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The Uniqueness of Equilibrium for Time-Inconsistent Stochastic Linear–Quadratic Control

Ying Hu* Hanqing Jin[†] Xun Yu Zhou[‡]

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Abstract

We prove the uniqueness of an equilibrium solution to a general time-inconsistent LQ control problem under mild conditions which ensure the existence of a solution. This is the first positive result on the uniqueness of the solution to a time inconsistent dynamic decision problem in continuous-time setting.

Key words. time-inconsistency, stochastic linear-quadratic control, uniqueness of equilibrium control, forward–backward stochastic differential equation, mean–variance portfolio selection.

AMS subject classification 91B51, 93E99, 60H10

1 Introduction

Time inconsistency in dynamic decision making is often observed in social systems and daily life. The study on time inconsistency by economists can be dated back to Strotz [10] in 1950’s, who initiated the formulation of time inconsistent decision making as a game between incarnations of the player herself.

The game formulation is quite easy to understand when time setting is (finitely) discrete. When time setting is continuous, the formulation can be generalized in different ways. It is still not clear which is the best one among different definitions of a solution to the time inconsistent decision problem. Mathematically, both the existence and the uniqueness of a solution make a definition more acceptable. Although it is common that a game problem

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admits multiple solutions, the time inconsistent decision problem is a decision problem for one player, and hence, even if the control in the solutions is not unique, an identical value process for all solutions sounds more reasonable.

Unfortunately to our best knowledge, neither existence nor uniqueness of a solution for general time inconsistent decision problem is available yet in continuous-time setting. Yong [12] and Ekeland and Pirvu [4] studied the existence of equilibrium solutions, with their own definitions for equilibrium solutions, for the time inconsistency arising from hyperbolic discounting. Grenadier and Wang [5] also studied the hyperbolic discounting problem in an optimal stopping model. In a Markovian system, Björk and Murgoci [2] proposed a definition of a general stochastic control problem with time inconsistent terms, and proposed some sufficient condition for a control to be a solution by a system of partial differential equations. They constructed some solutions for some examples including an LQ one, but it looks very hard to find not-too-harsh condition on parameters to ensure the existence of a solution. Björk, Murgoci and Zhou [3] also constructed an equilibrium for a mean-variance portfolio selection with state-dependent risk aversion. Basak and Chabakauri [1] studied the mean-variance portfolio selection problem and got more details on the constructed solution. In our previous paper [7], we generalized the discrete-time game formulation for an LQ control problem with time inconsistent terms in a non-Markovian system slightly different from the one in Björk and Murgoci, and constructed an equilibrium for quite general LQ control problem, including a non-Markovian system.

The study of the uniqueness is even less available in literature. Vieille and Weibull [11] studied the non-uniqueness for a finitely discrete system because of the non-uniqueness of optimal solution in backward steps. Otherwise we do not know any other results on the uniqueness for the solution in a continuous-time other than trivial cases (e.g., the time consistent control problem), neither positive results nor negative ones.

In this paper, we prove that the solution we obtained in [7] is unique. This result not only justifies our definition of a solution to the time inconsistent LQ control problem, but also justifies the game formulation of a time inconsistent dynamic decision problem. It also sheds a light on the search of conditions on the uniqueness of the problem.

The rest of this paper is organized as follows. In Section 2, we recall the formulation of the LQ control problem without time consistency studied in our previous work [7]. Then we develop an equivalent characterization of a solution by a system of stochastic differential equations in Section 3. Finally in Section 4, we prove that the solution obtained in [7] is the unique solution to our LQ control problem in cases when a solution is constructed in [7].

2 Problem Formulation

The Linear-Quadratic control problem without time consistency is the same as the one in our previous work [7]. Here we recall it for the convenience of reading, where we inherit notations for different spaces.

We consider the same LQ control problem with the controlled system in a finite time horizon $t \in [0, T]$,

$$(2.1) \quad dX_s = [A_s X_s + B'_s u_s + b_s] ds + \sum_{j=1}^d [C_s^j X_s + D_s^j u_s + \sigma_s^j] dW_s^j; \quad X_0 = x_0,$$

where $W = (W^1, \dots, W^d)'$ is a standard d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ with filtration generated by W , A is a bounded deterministic function on $[0, T]$ with value in $\mathbb{R}^{n \times n}$. The other parameters B, C^j, D^j are all essentially bounded adapted processes on $[0, T]$ with values in $\mathbb{R}^{l \times n}, \mathbb{R}^{n \times n}, \mathbb{R}^{n \times l}$, respectively; b and σ^j are stochastic processes in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$. The process $u \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$ is the control, and X is the state process valued in \mathbb{R}^n with initial value $x_0 \in \mathbb{R}^n$.

At any time t with the system state $X_t = x_t$, our aim is to minimize

$$(2.2) \quad \begin{aligned} J(t, x_t; u) &\triangleq \frac{1}{2} \mathbb{E}_t \int_t^T [\langle Q_s X_s, X_s \rangle + \langle R_s u_s, u_s \rangle] ds + \frac{1}{2} \mathbb{E}_t [\langle G X_T, X_T \rangle] \\ &\quad - \frac{1}{2} \langle h \mathbb{E}_t [X_T], \mathbb{E}_t [X_T] \rangle - \langle \mu_1 x_t + \mu_2, \mathbb{E}_t [X_T] \rangle \end{aligned}$$

over $u \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^l)$, where $X = X^{t, x_t, u}$, and $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$, Q and R are both given non-negative definite and essentially bounded adapted processes on $[0, T]$ with values in \mathbb{S}^n and \mathbb{S}^l , respectively, G, h, μ_1, μ_2 are all constants in $\mathbb{S}^n, \mathbb{S}^n, \mathbb{R}^{n \times n}$, and \mathbb{R}^n , and G is non-negative positive definite.

We also define an equilibrium by local perturbation. Given a control u^* . For any $t \in [0, T)$, $\varepsilon > 0$ and $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^l)$, define

$$(2.3) \quad u_s^{t, \varepsilon, v} = u_s^* + v \mathbf{1}_{s \in [t, t + \varepsilon)}, \quad s \in [t, T].$$

Definition 2.1 Let $u^* \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$ be a given control and X^* be the state process corresponding to u^* . The control u^* is called an equilibrium if

$$\liminf_{\varepsilon \downarrow 0} \frac{J(t, X_t^*; u^{t, \varepsilon, v}) - J(t, X_t^*; u^*)}{\varepsilon} \geq 0,$$

where $u^{t, \varepsilon, v}$ is defined by (2.3), for any $t \in [0, T)$ and $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^l)$.

Notice that here we changed \lim to \liminf in this definition, which makes no critical difference other than making the definition more robust.

3 An Equivalent Condition of Equilibrium Controls

In this section we present a general characterization for equilibria. This has been done partially in our previous paper [7] for the sufficiency by the second-order expansion in the local spike variation, in the same spirit of proving the stochastic Pontryagin's maximum principle [8, 9, 13], here we will study an equivalent condition for equilibria.

We also start with the obtained result from our previous paper [7]. Let u^* be a fixed control and X^* be the corresponding state process. For any $t \in [0, T)$, define in the time interval $[t, T]$ the processes $(p(\cdot; t), (k^j(\cdot; t))_{j=1, \dots, d}) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times (L^2_{\mathcal{F}}(t, T; \mathbb{R}^n))^d$ as the solution to

$$(3.1) \quad \begin{cases} dp(s; t) = -[A'_s p(s; t) + \sum_{j=1}^d (C_s^j)' k^j(s; t) + Q_s X_s^*] ds \\ \quad + \sum_{j=1}^d k^j(s; t) dW_s^j, \quad s \in [t, T], \\ p(T; t) = G X_T^* - h \mathbb{E}_t [X_T^*] - \mu_1 X_t^* - \mu_2; \end{cases}$$

Define $(P(\cdot; t), (K^j(\cdot; t))_{j=1, \dots, d}) \in L^2_{\mathcal{F}}(t, T; \mathbb{S}^n) \times (L^2_{\mathcal{F}}(t, T; \mathbb{S}^n))^d$ as the solution to

$$(3.2) \quad \begin{cases} dP(s; t) = -\left\{ A'_s P(s; t) + P(s; t) A_s \right. \\ \quad \left. + \sum_{j=1}^d [(C_s^j)' P(s; t) C_s^j + (C_s^j)' K^j(s; t) + K^j(s; t) C_s^j] + Q_s \right\} ds \\ \quad + \sum_{j=1}^d K^j(s; t) dW_s^j, \quad s \in [t, T], \\ P(T; t) = G. \end{cases}$$

The following estimate under local spike variation is from [7, Proposition 3.1].

Proposition 3.1 *For any $t \in [0, T)$, $\varepsilon > 0$ and $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^l)$, define $u^{t, \varepsilon, v}$ by (2.3). Then*

$$(3.3) \quad J(t, X_t^*; u^{t, \varepsilon, v}) - J(t, X_t^*; u^*) = \mathbb{E}_t \int_t^{t+\varepsilon} (\langle \Lambda(s; t), v \rangle + \frac{1}{2} \langle H(s; t) v, v \rangle) ds + o(\varepsilon)$$

where $\Lambda(s; t) \triangleq B_s p(s; t) + \sum_{j=1}^d (D_s^j)' k^j(s; t) + R_s u_s^*$ and $H(s; t) \triangleq R_s + \sum_{j=1}^d (D_s^j)' P(s; t) D_s^j$.

In the rest of this section, we will keep the notations Λ and H .

By the estimate in Proposition 3.1, it is straightforward to get the following characterization of an equilibrium.

Corollary 3.2 *A control $u^* \in L^2(0, T, \mathbb{R}^l)$ is an equilibrium if and only if*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [\Lambda(s; t)] ds = 0, \quad a.s., \quad \forall t \in [0, T).$$

Before going to the main result of this section, let us prove a key property for the solution to the system of BSDEs for $(p(\cdot; t), k(\cdot; t)) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times (L^2_{\mathcal{F}}(t, T; \mathbb{R}^n))^d$ and the special form of the process $\Lambda(s; t) \triangleq B_s p(s; t) + \sum_{j=1}^d (D_s^j)' k^j(s; t) + R_s u_s^*$.

Theorem 3.3 *For any pair of state and control processes (X^*, u^*) , the solution to (3.1) in $L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times (L^2_{\mathcal{F}}(t, T; \mathbb{R}^n))^d$ satisfies $k(s; t_1) = k(s; t_2)$ for a.e. $s \geq \max(t_1, t_2)$. Furthermore, there exist $\lambda_1 \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$, $\lambda_2 \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}^{l \times n})$ and $\xi \in L^2(\Omega; C(0, T; \mathbb{R}^n))$, such that*

$$\Lambda(s; t) = \lambda_1(s) + \lambda_2(s) \xi_t.$$

Proof: Define the function $\psi(\cdot)$ as the unique solution for the matrix-valued ordinary differential equation

$$d\psi(t) = \psi(t) A(t)' dt, \quad \psi(T) = I_n,$$

where I_n means the $n \times n$ identity matrix. It is well known that $\psi(\cdot)$ is invertible, and both $\psi(\cdot)$ and $\psi(\cdot)^{-1}$ are bounded.

Define $\hat{p}(s; t) = \psi(s) p(s; t) + h \mathbb{E}_t [X_T^*] + \mu_1 X_t^* + \mu_2$ and $\hat{k}^j(s; t) = \psi(s) k^j(s; t)$ for $j = 1, \dots, d$, then in the interval $s \in [t, T]$, $(\hat{p}(\cdot; t), \hat{k}(\cdot; t))$ satisfies

$$(3.4) \quad \begin{cases} d\hat{p}(s; t) = -[\sum_{j=1}^d \psi(s) (C_s^j)' \psi(s)^{-1} \hat{k}^j(s; t) + \psi(s) Q_s X_s^*] ds + \sum_{j=1}^d \hat{k}^j(s; t) dW_s^j, \\ \hat{p}(T; t) = G X_T^*. \end{cases}$$

From the uniqueness of solutions to Lipschitz BSDE, the solution $(\hat{p}(s; t), \hat{k}(s; t))$ does not depend on t , hence we denote its solution as $(\hat{p}(s), \hat{k}(s))$.

Now we have $k(s; t) = \psi(s)^{-1} \hat{k}(s) := k(s)$ and

$$p(s; t) = \psi(s)^{-1} \hat{p}(s) - \psi(s)^{-1} (h \mathbb{E}_t [X_T^*] + \mu_1 X_t^* + \mu_2) := p(s) + \psi(s)^{-1} \xi_t,$$

where $k(\cdot)$ is in $(L^2_{\mathcal{F}}(0, T; \mathbb{R}^n))^d$, $\xi_t := -h \mathbb{E}_t [X_T^*] - \mu_1 X_t^* - \mu_2$ defines the process $\xi \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$ and $p(s) := \psi(s)^{-1} \hat{p}(s)$ defines the process $p \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$. Hence

$$\begin{aligned} \Lambda(s; t) &= B_s p(s; t) + \sum_{j=1}^d (D_s^j)' k^j(s; t) + R_s u_s^* \\ &= B_s p(s) + \sum_{j=1}^d (D_s^j)' k^j(s) + R_s u_s^* + B_s \psi(s)^{-1} \xi_t \\ &:= \lambda_1(s) + \lambda_2(s) \xi_t, \end{aligned}$$

where $\lambda_1(s) := B_s p(s) + \sum_{j=1}^d (D_s^j)' k^j(s) + R_s u_s^*$ and $\lambda_2(s) := B_s \psi(s)^{-1}$. *Q.E.D.*

Theorem 3.4 *For any control $u \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$, the following two statements are equivalent:*

- (i) $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [\Lambda(s; t)] ds = 0, \quad a.s., \quad \forall t \in [0, T]$.
- (ii) $\Lambda(t; t) = 0, \quad a.s., \quad a.e.t \in [0, T]$.

Proof: We know $\Lambda(s; t) = \lambda_1(s) + \lambda_2(s) \xi_t$. Since λ_2 is bounded and ξ is continuous, we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbb{E}_t \left[\frac{1}{\varepsilon} \int_t^{t+\varepsilon} |\lambda_2(s) (\xi_s - \xi_t)| ds \right] &\leq c \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [|\xi_s - \xi_t|] ds \\ &= 0, \end{aligned}$$

where the last equality is because $\mathbb{E}_t [|\xi_s - \xi_t|]$ is a continuous function of s and vanishes at $s = t$.

By this fact, we know

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [\Lambda(s; t)] ds = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [\Lambda(s; s)] ds.$$

If (ii) holds, then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [\Lambda(s; t)] ds = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [\Lambda(s; s)] ds = 0.$$

If (i) holds, then $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [\Lambda(s; s)] ds = 0$. According to the following lemma 3.5, we know $\Lambda(s; s) = 0, \quad a.s., \quad a.e.t \in [0, T]$. *Q.E.D.*

Lemma 3.5 $Y \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$ is a given process. If $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds = 0$, a.e.t $\in [0, T)$, a.s., then $Y_t = 0$, a.e.t $\in [0, T)$, a.s..

Proof: Since $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^l)$ is a separable space, we can take a countable dense subset $\mathcal{D} \subset L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^l) \cap L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R}^l)$, such that for almost all t , we have

$$(3.5) \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [\langle Y_s, \eta \rangle] ds = \mathbb{E} [\langle Y_t, \eta \rangle], \quad \forall \eta \in \mathcal{D},$$

and $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [Y_s^2] ds = \mathbb{E} [Y_t^2]$.

For any $\eta \in \mathcal{D}$, define $\eta_s = \mathbb{E}_s[\eta]$, then $\mathbb{E} [\langle Y_s, \eta \rangle] = \mathbb{E} [\langle Y_s, \eta_s \rangle]$. Since

$$\begin{aligned} \left| \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [\langle Y_s, \eta_s - \eta_t \rangle] ds \right| &\leq \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sqrt{\int_t^{t+\varepsilon} \mathbb{E} [Y_s^2] ds \int_t^{t+\varepsilon} \mathbb{E} [(\eta_s - \eta_t)^2] ds} \\ &= \lim_{\varepsilon \downarrow 0} \sqrt{\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [Y_s^2] ds} \sqrt{\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [(\eta_s - \eta_t)^2] ds} \\ &\leq \lim_{\varepsilon \downarrow 0} \sqrt{\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [Y_s^2] ds} \sqrt{\sup_{s \in [t, t+\varepsilon]} \mathbb{E} [(\eta_s - \eta_t)^2]} \\ &\leq 2 \lim_{\varepsilon \downarrow 0} \sqrt{\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [Y_s^2] ds} \sqrt{\mathbb{E} [(\eta_{t+\varepsilon} - \eta_t)^2]} = 0. \end{aligned}$$

Hence for any $\eta \in \mathcal{D}$,

$$\begin{aligned} \mathbb{E} [\langle Y_t, \eta_t \rangle] &= \mathbb{E} [\langle Y_t, \eta \rangle] \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [\langle Y_s, \eta \rangle] ds \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [\langle Y_s, \eta_s \rangle] ds \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [\langle Y_s, \eta_t \rangle] ds \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [\langle \mathbb{E}_t [Y_s], \eta_t \rangle] ds \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\left\langle \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds, \eta_t \right\rangle \right]. \end{aligned}$$

Since

$$\begin{aligned} \mathbb{E} \left[\left(\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds \right)^2 \right] &\leq \mathbb{E} \left[\int_t^{t+\varepsilon} \frac{1}{\varepsilon^2} ds \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s]^2 ds \right] \\ &= \frac{1}{\varepsilon} \mathbb{E} \left[\int_t^{t+\varepsilon} \mathbb{E}_t [Y_s]^2 ds \right] \\ &\leq \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [Y_s^2] ds, \end{aligned}$$

and $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E} [Y_s^2] ds = \mathbb{E} [Y_t^2]$, there exists a constant $\delta_t > 0$, such that

$$\mathbb{E} \left[\left(\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds \right)^2 \right] < 2\mathbb{E} [Y_t^2], \quad \forall \varepsilon \in (0, \delta_t).$$

This implies that $\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds$ is uniformly integrable when $\varepsilon \in (0, \delta_t)$. Hence

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\left| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds \right| \right] = \mathbb{E} \left[\lim_{\varepsilon \downarrow 0} \left| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds \right| \right] = 0.$$

Then there exists a constant $c > 0$, such that

$$\begin{aligned} \left| \mathbb{E} \left[\left\langle \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds, \eta_t \right\rangle \right] \right| &\leq c \mathbb{E} \left[\left| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds \right| \right] \\ &\rightarrow 0, \end{aligned}$$

which means

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\left\langle \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t [Y_s] ds, \eta_t \right\rangle \right] = 0,$$

and hence $\mathbb{E} [\langle Y_t, \eta \rangle] = 0$, *a.e.t* $\in [0, T]$ for any $\eta \in \mathcal{D}$, which implies

$$Y_t = 0, \quad \textit{a.e.t} \in [0, T], \quad \textit{a.s.}$$

Q.E.D.

We summarize the main theorem into the following characterization for equilibria.

Theorem 3.6 *Given a control $u^* \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)$. Denote X^* as the state process of u^* , and $(p(\cdot; t), k(\cdot; t)) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times (L^2_{\mathcal{F}}(0, T; \mathbb{R}^n))^d$ as the unique solution for the BSDE (3.1), with $k(s) := k(s; t)$ according to Theorem 3.3. Define $\Lambda(\cdot; t) \triangleq B.p(\cdot; t) + \sum_{j=1}^d (D^j)' k(\cdot)^j + R.u^*$.*

Then u^ is an equilibrium if and only if*

$$\Lambda(t; t) = 0, \quad \textit{a.s.}, \quad \textit{a.e.} \quad t \in [0, T].$$

4 Uniqueness

Now we prove the system of equations (3.1) and the dynamics, together with the condition $\Lambda(t; t) = 0$, has a unique solution in cases studied in our previous paper [7], where we focus on the scalar system, i.e., $n = 1$ and the system is governed by the SDE

$$(4.1) \quad dX_s = [A_s X_s + B'_s u_s + b_s] ds + [C_s X_s + D_s u_s + \sigma_s]' dW_s; \quad X_0 = x_0.$$

In this case, we can rewrite the system of backward SDE $(p(\cdot; t), k(\cdot; t))$ into

$$(4.2) \quad \begin{cases} dp(s; t) = -[A_s p(s; t) + C'_s k(s; t) + Q_s X_s^*] ds + k(s; t)' dW_s, & s \in [t, T], \\ p(T; t) = G X_T^* - h \mathbb{E}_t [X_T^*] - \mu_1 X_t^* - \mu_2, \end{cases}$$

and the corresponding $\Lambda(s; t)$ is now in the form

$$\Lambda(s; t) \triangleq B_s p(s; t) + D'_s k(s; t) + R_s u_s^*.$$

4.1 Cases with Deterministic Parameters

In this section, we assume all parameters A, B, b, C, D, σ, Q and R are deterministic. We have constructed a solution to the joint SDEs and $\Lambda(t; t) = 0$ in [7] by solving the following system of ordinary differential equations:

$$(4.3) \quad \begin{cases} 0 = \dot{M} + (2A + |C|^2)M + Q \\ -M(B' + C'D)(R + MD'D)^{-1}[(M - N - \Gamma^{(1)})B + MD'C], \quad s \in [0, T], \\ M_T = G; \end{cases}$$

$$(4.4) \quad \begin{cases} 0 = \dot{N} + 2AN \\ -NB'(R + MD'D)^{-1}[(M - N - \Gamma^{(1)})B + MD'C], \quad s \in [0, T], \\ N_T = h; \end{cases}$$

$$(4.5) \quad \begin{cases} \dot{\Gamma}^{(1)} = -A\Gamma^{(1)}, \quad s \in [0, T], \\ \Gamma_T^{(1)} = \mu_1; \end{cases}$$

$$(4.6) \quad \begin{cases} 0 = \dot{\Phi} + \{A - [(M - N)B' + MC'D](R + MD'D)^{-1}B\}\Phi + (M - N)b \\ + C'M\sigma - [(M - N)B' + MC'D](R + MD'D)^{-1}MD'\sigma, \quad s \in [0, T], \\ \Phi_T = -\mu_2. \end{cases}$$

If this system of equations admits a solution $(M, N, \Gamma^{(1)}, \Phi)$, then the feedback control law

$$u_s^* = \alpha_s X_s^* + \beta_s$$

defines an equilibrium, where

$$\begin{aligned} \alpha_s &\triangleq -(R_s + M_s D'_s D_s)^{-1}[(M_s - N_s - \Gamma_s^{(1)})B_s + M_s D'_s C_s], \\ \beta_s &\triangleq -(R_s + M_s D'_s D_s)^{-1}(\Phi_s B_s + M_s D'_s \sigma_s). \end{aligned}$$

We assume the existence of $(M, N, \Gamma^{(1)}, \Phi)$ to the system of equations (4.3, 4.4, 4.5, 4.6), then we have an equilibrium constructed as above. In [7], there are several mild sufficient conditions for this assumption to hold. Now we claim that the equilibrium constructed above is the unique equilibrium.

Theorem 4.1 *In the case with deterministic parameters, when $(M, N, \Gamma^{(1)}, \Phi)$ exists, the equilibrium is unique.*

Proof: Suppose there is another equilibrium (X, u) , then the equation system (3.1), with X^* replaced by X , admits a solution $(p(s; t), k(s), X_s, u_s)$, which satisfies $B_s p(s; s) + D'_s k(s) + R_s u_s = 0$ for a.e. $s \in [0, T]$.

Define

$$\bar{p}(s; t) = p(s; t) - [M_s X_s - N_s \mathbb{E}_t [X_s] - \Gamma_s^{(1)} X_t + \Phi_s], \quad \bar{k}(s) = k(s) - M_s (C_s X_s + D_s u_s + \sigma_s).$$

By the equilibrium condition, we have

$$B_s(\bar{p}(s; s) + (M_s - N_s - \Gamma_s^{(1)})X_s + \Phi_s) + D'_s[\bar{k}(s) + M_s(C_s X_s + D_s u_s + \sigma_s)] + R_s u_s = 0.$$

Since $R_s + D'_s M_s D_s$ is invertible, we have

$$\begin{aligned} u_s &= -[R_s + D'_s M_s D_s]^{-1} [B_s \bar{p}(s; s) + D'_s \bar{k}(s) + (B_s(M_s - N_s - \Gamma_s^{(1)}) + D'_s M_s C_s)X_s \\ &\quad + B_s \Phi_s + D'_s M_s \sigma_s]. \end{aligned}$$

Hence

$$\begin{aligned} &d\bar{p}(s; t) \\ &= dp(s; t) - d[M_s X_s - N_s \mathbb{E}_t[X_s] - \Gamma_s^{(1)} X_t + \Phi_s] \\ &= -[A_s p(s; t) + C'_s k_s + Q_s X_s] ds + k'_s dW_s - d[M_s X_s - N_s \mathbb{E}_t[X_s] - \Gamma_s^{(1)} X_t + \Phi_s] \\ &= -\{A\bar{p}(s; t) + C'\bar{k}(s) - [C'MD + MB'] [R + D'MD]^{-1} [B\bar{p}(s; s) + D'\bar{k}(s)] \\ &\quad + NB' [R + D'MD]^{-1} \mathbb{E}_t [B\bar{p}(s; s) + D'\bar{k}(s)]\} ds + \bar{k}(s)' dW_s, \end{aligned}$$

where we suppress the subscript s for A, B, C, D, M, N, R , and we have used the equations for $M, N, \Gamma^{(1)}, \Phi$ in the last equality. It is easy to prove that $\mathbb{E} \left[\int_0^T |\bar{k}(s)|^2 ds \right] < +\infty$ and $\sup_{t \in [0, T]} \mathbb{E} [\sup_{s \geq t} |\bar{p}(s; t)|^2] < +\infty$.

We will prove in the next theorem that the equation for $(\bar{p}(s; t), \bar{k}(s))$ admits at most one solution in the space $\mathcal{L}_1 \times \mathcal{L}_2$, where

$$\mathcal{L}_1 := \left\{ X(\cdot; \cdot) : X(\cdot; t) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}), \sup_{t \in [0, T]} \mathbb{E} \left[\sup_{s \geq t} |X(s; t)|^2 \right] < +\infty \right\},$$

and

$$\mathcal{L}_2 := \left\{ Y(\cdot; \cdot) : Y(\cdot; t) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}^d), \sup_{t \in [0, T]} \mathbb{E} \left[\int_t^T |Y(s; t)|^2 ds \right] < +\infty \right\}.$$

Hence $\bar{p}(s; t) \equiv 0$ and $\bar{k}(s) \equiv 0$.

Finally, plugging $\bar{p} \equiv \bar{k} \equiv 0$ into u , we get the u being in the same form of feedback control law as the one specified by $M, N, \Gamma^{(1)}, \Phi$, and hence (X, u) is the same as we got before. *Q.E.D.*

For the uniqueness of $(\bar{p}(s; t), \bar{k}(s))$, we study a more general equation

$$(4.7) \quad \begin{cases} d\bar{p}(s; t) = -f(s, \bar{p}(s; t), \bar{p}(s; s), \mathbb{E}_t[l_1(s)\bar{p}(s; s)], \bar{k}(s; t), \mathbb{E}_t[l_2(s)\bar{k}(s; t)]) ds + \bar{k}(s; t)' dW_s, \\ \bar{p}(T; t) = 0, \end{cases}$$

where $l_i(s)$ are uniformly bounded, adapted vector process with finite dimension, $f(s, \dots)$ is uniformly Lipschitz for all variables except for s .

Theorem 4.2 Equation (4.7) admits at most one solution (\bar{p}, \bar{k}) in the space $\mathcal{L}_1 \times \mathcal{L}_2$.

Proof: For any $t \in [0, T]$, $s \in [t, T]$, by Itô's formula, we have

$$\begin{aligned} & |\bar{p}(s; t)|^2 + \int_s^T |\bar{k}(u; t)|^2 du \\ &= 2 \int_s^T \bar{p}(u; t) f(u, \bar{p}(u; t), \bar{p}(u; u), \mathbb{E}_t [l_1(u) \bar{p}(u; u)], \bar{k}(u; t), \mathbb{E}_t [l_2(u) \bar{k}(u; t)]) du \\ &\quad - 2 \int_s^T \bar{p}(u; t) \bar{k}(u; t)' dW_u. \end{aligned}$$

By this equality, there exists a constant $c_1 > 0$, such that

$$\begin{aligned} & \mathbb{E} [|\bar{p}(s; t)|^2] + \mathbb{E} \left[\int_s^T |\bar{k}(u; t)|^2 du \right] \\ & \leq c_1 \mathbb{E} \left[\int_s^T |\bar{p}(u; t)| (|\bar{p}(u; t)| + |\bar{k}(u; t)| + |\bar{p}(u; u)| + \mathbb{E}_t [|\bar{p}(u; u)|] + \mathbb{E}_t [|\bar{k}(u; t)|]) du \right] \\ & \leq c_2 \mathbb{E} \left[\int_s^T (|\bar{p}(u; t)|^2 + |\bar{p}(u; u)|^2) du \right] + \frac{1}{2} \mathbb{E} \left[\int_s^T |\bar{k}(u; t)|^2 du \right], \end{aligned}$$

where we have used the inequality $cxy \leq c^2x^2 + \frac{1}{4}y^2$ for any nonnegative c, x, y . Hence there exists a $c_3 > 0$ such that

$$(4.8) \quad \mathbb{E} [|\bar{p}(s; t)|^2] + \mathbb{E} \left[\int_s^T |\bar{k}(u; t)|^2 du \right] \leq c_3 \mathbb{E} \left[\int_s^T (|\bar{p}(u; t)|^2 + |\bar{p}(u; u)|^2) du \right].$$

Furthermore, we have for any $s \in [t, T]$ that

$$\begin{aligned} \mathbb{E} \left[|\bar{p}(s; t)|^2 + \int_s^T |\bar{k}(u; t)|^2 du \right] & \leq c_3(T-t) \left[\sup_{u \in [t, T]} \mathbb{E} [|\bar{p}(u; t)|^2] + \sup_{u \in [t, T]} \mathbb{E} [|\bar{p}(u; u)|^2] \right] \\ & \leq 2c_3(T-t) \sup_{t \leq u \leq s \leq T} \mathbb{E} [|\bar{p}(s; u)|^2]. \end{aligned}$$

Hence

$$(4.9) \quad \sup_{t \leq u \leq s \leq T} \mathbb{E} [|\bar{p}(s; u)|^2] \leq 2c_3(T-t) \sup_{t \leq u \leq s \leq T} \mathbb{E} [|\bar{p}(s; u)|^2].$$

Now take $\delta \in (0, 1/(4c))$, then for any $t \in [T - \delta, T]$, we have

$$\sup_{t \leq u \leq s \leq T} \mathbb{E} [|\bar{p}(s; u)|^2] \leq \frac{1}{2} \sup_{t \leq u \leq s \leq T} \mathbb{E} [|\bar{p}(s; u)|^2],$$

which means $\sup_{t \leq u \leq s \leq T} \mathbb{E} [|\bar{p}(s; u)|^2] = 0$, hence $\bar{p}(s; u) = 0, a.s.$ almost everywhere in $\{(s, u) : t \leq u \leq s \leq T\}$.

For $t \in [T - 2\delta, T - \delta]$ and $s \in [T - \delta, T]$, since $\bar{p}(u, u) = 0$ for any $u \in [s, T]$, by (4.8) we have

$$(4.10) \quad \mathbb{E} [|\bar{p}(s; t)|^2] + \mathbb{E} \left[\int_s^T |\bar{k}(u; t)|^2 du \right] \leq c_3 \mathbb{E} \left[\int_s^T |\bar{p}(u; t)|^2 du \right].$$

By Gronwall's inequality, we have $\bar{p}(s; t) = 0, \bar{k}(s; t) = 0$.

For $t \in [T - 2\delta, T - \delta]$ and $s \in [t, T - \delta]$, since we have $\bar{p}(T - \delta; t) = 0$, we can apply previous trick for the region $t \in [T - \delta, T]$ and $s \in [t, T]$ to confirm that $\bar{p} \equiv 0, \bar{k} \equiv 0$.

We can repeat the same analysis to $t \in [T - 3\delta, T - 2\delta]$, and again and again until time $t = 0$. *Q.E.D.*

4.2 Mean-Variance Equilibrium strategies in A Complete Market with Random Parameters

In this subsection, we consider the mean-variance investment problem in a complete financial market. In terms of the LQ control problem, we study the system governed by the SDE

$$(4.11) \quad \begin{cases} dX_s = r_s X_s ds + \theta'_s u_s ds + u'_s dW_s, & s \in [t, T], \\ X_t = x_t, \end{cases}$$

where r is the deterministic interest rate function, θ is the random risk premium process. The objective at time t with state $X_t = x_t$ is to minimize

$$(4.12) \quad \begin{aligned} J(t, x_t; u) &\triangleq \frac{1}{2} \text{Var}_t(X_T) - (\mu_1 x_t + \mu_2) \mathbb{E}_t[X_T] \\ &= \frac{1}{2} (\mathbb{E}_t[X_T^2] - (\mathbb{E}_t[X_T])^2) - (\mu_1 x_t + \mu_2) \mathbb{E}_t[X_T] \end{aligned}$$

with $\mu_1 \geq 0$.

In [7], we constructed an equilibrium by the solutions (M, U) , $(\Gamma^{(1)}, \gamma^{(1)})$, $(\Gamma^{(2)}, \gamma^{(2)})$, and $(\Gamma^{(3)}, \gamma^{(3)})$ for BSDEs:

$$(4.13) \quad \begin{cases} dM_s &= -[2rM + (\theta M + U)' \alpha] ds + U'_s dW_s, & M_T = 1, \\ d\Gamma_s^{(1)} &= -r\Gamma_s^{(1)} ds + (\gamma_s^{(1)})' dW_s, & \Gamma_T^{(1)} = \mu_1, \\ d\Gamma_s^{(2)} &= -[r\Gamma_s^{(2)} + (\theta M + U)' \beta] ds + (\gamma_s^{(2)})' dW_s, & \Gamma_T^{(2)} = -\mu_2, \\ d\Gamma_s^{(3)} &= -[r\Gamma_s^{(3)} + (\theta M + U)' \beta] ds + (\gamma_s^{(3)})' dW_s, & \Gamma_T^{(3)} = 0, \end{cases}$$

with

$$\begin{aligned} \alpha_s &\triangleq -M_s^{-1} (-\theta_s \Gamma_s^{(1)} + U_s - \gamma_s^{(1)}), \\ \beta_s &\triangleq -M_s^{-1} (\theta_s (\Gamma_s^{(2)} - \Gamma_s^{(3)}) + \gamma_s^{(2)}). \end{aligned}$$

In this case, the BSDE for $p(s; t)$ for a control u with state process X is

$$dp(s; t) = -r_s p(s; t) ds + k(s; t)' dW_s, \quad p(T; t) = X_T^* - \mathbb{E}_t[X_T^*] - \mu_1 X_t^* - \mu_2,$$

and the corresponding $\Lambda(s; t)$ is

$$\Lambda(s; t) = p(s; t) \theta_s + k(s; t).$$

It is proved in [7, Proposition 5.1] that the system of BSDEs admits a unique solution, both M and M^{-1} are *bounded*, and $U \cdot W$ is a BMO martingale. Furthermore, the feedback control policy

$$u_s^* = -M_s^{-1} \left[(U_s - \theta_s \mu_1 e^{\int_s^T r_v dv}) X_s^* + \Gamma_s \theta_s + \gamma_s^{(2)} \right]$$

defines a control in the space $L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$, which is an equilibrium for the mean-variance investment problem.

For any $q > 1$, define

$$\mathcal{L}_3(q) := \{X(\cdot; \cdot) : \text{For } \forall t \in [0, T], X(\cdot; t) \in L^q_{\mathcal{F}}(\Omega; C(t, T; \mathbb{R}))\},$$

and

$$\mathcal{L}_4(q) := \left\{ Y(\cdot) : Y \text{ is adapted, } \mathbb{E} \left[\left(\int_t^T |Y(s)|^2 ds \right)^{q/2} \right] < +\infty \right\}.$$

In this subsection, we claim that the equilibrium above is unique.

Theorem 4.3 *For the mean-variance problem with random θ , the equilibrium obtained is unique for the control problem.*

Proof: Suppose that there is another equilibrium, then there exists another solution for the equation system (5.4) in our previous paper, denoted as $p(s; t), k(s), X_s, u_s$ which satisfies $\theta_s p(s; s) + k(s) = 0$ for a.e. $s \in [0, T]$. We will also use the notation $M, U, \Gamma^{(1)}, \Gamma^{(2)}, \gamma^{(2)}, \Gamma^{(3)}$ and $\gamma^{(3)}$.

As we know from [7], $M, M^{-1}, \Gamma^{(1)}, \Gamma^{(2)}$ and $\Gamma^{(3)}$ are all bounded, $\gamma^{(2)} \cdot W$ and $U \cdot W$ are both BMO martingales.

By the fact that $U \cdot W$ is a BMO martingale, in view of John-Nirenberg inequality (see Kazamaki [6, Theorem 2.2, p.29]), we know that there exists a $\varepsilon > 0$ such that $\mathbb{E} \left[e^{\varepsilon \int_0^T |U_s|^2 ds} \right] < +\infty$, hence $\mathbb{E} \left[\left(\int_0^T |U_s|^2 ds \right)^q \right] < +\infty$ for any $q > 0$.

Define $\bar{p}(s; t) = p(s; t) - [M_s X_s + \Gamma_s^{(2)} - \mathbb{E}_t [M_s X_s + \Gamma_s^{(3)}] - \Gamma_s^{(1)} X_t]$ and $\bar{k}(s) = k(s) - [M_s u_s + U_s X_s + \gamma_s^{(2)}]$.

It is easy to check that $\bar{p}(\cdot; t) \in \mathcal{L}_3(2)$. Since $k \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$, $Mu + \gamma^{(2)} \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$, and for any $q \in (1, 2)$,

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T |U_s X_s|^2 ds \right)^{q/2} \right] &\leq \mathbb{E} \left[\sup_{s \in [0, T]} |X_s|^q \left(\int_0^T |U_s|^2 ds \right)^{q/2} \right] \\ &\leq \left(\mathbb{E} \left[\sup_{s \in [0, T]} |X_s|^2 \right] \right)^{q/2} \left(\mathbb{E} \left[\left(\int_0^T |U_s|^2 ds \right)^{q/(2-q)} \right] \right)^{1-q/2} \\ &< +\infty, \end{aligned}$$

together with $L^2_{\mathcal{F}}(0, T; \mathbb{R}^d) \subset \mathcal{L}_4(q)$ for any $q \in (1, 2)$ we can conclude that $\bar{k} \in \mathcal{L}_4(q)$ for any $q \in (1, 2)$.

Furthermore,

$$\begin{aligned} 0 &= \theta \bar{p}(s; s) + \bar{k}(s) + \theta[\Gamma_s^{(2)} - \Gamma_s^{(3)} - \Gamma_s^{(1)} X_s] + [M_s u_s + U_s X_s + \gamma_s^{(2)}], \\ u_s &= -M_s^{-1} [(U_s - \theta \Gamma_s^{(1)}) X_s + \theta \bar{p}(s; s) + \bar{k}(s) + \theta(\Gamma_s^{(2)} - \Gamma_s^{(3)}) + \gamma_s^{(2)}] \\ &= \alpha_s X_s + \beta_s - M_s^{-1} [\theta \bar{p}(s; s) + \bar{k}(s)], \end{aligned}$$

where α_s, β_s are the same as we defined in (5.14) in our previous paper.

Now we plug u_s into the calculation of $d\bar{p}(s; t)$,

$$(4.14) \quad \left\{ \begin{array}{l} d\bar{p}(s; t) = - \left\{ r_s \bar{p}(s; t) - (\theta_s + U_s M_s^{-1})' [\theta_s \bar{p}(s; s) + \bar{k}(s)] \right. \\ \quad \left. + \mathbb{E}_t [(\theta_s + U_s M_s^{-1})' [\theta_s \bar{p}(s; s) + \bar{k}(s)]] \right\} ds \\ \quad + \bar{k}(s)' dW_s, \\ \bar{p}(T; t) = 0. \end{array} \right.$$

We will prove in the next theorem that this equation admits at most one solution $(\bar{p}(s; t), \bar{k}(s))$ in the space $\mathcal{L}_3(q) \times \mathcal{L}_4(q)$ for some $q \in (1, 2)$, which means $\bar{p} \equiv 0$ and $\bar{k} \equiv 0$, and hence $p(s; t) = M_s X_s + \Gamma_s^{(2)} - \mathbb{E}_t [M_s X_s + \Gamma_s^{(3)}] + \Gamma_s^{(1)} X_t$, $k(s) = M_s u_s + U_s X_s + \gamma_s^{(2)}$. So we have $u_s = \alpha_s X_s + \beta_s$. Plugging this control to dX_s , we have the wealth process and the control (X, u) are the same. *Q.E.D.*

Theorem 4.4 *For any $q \in (1, 2)$, the equation (4.14) admits at most one solution $(\bar{p}, \bar{k}) \in \mathcal{L}_3(q) \times \mathcal{L}_4(q)$.*

Proof: Firstly, take $\mathbb{E}_t[\cdot]$ of the equation for \bar{p} , and notice that $\bar{k} \cdot W$ is a martingale, we get

$$\mathbb{E}_t [\bar{p}(s; t)] = \int_s^T r_\nu \mathbb{E}_t [\bar{p}(\nu; t)] d\nu,$$

which implies $\mathbb{E}_t [\bar{p}(s; t)] = 0$ for any $s \geq t$. Especially at $s = t$, we have $\bar{p}(t; t) = 0$, hence the equation (4.14) turns into

$$(4.15) \quad \left\{ \begin{array}{l} d\bar{p}(s; t) = - \left\{ r_s \bar{p}(s; t) - (\theta_s + \frac{U_s}{M_s})' \bar{k}(s) + \mathbb{E}_t \left[(\theta_s + \frac{U_s}{M_s})' \bar{k}(s) \right] \right\} ds + \bar{k}(s)' dW_s, \\ \bar{p}(T; t) = 0. \end{array} \right.$$

As r is deterministic and bounded, we can discount $\bar{p}(s; t)$ by $e^{-\int_s^T r_\nu d\nu}$ to remove the linear term $-r_s \bar{p}(s; t)$, and hence we can assume $r \equiv 0$ without loss of generality. Define $\tilde{p}(s; t) = \bar{p}(s; t) - \int_s^T \mathbb{E}_t [(\theta_\nu + U_\nu M_\nu^{-1})' \bar{k}(s)] d\nu$, then $\tilde{p}(T; t) = 0$ and

$$d\tilde{p}(s; t) = (\theta_s + U_s M_s^{-1})' \bar{k}(s) ds + \bar{k}(s)' dW_s.$$

In $\tilde{p}(\cdot; t)$, it is given that $\bar{p}(\cdot; t) \in \mathcal{L}_3(q)$. Furthermore, for any $\bar{q} \in (1, q)$, denote $\hat{q} = q/\bar{q}$,

and $1/\hat{p} + 1/\hat{q} = 1$, then

$$\begin{aligned}
& \sup_{s \in [t, T]} \mathbb{E} \left[\left| \int_s^T \mathbb{E}_t \left[(\theta_\nu + U_\nu M_\nu^{-1})' \bar{k}(\nu) \right] d\nu \right|^{\bar{q}} \right] \\
& \leq \mathbb{E} \left[\left(\int_t^T |(\theta_\nu + U_\nu M_\nu^{-1})' \bar{k}(\nu)| d\nu \right)^{\bar{q}} \right] \\
& \leq c_0 \mathbb{E} \left[\left(\int_t^T |\theta'_\nu \bar{k}(\nu)| d\nu \right)^{\bar{q}} \right] + c_0 \mathbb{E} \left[\left(\int_t^T M_\nu^{-1} |U'_\nu \bar{k}(\nu)| d\nu \right)^{\bar{q}} \right] \\
& \leq c_1 \mathbb{E} \left[\left(\int_t^T |\bar{k}(\nu)|^2 d\nu \right)^{\bar{q}/2} \right] + c_2 \mathbb{E} \left[\left(\int_t^T |U_\nu|^2 d\nu \right)^{\bar{q}/2} \left(\int_t^T |\bar{k}(\nu)|^2 d\nu \right)^{\bar{q}/2} \right] \\
& \leq c_3 + c_2 \left(\mathbb{E} \left[\left(\int_t^T |U_\nu|^2 d\nu \right)^{\bar{q}\hat{p}/2} \right] \right)^{1/\hat{p}} \left(\mathbb{E} \left[\left(\int_t^T |\bar{k}(\nu)|^2 d\nu \right)^{q/2} \right] \right)^{1/\hat{q}} \\
& < +\infty.
\end{aligned}$$

By this inequality, we have $\mathbb{E} [\sup_{s \in [t, T]} |\tilde{p}(s; t)|^{\bar{q}}] < +\infty$.

Define $\xi = \mathcal{E}(-(\theta_s + U_s M_s^{-1}) \cdot W)_T = e^{-\frac{1}{2} \int_0^T |\theta_s + \frac{U_s}{M_s}|^2 ds - \int_0^T (\theta_s + \frac{U_s}{M_s})' dW_s}$. Since U/M is a BMO, $\mathbb{E}[\xi] = 1$, and it can be used to define a new measure \mathbb{Q} by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi$, under which $\hat{W}_s = W_s + \int_0^s (\theta_v + U_v M_v^{-1}) dv$ is a standard Brownian motion. Furthermore,

$$d\tilde{p}(s; t) = \bar{k}(s) d\hat{W}_s, \quad \tilde{p}(T; t) = 0.$$

By Itô's formula,

$$\begin{aligned}
dM_s^{-1} &= -M_s^{-2} dM_s + M_s^{-3} U_s^2 ds \\
&= M_s^{-1} \left\{ \left[\theta \left(\frac{\Gamma_s^{(1)}}{M_s} - 1 \right) \frac{U_s}{M_s} + \frac{\Gamma_s^{(1)} |\theta_s|^2}{M} \right] ds - \frac{U_s}{M_s} dW_s \right\},
\end{aligned}$$

hence

$$M_T^{-1} = M_0^{-1} e^{-\int_0^T [\Gamma_s^{(1)} \frac{U'_s \theta_s}{M_s} - \Gamma_s^{(1)} \frac{|\theta_s|^2}{M_s} + \frac{1}{2} \frac{|U_s|^2}{M_s^2} - \Gamma_s^{(1)} \frac{U'_s \theta_s}{M_s}] ds - \int_0^T \frac{U'_s}{M_s} dW_s}.$$

Comparing ξ and M_T^{-1} , we can see

$$\xi M_T = M_0 e^{-\int_0^T \Gamma_s^{(1)} |\theta_s|^2 \frac{1}{M_s} ds} e^{-\int_0^T \Gamma_s^{(1)} \frac{\theta'_s U_s}{M_s} ds} e^{-\frac{1}{2} \int_0^T |\theta_s|^2 ds - \int_0^T \theta_s dW_s}.$$

It is obvious that $M_0 e^{-\int_0^T \Gamma_s^{(1)} |\theta_s|^2 \frac{1}{M_s} ds}$ is bounded, $e^{-\frac{1}{2} \int_0^T |\theta_s|^2 ds - \int_0^T \theta'_s dW_s} \in L^{\bar{q}}$ for any $\bar{q} > 1$. Furthermore, we know for any $\bar{q} > 1$, any $\varepsilon > 0$, there exists a constant $C > 0$, such that

$$\mathbb{E} \left[\left(e^{-\int_0^T \Gamma_s^{(1)} \frac{\theta_s U_s}{M_s} ds} \right)^{\bar{q}} \right] \leq C \mathbb{E} \left[e^{\varepsilon \int_0^T |U_s|^2 ds} \right].$$

As we claimed before, there exists a $\varepsilon > 0$ such that $\mathbb{E} \left[e^{\varepsilon \int_0^T |U_s|^2 ds} \right] < +\infty$, hence $e^{-\int_0^T \Gamma_s^{(1)} \frac{\theta_s U_s}{M_s} ds} \in L^{\bar{q}}$. This proves $\xi M_T \in L^{\bar{q}}$, and therefore, by the fact that M^{-1} is bounded, we know $\xi \in L^{\bar{q}}$ for any $\bar{q} > 1$.

Now for any $\bar{q} \in (1, q)$ and $\hat{q} \in (1, \bar{q})$, we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\sup_{s \in [t, T]} |\tilde{p}(s; t)|^{\hat{q}}\right] &= \mathbb{E}\left[\sup_{s \in [t, T]} |\tilde{p}(s; t)|^{\hat{q}} \xi\right] \\ &\leq \left(\mathbb{E}\left[\sup_{s \in [t, T]} |\tilde{p}(s; t)|^{\bar{q}}\right]\right)^{\hat{q}/\bar{q}} \left(\mathbb{E}\left[\xi^{\bar{q}/(\bar{q}-\hat{q})}\right]\right)^{(\bar{q}-\hat{q})/\bar{q}} \\ &< +\infty, \end{aligned}$$

which means $\tilde{p}(\cdot; t)$ is a \mathbb{Q} -martingale, and hence $\tilde{p} \equiv 0$.

Q.E.D.

5 Concluding Remarks

Surprisingly, the LQ control problem we studied in this paper admits a *unique* solution. The uniqueness of the solution implies the uniqueness of the value process, and the latter relieves the worry like “why an equilibrium is defined like that”, or “what should the decision maker do if there are multiple solutions”. Since equilibria are defined via perturbation for the game formulation of the control problem, we believe that the definition of equilibria, unlike the optimal solution, is not conceptually linked to the uniqueness of the value process. We believe that the uniqueness (of the value process) is from the LQ structure of the system and the objective. In our proof, the linear feedback structure of our equilibrium plays a key role, which sounds hard to be generalized for other time inconsistent control problem.

References

- [1] S. BASAK AND G. CHABAKAURI, *Dynamic mean–variance asset allocation*, Rev. Financial Stud., **23** (2010), 2970–3016.
- [2] BJÖRK T. AND A. MURGOCI, *A General Theory of Markovian Time Inconsistent Stochastic Control Problems*, 1694759, Social Science Research Network (SSRN), 2010. Available online at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1694759.
- [3] T. BJÖRK, A. MURGOCI, AND X.Y. ZHOU, *Mean-variance portfolio optimization with state dependent risk aversion*, Math. Finance, to appear. Published online Feb. 3, 2012; DOI: 10.1111/j.1467-9965.2011.00515.x.
- [4] I. EKELAND AND T. A. PIRVU, *Investment and consumption without commitment*, Math. Financial Economics, **2** (2008), 57–86.
- [5] S.R. GRENDADIER AND N. WANG, *Investment under uncertainty and time-inconsistent preferences*, J. Financial Economics, **84** (2007), 2-39.
- [6] KAZAMAKI N., *Continuous exponential martingales and BMO*. Berlin: Springer-Verlag, 1994.

- [7] Y. HU, H. JIN AND X. ZHOU, Time-Inconsistent Stochastic Linear–Quadratic Control, *SIAM Control*, **50** (2012) 1548–1572.
- [8] L.S. PONTRYAGIN, V.G. BOLTYANSKII, R.V. GAMKRELIDZE, AND E.F. MISHCHENKO, The Mathematical Theory of Optimal Processes, *John Wiley, New York*, 1962.
- [9] S. PENG, *A general stochastic maximum principle for optimal control problems*, *SIAM J. Control Optim.*, **28** (1990), 966–979.
- [10] R.H. STROTZ, *Myopia and inconsistency in dynamic utility maximization*. *The Review of Economic Studies* (1955), 165-180.
- [11] N. Vieille and J.W. Weibull, *Multiple solutions under quasi-exponential discounting*. *Economic Theory*, **39** (2009), 513-526.
- [12] J. YONG, *Time-inconsistent optimal control problems and the equilibrium HJB equation*. *Math. Control Relat. Fields*, **2** (2012), 271329.
- [13] J. YONG AND X.Y. ZHOU, **Stochastic Controls: Hamiltonian Systems and HJB Equations**, *Springer–Verlag, New York*, 1999.