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Claire Lacour

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Erratum to

“Nonparametric estimation of the stationary density and the transition density of a Markov chain” Stochastic Process. Appl. 118 (2008)

Claire Lacour

New proof of Proposition 1

The result of Proposition 1 is true but the proof must be modified in the following way. We replace Lemma 10 by

Lemma 10 *Under the assumptions of Proposition 1, and if (X_n) has an atom A ,*

$$\sum_{\lambda \in \Lambda_m} \mathbb{E} |S_j(\varphi_\lambda)|^2 \leq r_0^2 \mathbb{E}_A(\tau^2) D_m.$$

Proof of Lemma ??: Using a convex inequality, we can write

$$\sum_{\lambda \in \Lambda_m} \mathbb{E} |S_j(\varphi_\lambda)|^2 \leq \sum_{\lambda \in \Lambda_m} \mathbb{E}_\mu \left| \sum_{i=\tau+1}^{\tau(2)} \varphi_\lambda(X_i) \right|^2 \leq \sum_{\lambda \in \Lambda_m} \mathbb{E}_\mu \left((\tau(2) - \tau) \sum_{i=\tau+1}^{\tau(2)} \varphi_\lambda^2(X_i) \right)$$

Assumption M2 entails $\| \sum_{\lambda \in \Lambda_m} \varphi_\lambda \|_\infty \leq r_0^2 D_m$. Then

$$\sum_{\lambda \in \Lambda_m} \mathbb{E} |S_j(\varphi_\lambda)|^2 \leq \mathbb{E}_\mu \left((\tau(2) - \tau) \sum_{i=\tau+1}^{\tau(2)} r_0^2 D_m \right) \leq r_0^2 \mathbb{E}_\mu ((\tau(2) - \tau)^2) D_m$$

To conclude, recall that by the Markov property,

$$\begin{aligned} \mathbb{E}_\mu ((\tau(2) - \tau)^2) &= \sum_k \sum_{l>k} (l - k)^2 \mathbb{P}_\mu(\tau = k, \tau(2) = l) \\ &= \sum_k \sum_{l>k} (l - k)^2 \mathbb{P}(X_{k+1} \notin A, \dots, X_{l-1} \notin A, X_l \in A | X_k \in A) \mathbb{P}_\mu(X_1 \notin A, \dots, X_k \in A) \\ &= \sum_k \sum_{l>k} (l - k)^2 \mathbb{P}_A(X_1 \notin A, \dots, X_{l-k-1} \notin A, X_{l-k} \in A) \mathbb{P}_\mu(\tau = k) \\ &= \sum_k \sum_{j>0} j^2 \mathbb{P}_A(\tau = j) \mathbb{P}_\mu(\tau = k) = \mathbb{E}_A(\tau^2). \end{aligned}$$

□

We can then give the bound

$$\sum_{\lambda \in \Lambda_m} \mathbb{E}(\nu_n^{(3)}(\varphi_\lambda)^2) \leq \frac{r_0^2 \mathbb{E}_A(\tau^2) D_m}{n}.$$

Finally $\mathbb{E}\|f_m - \hat{f}_m\|^2 \leq CD_m/n$ with $C = 4[8r_0^2(\mathbb{E}_\mu(\tau^2) + \mu(A)\mathbb{E}_A(\tau^4)) + r_0^2\mathbb{E}_A(\tau^2)]$.

New proof of Theorem 3

The result of Theorem 3 is true but the proof must be modified in the following way. Proposition 12 must be replaced by:

Proposition 12 *Let (X_n) be a Markov chain which satisfies A1–A5 and $(S_m)_{m \in \mathcal{M}_n}$ be a collection of models satisfying M1–M3. We suppose that (X_n) has an atom A . Let $B(m, m') = \{t \in S_m + S_{m'}, \|t\| = 1\}$ and*

$$p(m, m') = K\mu(A)\mathbb{E}_A(\tau^2)r_0^2 \frac{\dim(S_m + S_{m'})}{n}$$

(where K is a numerical constant). Then

$$\sum_{m' \in \mathcal{M}_n} \mathbb{E} \left[\sup_{t \in B(m, m')} \nu_n^2(t) - p(m, m') \right]_+ = O(n^{-1}).$$

Remark 1 *This gives a penalty in Theorem 3 of the form*

$$\text{pen}(m) = K\mu(A)\mathbb{E}_A(\tau^2)r_0^2 \frac{D_m}{n}, \text{ for some } K > K_0$$

with K_0 a numerical constant. Note that this penalty is simpler than in the previous version of this theorem. In particular, it does not depend on $\|f\|_\infty$.

Remark 2 *As it can be seen in the proof, Assumption M1 can be relaxed, it is now sufficient to assume that each S_m is a linear subspace of $(L^\infty \cap L^2)([0, 1])$ with dimension $D_m \leq n$. This entails an improvement on the smoothness assumption for Corollary 5 : $\alpha > 0$ is sufficient. In the same way, M1' can be relaxed and the condition for Corollary 8 is only $\alpha > 0$.*

Proof of Proposition ??: The heart of the proof is to use Theorem 7 in ? which is a concentration inequality for Markov chains. In our case $T_1 = \tau(1) = \tau$ and $T_2 = \tau(2) - \tau(1)$. Let us check that our assumptions allow us to use this theorem.

- We can easily prove that our assumption A4 implies the Minorization Condition with $m = 1$ in ?. Indeed, since $\int h d\mu > 0$, there exists C with measure $\mu(C) > 0$ and $\delta > 0$ such that h is larger than δ on C . Then for all x in C and all event B , $P(x, B) \geq h(x)\nu(B) \geq \delta\nu(B)$. Moreover, fixing $x \in \mathbb{R}$, for n large enough, the ergodicity of the chain gives

$$|P^n(x, C) - \mu(C)| \leq \frac{\mu(C)}{2},$$

which implies $P^n(x, C) \geq \mu(C)/2 > 0$.

- As noticed at the very beginning of his Section 3.5, the assumption of finiteness of Orlicz norm of T_1 and T_2 , which is required to apply the theorem, is equivalent to existence of a number $s > 1$ such that

$$\mathbb{E}_\mu(s^\tau) < \infty, \quad \mathbb{E}_\nu(s^\tau) < \infty. \quad (1)$$

Now, we use condition A5 of geometric ergodicity. Theorem 15.4.2 in ? shows that there exists a full absorbing set S such that S is geometrically regular, i.e. $\sup_{x \in S} \mathbb{E}_x(s^\tau) < \infty$ for some $s > 1$ (depending on A). Since S is full absorbing, and μ is the limit distribution of the chain $\mu(S) = 1$. Moreover $\mu(C \cap S) > 0$, where C is the set introduced in the Minorization Condition. So we can find an $x \in C \cap S$ and $\delta\nu(S^c) \leq P(x, S^c) = 0$. Thus $\nu(S) = 1$ too. This implies condition (??).

Now we write an integrated version of the concentration inequality. We denote $\nu_n(t) = n^{-1} \sum_{i=1}^n [t(X_i) - \langle t, f \rangle]$ where f is the stationary density of the chain and we consider a countable class \mathcal{B} of measurable functions t . Let a and H such that

$$\sup_{t \in \mathcal{B}} \|t - \langle t, f \rangle\|_\infty \leq a, \quad \mathbb{E} \left(\sup_{t \in \mathcal{B}} |\nu_n(t)| \right) \leq H.$$

Let the variance term

$$\sigma^2 = \mathbb{E}_A(\tau)^{-1} \sup_{t \in \mathcal{B}} \mathbb{E}_A \left[\left(\sum_{i=1}^{\tau} t(X_i) - \langle t, f \rangle \right)^2 \right].$$

Then we prove the existence of a numerical constant $c > 0$ such that

$$\mathbb{E}[\sup_{t \in \mathcal{B}} |\nu_n(t)|^2 - cH^2]_+ \leq K_1 \left(\frac{1}{n^2} + \frac{\sigma^2}{n} e^{-K_2 \frac{nH^2}{\sigma^2}} + \frac{a^2(\log n)^2}{n^2} e^{-K_3 \frac{nH}{a \log n}} \right) \quad (2)$$

where K_1, K_2, K_3 are depending on the chain. Indeed we compute, for $c = 8K^2$,

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in \mathcal{B}} |\nu_n(t)|^2 - cH^2 \right]_+ &= \int_0^\infty P \left(\sup_{t \in \mathcal{B}} |\nu_n(t)|^2 \geq cH^2 + x \right) dx \\ &\leq \int_0^\infty P \left(\sup_{t \in \mathcal{B}} |\nu_n(t)| \geq \sqrt{c/2}H + \sqrt{x/2} \right) dx \leq \int_0^\infty P \left(Z \geq \sqrt{c/2}\mathbb{E}Z + n\sqrt{\frac{x}{2}} \right) dx \\ &\leq \int_0^\infty P \left(Z \geq K\mathbb{E}Z + K\mathbb{E}Z + n\sqrt{\frac{x}{2}} \right) dx \end{aligned}$$

where $Z = n \sup_{t \in \mathcal{B}} |\nu_n(t)|$. If $x \geq 2n^{-2}$, $t = K\mathbb{E}Z + n\sqrt{x/2} \geq 1$ so that we can apply Theorem 7. Moreover

$$\int_0^{2n^{-2}} P \left(Z \geq K\mathbb{E}Z + K\mathbb{E}Z + n\sqrt{\frac{x}{2}} \right) dx \leq 2n^{-2}.$$

Thus

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in \mathcal{B}} |\nu_n(t)|^2 - cH^2 \right]_+ \leq \frac{2}{n^2} \\ & + \int_0^\infty K \exp \left(-\frac{1}{K'} \min \left(\frac{[K\mathbb{E}Z + n\sqrt{x/2}]^2}{n\sigma^2}, \frac{K\mathbb{E}Z + n\sqrt{x/2}}{a \log n} \right) \right) dx \\ & \leq \frac{2}{n^2} + \frac{1}{K_2} e^{-\frac{K_2(\mathbb{E}Z)^2}{n\sigma^2}} \int_0^\infty e^{-\frac{K_2nx}{\sigma^2}} dx + \frac{1}{K_3} e^{-\frac{K_3\mathbb{E}Z}{a \log n}} \int_0^\infty e^{-\frac{K_3n\sqrt{x}}{a \log n}} dx \\ & \leq \frac{2}{n^2} + K_4 \frac{\sigma^2}{n} e^{-\frac{K_2nH^2}{\sigma^2}} + K_5 \frac{(a \log n)^2}{n^2} e^{-\frac{K_3nH}{a \log n}} \end{aligned}$$

This gives inequality (??). This result can be extended to a non-countable class \mathcal{B} with classical density arguments. So we apply it with $\mathcal{B} = B(m, m')$. Moreover, the result of ? is also true when replacing $\mathbb{E}Z = n\mathbb{E}(\sup_{t \in \mathcal{B}} |\nu_n(t)|)$ by $n\mathbb{E}(\sup_{t \in \mathcal{B}} |\nu'_n(t)|)$ with

$$\nu'_n(t) = \frac{1}{n} \sum_{j=1}^{\lfloor 3n/\mathbb{E}_A(\tau) \rfloor} S_j(t)$$

(see in the proof of Theorem 7, p 1020). Thus (??) is also valid with $H \geq \mathbb{E}(\sup_{t \in \mathcal{B}} |\nu'_n(t)|)$. It remains to compute a, H and σ^2 . We denote $D(m, m') = \max(D_m, D_{m'})$ the dimension of the space $S_m + S_{m'}$ (recall that the models are nested) and $(\varphi_\lambda)_{\lambda \in \Lambda(m, m')}$ an orthonormal basis of $S_m + S_{m'}$.

- Computation of a . If $t \in S_m + S_{m'}$, $\|t\|_\infty \leq r_0 \sqrt{D(m, m')} \|t\|$. Then $a = 2r_0 \sqrt{D(m, m')}$.
- Computation of H^2 . Since any $t \in B(m, m')$ can be written $t = \sum_{\lambda \in \Lambda(m, m')} a_\lambda \varphi_\lambda$,

$$\mathbb{E} \left(\sup_{t \in B(m, m')} \nu'_n(t)^2 \right) \leq \sum_{\lambda \in \Lambda(m, m')} \mathbb{E}(\nu'_n(\varphi_\lambda)^2) \leq \sum_{\lambda \in \Lambda(m, m')} \mathbb{E} \left(\left(\frac{1}{n} \sum_{j=1}^{\lfloor 3n/\mathbb{E}_A(\tau) \rfloor} S_j(\varphi_\lambda) \right)^2 \right).$$

Recall that the $S_j(t)$ are independent identically distributed and centered. Then, using (new) Lemma 10,

$$\mathbb{E} \left(\sup_{t \in B(m, m')} \nu'_n(t)^2 \right) \leq \frac{\lfloor 3n/\mathbb{E}_A(\tau) \rfloor}{n^2} r_0^2 \mathbb{E}_A(\tau^2) D(m, m').$$

Finally, since $\mu(A) = \mathbb{E}_A(\tau)^{-1}$, we set $H^2 = CD(m, m')/n$ with $C = 3\mu(A)\mathbb{E}_A(\tau^2)r_0^2$.

- Computation of σ^2 . We use the following inequality, given in ?, subsection 17.4.3:

$$\mu(A)\mathbb{E}_A \left[\left(\sum_{i=1}^{\tau} t(X_i) - \langle t, f \rangle \right)^2 \right] = 2 \int (t - \langle t, f \rangle) \hat{t} d\mu - \int (t - \langle t, f \rangle)^2 d\mu$$

where

$$\hat{t}(x) := \mathbb{E}_x \left(\sum_{i=0}^{\sigma_A} t(X_i) - \langle t, f \rangle \right)$$

and $\sigma_A = \inf\{n \geq 0, X_n \in A\}$. Then, since $\mu(A) = \mathbb{E}_A(\tau)^{-1}$,

$$\sigma^2 \leq \sup_{t \in B(m, m')} 2 \int (t - \langle t, f \rangle) \hat{t} d\mu \leq \sup_{t \in B(m, m')} 2 \left(\int (t - \langle t, f \rangle)^2 d\mu \int \hat{t}^2 d\mu \right)^{1/2}.$$

But $\int (t - \langle t, f \rangle)^2 d\mu \leq \int t^2 f \leq \|f\|_{\infty} \|t\|^2$ and

$$\hat{t}^2(x) \leq \mathbb{E}_x \left(\left(\sum_{i=0}^{\sigma_A} t(X_i) - \langle t, f \rangle \right)^2 \right) \leq 4\|t\|_{\infty}^2 \mathbb{E}_x((\sigma_A + 1)^2).$$

with $\mathbb{E}_x((\sigma_A + 1)^2) \leq \mathbb{E}_x((\tau + 1)^2)$. Then

$$\sigma^2 \leq 4\sqrt{\mathbb{E}_{\mu}((\tau + 1)^2)} \sqrt{\|f\|_{\infty}} \sup_{t \in B(m, m')} \|t\|_{\infty} \|t\|$$

so that

$$\sigma^2 \leq 4\sqrt{\mathbb{E}_{\mu}((\tau + 1)^2)} \sqrt{\|f\|_{\infty}} r_0 \sqrt{D(m, m')}.$$

Now, we can use inequality (??): it implies existence of positive constants K'_1, K'_2, K'_3 such that

$$\mathbb{E}[\sup_{t \in \mathcal{B}} |\nu_n(t)|^2 - cCD(m, m')/n]_+ \leq K'_1 \left(\frac{1}{n^2} + \frac{\sqrt{D(m, m')}}{n} e^{-K'_2 \sqrt{D(m, m')}} + \frac{D(m, m')(\log n)^2}{n^2} e^{-K'_3 \frac{\sqrt{n}}{\log n}} \right).$$

Using that $D(m, m') = \max(D_m, D'_m) \leq n$, we obtain that $\sum_{m' \in \mathcal{M}_n} \sqrt{D(m, m')} e^{-K'_2 \sqrt{D(m, m')}}$ and $\sum_{m' \in \mathcal{M}_n} D(m, m')(\log n)^2 n^{-1} e^{-K'_3 \frac{\sqrt{n}}{\log n}}$ are bounded. Moreover $|\mathcal{M}_n| n^{-2} = O(n^{-1})$. Thus

$$\sum_{m' \in \mathcal{M}_n} \mathbb{E}[\sup_{t \in \mathcal{B}} |\nu_n(t)|^2 - cCD(m, m')/n]_+ = O(n^{-1})$$

□

New proof of Theorem 9

The result of Theorem 9 is true but the proof must be modified in the following way. Recall that we denote $E_n = \{\|f - \tilde{f}\|_\infty \leq \chi/2\}$ and E_n^c its complementary. We have

$$\mathbb{E}\|\pi - \tilde{\pi}\|^2 \leq \frac{8}{\chi^2} \left(\mathbb{E}\|g - \tilde{g}\|^2 + \|\pi\|_\infty^2 \mathbb{E}\|f - \tilde{f}\|^2 \right) + (a_n + \|\pi\|_\infty)^2 P(E_n^c)$$

so that it is sufficient to bound $(a_n + \|\pi\|_\infty)^2 P(E_n^c)$. We have proven that, for n large enough,

$$P(E_n^c) \leq P\left(\|f_{\hat{m}} - \hat{f}_{\hat{m}}\|_\infty > \frac{\chi}{4}\right) \leq P\left(\|f_{\hat{m}} - \hat{f}_{\hat{m}}\| > \frac{\chi}{4r_0\sqrt{D_{\hat{m}}}}\right).$$

But

$$\|f_{\hat{m}} - \hat{f}_{\hat{m}}\| = \sup_{t \in S_{\hat{m}}, \|t\| \leq 1} \int t(\hat{f}_{\hat{m}} - f_{\hat{m}}) = \sup_{t \in S_{\hat{m}}, \|t\| \leq 1} \nu_n(t).$$

Let S_{m_0} the largest model with dimension $D_{m_0} \leq n^{1/4}$.

$$P(E_n^c) \leq P\left(\sup_{t \in S_{\hat{m}}, \|t\| \leq 1} \nu_n(t)^2 > \frac{\chi^2}{16r_0^2 D_{\hat{m}}}\right) \leq P\left(\sup_{t \in S_{m_0}, \|t\| \leq 1} \nu_n(t)^2 > \frac{\chi^2}{16r_0^2 D_{m_0}}\right).$$

As shown in the (new) proof of Proposition 12, our assumptions allow us to use Theorem 7 in ?. Then, reasoning as in the proof of Proposition 12, we can show the existence of a numerical constant $c > 0$ and constants depending on the chain $K_1, K_2, K_3 > 0$ such that

$$P\left(\sup_{t \in S_{m_0}, \|t\| \leq 1} \nu_n(t)^2 \geq \frac{c}{2} H^2\right) \leq K_1 \left(e^{-K_2 \sqrt{D_{m_0}}} + e^{-K_3 \sqrt{n}/\log(n)} \right)$$

where $H^2 = 3\mu(A)\mathbb{E}_A(\tau^2)r_0^2 D_{m_0}/n$. Now, for n large enough, since $D_{m_0}^2 = o(n)$,

$$\frac{\chi^2}{16r_0^2 D_{m_0}} \geq \frac{3c\mu(A)\mathbb{E}_A(\tau^2)r_0^2 D_{m_0}}{2n}.$$

Then

$$P(E_n^c) \leq P\left(\sup_{t \in S_{m_0}, \|t\| \leq 1} \nu_n(t)^2 \geq \frac{c}{2} H^2\right) \leq K_1 \left(e^{-K_2 \sqrt{D_{m_0}}} + e^{-K_3 \sqrt{n}/\log(n)} \right)$$

so that $(a_n + \|\pi\|_\infty)^2 P(E_n^c) = o(n^{-1})$. Note that it is sufficient to have $D_{m_0} = \lfloor n^{1/2-\epsilon} \rfloor$ to obtain the result.

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