in $W \cap \pi^{-1}(y_\nu)$. Then each x_ν is in the compact set $\bar{W} \cap \pi^{-1}(F)$. Up to pass to a sub-sequence we may assume that the sequence (x_ν) converges to a point $x \in \bar{W} \cap \pi^{-1}(F)$. Then $y := \pi(x)$ is in F and is the limit of the sequence (y_ν) ,

so we have $C_0 = \varphi(y)$ with $y \in F$. So the map φ is closed and then proper. Now the theorem 2.3.2¹⁴ implies that $N_\pi := \varphi(N)$ is a closed analytic subset which is an irreducible complex space and $\varphi : N \rightarrow N_\pi$ is a holomorphic homeomorphism. To complete the proof of the theorem we have to show that any germ of holomorphic function at a point $y \in N$ is the pull-back of a germ of holomorphic function of N_π at the point $\varphi(y)$.

So fix a point $y \in N$ and a holomorphic germ $f \in \mathcal{O}_{N,y}$. Choose a smooth point x of $|\varphi(y)|$. Let $E := (U, B, j)$ be a n -scale on M adapted to $\varphi(y)$ such that x is in $j^{-1}(U \times B)$ and such that $\varphi(y) \cap j^{-1}(U \times B) = k \cdot j^{-1}(U \times \{0\})$. Then we can assume that the holomorphic germ $f \in \mathcal{O}_{N,y}$ is define on the open set $\pi(j^{-1}(U \times B))$ (remember that π is open as it is a GF map). Now choose a function $\rho \in \mathcal{C}_c^\infty(U)$ such that $\int_U \rho(t).dt \wedge d\bar{t} = 1/k$. Define the (n, n) -form $\omega := f(\pi(j^{-1}(t, x))).\rho(t).dt \wedge d\bar{t}$ on $U \times B$; it has a B -proper support and it is $\bar{\partial}$ -closed. Then it induces by integration on cycles a holomorphic function on the open set $\Omega_k(E)$ ¹⁵ of $\mathcal{C}_n^f(M)$ via the integration map

$$\int_{\square} \omega : H(\bar{U}, \text{Sym}^k(B)) \rightarrow \mathbb{C} \quad \text{given by} \quad X \mapsto \int_X \omega.$$

On any $\varphi(z) = C \in \varphi(N) \cap \Omega_k(E)$ this holomorphic function takes the value $f(z)$, and this proves that $\varphi : N \rightarrow N_\pi$ is an isomorphism of complex spaces. \blacksquare

Note that, conversely, if $\varphi : N \rightarrow \mathcal{C}_n^f(M)$ is a proper holomorphic embedding, a necessary and sufficient condition for φ to be the fiber map of a GF homorphic map $\pi : M \rightarrow N$ is the fact that for y generic in N the cycle $\varphi(y)$ is reduced and that the projection $p : G \rightarrow M$ of the graph $G \subset N \times M$ of the f-analytic family parametrized by N is an isomorphism onto M . Of course this implies that for $y \neq y'$ in N the cycles $\varphi(y)$ and $\varphi(y')$ are disjoint.

3.2 Strongly quasi-proper map.

Recall now, for the convenience of the reader, the definition of a strongly quasi-proper map given in [B.13], using the terminology introduced above. Consider the following situation :

THE STANDARD SITUATION. Let $\pi : M \rightarrow N$ be a holomorphic map between a pure dimensional complex space M and an irreducible complex space N . Define the integer n as $n := \dim M - \dim N$. Assume that there is a closed analytic subset $\Sigma \in N$ with no interior point in N such that the restriction $\pi' : M \setminus \pi^{-1}(\Sigma) \rightarrow N \setminus \Sigma$ is a GF map. Let $\varphi' : N \setminus \Sigma \rightarrow \mathcal{C}_n^f(M)$ the holomorphic map classifying the fibers of π' (as n -cycles in M) and $\Gamma \subset (N \setminus \Sigma) \times M$ the graph of this family.

¹⁴In the proper case ; for the generalization of Remmert's direct image theorem with value in a open set in a Banach space see [B-M] ch.III th. 7.3.1.

¹⁵see [B.75] ch. IV or [B-M] ch.IV

REMARK. For a quasi-proper surjective map $\pi : M \rightarrow N$ between irreducible complex spaces, with $n := \dim M - \dim N$, there always exists a closed analytic subset $\Sigma \subset N$ with no interior point in N such that the following restriction of π $\pi' : M \setminus \pi^{-1}(\Sigma) \rightarrow N \setminus \Sigma$ is a GF map (see the end of the proof of the proposition 3.2.2 for details).

Definition 3.2.1 *In the standard situation described above we shall say that π is strongly quasi-proper (a **SQP map** for short) if and only if the closure $\bar{\Gamma}$ of Γ in $N \times \mathcal{C}_n^f(M)$ is proper over N .*

Of course we shall show that a SQP map is quasi-proper in the usual sense (see the proof of the next proposition).

We begin by an improvement of the criterium given in [B.13] in order that a holomorphic map $\pi : M \rightarrow N$ between irreducible complex spaces will be SQP.

Proposition 3.2.2 *Let $h : M \rightarrow N$ a holomorphic map between a pure dimensional complex space M and an irreducible complex space N . Put $n := \dim M - \dim N$. Assume that there exists a dense subset Λ in N such that $h^{-1}(\Lambda)$ is dense in M and such that for each $y \in \Lambda$ the fiber $h^{-1}(y)$ is non empty, reduced¹⁶ and of (pure) dimension n with finitely many irreducible components. Note $\gamma : \Lambda \rightarrow \mathcal{C}_n^f(M)$ the map defined by $\gamma(y) := h^{-1}(y)$ where the n -cycle $\gamma(y)$ is reduced. Let Γ the graph of the map γ and $\bar{\Gamma}$ the closure of Γ in $N \times \mathcal{C}_n^f(M)$. Our main assumption is now the following:*

- *The natural projection $\tau : \bar{\Gamma} \rightarrow N$ is proper.*

Then the map h is strongly quasi-proper.

PROOF. The empty n -cycle is open (and closed) in $\mathcal{C}_n^f(M)$ so any sequence converging to it is stationary. As Λ is dense in N and $\gamma(y) \neq \emptyset_n$ for $y \in \Lambda$, any fiber of τ cannot be equal to $\{\emptyset_n\}$. But $\tau(\bar{\Gamma}) = N$ as it is closed and contains Λ . This implies that $h(M) = \tau(\bar{\Gamma}) = N$ and h is surjective.

Our second step (and the main step in fact) will be the proof that the map h is quasi-proper. So fix a point $y \in N$ and let V be an open relatively compact neighbourhood of y in N . Fix $y' \in V$ and choose an irreducible component C of $h^{-1}(y')$ ¹⁷. Let x' be a (generic) point in C such that x' does not belong to any other irreducible component of $h^{-1}(y')$. Then we can choose a sequence $(x_\nu)_{\nu \geq 0}$ in $h^{-1}(\Lambda)$ converging to x' . For $\nu \gg 1$ we have $h(x_\nu) \in V$ so the cycles $\gamma(h(x_\nu))$ are in the compact set $p_2(\tau^{-1}(\bar{V}))$ of $\mathcal{C}_n^f(M)$. Up to pass to a sub-sequence, we can assume that the sequence $(\gamma(h(x_\nu)))$ converges to a cycle $\delta \in \mathcal{C}_n^f(M)$. As we have $x_\nu \in \gamma(h(x_\nu))$ for each ν we have $x' \in |\delta|$. By compactness of the subset $p(\tau^{-1}(\bar{V}))$ of $\mathcal{C}_n^f(M)$,

¹⁶We mean here reduced as a cycle ; this is equivalent to the fact that the natural structure of complex space on this fiber is generically reduced “in the algebraic sense”.

¹⁷From the surjectivity of h proved above, $h^{-1}(y')$ is not empty.

there exists a compact subset K in M such that any irreducible component of any cycle in $p(\tau^{-1}(\bar{V}))$ meets K . So this is the case for each irreducible component of any cycle $\gamma(h(x_\nu))$ and of δ . Let δ_0 be an irreducible component of δ containing x' . So δ_0 meets K . But $|\delta|$ is contained in $h^{-1}(y')$ because the condition for a n -cycle in $\mathcal{C}_n^f(M)$ to be contained in a fiber of h is a closed condition (see the proposition 2.1.7). As δ_0 is irreducible, contained in $h^{-1}(y')$ and contains x' , this implies $\delta_0 \subset C$ and so C meets K . So we have proved that for any $y \in N$ there exists an open neighbourhood V of y in N and a compact set K in M such that for any $y' \in V$ and any irreducible component C of $h^{-1}(y')$ the intersection $C \cap K$ is not empty. This is the definition of the quasi-properness of the map h .

Now we conclude that the subset Σ_0 of N of y such that $\dim h^{-1}(y) > n$ is a closed analytic set with no interior point because Σ_0 is the image by h of the closed analytic subset

$$Z := \{x \in M / \dim_x(h^{-1}(h(x))) > n\}$$

in M which is an union of irreducible components of fibers of h ; so the restriction of h to this analytic subset is still quasi-proper. So Kuhlmann's theorem gives the conclusion. Now let Σ_1 be the closed analytic subset of non normal points in N and let $N' := N \setminus (\Sigma_0 \cup \Sigma_1)$. Then the map $h' : h^{-1}(N') \rightarrow N'$ is quasi-proper equidimensional on a normal complex space. Then we have a holomorphic map

$$\varphi : N' \rightarrow \mathcal{C}_n^f(M)$$

classifying the generic fibers of h' . Of course, for $y \in \Lambda \cap N'$ we have $|\varphi(y)| = \gamma(y)$. But the subset T of points $y \in N'$ such that $\varphi(y)$ is not a reduced cycle is a closed analytic subset of N' (see the lemma 2.1.6) which has no interior point. So the dense subset Λ meets the dense open set $N' \setminus T$ in a dense subset in N . This implies that the closure of the graph of φ in $N \times \mathcal{C}_n^f(M)$ is equal to $\bar{\Gamma}$. Then, by definition, the map h is strongly quasi-proper. ■

An immediate corollary of this result is a first stability result of SQP maps.

Corollaire 3.2.3 *Let $\pi : M \rightarrow N$ be a strongly quasi-proper holomorphic map between irreducible complex spaces and $\tau : T \rightarrow N$ a holomorphic map from an irreducible complex space T such that $\tau(T)$ is not contained in A the set of points in N having "big" fibers for π . Then for any irreducible component $\tau^*(M)_i$ of $(T \times_N M)$ which surjects on T the pull-back $\tau^*(\pi)_i : \tau^*(M)_i \rightarrow T$ is strongly quasi-proper.* ■

We give now a characterization of SQP maps in term of GF maps, which is variant of the important theorem 2.4.4 of [B.13]; see [M.00] or the Appendix (section 4) for a complete proof.

Theorem 3.2.4 *Let $\pi : M \rightarrow N$ be a surjective holomorphic map between irreducible complex spaces. The map π is strongly quasi-proper if and only if there exists a modification $\sigma : \tilde{N} \rightarrow N$ such that the strict transform $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$ by σ of the map π is a GF map.*

PROOF. Assume first that π is a SQP map. Then there exists a modification $\sigma : \tilde{N} \rightarrow N$ and a holomorphic map $\tilde{\varphi} : \tilde{N} \rightarrow \mathcal{C}_n^f(M)$ extending holomorphically the map $\varphi : N \rightarrow \mathcal{C}_n^f(M)$ classifying the generic fibers of π (see section 4). Then define the holomorphic map $\psi : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$ by $\psi(\tilde{y}) := \{\tilde{y}\} \times \tilde{\varphi}(y)$ as a cycle in $\tilde{N} \times M$. The holomorphy is consequence of the product theorem for analytic families of cycles (see [B.75] or [B.M] ch.IV) and these cycles are in $\tilde{M} := M \times_{N, str} \tilde{N}$ because this is true for generic $\tilde{y} \in \tilde{N}$ and so for all $\tilde{y} \in \tilde{N}$ by continuity of ψ . Then we can conclude that $\tilde{\pi}$ is a GF map applying the lemma 3.1.2.

Assume now that we have a modification $\sigma : \tilde{N} \rightarrow N$ such that the strict transform $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$ of π by σ is a GF map. Then the composition of the holomorphic map $\psi : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$ classifying the fibers of $\tilde{\pi}$ with the direct image of cycles by the modification $\tilde{\sigma} : \tilde{M} \rightarrow M$ induced by the projection on M of the fiber product $M \times_{N, str} \tilde{N}$ gives a holomorphic map $\tilde{\varphi} : \tilde{N} \rightarrow \mathcal{C}_n^f(M)$. It is clear that, if Σ is the center of σ , the restriction of $\tilde{\varphi}$ is the map φ on $N \setminus \Sigma$. So, applying the theorem 3.1.9 we see that π is a SQP map because the closure of the graph of its fiber map is the image of \tilde{N} which is proper on N . ■

Remark that in the previous theorem we can replace GF by wGF thanks to lemma 3.1.3.

Proposition 3.2.5 *A quasi-proper surjective holomorphic map between irreducible complex spaces is SQP if and only if its fiber map, defined and holomorphic on a dense Zariski open set in M ¹⁸, is meromorphic. Moreover, if $\pi : M \rightarrow N$ is a quasi-proper surjective map such that its fiber map is meromorphic, then the strict transform $\tilde{\pi} : \tilde{M} \rightarrow N_\pi$ of π by the modification $\tau : N_\pi \rightarrow N$ given by the graph of the meromorphic fiber map (see the paragraph 1.4) is a GF map.*

PROOF. Consider a SQP holomorphic map $\pi : M \rightarrow N$. Then, thanks to the theorem 3.2.4, there exists a modification $\sigma : \tilde{N} \rightarrow N$ such that the strict transform $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$ of π by σ is a GF map. So there exists a holomorphic map $\tilde{\varphi} : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$ classifying the fibers of $\tilde{\pi}$. By composition with the direct image of cycles by the modification $\tilde{\sigma} : \tilde{M} \rightarrow M$ we obtain, thanks to the direct image theorem for cycles by a proper map (see [B.M] ch.IV), a holomorphic map $\varphi := \tilde{\sigma}_* \circ \tilde{\varphi} : \tilde{N} \rightarrow \mathcal{C}_n^f(M)$.

Conversely assume that $\pi : M \rightarrow N$ is a quasi-proper surjective holomorphic map between irreducible complex spaces such that the fiber map (defined on a dense Zariski open set) is meromorphic; let $\tau : N_\pi \rightarrow N$ be the modification given by the graph N_π of the meromorphic fiber map of π (see the paragraph 1.4). Then we have a holomorphic map $\tilde{\varphi} : N_\pi \rightarrow \mathcal{C}_n^f(M)$ which extends the fiber map. Then to prove that the strict transform $\tilde{\pi} : \tilde{M} \rightarrow N_\pi$ of π by τ is a GF map, consider the holomorphic map $\psi : N_\pi \rightarrow \mathcal{C}_n^f(M \times_N N_\pi)$ given by $\tilde{y} \mapsto \tilde{\varphi}(\tilde{y}) \times \{\tilde{y}\}$. It is holomorphic by

¹⁸It is enough to take the set of normal points in N intersected with the complement of the “big fibers” subset as in the remark following the definition of the standard situation (see the beginning of the paragraph 3.2).

the product theorem for analytic families of cycles (see [B.75] or [B-M] ch. IV). As for any \tilde{y} in N_π the cycle $\psi(\tilde{y})$ is in $\tilde{M} := M \times_{N, str} N_\pi$ this map is holomorphic with values in $\mathcal{C}_n^f(\tilde{M})$ and it is then easy to see that it is the (holomorphic) fiber map for $\tilde{\pi}$. So $\tilde{\pi}$ is a GF map. ■

Definition 3.2.6 *Let $\pi : M \rightarrow N$ be a SQP map. With the notations introduced above, the map $\tilde{\varphi} := \tau_* \circ \psi : N_\pi \rightarrow \mathcal{C}_n^f(M)$ will be called a **classifying map for the family of fibers of π** .*

The following proposition summarizes some basic properties of a SQP map.

Proposition 3.2.7 *Let $\pi : M \rightarrow N$ be a strongly quasi-proper holomorphic map between irreducible complex spaces. Then we have*

- i) *the map π is quasi-proper.*
- ii) *the locus Σ_0 of “big fibers” of π , $\Sigma_0 := \{y \in N \mid \dim \pi^{-1}(y) > n\}$, where $n := \dim M - \dim N$, is a closed analytic subset in N .*
- iii) *Let $\Sigma \subset N$ be the closed analytic nowhere dense subset which is the union of the “big fibers” subset and the subset of non normal points in N . There exists a holomorphic map $\varphi : N \setminus \Sigma \rightarrow \mathcal{C}_n^f(M)$ classifying the generic fibers of π and the map φ is meromorphic along Σ .*
- iv) *Let $\tau : N_\pi \rightarrow N$ the projection on N of the graph of the meromorphic map φ and $\tilde{\varphi} : N_\pi \rightarrow \mathcal{C}_n^f(M)$ its second projection. For each $y \in N$ we have*

$$\pi^{-1}(y) = \cup_{z \in \tau^{-1}(y)} |\tilde{\varphi}(z)|,$$

that is to say that any “big fiber” of π is filled up by limits of generic fibers.

PROOF. The properties i) and ii) has been proved in the proof of the proposition 3.2.2. The point iii) is proved in the proposition 3.2.5. To prove iv) consider a point $x \in M$ and consider a sequence (x_ν) in $\pi^{-1}(N \setminus \Sigma)$ converging to x , where $N \setminus \Sigma$ is a dense open set where φ is holomorphic. This exists by irreducibility of M . Then $y_\nu := \pi(x_\nu)$ converges to $y := \pi(x)$ and $\varphi(y_\nu)$, up to pass to a sub-sequence (τ is proper), converges to a point $\tilde{y} \in \tau^{-1}(x)$. We have $x_\nu \in |\varphi(y_\nu)|$ for each ν so that x is in $|\tilde{\varphi}(\tilde{y})|$. ■

REMARK. For an equidimensional surjective holomorphic map between irreducible complex spaces to be strongly quasi-proper is equivalent to be a quasi-proper map. But if a surjective holomorphic map is not equidimensional, the condition to be quasi-proper is not strong enough; in particular it is not stable by taking the strict transform of $\pi : M \rightarrow N$ by a modification on N . Of course, our definition of a SQP map will be stable by such an operation. But we shall see that this notion enjoys

many other stability properties.

The reader can also see interesting applications of this notion in [B.13] in the construction of meromorphic quotients for a large class of non proper analytic equivalence relations.

EXAMPLE. Let $Y := \{((a, b), (x, y)) \in \mathbb{C}^2 \times \mathbb{C}^2 / a.x^2 + b.x - a^2.y^2 = 0\}$ and let $\pi : Y \rightarrow \mathbb{C}^2$ the map induced by the first projection. Then we have the following properties

- I) The (algebraic) hypersurface Y of \mathbb{C}^4 is irreducible (in fact normal and connected).
- II) The map $\pi : Y \rightarrow \mathbb{C}^2$ is quasi-proper.
- III) The map π is not strongly quasi-proper.
- IV) More precisely, after blowing-up the origin in \mathbb{C}^2 the strict transform of π is no longer quasi-proper.

PROOF OF I). The critical set of the polynomial $P(a, b, x, y)$ is given by the following equations

$$2a.x + b = 0, \quad 2a^2.y = 0, \quad x^2 - 2a.y^2 = 0, \quad x = 0 \quad (1)$$

So the subset $S := \{a = b = x = 0\} \cup \{x = y = b = 0\}$ which is one dimensional and is contained in Y is the singular subset of Y . So the singular set of Y is exactly S . As it has codimension 2 in Y , the hypersurface Y is normal. We shall see that each fiber of π is connected and then the fact that $\mathbb{C}^2 \times \{0\}$ is a section of π implies that Y is connected. So Y is irreducible.

PROOF OF II). First we shall describe the fibers of π . For $a.b \neq 0$ the fiber $\pi^{-1}(a, b)$ is a smooth conic containing the origin in \mathbb{C}^2 . For $a \neq 0$ and $b = 0$ the fiber $\pi^{-1}(a, 0)$ is the union of two distinct lines through the origin. For $a = 0$ and $b \neq 0$ the fiber $\pi^{-1}(0, b)$ is the line $x = 0$ which also contains the origin. Finally the fiber $\pi^{-1}(0, 0)$ is \mathbb{C}^2 . So each fiber is connected and contains the origin. Then the π -proper set $\mathbb{C}^2 \times \{0\}$ meets every irreducible component of any fiber of π , so this map is quasi-proper.

PROOF OF III). Consider now the map $f_s : \mathbb{C} \rightarrow \mathbb{C}^2$ given by $f_s(b) := (b.s, b)$ where s is a non zero complex number. Then for $b \neq 0$ the fiber of π at the point $f_s(b)$ is given by the equation $b.s.x^2 + b.x - b^2.s^2.y^2 = 0$ and assuming that $b \neq 0$ this is equivalent to

$$x.(s.x + 1) - b.s^2.y^2 = 0.$$

Now, for $s \neq 0$ fixed, the limit of this non degenerate conic when b goes to 0 is the union of the lines $\{x = 0\}$ and $\{x = -1/s\}$. And it is clear that when $|s|$ goes to 0, the component $\{x = -1/s\}$ gets out of any compact set in \mathbb{C}^2 . This phenomenon of escape at infinity near the point $(0, 0)$ in the target of π implies that the map π cannot be strongly quasi-proper.

PROOF OF IV). Consider now the blow-up $\tau : X \rightarrow \mathbb{C}^2$ of the (reduced) origin in \mathbb{C}^2 . The complex manifold X is the sub-manifold

$$X := \{((a, b), (\alpha, \beta)) \in \mathbb{C}^2 \times \mathbb{P}_1 / a.\beta = b.\alpha\}.$$

It will be enough to show that the strict transform of π over the chart $\{\beta \neq 0\}$ of X is not quasi-proper to achieve our goal. So let $s := \alpha/\beta$. Then we have coordinates $(s, b) \in \mathbb{C}^2$ for this chart on X . The total transform of Y is given by the equation

$$s.b.x^2 + b.x - s^2.b^2.y^2 = 0$$

and, as the function b is not generically zero on the strict transform \tilde{Y} of Y by τ , we have

$$\tilde{Y}_{\beta \neq 0} = \{((s, b), (x, y)) \in \mathbb{C}^2 \times \mathbb{C}^2 / x.(s.x + 1) - b.s^2.y^2 = 0\}.$$

So the fiber of the strict transform $\tilde{\pi}$ at the point $(s, 0)$ is the union of the two lines $\{x = 0\}$ and $\{x = -1/s\}$ for $s \neq 0$. Then it is clear that this map is not quasi-proper as an irreducible component of the fiber at $(0, s)$, $s \neq 0$ gets out of any compact set in \mathbb{C}^2 when $s \neq 0$ goes to 0. \square

IMPORTANT REMARK. The previous example is **algebraic**. And in opposition of the context where the notion of a strongly quasi-proper map has been introduced (in complex analytic geometry), this notion makes sense in algebraic geometry and is, of course, related to the behaviour of the limits of generic fibers of a map in some compactification of the map. But it is independent of the chosen compactification and so it has to be considered also in algebraic geometry.

Note that the phenomenon of “escape at infinity” happens in the algebraic setting, but not the phenomenon of “infinite breaking”¹⁹, which is purely transcendental. \square

In order to prove some more stability results for SQP maps, we shall use the following two lemma.

Lemma 3.2.8 *Let $\alpha : M \rightarrow N$ and $\beta : N \rightarrow P$ be holomorphic surjective maps between irreducible complex spaces and let $\sigma : \tilde{P} \rightarrow P$ be a holomorphic dominant*

¹⁹We say that we have an infinite breaking when the limit in the sense of $\mathcal{C}_n^{loc}(M)$ of a sequence (C_ν) of irreducible cycles has infinitely many irreducible components.

map. Let $\tilde{\beta} : \tilde{N} \rightarrow N$ the strict transform of β by σ and let $\tilde{\alpha} : \tilde{M} \rightarrow \tilde{N}$ the strict transform of α by $\tilde{\sigma}$ the projection of \tilde{N} to N . Then the strict transform of $\beta \circ \alpha$ by σ is the composition of the strict transform of β by σ and the strict transform of α by $\tilde{\sigma}$.

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\tilde{\alpha}} & \tilde{N} & \xrightarrow{\tilde{\beta}} & \tilde{P} \\ \tau \downarrow & & \tilde{\sigma} \downarrow & & \sigma \downarrow \\ M & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & P \end{array}$$

PROOF. For the “usual” fiber product we have a canonical isomorphism

$$M \times_N (N \times_P \tilde{P}) \simeq M \times_P \tilde{P}$$

given by the obvious projection : the inverse is given by $(x, \tilde{z}) \mapsto (x, (\alpha(x), \sigma(\tilde{z})))$. Then our hypothesis allows to see that the strict transforms correspond via this isomorphism. \blacksquare

Lemma 3.2.9 *Let M, N, \tilde{N}, P be irreducible complex spaces, let $\alpha : M \rightarrow N$ and $\tau : \tilde{N} \rightarrow N$ be holomorphic surjective maps and $\sigma : P \rightarrow N$ a dominant holomorphic map. Note α_1 and α_2 the strict transforms of α by τ and σ respectively. Let $P_1 := \tilde{N} \times_{N, str} P$, and note $\tau' : P_1 \rightarrow \tilde{N}$ and $\sigma' : P_1 \rightarrow P$ the projections. Then note α'_1 and α'_2 the strict transforms of α_1 and α_2 by τ' and σ' respectively.*

$$\begin{array}{ccccc} & & M_1 & \xrightarrow{\tilde{\tau}} & M & \xleftarrow{\tilde{\sigma}} & M_2 & & \\ & & \downarrow \alpha_1 & & \downarrow \alpha & & \downarrow \alpha_2 & & \\ M_1 \times_{\tilde{N}} P_1 & & \tilde{N} & \xrightarrow{\tau} & N & \xleftarrow{\sigma} & P & & M_2 \times_P P_1 \\ & & \swarrow \tau' & & \searrow \sigma' & & & & \\ & & & & P_1 & & & & \\ & & \swarrow \alpha'_1 & & & & \swarrow \alpha'_2 & & \end{array}$$

Then $M_1 \times_{\tilde{N}, str} P_1$ is canonically isomorphic to $M_2 \times_{P, str} P_1$.

PROOF. Again we have a canonical isomorphism

$$(M \times_N \tilde{N}) \times_{\tilde{N}} (\tilde{N} \times_N P_1) \simeq (M \times_N P) \times_P (P \times_N \tilde{N})$$

given by the map $((x, \tilde{y}), (\tilde{y}, (\tilde{y}, z))) \mapsto ((x, z), (z, \tilde{y}))$; the inverse map is given by

$$((x, z), (z, \tilde{y})) \mapsto ((x, \tilde{y}), (\tilde{y}, (\tilde{y}, z))).$$

Again our hypothesis allows to see that the strict transforms correspond via this isomorphism. \blacksquare

Proposition 3.2.10 *Let M, N, P be irreducible complex spaces, $\alpha : M \rightarrow N$ be a holomorphic SQP map and $\sigma : P \rightarrow N$ a holomorphic dominant map. Then the strict transform $\alpha_2 : M_2 \rightarrow P$ of α by σ is SQP map.*

PROOF. By assumption, there exists a modification $\tau : \tilde{N} \rightarrow N$ such that the strict transform $\alpha_1 : M_1 \rightarrow \tilde{N}$ is a GF map. If $\tau' : P_1 \rightarrow \tilde{N}$ is the strict transform of σ , it is again dominant and we can apply the lemma 3.2.9 to obtain that the strict transform of α_1 by τ' is a GF map. Now, as the lemma 3.2.9 gives that $\alpha'_2 \simeq \alpha'_1$ the conclusion follows thanks to the lemma 3.1.7.

Proposition 3.2.11 *Let M, N, P be irreducible complex spaces, $\alpha : M \rightarrow N$ be a holomorphic wGF map and $\beta : N \rightarrow P$ a holomorphic SQP map. Then the composed map $\beta \circ \alpha$ is a SQP map.*

PROOF. Let $\sigma : \tilde{P} \rightarrow P$ a modification such that the strict transform $\tilde{\beta} : \tilde{N} \rightarrow \tilde{P}$ is a GF map. Note $\tilde{\sigma} : \tilde{N} \rightarrow N$ the modification induced by the projection. Then the strict transform $\tilde{\alpha}$ of α by σ is a wGF map, thanks to lemma 3.1.5, and the composed map $\tilde{\beta} \circ \tilde{\alpha}$ is then a wGF map using the lemma 3.1.6 ; then $\beta \circ \alpha$ is a SQP map by the remark following the proof of the proposition 3.2.4 and we conclude using the lemma 3.2.8. ■

REMARKS.

1. Note that we don't need that the map β is holomorphic in the proof above : if we compose a wGF map with a meromorphic strongly quasi-proper map²⁰ we obtain a meromorphic strongly quasi-proper map.
2. It is not true that the composition of a modification with a SQP map is again a SQP map. The example below shows that the composition of a modification with a GF map is not quasi-proper in general.

EXAMPLE. Let $\alpha : \mathbb{C}^2 \rightarrow \mathbb{C}$ the projection on the first coordinate. Then α is clearly a GF map. Consider now $\tau : X \rightarrow \mathbb{C}^2$ the blow-up in \mathbb{C}^2 of the reduced ideal defining the closed analytic subset $\mathbb{Z} \times \{0\}$. Then the map $\alpha \circ \tau : X \rightarrow \mathbb{C}$ has a fiber at 0 which is not a finite type cycle. So this map is not quasi-proper but nevertheless equidimensional. □

We conclude this section by another important stability result for strongly quasi-proper holomorphic maps.

Theorem 3.2.12 *Let $\alpha : M \rightarrow N$ be a SQP holomorphic map. Let $Z \subset N$ be a closed analytic irreducible subset in N and X an irreducible component of $\alpha^{-1}(Z)$ which is dominant on Z for the map $\alpha|_X$ induced by α . Then the map $\alpha|_X : X \rightarrow Z$ is SQP.*

The proof of this result will use the following proposition.

²⁰We say that a meromorphic map $h : N \dashrightarrow P$ is strongly quasi-proper if the holomorphic map $\tilde{h} : \tilde{N} \rightarrow P$ which is the projection on P of the graph $\tilde{N} \subset N \times P$ of h is strongly quasi-proper.

Proposition 3.2.13 *Let $\alpha : M \rightarrow N$ be a holomorphic GF map where M is a pure dimensional complex space and N a smooth connected manifold and define $n := \dim M - \dim N$. Then for each integer $q \geq 0$ there is a holomorphic map $\Phi_q : \mathcal{C}_q(N) \rightarrow \mathcal{C}_{n+q}^f(M)$ which is given by $\Phi_q(C) := (p_2)_*(p_1^*(C))$, where $p_1 : G \rightarrow N$ and $p_2 : G \rightarrow M$ are the projections of the graph $G \subset N \times M$ of the f -analytic family of fibers of α .*

REMARKS.

1. As we know that p_2 is an isomorphism, the map $(p_2)_* : \mathcal{C}_{n+q}^f(G) \rightarrow \mathcal{C}_{n+q}^f(M)$ is also bi-holomorphic in a obvious sense (see definition 2.1.4). So we may identify M and G in the proof of the proposition.
2. As we assume that N is smooth, the cycle $p_1^*(C)$ is well defined as a cycle in G^{21} as soon as the codimension of $G \cap (|C| \times M)$ in $N \times M$ is the sum of codimensions of C in N and of G in $N \times M$, so is equal to $\dim N + \dim N - q$. This means that its dimension is equal to $\dim M - \dim N + q = n + q$. But as the fibers of α have pure dimension n this is true for any q -cycle in N .

PROOF OF THE PROPOSITION 3.2.13. Thanks to the remark 2. above the holomorphy of the map $j \circ \Phi_q : \mathcal{C}_q(N) \rightarrow \mathcal{C}_{n+q}^{loc}(M)$, where $j : \mathcal{C}_{n+q}^f(M) \rightarrow \mathcal{C}_{n+q}^{loc}(M)$ is the obvious map, is an immediate consequence of the variant for the pull-back of the intersection theorem for analytic families of cycles (see [B.75] chapter VI or [B-M 2] chapter VII). The only points to show are :

1. for each compact q -cycle $C \in \mathcal{C}_q(N)$ the $(n + q)$ -cycle $\Phi_q(C)$ is a finite type cycle;
2. the map Φ_q is continuous for the topology of $\mathcal{C}_{n+q}^f(M)$.

Fix $C_0 \in \mathcal{C}_q(N)$ and choose a relatively compact open set $V \subset N$ such that $|C_0| \subset V$. As \bar{V} is compact in N there exists a relatively compact open set W in M such that any irreducible component of the fiber $\alpha^{-1}(y)$ for any $y \in V$ meets W . Take now an irreducible component D of $\Phi_q(C')$ where $C' \in \mathcal{C}_q(N)$ such that $|C'| \subset V$. We have $|\Phi_q(C')| = |p_1^*(C')| = G \cap (|C'| \times M) \simeq \alpha^{-1}(|C'|)$. As the restriction of α to $\alpha^{-1}(|C'|)$ has pure n -dimensional fibers over $|C'|$ its restriction to D is a wGF map, thanks to the remark 2 following the definition 3.1.4, and D is an union of irreducible components of fibers of α over $|C'|$. But each such irreducible component meets W because $|C'| \subset V$. So D meets W , and then $\Phi_q(C')$ is a finite type cycle for any such C' .

Now take a sequence $(C_\nu)_{\nu \geq 1}$ in $\mathcal{C}_q(N)$ converging to C_0 . Then it is clear from the discussion above that the sequence $(\Phi_q(C_\nu))$ converges to $\Phi_q(C_0)$ in $\mathcal{C}_{n+q}^{loc}(M)$ and that we have a relatively compact open set W in M such that any irreducible component of any $\Phi_q(C_\nu)$ meets W . So the proposition 2.1.2 implies the convergence of $(\Phi_q(C_\nu))$ to $\Phi_q(C_0)$ in $\mathcal{C}_{n+q}^f(M)$. This is the continuity of Φ_q at C_0 . ■

²¹see [B.75] ch.VI.

PROOF OF THE THEOREM 3.2.12. We begin by the description of such an X :
 Let $\tau : \tilde{N} \rightarrow N$ a modification such that there exists $\varphi : \tilde{N} \rightarrow \mathcal{C}_n^f(M)$ a holomorphic map classifying the generic fibers of α . Let \tilde{Z} an irreducible component of $\tau^{-1}(Z)$ which is surjective onto Z and let $Y := \alpha^{-1}(Z) \times_{Z, str} \tilde{Z}$. The projection $p_2 : Y \rightarrow \tilde{Z}$ is a GF map and the projection $p_1 : Y \rightarrow \alpha^{-1}(Z)$ is proper. Let \tilde{X} be an irreducible component of Y and define $X := p_1(\tilde{X})$. As p_1 is proper X is a closed analytic irreducible subset in $\alpha^{-1}(Z)$. As \tilde{X} is surjective on \tilde{Z} and contains a non empty open set in Y , X is an irreducible component of $\alpha^{-1}(Z)$ and is dominant on Z .
 Conversely, if X is an irreducible component of $\alpha^{-1}(Z)$ and is dominant on Z then $X \times_{Z, str} \tau^{-1}(Z)$ has an irreducible component \tilde{X} surjective on X (p_1 and τ are proper) and $\tilde{Z} := p_2(\tilde{X})$ is an irreducible component of $\tau^{-1}(Z)$ which is surjective on Z .

Now fix X, \tilde{X} and \tilde{Z} as above. Using the geometric flattning theorem (for a proper map see [B-M] chapter IV corollary 9.3.1) and Hironaka desingularization theorem, we can find a modification $\sigma : \hat{Z} \rightarrow \tilde{Z}$ such that we have the following situation :

- i) The proper map $\tilde{\tau} := \tau \circ \sigma : \hat{Z} \rightarrow Z$ is equidimensionnal with \hat{Z} smooth and connected.
- ii) The strict transform $\hat{\alpha} : \hat{X} \rightarrow \hat{Z}$ of p_1 by σ is a GF map.
- iii) The projection $p : \hat{X} \rightarrow X$ is proper and surjective.

$$\begin{array}{ccccccc}
 & & & \hat{X} & \xrightarrow{\hat{\alpha}} & \hat{Z} & \\
 & & & \downarrow & & \downarrow \sigma & \\
 & & Y & \longleftarrow & \tilde{X} & \xrightarrow{p_1} & \tilde{Z} & \longrightarrow & \tilde{N} \\
 & & \downarrow p_2 & & \downarrow & & \downarrow & & \downarrow \tau \\
 M & \longleftarrow & \alpha^{-1}(Z) & \longleftarrow & X & \xrightarrow{\alpha|_X} & Z & \longrightarrow & N
 \end{array}$$

Then we have a holomorphic map $Z_1 \rightarrow \mathcal{C}_q(\hat{Z})$, where Z_1 is the normalization of Z , classifying the generic fibers of $\tilde{\tau}$. Composing this map with the holomorphic map $\Psi_q : \mathcal{C}_q(\hat{Z}) \rightarrow \mathcal{C}_{n+q}^f(M)$ associated via the proposition 3.2.13 to the holomorphic map $\psi : \hat{Z} \rightarrow \mathcal{C}_n^f(M)$ defined as $\psi := \varphi|_{\hat{Z}} \circ \tilde{\tau}$, gives a holomorphic map

$$\hat{Z} \rightarrow \mathcal{C}_{n+q}^f(X) \subset \mathcal{C}_{n+q}^f(M)$$

which classifies the generic fibers of the restriction $\alpha|_X : X \rightarrow Z$. This complete the proof. ■

4 Appendix

4.1 D. Mathieu's flattening theorem.

Our goal is to give a proof of the following result of D. Mathieu (see [M. 00]) using only the proposition 3.2.2, its corollary 3.2.3 and the generalization of Remmert's proper direct image theorem with values in a Banach open set (see [B-M] ch.III).

We shall consider a surjective holomorphic map $\pi : M \rightarrow N$ between irreducible complex spaces which is quasi-proper (in the usual sense). So we are in the standard situation described at the beginning of the paragraph 3.2. With the notations introduced there we shall assume the condition (which is our definition of a SQP map) :

- The projection $p : \bar{\Gamma} \rightarrow N$ is proper, (ⓐ)

where we recall that Γ is the graph of the holomorphic map $\varphi : N \setminus \Sigma \rightarrow \mathcal{C}_n^f(M)$ classifying a f-analytic family of cycles in M , for y generic in $N \setminus \Sigma$, and $\bar{\Gamma}$ its closure in $N \times \mathcal{C}_n^f(M)$.

Theorem 4.1.1 *Let $\pi : M \rightarrow N$ a quasi-proper surjective holomorphic map between irreducible complex spaces and we put $n := \dim M - \dim N$. Assume that (ⓐ) is satisfied. Then there exists a modification $\tau : \tilde{N} \rightarrow N$ such that the strict transform $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$ is a geometrically f-flat map.*

The proof has two steps.

The first step is to prove that it is enough to prove the result locally on Σ . This will use the proper direct image theorem with values in $\mathcal{C}_n^f(M)$ which is a simple consequence (see the theorem 2.3.2) of the generalization of Remmert's theorem with values in a Banach space (see [B-M] ch.III). This step is given by the proposition 4.1.2.

The second step is devoted to the proof that the result is locally true on Σ . This will be done by induction on the dimension of the fibers of π . The induction step is given by the proposition 4.1.4.

Proposition 4.1.2 *Assume that we have a modification $\tau : \tilde{N} \rightarrow N$ such that the conclusion of the theorem is valid. Then $\bar{\Gamma} \subset N \times \mathcal{C}_n^f(M)$ is a closed analytic subset and the sheaf of holomorphic functions on $N \times \mathcal{C}_n^f(M)$ induces on $\bar{\Gamma}$ a structure of reduced complex space.*

PROOF. Let $\tilde{\varphi} : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$ be the holomorphic map classifying the fibers of the strict transform $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$ of π by the modification $\tau : \tilde{N} \rightarrow N$. The composition with the direct image $\tau_* : \mathcal{C}_n^f(\tilde{M}) \rightarrow \mathcal{C}_n^f(M)$ of cycles by the proper map τ gives a holomorphic map $\psi := \tau_* \circ \tilde{\varphi}$. Then define $\chi := (\tau, \psi) : \tilde{N} \rightarrow N \times \mathcal{C}_n^f(M)$. On a dense open set in \tilde{N} the holomorphic map χ takes its values in Γ . So, if we prove that χ is a proper map we can conclude, first that its image is $\bar{\Gamma}$ and then using the

proper direct image in this context we shall conclude that $\bar{\Gamma}$ is a (locally finite dimensional) reduced complex space with the sheaf of holomorphic functions induced from $N \times \mathcal{C}_n^f(M)$.

As each fiber of χ is a closed subset in a fiber of τ , the fibers are compact. We have now to prove that the map χ is closed. Let F be a closed set in \tilde{N} and choose a sequence (\tilde{y}_ν) be a sequence in F such that the sequence $(\chi(\tilde{y}_\nu))$ converges to $(y, C) \in N \times \mathcal{C}_n^f(M)$. Then, up to pass to a sub-sequence, we can assume that the sequence (\tilde{y}_ν) converges to some $\tilde{y} \in \tau^{-1}(y) \cap F$ by the properness of τ . Then by continuity of χ we have $\chi(\tilde{y}) = (y, C)$ and so $\chi(\tilde{y})$ is in $\chi(F)$. ■

Corollaire 4.1.3 *Let $\pi : M \rightarrow N$ be a holomorphic map between irreducible complex spaces, and assume that we are in the standard situation. Assume also that for each point σ in Σ there exists an open neighbourhood V_σ of σ in N and a modification $N_\sigma \rightarrow V_\sigma$ and a holomorphic map $\varphi_\sigma : N_\sigma \rightarrow \mathcal{C}_n^f(M)$ extending $\varphi|_{V_\sigma \setminus \Sigma}$. Then there exists a unique modification $\tau : \tilde{N} \rightarrow N$ with a holomorphic map $\tilde{\varphi} : \tilde{N} \rightarrow \mathcal{C}_n^f(M)$ extending φ and with the following universal property :*

- *For any modification $\tau_U : N_U \rightarrow U$ of an open set U in N on which there exists a holomorphic map $\varphi_U : N_U \rightarrow \mathcal{C}_n^f(M)$ extending φ on $U \setminus \Sigma$, there exists a unique holomorphic map $\theta_U : N_U \rightarrow \tau^{-1}(U) \subset \tilde{N}$ compatible with the projections on U , such that $\varphi_U = \tilde{\varphi} \circ \theta_U$.*

PROOF. This corollary is an obvious consequence of the definition of a meromorphic map of N with values in $\mathcal{C}_n^f(M)$, the definition of its graph, and the universal property of the projection of this graph on N because the content of the previous proposition is precisely the meromorphy of φ along Σ . Then the universal property gives the patching for free. ■

Note that we only use the direct image theorem (with values in $\mathcal{C}_n^f(M)$) only in the proper case for the results above (and also for defining the graph of a meromorphic map with values in $\mathcal{C}_n^f(M)$).

Proposition 4.1.4 *Assume that in the situation of the theorem 4.1.1 the map π satisfies (ⓐ). Let $y_0 \in N$ and assume that $\dim \pi^{-1}(y_0) = n + k$ with $k \geq 1$, then there exists an open neighbourhood V of y_0 in N and a modification $\tau : \tilde{V} \rightarrow V$ such that the strict transform $\tilde{\pi}_V : \tilde{M}_V \rightarrow \tilde{V}$ has fibers of dimension at most $n + k - 1$, where we denote $M_V := \pi^{-1}(V)$ and $\pi_V : M_V \rightarrow V$ the restriction of π .*

First note that the following lemma has already been proved in the proof of the proposition 3.2.2.

Lemma 4.1.5 *Let $\pi : M \rightarrow N$ be a holomorphic map between irreducible complex spaces. Assume that there exists a closed analytic subset Σ with no interior point*

in N such that the restriction of π to $M \setminus \pi^{-1}(M) \rightarrow N \setminus \Sigma$ is quasi proper and n -equidimensional, and a holomorphic map $\varphi : N \setminus \Sigma \rightarrow \mathcal{C}_n^f(\tilde{M})$ such that for y generic in $N \setminus \Sigma$ we have $\varphi(y) = |\varphi(y)| = \pi^{-1}(y)$. Assume that the condition (©) is satisfied. Then π is quasi-proper. ■

Remark that, thanks to the corollary 3.2.3 we keep the hypothesis (©) for $\tilde{\pi}_V$, and then, thanks to the lemma 4.1.5, the map $\tilde{\pi}$ is again quasi-proper. Remark that it is not true in general that the strict transform by a modification of a quasi-proper map is again quasi-proper (see the example following the proposition 3.2.7).

PROOF OF THE PROPOSITION 4.1.4. As π is quasi-proper, there exists only finitely many irreducible components of $\pi^{-1}(y_0)$. Let $\Gamma_1, \dots, \Gamma_N$ be the irreducible components of $\pi^{-1}(y_0)$ which have dimension $n + k$. Choose on each of them a generic point $x_i, i \in [1, N]$, and adapted $(n + k)$ -scales $E_1, \dots, E_N, E_i = (U_i, B_i, j_i)$ with the following properties:

- i) The polydisc U_i and B_i contain the origin ; let $pr_i : U_i \times B_i \rightarrow U_i$ be the projection.
- ii) The point x_i is in the center of E_i and $pr_i(j_i(x_i)) = (0, 0)$.
- iii) $j_i(\Gamma_i \cap j_i^{-1}(U_i \times B_i)) = U_i \times \{0\}$ and $\deg_{E_i}(\Gamma_i) = 1$.

Let V be an open neighbourhood of y_0 in N such that $\pi^{-1}(V) \cap j_i^{-1}(\bar{U}_i \times \partial B_i) = \emptyset$ for each $i \in [1, N]$. This exists because the compact set $\cup_{i=1}^N j_i^{-1}(\bar{U}_i \times \partial B_i)$ does not meet $\pi^{-1}(y_0)$. Let $W_i := \pi^{-1}(V) \cap j_i^{-1}(U_i \times B_i)$ and consider the holomorphic map $\theta_i : W_i \rightarrow V \times U_i$ given by $(\pi, pr_i \circ j_i)$. As $\theta_i^{-1}(y_0, 0) = \{x_i\}$, up to shrink V , we can assume that each map θ_i is proper and finite. We know that for generic y in V the analytic set $\theta_i(\pi^{-1}(y) \cap W_i)$ has dimension n so cannot contains $\{y\} \times U_i$; then the closed analytic set

$$Z_i := \{y \in V / \{y\} \times U_i \subset \theta_i(\pi^{-1}(y) \cap W_i)\}$$

has empty interior in V . Let \mathcal{I}_i be a coherent ideal in \mathcal{O}_V such the strict transform of $\pi : W_i \rightarrow V$ by the blowing-up of \mathcal{I}_i has no longer a fiber of dimension $\geq n + k$. See [B-M] ch.III proposition 6.1.5 for the existence of such a coherent ideal with $Supp(\mathcal{O}_V/\mathcal{I}_i) \subset Z_i$. So consider the blow-up $\tau : \tilde{V} \rightarrow V$ in V of the ideal \mathcal{I} which is the product of the \mathcal{I}_i for $i \in [1, N]$. It will give a strict transform $\tilde{\pi}_V : \tilde{M}_V \rightarrow \tilde{V}$ of $\pi|_V$ by τ whose fibers has dimension at most $n + k - 1$, concluding the proof. ■

PROOF OF THE THEOREM 4.1.1. First we shall consider the case where all fibers of π have dimension at most n . The lemma 4.1.5 gives that π is quasi-proper and so its image is closed. So the map π is surjective and n -equidimensional. If $\nu : \tilde{N} \rightarrow N$ is the normalization map, then the strict transform $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$ of π is again n -equidimensional and surjective and so we have a holomorphic map

$\tilde{\varphi} : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$ classifying the fibers of $\tilde{\pi}$, meaning that $\tilde{\pi}$ is a GF map.

Assume now that the local version of the theorem is proved when the maximal dimension of a fiber is $n + k - 1$, with $k \geq 1$. Then the proposition 4.1.4 shows that this local version is also true when the maximal dimension of a fiber is $n + k$. This proves the local version of the theorem. But then the corollary 4.1.3 shows that the local version of the theorem implies the theorem itself, concluding the proof. ■

REMARK. The corollary gives a more precise result than what we state in the theorem 4.1.1 : there exists a natural modification of N which factorizes any modification of an open set U in N such that the map $\varphi_{U \setminus \Sigma}$ extends holomorphically on it. So the map φ is meromorphic along Σ and the natural modification of N is simply the graph of this meromorphic map (see 2.4.6).

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