



HAL
open science

Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting

T Kruse, A Popier

► **To cite this version:**

T Kruse, A Popier. Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting. 2015. hal-01139364v1

HAL Id: hal-01139364

<https://hal.science/hal-01139364v1>

Preprint submitted on 4 Apr 2015 (v1), last revised 27 Dec 2015 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting.

T. Kruse ^{*}, A. Popier [†]

April 4, 2015

Abstract

We study the existence of a minimal supersolution for backward stochastic differential equations when the terminal data can take the value $+\infty$ with positive probability. We deal with equations on a general filtered probability space and with generators satisfying a general monotonicity assumption. With this minimal supersolution we then solve an optimal stochastic control problem related to portfolio liquidation problems. We generalize the existing results in three directions: firstly there is no assumption on the underlying filtration (except completeness and quasi-left continuity), secondly we relax the terminal liquidation constraint and finally the time horizon can be random.

Introduction

This paper is devoted to the study of backward stochastic differential equations (BSDEs) with *singular* terminal condition. We adopt from [22] and [23] the notion of a weak (super)solution (Y, ψ, M) to a BSDE of the following form

$$dY_t = -\tilde{f}(t, Y_t, \psi_t)dt + \int_{\mathcal{Z}} \psi_t(z) \tilde{\pi}(dz, dt) + dM_t, \quad (1)$$

where $\tilde{\pi}$ is a compensated Poisson random measure on a filtered probability $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$. The filtration \mathbb{F} is supposed to be complete, right continuous and quasi-left continuous. In particular, it can support a Brownian motion orthogonal to $\tilde{\pi}$. The solution component M is required to be a local martingale orthogonal to $\tilde{\pi}$. The function $\tilde{f} : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is called the *driver* (or *generator*) of the BSDE. The particularity here is that we allow the *terminal condition* ξ to be *singular*. More precisely, for a stopping time τ and a \mathcal{F}_τ -measurable random variable ξ that takes the value $+\infty$ with positive probability, we impose that $\liminf_{t \rightarrow \tau} Y_t \geq \xi$.

^{*}Laboratoire de Mathématiques et Modélisation d'Évry, Université d'Évry Val d'Essonne, 23 Boulevard de France, 91037 Évry, France.

e-mail: thomas.kruse@univ-evry.fr

[†]Laboratoire Manceau de Mathématiques, Université du Maine, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France.

e-mail: alexandre.popier@univ-lemans.fr

Thomas Kruse acknowledges the financial support from the French Banking Federation through the Chaire "Markets in Transition".

In our first main result (Theorem 1) we establish existence of a *minimal* weak supersolution to (1). This supersolution is constructed via approximation from below. For each $L > 0$ we consider a truncated version of (1) with terminal condition $\xi \wedge L$. We impose that the driver \tilde{f} satisfies a monotonicity assumption in the y -variable and is Lipschitz continuous with respect to ψ . Then existence, uniqueness and comparison results for a solution (Y^L, ψ^L, M^L) to the truncated BSDE can be deduced from [17], where the theory of BSDEs with a monotone driver in a general filtration has been developed. We obtain the minimal supersolution (Y, ψ, M) with singular terminal condition by passing to the limit $L \rightarrow \infty$. The crucial task is to establish suitable a priori estimates for Y^L guaranteeing that when passing to the limit the solution Y does *not* explode before time τ . To this end, we impose that \tilde{f} decreases at least polynomially with random coefficient in the y -variable. In the case where τ is deterministic this condition suffices to ensure boundedness of Y^L . When τ is random, we restrict attention to first exit of diffusions from a regular set.

Our findings generalize some results from [3], [22] and [23] (see also [10] for a treatise on BSPDEs) to the case of a general driver \tilde{f} . Indeed, in the previously mentioned papers \tilde{f} is assumed to be a polynomial function of y plus possibly a particular function of ψ . We note that our results can be extended to the case where the driver is additionally a Lipschitz continuous function of a variable Z , which represents the integrand in the martingale representation w.r.t. a Brownian motion (c.f. Remark 5).

Since the seminal paper by Pardoux and Peng [19] BSDEs have proved to be a powerful tool to solve stochastic optimal control problems (see e.g. the survey article [6] or the book [20]). In the second part of the paper we use the notion of weak supersolutions to provide a purely probabilistic solution of a stochastic control problem with a terminal constraint on the controlled process. More precisely, we consider the problem of minimizing the functional

$$J(X) = \mathbb{E} \left[\int_0^\tau \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^p + \int_{\mathcal{Z}} \lambda_s(z) |\beta_s(z)|^p \mu(dz) \right) ds + \xi \mathbf{1}_{\xi < \infty} |X_T|^p \middle| \mathcal{F}_t \right] \quad (2)$$

over all progressively measurable processes X that satisfy the dynamics

$$X_s = x + \int_0^s \alpha_u du + \int_0^s \int_{\mathcal{Z}} \beta_u(z) \pi(dz, du)$$

and the terminal state constraint

$$X_\tau \mathbf{1}_{\xi = \infty} = 0.$$

Here $p > 1$ and the processes η, γ and λ are non negative progressively measurable. In the cases where τ is deterministic or a first exit time, we characterize optimal strategies and the value function of this control problem with the BSDE

$$dY_t = (p-1) \frac{Y_t^q}{\eta_t^{q-1}} dt + \Theta(t, Y_t, \psi_t) dt - \gamma_t dt + \int_{\mathcal{Z}} \psi_t(z) \tilde{\pi}(dz, dt) + dM_t \quad (3)$$

with $\liminf_{t \rightarrow T} Y_t \geq \xi$. Here $q > 1$ is the Hölder conjugate of p and Θ is a Lipschitz continuous function given by (22). We provide sufficient conditions on the coefficient processes η, γ and λ such that Theorem 1 ensures existence of a minimal weak supersolution to (3) and carry out a verification that is based on a penalization argument.

The analysis of optimal control problems with state constraints on the terminal value is motivated by models of optimal portfolio liquidation under stochastic price impact. The traditional assumption that all trades can be settled without impact on market dynamics is not always appropriate when investors need to close large positions over short time periods. In recent years models of optimal portfolio liquidation have been widely developed, see, e.g. [1], [2], [7], [8], [12], or [16], among many others.

Variants of the position targeting problem (2) have been studied in [3], [4], [25], [10] or [11]. In this framework the state process X denotes the agent's position in the financial market. She has two means to control her position. At each point in time t she can trade in the primary venue at a rate α_t which generates costs $\eta_t|\alpha_t|^p$ incurred by the stochastic price impact parameter η_t . Moreover, she can submit passive orders to a secondary venue ("dark pool"). These orders get executed at the jump times of the Poisson random measure π and generate so called slippage costs $\int_{\mathcal{Z}} \lambda_t(z)|\beta_t(z)|^p \mu(dz)$. We refer to [16] for a more detailed discussion. The term $\gamma_t|X_t|^p$ can be understood as a measure of risk associated to the open position. $J(X)$ thus represents the overall expected costs for closing an initial position x over the time period $[0, \tau]$ using strategy X .

Our approach allows to incorporate some novel features into optimal liquidation models. First, we do not impose any assumption on the filtration (except quasi-left continuity). For the financial model, this means that the noise is not necessarily generated by a Brownian motion. Moreover, the liquidation constraint is relaxed in the following way. Instead of enforcing the condition $X_\tau = 0$ a.s., that is the position has to be closed imperatively, our model is flexible enough to allow for a specification of a set of market scenarios $\mathcal{S} \subset \mathcal{F}_\tau$ where liquidation is mandatory: $X_\tau \mathbf{1}_\mathcal{S} = 0$. On the complement \mathcal{S}^c a penalization depending on the remaining position size can be implemented. This terminal constraint is described by the \mathcal{F}_τ -measurable non negative random variable ξ such that $\mathcal{S} = \{\xi = +\infty\}$. Thus for a binding liquidation $X_\tau = 0$, we take $\xi = +\infty$ a.s. For excepted scenarios, we can consider $\xi = \infty \mathbf{1}_\mathcal{S}$ with for example $\mathcal{S} = \{\max_{t \in [0, T]} \eta_t \leq H\}$ or $\mathcal{S} = \{\int_0^T \eta_t dt \leq H\}$ for a given threshold $H > 0$. This means that liquidation is only mandatory if the maximal price impact (or the average price impact) is small enough throughout the liquidation period. If the illiquidity of the market is too high, the trader has not obligatorily to close his position. Finally, our model allows for a random time horizon τ . For example, one can consider *price-sensitive* liquidation periods where the position has to be closed before the first time when the unaffected market price S (a diffusion) falls below some threshold level $K > 0$, i.e. $\tau = \inf\{t \geq 0 | S_t \leq K\}$.

The paper is decomposed as follows. In the first section, we give the mathematical setting and present the main results concerning the BSDE (1) and the control problem (2). The set of assumptions will differ in the two cases τ deterministic and τ random. In the next section, we construct a supersolution of the BSDE (1) using truncation arguments as in [22] or [3] and we prove that this solution is minimal. As mentioned before the main difficulty is to control the sequence of solutions for the truncated BSDE (see Propositions 2 and 5). In Section 3 we use the previous results to obtain a minimal supersolution for BSDE (3) and we verify that this solution gives the value function and an optimal control for the optimal position targeting problem.

1 Problem formulation and main results

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$. The filtration is assumed to be complete and right continuous. Moreover, we assume that \mathbb{F} is quasi-left continuous, which means that for every sequence (τ_n) of \mathbb{F} stopping times such that $\tau_n \nearrow \tilde{\tau}$ for some stopping time $\tilde{\tau}$ we have $\bigvee_{n \in \mathbb{N}} \mathcal{F}_{\tau_n} = \mathcal{F}_{\tilde{\tau}}$. We assume that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ supports a Poisson random measure π with intensity $\mu(dz)dt$ on the space $\mathcal{Z} \subset \mathbb{R}^d \setminus \{0\}$. The measure μ is σ -finite on \mathcal{Z} such that

$$\int_{\mathcal{Z}} (1 \wedge |z|^2) \mu(dz) < +\infty.$$

By \mathcal{P} we denote the predictable σ -field on $\Omega \times \mathbb{R}_+$. We set $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathcal{Z})$ where $\mathcal{B}(\mathcal{Z})$ is the Borelian σ -field on \mathcal{Z} . On $\tilde{\Omega} = \Omega \times [0, T] \times \mathcal{Z}$, a function that is $\tilde{\mathcal{P}}$ -measurable, is called predictable. $G_{loc}(\pi)$ is the set of $\tilde{\mathcal{P}}$ -measurable functions ψ on $\tilde{\Omega}$ such that for any $t \geq 0$ a.s.

$$\int_0^t \int_{\mathcal{Z}} (|\psi_s(z)|^2 \wedge |\psi_s(z)|) \mu(dz) ds < +\infty.$$

For any stopping time $\tilde{\tau}$ and $m > 1$, the set $L_{\pi}^m(0, \tilde{\tau})$ contains all processes $\psi \in G_{loc}(\mu)$ such that

$$\mathbb{E} \left[\int_0^{\tilde{\tau}} \int_{\mathcal{Z}} |\psi_s(z)|^m \mu(dz) ds \right] < +\infty.$$

By $L_{\mu}^m = L^m(\mathcal{Z}, \mu; \mathbb{R}^d)$ we denote the set of measurable functions $\psi : \mathcal{Z} \rightarrow \mathbb{R}^d$ such that

$$\|\psi\|_{L_{\mu}^m}^m = \int_{\mathcal{Z}} |\psi(z)|^m \mu(dz) < +\infty.$$

By \mathcal{M}^{\perp} we denote the set of càdlàg local martingales orthogonal to $\tilde{\pi}$. If $M \in \mathcal{M}^{\perp}$ then $\mathbb{E}(\Delta M * \pi | \tilde{\mathcal{P}}) = 0$, where the product $*$ denotes the integral process (see II.1.5 in [13]). For any stopping time $\tilde{\tau}$ the set $\mathcal{M}^m(0, \tilde{\tau})$ is the subset of all martingales such that $\mathbb{E} \left[\langle M \rangle_{\tilde{\tau}}^{m/2} \right] < +\infty$. Finally, for $m > 1$, $\mathbb{D}^m(0, \tilde{\tau})$ is the set of all progressively measurable càdlàg processes F such that $\mathbb{E} \left[\sup_{t \in [0, \tilde{\tau}]} |F_t|^m \right] < +\infty$. The set $\mathbb{H}^m(0, \tilde{\tau})$ contains all progressively measurable càdlàg processes F such that $\mathbb{E} \left[\int_0^{\tilde{\tau}} |F_t|^m dt \right] < +\infty$.

Minimal supersolutions to singular BSDEs

Let us τ denote a \mathbb{F} -stopping time and let ξ be a \mathcal{F}_{τ} -measurable random variable. We explicitly allow ξ to take the value $+\infty$ with positive probability. We denote by \mathcal{S} the set $\{\xi = +\infty\}$ and we assume that $\xi^- \in L^2(\Omega)$. Our first aim is to establish existence of a solution to BSDE (1) with singular terminal condition ξ . We will distinguish two cases.

In the first case we assume that τ is deterministic, i.e. $\tau = T$ a.s. for some constant $T > 0$. In this case we set $\tau_{\varepsilon} = T - \varepsilon$ for $\varepsilon > 0$.

In the second case we assume that τ is given as a first exit time of a diffusion. More precisely, we assume that the filtration \mathbb{F} supports a d -dimensional Brownian motion W which is orthogonal to π and we introduce a forward process Γ in \mathbb{R}^d , that is a solution to the stochastic differential equation

$$d\Gamma_t = b(\Gamma_t)dt + \sigma(\Gamma_t)dW_t \tag{4}$$

with some initial value $\Gamma_0 \in \mathbb{R}^d$. The functions $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ satisfy a global Lipschitz condition: there exists some $K > 0$ such that

$$\forall x, y \in \mathbb{R}^d \quad \|\sigma(x) - \sigma(y)\| + \|b(x) - b(y)\| \leq K\|x - y\|.$$

Under this assumption there exists a unique strong solution Γ to (4). Let D be an open bounded subset of \mathbb{R}^d , whose boundary is at least of class C^2 (see for example [9], section 6.2, for the definition of a regular boundary). From now on Γ_0 is fixed and supposed to be in D . We define the stopping time τ as the first exit time of D , i.e.

$$\tau = \tau_D = \inf\{t \geq 0, \Gamma_t \notin D\}. \quad (5)$$

Moreover, we impose the following integrability condition on τ_D . There exists $\beta > 0$ such that

$$\mathbb{E}e^{\beta\tau_D} < +\infty. \quad (6)$$

Notice that these assumptions hold if for example σ is uniformly elliptic, i.e. there exists a constant $\alpha > 0$ such that $(\sigma\sigma^*)(x) \geq \alpha \text{Id}$ for all $x \in \mathbb{R}^d$ (see [5], Corollary III.3.2 and [21], Theorem II.2.1). In this second case we set

$$\tau_\varepsilon = \inf\{t \geq 0, \text{dist}(\Gamma_t) \leq \varepsilon\},$$

where $\text{dist}(\Gamma_t)$ denotes the distance between the position of Γ at time t and the boundary of D .

Since we possibly have $\mathbb{P}[\xi = \infty] > 0$ we need to specify a weak notion of solutions to (1) (and (3)). We relax the usual definition of a solution to a BSDE by only requiring that (1) holds up to time τ_ε for all $\varepsilon > 0$.

Definition 1 (Weak supersolution) *We say that a triple of processes (Y, ψ, M) is a supersolution to the BSDE (1) with singular terminal condition $Y_\tau = \xi$ if it satisfies:*

1. $M \in \mathcal{M}^\perp$, $\psi \in G_{loc}(\pi)$ and there exists some $\ell > 1$ such that for all $t \geq 0$ and $\varepsilon > 0$:

$$\mathbb{E} \left(\sup_{s \in [0, t]} |Y_{s \wedge \tau_\varepsilon}|^\ell + \int_0^{t \wedge \tau_\varepsilon} \int_{\mathcal{Z}} |\psi_s(z)|^\ell \mu(dz) ds + [M]_{t \wedge \tau_\varepsilon}^{\ell/2} \right) < +\infty;$$

2. Y^- belongs to $\mathbb{D}^2(0, \tau)$.

3. for all $0 \leq s \leq t$ and $\varepsilon > 0$:

$$\begin{aligned} Y_{s \wedge \tau_\varepsilon} &= Y_{t \wedge \tau_\varepsilon} + \int_{s \wedge \tau_\varepsilon}^{t \wedge \tau_\varepsilon} \tilde{f}(u, Y_u, \psi_u) du \\ &\quad - \int_{s \wedge \tau_\varepsilon}^{t \wedge \tau_\varepsilon} \int_{\mathcal{Z}} \psi_u(z) \tilde{\pi}(dz, du) - \int_{s \wedge \tau_\varepsilon}^{t \wedge \tau_\varepsilon} dM_u. \end{aligned}$$

4. On the set $\{t \geq \tau\}$: $Y_t = \xi$, $\Psi = M = 0$ a.s. and $\liminf_{t \rightarrow \infty} Y_{t \wedge \tau_\varepsilon} \geq \xi$.

We say that (Y, ψ, M) is a minimal supersolution to the BSDE (1) if for any other supersolution (Y', ψ', M') we have $Y_t \leq Y'_t$ a.s. for any $t > 0$.

To establish existence of a minimal supersolution to BSDE (1) we impose the following conditions.

Assumption 1 (A and A') *If $\tau = T$ is deterministic, we say that Conditions (A) are satisfied if all seven hypotheses (A1) to (A7) hold.*

If τ is a stopping time given by (5), we say that Conditions (A') are satisfied if all hypotheses (A1), (A2), (A3'), (A4'), (A5), (A6') and (A7) hold.

The reader is referred to Sections 2.1 and 2.2 for the precise statement of the Assumptions (A1) to (A7) and (A3'), (A4'), (A6'), respectively. Our first main result is the following.

Theorem 1 *Under Assumptions (A) or (A') there exists a minimal supersolution (Y, ψ, M) to (1) with singular terminal condition $Y_\tau = \xi$.*

Optimal position targeting

Let us now describe our stochastic control problem. We fix some $p > 1$ and denote by $q = 1/(1 - 1/p)$ its Hölder conjugate. For any $t \leq \tau$ and $x \geq 0$, we denote by $\mathcal{A}_0^S(t, x)$ the set of progressively measurable processes X that satisfy the dynamics

$$X_s = x + \int_t^s \alpha_u du + \int_t^s \int_{\mathcal{Z}} \beta_u(z) \pi(dz, du) \quad (7)$$

for any $s \in [t, \tau]$ and for some $\alpha \in L^1(t, \tau)$ a.s. and $\beta \in G_{loc}(\pi)$. Moreover, we impose that every $X \in \mathcal{A}_0^S(t, x)$ satisfies almost surely the terminal state constraint

$$X_\tau \mathbf{1}_S = 0. \quad (8)$$

We consider the stochastic control problem to minimize the functional

$$J(t, X) = \mathbb{E} \left[\int_t^\tau \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^p + \int_{\mathcal{Z}} \lambda_s(z) |\beta_s(z)|^p \mu(dz) \right) ds + \xi \mathbf{1}_{S^c} |X_T|^p \middle| \mathcal{F}_t \right] \quad (9)$$

over all $X \in \mathcal{A}_0^S(t, x)$. Here, the measure μ is finite and ξ is supposed to be non negative. The coefficient processes $(\eta_t)_{t \geq 0}$, $(\gamma_t)_{t \geq 0}$ and $(\lambda_t)_{t \geq 0}$ are nonnegative progressively measurable càdlàg processes. The process λ is $\tilde{\mathcal{P}}$ -measurable with values in $[0, +\infty]$.

We introduce the random field v that represents for each initial condition (t, x) the minimal value of J

$$v(t, x) = \operatorname{ess\,inf}_{X \in \mathcal{A}_0^S(t, x)} J(t, X). \quad (10)$$

We follow the convention that the infimum over the empty set is equal to ∞ . We will characterize the value function v and the optimal control in terms of minimal supersolutions to the BSDE (3) with singular terminal condition ξ . Again we distinguish two cases. In the first case we assume that τ is deterministic and impose some nice integrability assumptions on the coefficient processes $(\eta_t)_{t \geq 0}$ and $(\gamma_t)_{t \geq 0}$.

Assumption 2 (B1) *The stopping time τ is a.s. equal to a deterministic constant $T > 0$. The process η is positive such that*

$$\mathbb{E} \left[\int_0^T \eta_t^2 dt \right] < \infty \quad \text{and} \quad \mathbb{E} \left[\int_0^T \frac{1}{\eta_t^{q-1}} dt \right] < \infty.$$

γ is non negative and satisfies (A6).

To ensure existence of a minimal supersolution to (3) if $\tau = \tau_D$ is given by (5) we need to impose some stronger boundedness conditions on η and γ compared to (B1).

Assumption 3 (B2) *We have $\tau = \tau_D$ and the processes η and γ are bounded from above, η is positive and satisfies the integrability conditions*

$$\mathbb{E} \left[\int_0^n \frac{1}{\eta_t^{q-1}} dt \right] < \infty \tag{11}$$

for all $n \in \mathbb{N}$ and for some $m > 1$

$$\mathbb{E} \left[\int_0^\tau \frac{1}{\eta_t^{m(q-1)}} dt \right] < +\infty. \tag{12}$$

The next result is a consequence of Theorem 1.

Corollary 1 *Under Assumptions (B1) or (B2), the singular BSDE (3) has a minimal non negative weak supersolution (Y, ψ, M) .*

In our second main result we characterize the value function v and optimal strategies in terms of the the minimal supersolution to (3).

Theorem 2 *Let Assumption (B1) or (B2) hold and let (Y, ψ, M) denote the minimal supersolution to (3) with singular terminal condition $Y_\tau = \xi$. Then we have $v(t, x) = Y_t x^p$. Moreover, the process X satisfying the linear dynamics*

$$X_s = x - \int_t^s \left(\frac{Y_u}{\eta_u} \right)^{q-1} X_u du - \int_s^t X_{u-} \int_{\mathcal{Z}} \zeta_u(z) \pi(dz, du),$$

with

$$\zeta_u(z) = \frac{(Y_{u-} + \psi_u(z))^{q-1}}{[(Y_{u-} + \psi_u(z))^{q-1} + \lambda_u(z)^{q-1}]}$$

belongs to $\mathcal{A}_0^S(t, x)$ and is optimal in (10).

[give explicit form of X](#)

2 Existence of minimal supersolutions for the singular BSDE

In this section we establish existence of a minimal supersolution (Y, ψ, M) to the BSDE (1) with singular terminal condition.

2.1 Deterministic terminal times

Let us specify the conditions on the driver $\tilde{f} : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$. For notational convenience we write $\tilde{f}(t, y, \psi) = f(t, y, \psi) + \gamma_t$ with $\gamma_t = \tilde{f}(t, 0, 0)$. In particular, we have $f(t, 0, 0) = 0$.

- The function $y \mapsto f(t, y, \psi)$ is continuous and monotone: there exists $\chi \in \mathbb{R}$ such that a.s. and for any $t \in [0, T]$ and $\psi \in L_\mu^2$

$$(f(t, y, \psi) - f(t, y', \psi))(y - y') \leq \chi(y - y')^2. \quad (\text{A1})$$

- There exists a progressively measurable process $\kappa = \kappa^{y, \psi, \phi} : \Omega \times \mathbb{R}_+ \times \mathcal{Z} \rightarrow \mathbb{R}$ such that

$$f(t, y, \psi) - f(t, y, \phi) = \int_{\mathcal{Z}} (\psi(z) - \phi(z)) \kappa_t^{y, \psi, \phi}(z) \mu(dz) \quad (\text{A2})$$

with $\mathbb{P} \otimes Leb \otimes \mu$ -a.e. for any (y, ψ, ϕ) , $-1 \leq \kappa_t^{y, \psi, \phi}(z)$ and $|\kappa_t^{y, \psi, \phi}(z)| \leq \vartheta(z)$ where $\vartheta \in L_\mu^2$.

- For every $n > 0$ the function

$$\sup_{|y| \leq n} |f(t, y, 0)| \in L^1((0, T)), \quad \text{a.s.} \quad (\text{A3})$$

- The negative parts of ξ and γ are square integrable

$$\xi^- \in L^2(\Omega) \text{ and } \gamma^- \in L^2((0, T) \times \Omega). \quad (\text{A4})$$

Conditions (A1)-(A4) will ensure existence and uniqueness of the solution for a version of BSDE (1), where the terminal condition ξ is replaced by $\xi \wedge L$ for some $L > 0$. We obtain the minimal supersolution from Theorem 1 with singular terminal condition ξ by letting the truncation L tend to ∞ . To ensure that in the limit $L \rightarrow \infty$ the solution component Y attains the value ∞ on \mathcal{S} at time T but is finite before time T , we have to impose some further growth behavior on f . We assume that f decreases at least polynomially in the y -variable. More precisely, we suppose that there exists a constant $r > 1$ and a positive process a such that for any $y \geq 0$

$$f(t, y, \psi) \leq -\frac{\rho - 1}{a_t} |y|^r + f(t, 0, \psi). \quad (\text{A5})$$

ρ is the Hölder conjugate of r . Moreover, in order to derive the a priori estimate, the following assumptions will hold.

- There exists $\ell > 1$ such that

$$\mathbb{E} \int_0^T [a_s^{\rho-1} + (T-s)^\rho \gamma_s^+]^\ell ds < +\infty. \quad (\text{A6})$$

- If ι is the Hölder conjugate of ℓ , let $k > \max(\iota, 2)$ and

$$\int_{\mathcal{Z}} |\vartheta(z)|^k \mu(dz) < +\infty. \quad (\text{A7})$$

In the rest of this section, the generator \tilde{f} satisfies **Conditions (A)**.

Remark 1 In (A1) we can suppose w.l.o.g. that $\chi = 0$. Indeed if (Y, ψ, M) is a solution of (1) then $(\bar{Y}, \bar{\psi}, \bar{M})$ with

$$\bar{Y}_t = e^{\chi t} Y_t, \quad \bar{\psi}_t = e^{\chi t} \psi_t, \quad d\bar{M}_t = e^{\chi t} dM_t$$

satisfies an analogous BSDE with terminal condition $\bar{\xi} = e^{\chi T} \xi$, and generator

$$\bar{f}(t, y, \psi) = \left[e^{\chi t} \tilde{f}(t, e^{-\chi t} y, e^{-\chi t} \psi) - \chi y \right]$$

and \bar{f} satisfies the same assumptions with $\chi = 0$. Hence in the rest of the paper, we will suppose that $\chi = 0$.

Remark 2 The second condition (A2) implies that f is Lipschitz continuous w.r.t. ψ uniformly in ω , t and y :

$$|f(t, y, \psi) - f(t, y, \phi)| \leq \|\vartheta\|_{L^2_\mu} \|\psi - \phi\|_{L^2_\mu}.$$

Remark 3 It follows from Condition (A3) and (A5) that the process $1/a$ must be in $L^1(0, T)$ a.s.

$$\int_0^T \frac{1}{a_t} dt < +\infty. \quad (13)$$

To prove existence of a weak supersolution we proceed as in [3] by truncation. For any $L \geq 0$ we consider the BSDE

$$dY_t^L = -f^L(t, Y_t^L, \psi_t^L) dt + \int_{\mathcal{Z}} \psi_t^L(z) \tilde{\pi}(dz, dt) + dM_t^L \quad (14)$$

with bounded terminal condition $Y_T^L = \xi \wedge L$ and where

$$f^L(t, y, \psi) = f(t, y, \psi) + \gamma_t \wedge L. \quad (15)$$

Proposition 1 Under Assumptions (A), there exists for every $L > 0$ a solution (Y^L, ψ^L, M^L) to (14) with $Y^L \in \mathbb{D}^2(0, T)$, $\psi^L \in L^2_\pi(0, T)$, $M^L \in \mathcal{M}^2(0, T) \cap \mathcal{M}^\perp$. Moreover there exists a process \bar{Y} in $\mathbb{D}^2(0, T)$, independent of L , such that a.s. for any $t \in [0, T]$, $\bar{Y}_t \leq Y_t^L$.

Proof. From our hypotheses on f and γ , it follows that f^L is monotone w.r.t. y , Lipschitz continuous w.r.t. ψ , and $f^L(t, 0, 0) = \gamma_t \wedge L \in L^2((0, T) \times \Omega)$. Moreover for every $n > 0$ and $|y| \leq n$:

$$|f^L(t, y, 0) - f^L(t, 0, 0)| = |f(t, y, 0)| \leq \sup_{|y| \leq n} |f(t, y, 0)|.$$

By Assumption (A3), the mapping $t \mapsto \sup_{|y| \leq n} |f(t, y, 0)|$ is in $L^1(0, T)$ a.s. From Theorem 1 in [17] there exists a unique solution (Y^L, ψ^L, M^L) to (14) with terminal condition $\xi \wedge L$. This solution satisfies

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^L|^2 + \int_0^T \int_{\mathcal{Z}} (\psi_t^L(z))^2 \mu(dz) dt + [M^L]_T \right] < +\infty.$$

Next, we construct the lower bound \bar{Y} . Let us take $\zeta = -\xi^-$ and $g(t, y, \psi) = f(t, y, \psi) - (\gamma_t)^-$. The solution $(\bar{Y}, \bar{\psi}, \bar{M})$ with $\bar{Y} \in \mathbb{D}^2(0, T)$ of the BSDE with data (ζ, g) does not depend on L , and by comparison (Proposition 4 in [17]) we have $\bar{Y}_t \leq Y_t^L$ a.s. for any $t \in [0, T]$. In particular, if $(\gamma_t)^- = \xi^- = 0$, then $\bar{Y}_t = 0$, and Y_t^L is non negative. \square

Proposition 2 *For every $t \in [0, T]$ the random variable Y_t^L is bounded from above by $L(1 + T)$ and for $t \in [0, T]$ the following estimate holds:*

$$Y_t^L \leq \frac{K_\vartheta}{(T-t)^\rho} \left[\mathbb{E} \left(\int_t^T (a_s^{\rho-1} + (T-s)^\rho \gamma_s^+)^\ell ds \middle| \mathcal{F}_t \right) \right]^{1/\ell} \quad (16)$$

where K_ϑ is a constant depending only on ϑ .

Proof. Let us first consider the triple $(A_t, B_t, C_t) = (L(1 + (T-t)), 0, 0)$. It is the solution of the BSDE with terminal condition $L \geq \xi \wedge L$ and generator equal to L . Since f is monotone from (A1), $f(t, A_t, B_t) \leq f(t, 0, 0) = 0$, thus by the definition (15) of f^L we have $f^L(t, A_t, B_t) \leq L$. By comparison principle we obtain $Y_t^L \leq A_t \leq L(T+1)$ a.s. for any $t \in [0, T]$,

This upper bound depends on L . Next, we verify (16) which is an a priori estimate independent of L . We consider the driver

$$h(t, y, \psi) = b_t^L - \rho \frac{1}{T-t} y + f(t, 0, \psi).$$

with $b_t^L = \frac{a_t^{\rho-1}}{(T-t)^\rho} + (\gamma_t^+ \wedge L)$. Let $\varepsilon > 0$ and denote by $(\mathcal{Y}^{\varepsilon, L}, \phi^{\varepsilon, L}, N^{\varepsilon, L})$ the solution process of the linear BSDE on $[0, T - \varepsilon]$ with driver h and terminal condition $\mathcal{Y}_{T-\varepsilon}^{\varepsilon, L} = Y_{T-\varepsilon}^{L,+} \geq 0$. Recall that

$$f(t, 0, \psi) = \int_{\mathcal{Z}} \psi(z) \kappa_t^{0, \psi, 0}(z) \mu(dz).$$

Hence by the solution formula for linear BSDE we have

$$\mathcal{Y}_t^{\varepsilon, L} = \mathbb{E} \left[\Gamma_{t, T-\varepsilon} Y_{T-\varepsilon}^{L,+} + \int_t^{T-\varepsilon} \Gamma_{t,s} b_s^L ds \middle| \mathcal{F}_t \right]$$

where for $t \leq s \leq T - \varepsilon$

$$\Gamma_{t,s} = \exp \left(- \int_t^s \frac{\rho}{T-u} du \right) V_{t,s}^{\varepsilon, L} = \left(\frac{T-s}{T-t} \right)^\rho V_{t,s}^{\varepsilon, L}$$

and

$$V_{t,s}^{\varepsilon, L} = 1 + \int_t^s \int_{\mathcal{Z}} V_{t,u}^{\varepsilon, L} \kappa_u^{0, \phi^{\varepsilon, L}, 0}(z) \tilde{\pi}(dz, du). \quad (17)$$

Hence

$$\mathcal{Y}_t^{\varepsilon, L} = \frac{1}{(T-t)^\rho} \mathbb{E} \left[\varepsilon^\rho V_{t, T-\varepsilon}^{\varepsilon, L} Y_{T-\varepsilon}^{L,+} + \int_t^{T-\varepsilon} V_{t,s}^{\varepsilon, L} c_s^L ds \middle| \mathcal{F}_t \right]$$

where

$$c_t^L = (T-t)^\rho b_t^L = a_t^{\rho-1} + (T-t)^\rho (\gamma_t^+ \wedge L).$$

Now $c^L \geq 0$ and therefore $\mathcal{Y}_t^{\varepsilon,L} \geq 0$ a.s. for every $t \in [0, T]$. Hence from Condition (A5)

$$f^L(t, \mathcal{Y}_t^{\varepsilon,L}, \phi_t^{\varepsilon,L}) \leq -\frac{\rho-1}{a_t}(\mathcal{Y}_t^{\varepsilon,L})^r + f^L(t, 0, \phi_t^{\varepsilon,L}).$$

It follows that

$$\begin{aligned} f^L(t, \mathcal{Y}_t^{\varepsilon,L}, \phi_t^{\varepsilon,L}) &\leq h(t, \mathcal{Y}_t^{\varepsilon,L}, \phi_t^{\varepsilon,L}) - \frac{\rho-1}{a_t}(\mathcal{Y}_t^{\varepsilon,L})^r - \frac{a_t^{\rho-1}}{(T-t)^\rho} + \frac{\rho}{T-t}\mathcal{Y}_t^{\varepsilon,L} \\ &\leq h(t, \mathcal{Y}_t^{\varepsilon,L}, \phi_t^{\varepsilon,L}), \end{aligned}$$

where we used the inequality $d^\rho + (\rho-1)y^r - \rho dy \geq 0$ which holds for all $d, y \geq 0$. The comparison theorem implies $Y_t^L \leq \mathcal{Y}_t^{\varepsilon,L}$ for all $t \in [0, T-\varepsilon]$ and $\varepsilon > 0$.

Recall once again from Condition (A7) that $V_{t,\cdot}^{\varepsilon,L}$ belongs to $\mathbb{H}^k(0, T-\varepsilon)$ for $k \geq 2$. From the upper bound $Y_t^L \leq A_t \leq L(T+1)$ and from the integrability property of $V_{t,\cdot}^{\varepsilon,L}$, with dominated convergence, by letting $\varepsilon \downarrow 0$ we obtain a.s.

$$\mathbb{E} \left[\varepsilon^\rho V_{t, T-\varepsilon}^{\varepsilon,L} Y_{T-\varepsilon}^{L,+} \middle| \mathcal{F}_t \right] \longrightarrow 0.$$

From Assumption (A7) and by the proof of Proposition A.1 in [24], there exists a constant K_ϑ such that a.s.

$$\left[\int_t^{T-\varepsilon} (V_{t,s}^{\varepsilon,L})^k \middle| \mathcal{F}_t \right] \leq K_\vartheta.$$

From Hypothesis (A6), the process $(c_t^L, 0 \leq t \leq T)$ belongs to $\mathbb{H}^\ell(0, T)$. Therefore by Hölder inequality we obtain

$$\mathbb{E} \left[\int_t^{T-\varepsilon} V_{t,s}^{\varepsilon,L} c_s^L ds \middle| \mathcal{F}_t \right] \leq K_\vartheta \mathbb{E} \left[\int_t^T (c_s^L)^\ell ds \middle| \mathcal{F}_t \right]^{1/\ell}.$$

Hence we can pass to the limit as $\varepsilon \downarrow 0$

$$Y_t^L \leq \frac{K_\vartheta}{(T-t)^\rho} \mathbb{E} \left[\int_t^T (c_s^L)^\ell ds \middle| \mathcal{F}_t \right]^{1/\ell}.$$

Assumption (A6) implies by monotone convergence for $L \rightarrow \infty$

$$Y_t^L \leq \frac{K_\vartheta}{(T-t)^\rho} \mathbb{E} \left[\int_t^T (a_s^{\rho-1} + (T-s)^\rho \gamma_s^+)^\ell ds \middle| \mathcal{F}_t \right]^{1/\ell} < +\infty$$

Thus we obtain the upper bound in (16). \square

Next, we show that by passing to the limit $L \rightarrow \infty$ we obtain a supersolution of (1) with singular terminal condition ξ .

Theorem 3 *Assume that Assumptions (A) hold. Let (Y^L, ψ^L, M^L) be the solution of BSDE (14) obtained in Proposition 1. Then there exists a process (Y, ψ, M) such that for every $0 \leq t < T$, Y^L converges to Y in $\mathbb{D}^\ell(0, t)$, ψ^L converges in $L^\ell_\pi([0, t])$ to ψ and M^L converges in $\mathcal{M}^\ell(0, t)$ to M . Moreover, Y^- belongs to $\mathbb{D}^2(0, T)$. The limit process (Y, ψ, M)*

is a weak supersolution for the BSDE (1) with singular terminal condition ξ . Moreover Y satisfies the estimate (16)

$$Y_t \leq \frac{K_\vartheta}{(T-t)^\rho} \mathbb{E} \left[\int_t^T (a_s^{\rho-1} + (T-s)^\rho \gamma_s^+)^\ell ds \middle| \mathcal{F}_t \right]^{1/\ell}.$$

Remark 4 Observe that the upper bound in (16) can be simplified if we assume that $f(t, 0, \psi)$ is bounded, i.e. that there exists a constant K_f such that a.s. for any t and ψ ,

$$f(t, 0, \psi) \leq K_f. \quad (\text{A7}')$$

Under this condition one can show that

$$Y_t^L \leq \frac{1}{(T-t)^\rho} \mathbb{E} \left[\int_t^T (a_s^{\rho-1} + (T-s)^\rho \gamma_s) ds \middle| \mathcal{F}_t \right] + \frac{K_f}{\rho+1} (T-t). \quad (18)$$

The constants K_ϑ and $\ell > 1$ in (16) come from the growth condition on f w.r.t. ψ and from the lack of an estimate of ψ^L independent of L . Under (A7') Estimate (18) is then also an upper bound for Y .

Proof. The comparison result (see Proposition 4 in [17]) yields that $Y^L \leq Y^N$ if $N > L$. Hence, for all $t \leq T$ we can define Y_t as the increasing limit of Y_t^L as $L \rightarrow \infty$. Recall that Y^L is bounded from below uniformly in L by some process $\bar{Y} \in \mathbb{D}^2(0, T)$. Thus immediately $Y^- \in \mathbb{D}^2(0, T)$.

By Equation (16) for fixed $t < T$ the family of random variables $(Y_t^L, L \geq 0)$ is bounded from above:

$$Y_t^{L,+} \leq \frac{K_\vartheta}{(T-t)^\rho} \mathbb{E} \left[\int_t^T (a_s^{\rho-1} + (T-s)^\rho \gamma_s^+)^\ell ds \middle| \mathcal{F}_t \right]^{1/\ell}.$$

Once again by Assumption (A6), the random variable on the right hand side of (16) is in $L^\ell(\Omega)$. W.l.o.g. we suppose that $\ell \leq 2$. By dominated convergence, Y_t^L converges to Y_t in $L^\ell(\Omega)$ for $t < T$.

Since the filtration is quasi-left continuous, we have: $\lim_{t \nearrow T} Y_t^L = \xi \wedge L$. Indeed, in Equation (14), using Fubini's theorem for conditional expectation, the only discontinuous term could be the martingale term M . But the assumption on the filtration shows that M has no jump at time T (see [14], Proposition 25.19). Now for any $L \geq 0$ we have

$$\liminf_{t \uparrow T} Y_t \geq \liminf_{t \uparrow T} Y_t^L = \xi \wedge L,$$

which gives the desired inequality $\liminf_{t \nearrow T} Y_t \geq \xi$. In particular, $(\liminf_{t \nearrow T} Y_t) \mathbf{1}_S = +\infty$.

For the convergence of (ψ^L, M^L) let $0 \leq s \leq t < T$. For L and N nonnegative, we put

$$\widehat{Y}_s = Y_s^N - Y_s^L, \quad \widehat{\psi}_s(z) = \psi_s^N(z) - \psi_s^L(z), \quad \widehat{M}_s = M_s^N - M_s^L.$$

Then Itô's formula implies

$$\begin{aligned}
|\widehat{Y}_s|^\ell &\leq |\widehat{Y}_t|^\ell + \ell \int_s^t |\widehat{Y}_u|^{\ell-1} \check{Y}_u (f^N(u, Y_u^N, \psi_u^N) - f^L(u, Y_u^L, \psi_u^L)) du \\
&\quad - \ell \int_s^t |\widehat{Y}_{u-}|^{\ell-1} \check{Y}_{u-} d\widehat{M}_u - \ell \int_s^t |\widehat{Y}_u|^{\ell-1} \check{Y}_u \int_{\mathcal{Z}} \widehat{\psi}_u(z) \tilde{\pi}(dz, du) \\
&\quad - \int_s^t \int_{\mathcal{Z}} \left[|\widehat{Y}_{u-} + \widehat{\psi}_u(z)|^\ell - |\widehat{Y}_{u-}|^\ell - \ell |\widehat{Y}_{u-}|^{\ell-1} \check{Y}_{u-} \psi_u(z) \right] \pi(dz, du) \\
&\quad - \sum_{0 < s \leq t} \left[|\widehat{Y}_{u-} + \Delta \widehat{M}_u|^\ell - |\widehat{Y}_{u-}|^\ell - \ell |\widehat{Y}_{u-}|^{\ell-1} \check{Y}_{u-} \Delta \widehat{M}_u \right] \\
&\quad - c(\ell) \int_s^t |\widehat{Y}_u|^{\ell-2} \mathbf{1}_{\widehat{Y}_u \neq 0} d[M]_u^c.
\end{aligned}$$

Here $\check{x} = |x|^{-1} x \mathbf{1}_{x \neq 0}$ and $c(\ell) = \ell(\ell-1)/2$. For the term containing the generators we have

$$\begin{aligned}
&|\widehat{Y}_u|^{\ell-1} \check{Y}_u (f^N(u, Y_u^N, \psi_u^N) - f^L(u, Y_u^L, \psi_u^L)) \\
&\leq |\widehat{Y}_u|^{\ell-1} \check{Y}_u (f^N(u, Y_u^N, \psi_u^N) - f^L(u, Y_u^N, \psi_u^N)) = |\widehat{Y}_u|^{\ell-1} \check{Y}_u (\gamma_u \wedge N - \gamma_u \wedge L) \\
&\leq |\widehat{Y}_u|^{\ell-1} |\gamma_u \wedge N - \gamma_u \wedge L|
\end{aligned}$$

where we used monotonicity of f^L w.r.t. y . We define

$$X = |\widehat{Y}_t|^\ell + \ell \int_0^t |\widehat{Y}_u|^{\ell-1} |\gamma_u \wedge N - \gamma_u \wedge L| du.$$

From Lemma 8 in [17] we obtain for every $s \in [0, t]$:

$$\begin{aligned}
&|\widehat{Y}_s|^\ell + c(\ell) \sum_{s < u \leq t} |\Delta M_u|^2 (|Y_{u-}|^2 \vee |Y_{u-} + \Delta M_u|^2)^{\ell/2-1} \mathbf{1}_{Y_{u-} \neq 0} \\
&\quad + c(\ell) \int_s^t \int_{\mathcal{Z}} |\psi_u(z)|^2 (|Y_{u-}|^2 \vee |Y_{u-} + \psi_u(z)|^2)^{\ell/2-1} \mathbf{1}_{Y_{u-} \neq 0} \pi(dz, du) \\
&\quad + c(\ell) \int_s^t |\widehat{Y}_u|^{\ell-2} \mathbf{1}_{\widehat{Y}_u \neq 0} d[M]_u^c \\
&\leq X - \ell \int_s^t |\widehat{Y}_{u-}|^{\ell-1} \check{Y}_{u-} d\widehat{M}_u - \ell \int_s^t |\widehat{Y}_u|^{\ell-1} \check{Y}_u \int_{\mathcal{Z}} \widehat{\psi}_u(z) \tilde{\pi}(dz, du).
\end{aligned}$$

Next we proceed as in Proposition 3 in [17] and we prove that there exists a constant C_ℓ depending only on ℓ such that $\mathbb{E} \left(\sup_{s \in [0, t]} |\widehat{Y}_s|^\ell \right) \leq C_\ell \mathbb{E}(X)$, and we apply Young's inequality to obtain that

$$C_\ell \mathbb{E}(X) \leq C_\ell \mathbb{E} |\widehat{Y}_t|^\ell + \frac{1}{2} \mathbb{E} \left(\sup_{s \in [0, t]} |\widehat{Y}_s|^\ell \right) + \bar{C}_\ell \mathbb{E} \int_0^t |\gamma_u \wedge N - \gamma_u \wedge L|^\ell du.$$

Now we follow the scheme of the first part of the proof of Proposition 3 in [17] which yields

$$\mathbb{E} \left(\int_0^t \int_{\mathcal{Z}} |\widehat{\psi}_u(z)|^2 \mu(dz) du \right)^{\ell/2} + \mathbb{E} [\widehat{M}]_t^{\ell/2} \leq \tilde{C}_\ell \mathbb{E} |\widehat{Y}_t|^\ell + \tilde{C}_\ell \mathbb{E} \int_0^t |\gamma_u \wedge N - \gamma_u \wedge L|^\ell du.$$

Since $\gamma \in \mathbb{H}^\ell(0, t)$ (see Condition (A6)), the right-hand side converges to zero as N and L go to $+\infty$. Then (ψ^L) is a Cauchy sequence in $L_\pi^\ell(0, t)$ and converges to $\psi \in L_\pi^\ell(0, t)$ for every $t < T$. The same holds for the sequence (M^L) in $\mathcal{M}^\ell(0, t)$. Moreover the previous inequality yields that $\mathbb{E} \left(\sup_{0 \leq s \leq t} |Y_s|^\ell \right) < +\infty$.

Finally, taking the limit as L goes to ∞ in (14) implies that (Y, ψ, M) satisfies (1) for every $0 \leq s \leq t < T$. \square

To finish the proof of Theorem 1 let us prove minimality of the limit process.

Proposition 3 *The solution obtained in Theorem 3 is minimal. If (Y', ψ', M') is another weak supersolution of (1) with terminal condition ξ , then $Y'_t \geq Y_t$ a.s. for all $t \in [0, T]$.*

Proof. Fix $L > 0$ and let (Y^L, ψ^L, M^L) denote the solution of (14) with terminal condition $Y_T^L = \xi \wedge L$. Let (Y', ψ', M') be a weak supersolution of (1) in the sense of Definition 1. Set

$$\widehat{Y}_s = Y'_s - Y_s^L, \quad \widehat{\psi}_s(z) = \psi'_s(z) - \psi_s^L(z), \quad \widehat{M}_s = M'_s - M_s^L.$$

We have

$$f(t, Y'_t, \psi'_t) - f(t, Y_t^L, \psi_t^L) = -b_t \widehat{Y}_t + \int_{\mathcal{Z}} \widehat{\psi}_t(z) \kappa_t^{Y_t^L, \psi_t^L, \psi_t^L} \mu(dz)$$

with

$$-b_t = \frac{f(t, Y'_t, \psi'_t) - f(t, Y_t^L, \psi_t^L)}{\widehat{Y}_t} \mathbf{1}_{\widehat{Y}_t \neq 0}.$$

Note that from Condition (A1) $-b_t \geq \chi = 0$. For every $t < T$ the process $(\widehat{Y}, \widehat{\psi}, \widehat{M})$ solves the linear BSDE

$$\begin{aligned} d\widehat{Y}_s &= \left[-b_s \widehat{Y}_s - (\gamma_s - L)^+ + \int_{\mathcal{Z}} \widehat{\psi}_s(z) \kappa_s^{Y_s^L, \psi_s^L, \psi_s^L} \mu(dz) \right] ds \\ &\quad + \int_{\mathcal{Z}} \widehat{\psi}_s(z) \widetilde{\pi}(dz, ds) + d\widehat{M}_s \end{aligned}$$

on $[0, t]$ with terminal condition $\widehat{Y}_t = Y'_t - Y_t^L$. From Lemma 9 in [17], we have

$$\widehat{Y}_s = \mathbb{E} \left[\widehat{Y}_t \Gamma_{s,t} + \int_s^t \Gamma_{s,u} (\gamma_u - L)^+ du \middle| \mathcal{F}_s \right]$$

where $\Gamma_{s,t} = \exp \left(- \int_s^t b_u du \right) \zeta_{s,t}$ with $\zeta_{s,s} = 1$ and

$$d\zeta_{s,t} = \zeta_{s,t-} \int_{\mathcal{Z}} \kappa_t^{Y_t^L, \psi_t^L, \psi_t^L} \widetilde{\pi}(dz, dt).$$

Our assumptions ensure that ζ is non negative and belongs to $\mathbb{H}^k(0, T)$. From Proposition 2 we have $Y_t^L \leq (1+T)L$ and hence $\widehat{Y}_t \geq -((Y'_t)^- + (1+T)L)$. Thus $\widehat{Y}_t \Gamma_{s,t}$ is bounded from below by a process in $\mathbb{D}^m(0, T)$ for some $m > 1$. We can apply Fatou's lemma to obtain

$$\widehat{Y}_s = \liminf_{t \nearrow T} \mathbb{E} \left[\widehat{Y}_t \Gamma_{s,t} + \int_s^t \Gamma_{s,u} (\gamma_u - L)^+ du \middle| \mathcal{F}_s \right] \geq \mathbb{E} \left[\liminf_{t \nearrow T} \widehat{Y}_t \Gamma_{s,t} \middle| \mathcal{F}_s \right] \geq 0.$$

Finally, $Y'_s \geq Y_s^L$ for any $s \in [0, T]$ and $L \geq 0$. Taking the limit as L goes to ∞ yields the claim. \square

Remark 5 Note that all these results can be extended immediately if we assume that the filtration supports also a Brownian motion W and if our singular BSDE has form

$$dY_t = \tilde{f}(t, Y_t, Z_t, \psi_t)dt + Z_t dW_t + \int_{\mathcal{Z}} \psi_t(z) \tilde{\pi}(dz, dt) + dM_t,$$

where \tilde{f} satisfies Conditions (A) and is supposed to be Lipschitz in z .

2.2 Random terminal times

In this section we consider the case where the terminal time τ is random. Again we proceed via truncation of the terminal condition to obtain a family of solutions $(Y^L)_{L>0}$ to (14) with bounded terminal condition $Y_\tau = \xi \wedge L$. The crucial difference to the case of a deterministic terminal time is the derivation of a uniform upper bound for the family of processes (Y^L) (cf. Inequality (16)). Example 1 below shows that in general such an upper bound does not exist and that there exist stopping times τ such that the sequence (Y_t^L) converges to ∞ as $L \rightarrow \infty$ for $t < \tau$. Consequently one has to restrict the class of terminal times. Here we draw inspiration from [23], where BSDEs with random terminal time and singular terminal condition have been studied for the first time, and consider the case where τ is given by a first exit time $\tau = \tau_D$ of a diffusion Γ from a set D as described in Section 1. As in [23] we derive a Keller-Osserman type inequality (c.f. (21) and see [15, 18]). Using analytical properties of the diffusion near the boundary ∂D , allows us to bound at each time t the value of process Y_t^L against the distance of the diffusion Γ to the boundary ∂D .

The terminal time τ is given by (5) and again we decompose the generator \tilde{f} as $\tilde{f}(t, y, \psi) = f(t, y, \psi) + \gamma_t$ with $\gamma_t = \tilde{f}(t, 0, 0)$. Conditions **(A)** hold but with the following modifications (denoted by **(A')** in the rest of the paper).

- f satisfies (A1), (A2), (A5) and (A7). $K = \|\vartheta\|_{L_\mu^2}$ is the Lipschitz constant of f w.r.t. ψ .
- Assumption (A3) becomes: for every $j > 0$ and $n \geq 0$, the process

$$U_t(j) = \sup_{|y| \leq j} |f(t, y, 0)|$$

is in $L^1((0, n))$ a.s. and there exists $m > 1$ such that

$$\mathbb{E} \int_0^\tau |U_t(j)|^m dt < +\infty. \quad (\text{A3}')$$

- Condition (A4) on the negative part of ξ and γ is reinforced:

$$\xi^- \text{ and } \gamma^- \text{ are bounded.} \quad (\text{A4}')$$

- (A6) is replaced by:

$$a \text{ and } \gamma \text{ are bounded.} \quad (\text{A6}')$$

Recall that we have $f(t, 0, 0) = 0$. Note that Hypotheses (A3') and (A5) imply that

$$\mathbb{E} \int_0^\tau \frac{1}{a_s^m} ds < +\infty. \quad (19)$$

Proposition 4 *Assume that Assumptions (A') hold. Then there exists for each $L > 0$ a solution $(Y^L, \psi^L, M^L) \in \mathbb{D}^2(0, \tau) \times L_\pi^2(0, \tau) \times \mathcal{M}^2(0, \tau)$ to the BSDE (14) with terminal condition $Y_\tau^L = \xi \wedge L$.*

Proof. The driver f^L (c.f. (15)) of the BSDE (14) satisfies the monotonicity condition

$$(f^L(t, y, \psi) - f^L(t, y', \psi))(y - y') \leq 0$$

a.s. for any $(t, y, y', \psi) \in [0, T] \times \mathbb{R}^2 \times L_\mu^2$. Moreover, f^L is Lipschitz continuous w.r.t. ψ . By Condition (A3') f^L satisfies

$$\forall j > 0, \forall n \in \mathbb{N}, \quad \sup_{|y| \leq j} (|f^L(t, y, 0) - f^L(t, 0, 0)|) \in L^1(\Omega \times (0, n)).$$

Moreover $\xi \wedge L$ and $f^L(t, 0, 0) = \gamma_t \wedge L$ are bounded (Condition (A6')). From Assumption (6), we deduce that

$$\mathbb{E} \int_0^\tau e^{\delta\sigma t} (|\xi \wedge L|^\sigma + |f^L(t, 0, 0)|^\sigma) dt < +\infty, \quad (20)$$

for any $\delta > 0$ and $\sigma > 1$ such that $\delta\sigma < \beta$.

Next, let $\xi_t^L = \mathbb{E}[\xi \wedge L | \mathcal{F}_t]$ and let (Γ, l, N) be given by the martingale representation of $\xi \wedge L$

$$\xi \wedge L = \mathbb{E}[\xi \wedge L] + \int_0^\infty \Gamma_s dW_s + \int_0^\infty \int_{\mathcal{Z}} l_s(z) \tilde{\pi}(dz, ds) + N_\tau.$$

Since $\xi \wedge L$ is bounded (by L for L large enough), the process ξ_t is also bounded by L . Using Conditions (A1) and (A2) we obtain for some constant C (depending on σ) which will change from line to line:

$$\begin{aligned} \mathbb{E} \left[\int_0^\tau e^{\delta\sigma t} |f^L(t, \xi_t, l_t)|^\sigma dt \right] &\leq C \mathbb{E} \left[\int_0^\tau e^{\delta\sigma t} |f(t, \xi_t, l_t)|^\sigma dt \right] + C \mathbb{E} \int_0^\tau e^{\delta\sigma t} |\gamma_t \wedge L|^\sigma dt \\ &\leq CK^\chi \mathbb{E} \left[\int_0^\tau e^{\delta\sigma t} \|l_t\|_{L_\mu^\sigma}^\sigma dt \right] + C \mathbb{E} \left[\int_0^\tau e^{\delta\sigma t} |U_t(L)|^\sigma dt \right] \\ &\quad + C \mathbb{E} \int_0^\tau e^{\delta\sigma t} |\gamma_t \wedge L|^\sigma dt. \end{aligned}$$

Since γ is bounded, with the same parameters δ and σ used for (20), the last term is finite. By Hölder inequality, for any $a > 1$ and $b > 1$ such that $(a-1)(b-1) = 1$

$$\mathbb{E} \left[\int_0^\tau e^{\delta\sigma t} \|l_t\|_{L_\mu^\sigma}^\sigma dt \right] \leq \left(\mathbb{E} \int_0^\tau e^{\delta\sigma at} dt \right)^{1/a} \left(\mathbb{E} \int_0^\tau \|l_t\|_{L_\mu^{\sigma b}}^{\sigma b} dt \right)^{1/b}.$$

But since $\xi \wedge L$ is bounded, the process l coming from the martingale representation is in any $L_\pi^m(0, \tau)$, $m > 1$. Hence choosing a close enough to 1, this term is also finite. We proceed similarly for the remaining term using Hypothesis (A3') with $\sigma b \leq m$:

$$\mathbb{E} \left[\int_0^\tau e^{\delta\sigma t} |U_t(L)|^\sigma dt \right] \leq \left(\mathbb{E} \int_0^\tau e^{\delta\sigma at} dt \right)^{1/a} \left(\mathbb{E} \int_0^\tau |U_t(L)|^{\sigma b} dt \right)^{1/b}.$$

Choosing $\delta > 0$ small enough, we can find $\sigma > 1$ and $b > 1$ such that $\sigma b \leq m$ and $\delta \sigma a < \beta$.

Hence all assumptions of Theorem 3 and Remark 1 in [17] are satisfied and there exists a solution (Y^L, ψ^L, M^L) to the BSDE (14) with terminal condition $Y_\tau = \xi \wedge L$. More precisely for any $0 \leq t \leq T$

$$\begin{aligned} Y_{t \wedge \tau}^L &= Y_{T \wedge \tau}^L + \int_{t \wedge \tau}^{T \wedge \tau} [f(s, Y_s^L, \psi_s^L) + (\gamma_s \wedge L)] ds \\ &\quad - \int_{t \wedge \tau}^{T \wedge \tau} \int_{\mathcal{Z}} \psi_s^L(z) \tilde{\pi}(dz, ds) - \int_{t \wedge \tau}^{T \wedge \tau} dM_s^L, \end{aligned}$$

and $Y_t^L = \xi \wedge L$ on the set $\{t \geq \tau\}$. \square

Observe that the proof of Proposition 4 does not use the fact that τ is a first hitting time but works for every stopping time τ that satisfies the integrability conditions (6) and (A3'). The next example shows that further assumptions on τ are necessary in order to ensure that the family Y^L is uniformly bounded from above. Therefore we will assume the particular form (5) of τ in the sequel.

Example 1 Assume that $\tilde{f}(t, y, \psi) = -|y|^2$ and $\xi = \infty$. We assume that the filtration \mathcal{F} supports a stopping time τ such that $E[\frac{1}{\tau}] = \infty$ and that satisfies the integrability conditions (6) and (12). This holds for example for all stopping times that have a continuous density function f on \mathbb{R}_+ with $f(0) > 0$. In particular, one can take τ to be the first jump time of a Poisson process, in which case τ is exponentially distributed. For each $L > 0$ let Y^L denote the solution to BSDE (14) constructed in Proposition 4. Next, we derive a lower bound for Y^L . To this end let $X_t = \exp(-\int_0^t Y_s^L ds)$. From Itô's formula we obtain

$$dY_t^L X_t^2 = -(Y_t^L X_t)^2 dt + Z_t^L X_t^2 dW_t.$$

In particular, this implies $Y_0^L = E[\int_0^\tau \dot{X}_s^2 ds + LX_\tau^2]$. Next, fix a realization $\omega \in \Omega$.

Consider the deterministic control problem of minimizing the functional $\int_0^{\tau(\omega)} \dot{x}^2(s) ds + Lx^2(\tau(\omega))$ over functions $x : [0, \tau(\omega)] \rightarrow \mathbb{R}$ starting in $x(0) = 1$ and being absolutely continuous. Using Pontryagin's maximum principle one can show that the trajectory $x(s) = \frac{\tau(\omega) - s + 1/L}{\tau(\omega) + 1/L}$ is optimal in this deterministic problem. In particular, it follows that

$$\int_0^{\tau(\omega)} \dot{x}^2(s) ds + Lx^2(\tau(\omega)) = \frac{1}{\tau(\omega) + 1/L} \leq \int_0^{\tau(\omega)} \dot{X}_s^2(\omega) ds + LX_{\tau(\omega)}^2(\omega)$$

Taking expectations yields $Y_0^L \geq E[\frac{1}{\tau + 1/L}]$ and consequently we have by monotone convergence $\liminf_{L \rightarrow \infty} Y_0^L \geq E[\frac{1}{\tau}] = \infty$.

The preceding example shows that we cannot expect to obtain a finite supersolution to (1) with singular terminal condition and random terminal time if the terminal time occurs too suddenly. Therefore we restrict here attention to the case where τ is the first hitting time of a diffusion. We introduce the signed distance function $\text{dist} : \mathbb{R}^d \rightarrow \mathbb{R}$ of D , which is defined by $\text{dist}(x) = \inf_{y \notin D} \|x - y\|$ if $x \in D$ and $\text{dist}(x) = -\inf_{y \in D} \|x - y\|$ if $x \notin D$. Then we have the following result.

Proposition 5 *Under Assumptions (A') the solution processes Y^L constructed in Proposition 4 are bounded uniformly in L . There exists a process $\bar{Y} \in \mathbb{D}^2(0, \tau)$ and a constant C such that:*

$$\bar{Y}_{t \wedge \tau} \leq Y_{t \wedge \tau}^L \leq \frac{C}{\text{dist}(\Gamma_{t \wedge \tau})^{\rho-1}}. \quad (21)$$

Proof. First observe that the lower bound of Y^L follows as in Proposition 1 from a comparison theorem with a BSDE with terminal condition $-\xi^-$ and driver $f(t, y, \psi) - \gamma^-$.

For the upper bound, let $\mu > 0$ and introduce the set $\Gamma_\mu = \{x \in \mathbb{R}^d, |\text{dist}(x)| \leq \mu\}$. Then it follows from Lemma 14.16 in [9] that there exists a positive constant μ such that $\text{dist} \in C^2(\Gamma_\mu)$. Since D is bounded there exists a constant $R > 0$ such that $0 \leq \text{dist}(x) \leq R$ for all $x \in \bar{D}$. Let $\varphi \in C^\infty(\mathbb{R}^d, [0, 1])$ with $\varphi = 1$ on $\mathbb{R}^d \setminus \Gamma_\mu$ and $\varphi = 0$ on $\Gamma_{\mu/2}$. For $0 < \epsilon \leq 1$ we define a function $g \in C^2(\mathbb{R}^d, \mathbb{R}_+)$ such that $g = (1 - \varphi)\text{dist} + R\varphi + \epsilon$ on \bar{D} . Since $g \geq \epsilon$ on \bar{D} , there exists a function $\Phi \in C^2(\mathbb{R}^d, \mathbb{R}_+)$ satisfying $\Phi = Cg^{-2(\rho-1)}$ on \bar{D} for any $C > 0$. Observe that Φ is bounded from above by $C\text{dist}^{-2(\rho-1)}$. Next we apply Itô's formula to the process $\Phi(\Gamma_{t \wedge \tau})$. For every $t < \tau$ this yields

$$\begin{aligned} d\Phi(\Gamma_t) &= (\rho - 1) \frac{\Phi^r(\Gamma_t)}{a_t} dt + \nabla\Phi(\Gamma_t)\sigma(\Gamma_t)dW_t \\ &\quad + \left(\nabla\Phi(\Gamma_t)b(\Gamma_t) + \frac{1}{2} \text{Trace}(\sigma\sigma^*(\Gamma_t)D^2\Phi(\Gamma_t)) - (\rho - 1) \frac{\Phi^r(\Gamma_t)}{a_t} \right) dt \\ &= \left[(\rho - 1) \frac{\Phi^r(\Gamma_t)}{a_t} - \gamma_t \right] dt + \nabla\Phi(\Gamma_t)\sigma(\Gamma_t)dW_t \\ &\quad + \left[\gamma_t + \nabla\Phi(\Gamma_t)b(\Gamma_t) + \frac{1}{2} \text{Trace}(\sigma\sigma^*(\Gamma_t)D^2\Phi(\Gamma_t)) - (\rho - 1) \frac{\Phi^r(\Gamma_t)}{a_t} \right] dt. \end{aligned}$$

On \bar{D} we have

$$\begin{aligned} \Phi^r &= C^r g^{-2r(\rho-1)} = C^r g^{-2\rho} \\ \nabla\Phi &= -2(\rho - 1)Cg^{-2\rho+1}\nabla g \\ \frac{\partial^2\Phi}{\partial x_i \partial x_j} &= -2(\rho - 1)(-2\rho + 1)Cg^{-2\rho} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} - 2(\rho - 1)Cg^{-2\rho+1} \frac{\partial^2 g}{\partial x_i \partial x_j} \end{aligned}$$

For $t \leq \tau$ let

$$\begin{aligned} G_t &= \nabla\Phi(\Gamma_t)b(\Gamma_t) + \frac{1}{2} \text{Trace}(\sigma\sigma^*(\Gamma_t)D^2\Phi(\Gamma_t)) - (\rho - 1) \frac{\Phi^r(\Gamma_t)}{a_t} \\ &= -(\rho - 1)Cg^{-2\rho}(\Gamma_t) \left(\frac{C^{r-1}}{a_t} + 2g\nabla gb + (-2\rho + 1)\|\sigma\nabla g\|^2 + g \text{Trace}(\sigma\sigma^*D^2g) \right) (\Gamma_t) \\ &= -(\rho - 1)Cg^{-2\rho}(\Gamma_t)\Xi(\Gamma_t) \end{aligned}$$

Since \bar{D} is compact, the functions b and σ are bounded on \bar{D} . Moreover, the functions $g, \nabla g$ and D^2g are bounded on \bar{D} uniformly in ϵ . By Assumption (A6'), the processes a and γ are bounded from above. There exists $C > 0$ which does not depend on ϵ such that for every $t \geq 0$ and on \bar{D} we have $\Xi(\Gamma_t) \geq 1$ and

$$(\rho - 1)Cg^{-2\rho} \geq \gamma_t.$$

Hence, we obtain $G_t + \gamma_t \leq 0$.

Next, choose $\epsilon \leq (C/L)^{(q-1)/2}$. Since the process $\Phi(\Gamma)$ satisfies

$$\begin{aligned}\Phi(\Gamma_{t \wedge \tau}) &= \Phi(\Gamma_{T \wedge \tau}) + \int_{t \wedge \tau}^{T \wedge \tau} \left[-(\rho - 1) \frac{\Phi^r(\Gamma_s)}{a_s} + \gamma_s \right] ds \\ &\quad + \int_{t \wedge \tau}^{T \wedge \tau} \mathcal{G}_s ds - \int_{t \wedge \tau}^{T \wedge \tau} \Phi(\Gamma_s) \sigma(\Gamma_s) dW_s\end{aligned}$$

for all $0 \leq t \leq T$, with $\mathcal{G}_s \geq 0$, and $\Phi(\Gamma_{T \wedge \tau}) = \frac{C}{\epsilon^{2(p-1)}}$ on $\{T \geq \tau\}$, a comparison principle (c.f. Remark 3 in [17]) and Condition (A5) imply: $Y_{t \wedge \tau}^{L,+} \leq \Phi(\Gamma_{t \wedge \tau}) \leq C \text{dist}^{-2(\rho-1)}(\Gamma_{t \wedge \tau})$. \square

Now as in Section 2, we can define a process Y as the limit of the increasing sequence Y^L to obtain the minimal supersolution of (1). The next proposition completes the proof of Theorem 1 in the random terminal time case.

Proposition 6 *Suppose that Assumptions (A') are in force and let (Y^L, ψ^L, M^L) denote the solution of BSDE (14) obtained in Proposition 4. Then there exists a process (Y, ψ, M) such that Y_t^L converges a.s. to Y_t , ψ^L converges in $L_\pi^2(0, \tau_\epsilon)$ to ψ and M^L converges in $\mathcal{M}^2(0, \tau_\epsilon)$ to M . The limit process (Y, ψ, M) is the minimal supersolution for the BSDE (1) with terminal condition ξ such that $Y^- \in \mathcal{D}^2(0, \tau)$.*

Proof. We proceed as in the proof of Theorem 3. We outline the main steps. First observe that Y_t^L converges a.s. to a limit process Y by a comparison principle (c.f. Remark 3 in [17]). Recall the definition of the stopping times τ_ϵ , $\epsilon > 0$, $\tau_\epsilon = \inf\{t \geq 0, \text{dist}(\Gamma_t) \leq \epsilon\}$. We have $\text{dist}(\Gamma_{t \wedge \tau_\epsilon}) \geq \epsilon$ for ϵ small enough. Moreover τ_ϵ converges to τ when ϵ goes to zero. Using this sequence of times τ_ϵ , the whole sequence (Y^L, ψ^L, M^L) converges to (Y, ψ, M) on $\mathcal{D}^2(0, \tau_\epsilon) \times L_\mu^2(0, \tau_\epsilon) \times \mathcal{M}^2(0, \tau_\epsilon)$. The main argument is that by Proposition 5 on the interval $(0, \tau_\epsilon)$, the process Y^L is uniformly bounded by $C/\epsilon^{2(p-1)}$. Moreover (Y, ψ, M) satisfies for any $\epsilon > 0$ and any $0 \leq t \leq T$

$$\begin{aligned}Y_{t \wedge \tau_\epsilon} &= Y_{T \wedge \tau_\epsilon} + \int_{t \wedge \tau_\epsilon}^{T \wedge \tau_\epsilon} [f(s, Y_s, \psi_s) + \gamma_s] ds \\ &\quad - \int_{t \wedge \tau_\epsilon}^{T \wedge \tau_\epsilon} \int_{\mathcal{Z}} \psi_s(z) \tilde{\pi}(dz, ds) - \int_{t \wedge \tau_\epsilon}^{T \wedge \tau_\epsilon} dM_s.\end{aligned}$$

Since the filtration is supposed to be left-continuous, we have a.s. $\lim_{t \rightarrow +\infty} Y_{t \wedge \tau}^L = \xi \wedge L$. Therefore we obtain the following behaviour of Y at the terminal time $\liminf_{t \rightarrow +\infty} Y_{t \wedge \tau} \geq \xi$. Finally, minimality of the solution follows by the same arguments as in Proposition 3. \square

3 Back to the control problem

In this section we first conclude from Theorem 1 that there exists a minimal supersolution to (3). We then consider a variant of the minimization problem (10), where we omit the constraint $X_\tau \mathbf{1}_{\mathcal{S}} = 0$ on the set $\mathcal{A}_0(t, x)$ of admissible controls but penalize any non zero terminal state by $L|X_\tau|^p$. We show that optimal controls for this unconstrained minimization problem admit a representation in terms of the solutions Y^L of a truncated version of (3). We then use this result to derive an optimal control for (10).

3.1 Associated BSDE

We consider the singular BSDE (3)

$$dY_t = (p-1) \frac{Y_t^q}{\eta_t^{q-1}} dt + \Theta(t, Y_t, \psi_t) dt - \gamma_t dt + \int_{\mathcal{Z}} \psi_t(z) \tilde{\pi}(dz, dt) + dM_t$$

where the function Θ is given by

$$\Theta(t, y, \psi) = \int_{\mathcal{Z}} (y + \psi(z)) \left(1 - \frac{\lambda_t(z)}{((y + \psi(z))^{q-1} + \lambda_t(z)^{q-1})^{p-1}} \right) \mathbf{1}_{y+\psi(z) \geq 0} \mu(dz). \quad (22)$$

Recall that here μ is supposed to be a finite measure, thus Θ is well-defined. This BSDE is a special case of (1) with generator \tilde{f} given by

$$\tilde{f}(t, y, \psi) = f(t, y, \psi) + \gamma_t = -(p-1) \frac{y|y|^{q-1}}{\eta_t^{q-1}} - \Theta(t, y, \psi) + \gamma_t.$$

For simplicity we denote by ϖ the function

$$\varpi(t, y, \phi) = (y + \phi) \left(1 - \frac{\lambda_t(z)}{((y + \phi)^{q-1} + \lambda_t(z)^{q-1})^{p-1}} \right) \mathbf{1}_{y+\phi \geq 0}$$

such that

$$\Theta(t, y, \psi) = \int_{\mathcal{Z}} \varpi(t, y, \psi(z)) \mu(dz).$$

We have to prove that \tilde{f} satisfies Conditions **(A)** (respectively **(A')**) if (B1) (or (B2)) holds. A simple computation proves that for a fixed $(t, \psi) \in [0, T] \times L_{\mu}^2$ and $z \in \mathcal{Z}$, the function $y \mapsto \varpi(t, y, \psi(z))$ is non decreasing and of class C^1 on \mathbb{R} with a derivative bounded by 1

$$\frac{\partial \varpi}{\partial y}(t, y, \psi(z)) = \left(1 - \frac{\lambda_t(z)^q}{((y + \psi(z))^{q-1} + \lambda_t(z)^{q-1})^p} \right) \mathbf{1}_{y+\psi(z) \geq 0}.$$

Since $\eta > 0$, the condition (A1) is satisfied.

From the same argument the function ϖ is Lipschitz continuous w.r.t. $\psi(z)$ and hence we obtain

$$|\Theta(t, y, \psi) - \Theta(t, y, \psi')| \leq \int_{\mathcal{Z}} |\psi(z) - \psi'(z)| \mu(dz) \leq \mu(\mathcal{Z})^{1/2} \|\psi - \psi'\|_{L_{\mu}^2}.$$

Moreover for any $(t, y, \psi, \psi') \in [0, T] \times \mathbb{R} \times (L_{\mu}^2)^2$ we have

$$\begin{aligned} f(t, y, \psi) - f(t, y, \psi') &= -\Theta(t, y, \psi) + \Theta(t, y, \psi') = \int_{\mathcal{Z}} (\varpi(t, y, \psi'(z)) - \varpi(t, y, \psi(z))) \mu(dz) \\ &= \int_{\mathcal{Z}} (\psi(z) - \psi'(z)) \kappa_t^{y, \psi, \psi'}(z) \mu(dz) \end{aligned}$$

where

$$\kappa_t^{y, \psi, \psi'}(z) = -\frac{\varpi(t, y, \psi(z)) - \varpi(t, y, \psi'(z))}{\psi(z) - \psi'(z)} \mathbf{1}_{\psi(z) \neq \psi'(z)}.$$

Since ϖ is non decreasing in ψ with derivative bounded from above by 1, we obtain $-1 \leq \kappa_t^{y,\psi,\psi'} \leq 0$. Thus Conditions (A2) and (A7) hold for any $k \geq 1$. We can even note that (A7') (cf. Remark 4) is true with $K_f = 0$. For every $r > 0$ and $|y| \leq r$ we have

$$|f(t, y, 0) - f(t, 0, 0)| = (p-1) \frac{|y|^q}{\eta_t^{q-1}} + |\Theta(t, y, 0)| \leq (p-1) \frac{|r|^q}{\eta_t^{q-1}} + \mu(\mathcal{Z})|r| =: U_t(r).$$

By Assumption (B1), the mapping $t \mapsto U_t(r)$ is in $L^1(0, T)$ a.s. and Condition (A3) holds. Finally since $\Theta \geq 0$, Condition (A5) is satisfied with $r = q$ and $a_t = \eta_t^{q-1}$ and (A6) holds if Assumption (B1) is assumed.

A similar computation shows that under (B2), Conditions (A3'), (A4') and (A6') hold.

Hence Corollary 1 is a direct consequence of Theorem 1. Moreover, by Proposition 1 (respectively Proposition 4) there exists a solution (Y^L, ψ^L, M^L) of the truncated BSDE

$$dY_t^L = (p-1) \frac{(Y_t^L)^{1+q}}{\eta_t^q} dt + \Theta(t, Y_t^L, \psi_t^L) dt - \gamma_t dt + \int_{\mathcal{Z}} \psi_t^L(z) \tilde{\pi}(dz, dt) + dM_t^L \quad (23)$$

with terminal condition $Y_\tau^L = \xi \wedge L$. The process (Y, ψ, M) is the limit as L goes to $+\infty$ of (Y^L, ψ^L, M^L) and is the minimal (super-)solution of the BSDE (3).

3.2 Penalization

For some $L > 0$ we consider the unconstrained minimization problem:

$$\begin{aligned} v^L(t, x) &= \inf_{X \in \mathcal{A}(t, x)} J^L(t, X) \\ &= \inf_{X \in \mathcal{A}(t, x)} \mathbb{E} \left[\int_t^\tau \left(\eta_s |\alpha_s|^p + (\gamma_s \wedge L) |X_s|^p + \int_{\mathcal{Z}} \lambda_s(z) |\beta_s(z)|^p \mu(dz) \right) ds \right. \\ &\quad \left. + (\xi \wedge L) |X_\tau|^p \middle| \mathcal{F}_t \right] \end{aligned} \quad (24)$$

where $\mathcal{A}(t, x)$ is the set of all progressively measurable processes X of the form (7). Here no terminal constraint is imposed on X .

Proposition 7 *Let Assumption (B1) or (B2) hold and let (Y^L, ψ^L, M^L) be the solution to (23) with terminal condition $Y_\tau = \xi \wedge L$. Then the process X^L satisfying the linear dynamics*

$$X_s^L = x - \int_t^s \left(\frac{Y_r^L}{\eta_r} \right)^{q-1} X_r^L dr - \int_s^t X_{r-}^L \int_{\mathcal{Z}} \zeta_r^L(z) \pi(dz, dr),$$

with

$$\zeta_r^L(z) = \frac{(Y_{r-}^L + \psi_r(z))^{q-1}}{[(Y_{r-}^L + \psi_r^L(z))^{q-1} + \lambda_r(z)^{q-1}]}$$

is optimal in (10). Moreover, we have $v^L(t, x) = Y_t^L |x|^p$.

To prove Proposition 7 we will make use of the two following auxiliary results. The first lemma shows that we can without loss of generality restrict attention to monotone strategies. To this end we introduce the set $\mathcal{D}(t, x)$, the subset of $\mathcal{A}(t, x)$ containing only processes X that have nonincreasing sample paths (i.e. $\alpha_t \leq 0$ and $\beta_t(z) \leq 0$), and that remain nonnegative.

Lemma 1 *Let $x \geq 0$. Every control $X \in \mathcal{A}(t, x)$ can be modified to a control $\underline{X} \in \mathcal{D}(t, x)$ such that $J^L(t, X) \geq J^L(t, \underline{X})$. In particular, $v^L(t, x) = \inf_{X \in \mathcal{D}(t, x)} J^L(t, X)$.*

Proof. We consider the solution of the following SDE

$$\tilde{X}_s = x - \int_t^s \alpha_u^- du - \int_t^s \int_{\mathcal{Z}} \beta_s(z)^- \pi(dz, ds),$$

where x^- denotes the negative part of x . This process is nonincreasing and satisfies $\tilde{X}_s \leq X_s$. Then we define

$$\underline{X}_s = \tilde{X}_s \vee 0 = (\tilde{X}_s)^+.$$

By Tanaka's formula we have

$$\underline{X}_s = x - \int_t^s \mathbf{1}_{\tilde{X}_u > 0} \alpha_u^- du - \int_t^s \int_{\mathcal{Z}} \mathbf{1}_{\tilde{X}_{u^-} > 0} (\beta_u(z)^- \wedge (\tilde{X}_{u^-})^+) \pi(dz, ds).$$

We define

$$\hat{\alpha}_s = -\mathbf{1}_{\tilde{X}_s > 0} \alpha_s^-, \quad \hat{\beta}_s(z) = -\mathbf{1}_{\tilde{X}_{s^-} > 0} (\beta_s(z)^- \wedge (\tilde{X}_{s^-})^+).$$

Then \underline{X} belongs to $\mathcal{D}(t, x)$. Moreover we have

$$|\hat{\alpha}_s| \leq |\alpha_s|, \quad |\hat{\beta}_s(z)| \leq |\beta_s(z)|, \quad 0 \leq \underline{X}_s \leq |X_s|$$

which implies that $J^L(t, X) \geq J^L(t, \underline{X})$. □

The second lemma provides the dynamics of two auxiliary processes.

Lemma 2 *Let Assumptions (B1) or (B2) hold and let (Y^L, ψ^L, M^L) be the solution of (23). Let $X^L \in \mathcal{A}(t, x)$ be the strategy from Proposition 7. Then we have*

$$d(\eta_s |\alpha_s^L|^{p-1}) = (X_{s^-}^L)^{p-1} dM_s^L - (\gamma_s \wedge L) |X_s^L|^{p-1} ds - \int_{\mathcal{Z}} \phi_s(z) \tilde{\pi}(dz, ds),$$

with $\phi_s(z) = Y_s^L |X_{s^-}^L|^{p-1} - \lambda_s(z) |\beta_s^L(z)|^{p-1}$. Moreover, we have

$$\begin{aligned} d(Y_s^L (X_s^L)^p) &= - \left[\eta_s |\alpha_s^L|^p + \gamma_s^L (X_s^L)^p + \int_{\mathcal{Z}} \lambda_s(z) |\beta_s^L(z)|^p \mu(dz) \right] ds \\ &\quad + (X_{s^-}^L)^p dM_s^L + (X_{s^-}^L)^p \int_{\mathcal{Z}} (Y_{s^-}^L + \psi_s^L(z)) \left[(1 - \zeta_s^L(z))^p - 1 \right] \tilde{\pi}(dz, ds) \end{aligned}$$

Proof. To simplify notation we set $\gamma_s^L = \gamma_s \wedge L$. Recall that X^L and Y^L satisfy the following dynamics

$$\begin{aligned} dX_s^L &= -\frac{(Y_s^L)^{q-1}}{\eta_s^{q-1}} X_s^L ds - \int_{\mathcal{Z}} X_{s^-}^L \zeta_s^L(z) \pi(dz, ds), \\ dY_s^L &= \left[(p-1) \frac{(Y_s^L)^q}{\eta_s^{q-1}} + \vartheta(s, Y_s^L, \psi_s^L) - \gamma_s^L \right] ds + \int_{\mathcal{Z}} \psi_s^L(z) \tilde{\pi}(dz, ds) + dM_s^L \end{aligned}$$

Let

$$\Xi_s = \eta_s |\alpha_s^L|^{p-1} + \int_t^s \gamma_u^L |X_u^L|^{p-1} du = Y_s^L |X_s^L|^{p-1} + \int_t^s \gamma_u^L |X_u^L|^{p-1} du.$$

Applying the integration by parts formula to Ξ results in

$$\begin{aligned}
d\Xi_s &= (X_{s^-}^L)^{p-1} dY_s^L + Y_{s^-}^L d((X_s^L)^{p-1}) + d[Y^L, (X^L)^{p-1}]_s + \gamma_s^L |X_s^L|^{p-1} ds \\
&= (X_{s^-}^L)^{p-1} dY_s^L + Y_{s^-}^L (X_{s^-}^L)^{p-1} \left(-(p-1) \frac{(Y_s^L)^{q-1}}{\eta_s^{q-1}} \right) ds \\
&\quad + Y_{s^-}^L (X_{s^-}^L)^{p-1} \int_{\mathcal{Z}} \left((1 - \zeta_s^L(z))^{p-1} - 1 \right) \mu(dz) ds \\
&\quad + Y_{s^-}^L (X_{s^-}^L)^{p-1} \int_{\mathcal{Z}} \left((1 - \zeta_s^L(z))^{p-1} - 1 \right) \tilde{\pi}(dz, ds) \\
&\quad + (X_{s^-}^L)^{p-1} \int_{\mathcal{Z}} \psi_s^L(z) \left((1 - \zeta_s^L(z))^{p-1} - 1 \right) \pi(dz, ds) + p\gamma_s^L |X_s^L|^{p-1} ds \\
&= (X_{s^-}^L)^{p-1} \Theta(s, Y_s^L, \psi_s^L) ds + (X_{s^-}^L)^{p-1} \int_{\mathcal{Z}} (Y_{s^-}^L + \psi_s^L(z)) \left((1 - \zeta_s^L(z))^{p-1} - 1 \right) \mu(dz) ds \\
&\quad (X_{s^-}^L)^{p-1} dM_s^L + (X_{s^-}^L)^{p-1} \int_{\mathcal{Z}} (Y_{s^-}^L + \psi_s^L(z)) \left((1 - \zeta_s^L(z))^{p-1} - 1 \right) \tilde{\pi}(dz, ds) \\
&= (X_{s^-}^L)^{p-1} dM_s^L + (X_{s^-}^L)^{p-1} \int_{\mathcal{Z}} (Y_{s^-}^L + \psi_s^L(z)) \left((1 - \zeta_s^L(z))^{p-1} - 1 \right) \tilde{\pi}(dz, ds)
\end{aligned}$$

from the definition of ζ^L and Θ (see Equation (22)). Moreover we have

$$(Y_{s^-}^L + \psi_s^L(z)) \left[(1 - \zeta_s^L(z))^{p-1} - 1 \right] = \lambda_s(z) \zeta_s^L(z)^{p-1} - (Y_{s^-}^L + \psi_s^L(z)),$$

which yields the first claim.

For the second equation we apply the integration by parts formula to the process $Y^L(X^L)^p$ to obtain

$$\begin{aligned}
d(Y_s^L(X_s^L)^p) &= (X_{s^-}^L)^p dY_s^L + Y_{s^-}^L d((X_s^L)^p) + d[Y^L, (X^L)^p]_s \\
&= - \left[\eta_s(X_s^L)^p \frac{(Y_s^L)^q}{\eta_s^q} + \gamma_s^L (X_s^L)^p \right] ds + (X_{s^-}^L)^p dM_s^L \\
&\quad + (X_{s^-}^L)^p \Theta(s, Y_s^L, \psi_s^L) ds \\
&\quad + (X_{s^-}^L)^p \int_{\mathcal{Z}} (Y_{s^-}^L + \psi_s^L(z)) \left[(1 - \zeta_s^L(z))^p - 1 \right] \mu(dz) ds \\
&\quad + (X_{s^-}^L)^p \int_{\mathcal{Z}} (Y_{s^-}^L + \psi_s^L(z)) \left[(1 - \zeta_s^L(z))^p - 1 \right] \tilde{\pi}(dz, ds).
\end{aligned}$$

But note that

$$|\alpha_s^L|^p = \left| \frac{(Y_s^L)^{q-1}}{\eta_s^{q-1}} X_s^L \right|^p = \frac{(Y_s^L)^q}{\eta_s^q} (X_s^L)^p,$$

and from the very definition (22) of Θ

$$\begin{aligned}
& \Theta(s, Y_s^L, \psi_s^L) + \int_{\mathcal{Z}} (Y_s^L + \psi_s^L(z)) \left[(1 - \zeta_s^L(z))^p - 1 \right] \mu(dz) \\
&= \int_{\mathcal{Z}} (Y_s^L + \psi_s^L(z)) \left[\left(\frac{\lambda_s(z)^{q-1}}{[(Y_{s^-}^L + \psi_s^L(z))^{q-1} + \lambda_s(z)^{q-1}]} \right)^p \right. \\
&\quad \left. - \frac{\lambda_s(z)}{(|Y_s^L + \psi_s^L(z)|^{q-1} + \lambda_s(z)^{q-1})^{p-1}} \right] \mu(dz) \\
&= - \int_{\mathcal{Z}} (Y_s^L + \psi_s^L(z)) \frac{\lambda_s(z)}{[(Y_{s^-}^L + \psi_s^L(z))^{q-1} + \lambda_s(z)^{q-1}]^p} \left[(Y_s^L + \psi_s^L(z))^{q-1} \right] \mu(dz) \\
&= - \int_{\mathcal{Z}} \lambda_s(z) |\zeta_s(z)|^p \mu(dz).
\end{aligned}$$

□

We close this section with the proof of Proposition 7.

Proof of Proposition 7. We omit the superscript L in the sequel. Take another process \bar{X} in $\mathcal{D}_0^S(t, x)$. Use the convexity of the function $x \mapsto |x|^p$ and $\alpha_s \leq 0$ to obtain

$$\begin{aligned}
& \int_t^T (\eta_s (|\alpha_s|^p - |\bar{\alpha}_s|^p)) ds \leq -p \int_t^T \eta_s |\alpha_s|^{p-1} (\alpha_s - \bar{\alpha}_s) ds \\
&= -p \int_t^T \eta_s |\alpha_s|^{p-1} (dX_s - d\bar{X}_s) + p \int_t^T \int_{\mathcal{Z}} \eta_s |\alpha_s|^{p-1} (\beta_s(z) - \bar{\beta}_s(z)) \pi(dz, ds) \\
&= \mathcal{I}_t^1 + \mathcal{I}_t^2
\end{aligned} \tag{25}$$

By integration by parts on the first integral and using Lemma 2 and boundedness of X and \bar{X} (see Lemma 1), we obtain

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_t} \mathcal{I}_t^1 &= -p \mathbb{E}^{\mathcal{F}_t} [\eta_T |\alpha_T|^{p-1} (X_T - \bar{X}_T)] + p \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T (X_s - \bar{X}_s) d(\eta_s |\alpha_s|^{p-1}) \right] \\
&\quad - p \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T \int_{\mathcal{Z}} (\beta_s(z) - \bar{\beta}_s(z)) \phi_s(z) \pi(dz, ds) \right] \\
&= -p \mathbb{E}^{\mathcal{F}_t} [Y_T X_T^{p-1} (X_T - \bar{X}_T)] - p \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T (\gamma_s \wedge L) |X_s^L|^{p-1} (X_s - \bar{X}_s) ds \right] \\
&\quad - p \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T \int_{\mathcal{Z}} (\beta_s(z) - \bar{\beta}_s(z)) \phi_s(z) \mu(dz) ds \right]
\end{aligned}$$

where ϕ is defined as in Lemma 2. Using again convexity of $x \mapsto |x|^p$ yields

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_t} \mathcal{I}_t^1 &\leq -\mathbb{E}^{\mathcal{F}_t} [(\xi \wedge L) (X_T^p - \bar{X}_T^p)] - \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T (\gamma_s \wedge L) (X_s^p - \bar{X}_s^p) ds \right] \\
&\quad - p \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T \int_{\mathcal{Z}} (\beta_s(z) - \bar{\beta}_s(z)) \phi_s(z) \mu(dz) ds \right].
\end{aligned} \tag{26}$$

Moreover we have

$$\mathbb{E}^{\mathcal{F}_t} \mathcal{I}_t^2 = p \mathbb{E}^{\mathcal{F}_t} \int_t^T \int_{\mathcal{Z}} \eta_s |\alpha_s|^{p-1} (\beta_s(z) - \bar{\beta}_s(z)) \mu(dz) ds \tag{27}$$

Now, using (25), (26) and (27) we obtain

$$\begin{aligned} J(t, X) - J(t, \bar{X}) &\leq \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T \int_{\mathcal{Z}} p(\beta_s(z) - \bar{\beta}_s(z)) (\phi_s(z) - \eta_s |\alpha_s|^{p-1}) \mu(dz) ds \right] \\ &\quad + \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T \int_{\mathcal{Z}} \lambda_s(z) (|\beta_s(z)|^p - |\bar{\beta}_s(z)|^p) \mu(dz) ds \right]. \end{aligned}$$

Now recall that $\eta_s |\alpha_s|^{p-1} = Y_s^L |X_s^L|^{p-1}$. From the definition of ϕ_s and from convexity of $x \mapsto |x|^p$ we obtain:

$$J(t, X) - J(t, \bar{X}) \leq \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T \int_{\mathcal{Z}} p Y_s^L (\beta_s(z) - \bar{\beta}_s(z)) (|X_{s-}^L|^{p-1} - |X_s^L|^{p-1}) \mu(dz) ds \right]$$

and therefore $J(t, X) - J(t, \bar{X}) \leq 0$.

It remains to verify the identity $v^L(t, x) = Y_t^L |x|^p$. But from Lemma 2 we deduce that

$$\begin{aligned} Y_t^L |x|^p &= \mathbb{E}^{\mathcal{F}_t} \int_t^T \left[\eta_u |\alpha_u^L|^p + \gamma_u^L (X_u^L)^p + \int_{\mathcal{Z}} \lambda_u(z) |\beta_u^L(z)|^p \mu(dz) \right] du + \mathbb{E}^{\mathcal{F}_t} (Y_T^L |X_T^L|^p) \\ &= J(X) = v^L(t, x). \end{aligned}$$

□

3.3 Solving the constrained problem

This section is devoted to the proof of Theorem 2. For the convenience of the reader we restate the result here.

Theorem 4 *Let Assumptions (B1) or (B2) hold and let (Y, ψ, M) be the minimal solution to (3) with singular terminal condition $Y_T = \xi$ from Corollary 1. Then $v(t, x) = Y_t |x|^p$. Moreover the control*

$$X_s^* = x \exp \left[- \int_t^s \left(\frac{Y_u}{\eta_u} \right)^{q-1} du \right] \exp \left[\int_t^s \int_{\mathcal{Z}} \ln(1 - \zeta_u(z)) \pi(dz, du) \right] \quad (28)$$

with

$$\zeta_t(z) = \frac{(Y_{t-} + \psi_t(z))^{q-1}}{[(Y_{t-} + \psi_t(z))^{q-1} + \lambda_t(z)^{q-1}]}$$

belongs to $\mathcal{A}_0^S(t, x)$ and is optimal in (10).

Proof. Let us remark that

$$\begin{aligned} X_s^* &= x + \int_t^s \alpha_u^* du + \int_t^s \int_{\mathcal{S}} \beta_u^*(z) \pi(du, dz) \\ &= x - \int_t^s X_u^* \left(\frac{Y_u}{\eta_u} \right)^{q-1} du - \int_t^s \int_{\mathcal{Z}} X_{u-}^* \zeta_u(z) \pi(du, dz). \end{aligned}$$

Observe that Y and Y^L satisfy the same dynamics before time T . Hence, the results from Lemma 2 remain to hold true if Y^L and X^L are replaced by Y and X^* . In particular, it follows that the process $Y |X^*|^{p-1} + \int_0^\cdot \gamma_u |X_u^*|^{p-1} du$ is a nonnegative local martingale on

$[0, T)$. Consequently, it is a nonnegative supermartingale and thus converges almost surely in \mathbb{R} as t goes to T . Since Y satisfies the terminal condition $\liminf_{t \nearrow T} Y_t \mathbf{1}_{\mathcal{S}} = \infty$ we have that

$$0 \leq X_t^* = \left(\frac{\Xi_t - p \int_0^t \gamma_u |X_u^*|^{p-1} du}{pY_t} \right)^{q-1} \leq \left(\frac{\Xi_t}{pY_t} \right)^{q-1} \rightarrow 0$$

a.s. on \mathcal{S} when t goes T . It follows that $X^* \in \mathcal{A}_0^{\mathcal{S}}(t, x)$.

Appealing once more to Lemma 2 we observe that for $0 \leq t < T$

$$\begin{aligned} d(Y_t(X_t^*)^p) &= -[\eta_t |\alpha_t^*|^p + \gamma_t (X_t^*)^p] dt - \int_{\mathcal{Z}} \lambda_t(z) |\beta_t^*(z)|^p \mu(dz) dt \\ &\quad + (X_{t-}^*)^p dM_t + (X_{t-}^*)^p \int_{\mathcal{Z}} (Y_{t-} + \psi_t(z)) [(1 - \zeta_t(z))^p - 1] \tilde{\pi}(dz, dt) \end{aligned}$$

Since $M \in \mathcal{M}^2(0, t)$ and $|X_t^*| \leq x$ we can deduce for $t \leq s < T$

$$\begin{aligned} Y_t |x|^p &= \mathbb{E}^{\mathcal{F}_t} \int_t^s \left[\eta_u |\alpha_u^*|^p + \gamma_u (X_u^*)^p + \int_{\mathcal{Z}} \lambda_u(z) |\beta_u^*(z)|^p \mu(dz) \right] du + \mathbb{E}^{\mathcal{F}_t} (Y_s |X_s|^p) \\ &\geq \mathbb{E}^{\mathcal{F}_t} \int_t^s \left[\eta_u |\alpha_u^*|^p + \gamma_u (X_u^*)^p + \int_{\mathcal{Z}} \lambda_u(z) |\beta_u^*(z)|^p \mu(dz) \right] du. \end{aligned} \quad (29)$$

Taking the limit as s goes to T and appealing to monotone convergence theorem yields $Y_t |x|^p \geq J(t, X^*)$. Next, note that for every $X \in \mathcal{A}_0^{\mathcal{S}}(t, x)$ we have $J(t, X) \geq J^L(t, X)$. This implies $v(t, x) \geq v^L(t, x)$ for every $L > 0$. By Proposition 7 we have $Y_t^L |x|^p = v^L(t, x)$. Minimality of Y implies

$$Y_t |x|^p = \lim_{L \nearrow \infty} Y_t^L |x|^p = \lim_{L \nearrow \infty} v^L(t, x) \leq v(t, x).$$

Consequently we obtain with Equation (29)

$$Y_t |x|^p \geq J(t, X^*) \geq v(t, x) \geq Y_t |x|^p$$

and thus optimality of X^* . □

References

- [1] R. Almgren. Optimal trading with stochastic liquidity and volatility. *SIAM Journal on Financial Mathematics*, 3(1):163–181, 2012.
- [2] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.
- [3] S. Ankirchner, M. Jeanblanc, and T. Kruse. BSDEs with Singular Terminal Condition and a Control Problem with Constraints. *SIAM J. Control Optim.*, 52(2):893–913, 2014.
- [4] S. Ankirchner and T. Kruse. Optimal position targeting with stochastic linear-quadratic costs. *To appear in Banach Center Publications*, 2013.

- [5] R.F. Bass. *Diffusions and elliptic operators*. Probability and its Applications (New York). Springer-Verlag, New York, 1998.
- [6] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71, 1997.
- [7] P. A. Forsyth, J. S. Kennedy, S.T. Tse, and H. Windcliff. Optimal trade execution: a mean quadratic variation approach. *Journal of Economic Dynamics and Control*, 36(12):1971–1991, 2012.
- [8] J. Gatheral and A. Schied. Optimal trade execution under geometric brownian motion in the almgren and chris framework. *International Journal of Theoretical and Applied Finance*, 14(03):353–368, 2011.
- [9] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [10] P. Graewe, U. Horst, and J. Qiu. A Non-Markovian Liquidation Problem and Backward SPDEs with Singular Terminal Conditions. *ArXiv e-prints*, 2013.
- [11] P. Graewe, U. Horst, and E. Séré. Smooth solutions to portfolio liquidation problems under price-sensitive market impact. *ArXiv e-prints*, 2013.
- [12] U. Horst and F. Naujokat. When to cross the spread? trading in two-sided limit order books. *SIAM Journal on Financial Mathematics*, 5(1):278–315, 2014.
- [13] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.
- [14] O. Kallenberg. *Foundations of modern probability*. Springer: Berlin, 2002.
- [15] J. B. Keller. On solutions of $\Delta u = f(u)$. *Comm. Pure Appl. Math.*, 10:503–510, 1957.
- [16] P. Kratz and T. Schöneborn. Portfolio liquidation in dark pools in continuous time. *Mathematical Finance*, 2013.
- [17] T. Kruse and A. Popier. Bsdes with jumps in a general filtration. *ArXiv e-prints*, December 2014.
- [18] R. Osserman. On the inequality $\Delta u \geq f(u)$. *Pacific J. Math.*, 7:1641–1647, 1957.
- [19] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.
- [20] H. Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media, 2009.
- [21] R.G. Pinsky. *Positive harmonic functions and diffusion*, volume 45 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

- [22] A. Popier. Backward stochastic differential equations with singular terminal condition. *Stochastic Process. Appl.*, 116(12):2014–2056, 2006.
- [23] A. Popier. Backward stochastic differential equations with random stopping time and singular final condition. *Ann. Probab.*, 35(3):1071–1117, 2007.
- [24] M.-C. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Process. Appl.*, 123(8):3328–3357, 2013.
- [25] A. Schied. A control problem with fuel constraint and dawson–watanabe superprocesses. *The Annals of Applied Probability*, 23(6):2472–2499, 2013.