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# On Some Transmission Problems Set in a Biological Cell, Analysis and Resolution\*

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## Abstract

Some transmission problems are set in bodies with a crown of small thickness  $\varepsilon > 0$ . For instance, those concerning the conductivity in the biological cell. By a natural change of variables, we transform them in transmission problems set in two cylindrical bodies  $]-\infty, 0[ \times ]-\pi, \pi[$  and  $]0, \delta[ \times ]-\pi, \pi[$ , (where  $\delta = \ln(1 + \varepsilon)$ ) and then, in some general elliptic abstract differential equations  $(P^\delta)_{\delta > 0}$ . The goal of this first work is to give a complete study of these problems  $(P^\delta)_{\delta > 0}$  for every  $\delta > 0$ . Existence, uniqueness and maximal regularity results are obtained for the classical solutions essentially by using the semigroups theory.

*Key Words and Phrases.* Biological cells, Transmission problems, Thin layers, Analytic semigroups.

*AMS subject Classifications.* 34K10, 34K30, 35J25, 35J40, 47A60.

## 1 Introduction

Let us consider the model of a biological cell, constituted of an homogeneous cytoplasm  $\Omega_-^*$  (of boundary  $\Gamma^*$ ) centered at  $(0, 0)$  with radius one micrometer surrounded by a thin membrane  $\Omega_+^{*\varepsilon}$  (of boundary  $\Gamma_+^{*\varepsilon}$ ) with thickness of few nanometers  $\varepsilon > 0$ . The electric potential in this cell  $\Omega^{*\varepsilon} = \overline{\Omega_-^*} \cup \Omega_+^{*\varepsilon}$  verifies the

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following problem

$$(P^\varepsilon) \begin{cases} \nabla \cdot (\mu \nabla w^\varepsilon) = \mu h^\varepsilon & \text{in } \Omega^{*\varepsilon} \\ \frac{\partial w^\varepsilon}{\partial n} = l_+^\varepsilon & \text{on } \Gamma_+^{*\varepsilon} \\ \int_{\Gamma^*} w^\varepsilon(\sigma) d\sigma = 0, \end{cases}$$

where

$$\mu = \begin{cases} \mu_- & \text{in } \Omega_-^* \\ \mu_+ & \text{in } \Omega_+^{*\varepsilon}, \end{cases}$$

(typically about 1 S/m (Siemens per meter),  $5 \times 10^{-7}$  S/m respectively) are the conductivity positive coefficients of the two bodies  $\Omega_-^*$ ,  $\Omega_+^{*\varepsilon}$  depending possibly on  $\varepsilon$ , and the electric charge density

$$h^\varepsilon = \begin{cases} h_- & \text{in } \Omega_-^* \\ h_+^\varepsilon & \text{in } \Omega_+^{*\varepsilon}, \end{cases}$$

is taken, for instance, in space  $L^p(\Omega^{*\varepsilon})$ ,  $1 < p < \infty$ , that is  $h_- \in L^p(\Omega_-^*)$ ,  $h_+^\varepsilon \in L^p(\Omega_+^{*\varepsilon})$  and  $\partial/\partial n$  denotes the outward normal derivative,  $l_+^\varepsilon$  is the electric field imposed on the boundary  $\Gamma_+^{*\varepsilon}$ . The Neumann boundary condition on  $\Gamma_+^{*\varepsilon}$  implies the following compatibility condition on  $l_+^\varepsilon$  and  $(\mu h^\varepsilon)$

$$\int_{\Omega^{*\varepsilon}} (\mu h^\varepsilon)(x, y) dx dy + \int_{\Gamma_+^{*\varepsilon}} \mu_+ l_+^\varepsilon(\sigma) d\sigma = 0.$$

We will see that the gauge condition

$$\int_{\Gamma^*} w^\varepsilon(\sigma) d\sigma = 0,$$

is imposed to have the uniqueness of the solution. Problem  $(P^\varepsilon)$  can be written in the complete form

$$(P_{x,y}^\varepsilon) \begin{cases} (eq.1) & \Delta w_-^\varepsilon(x, y) = h_-(x, y) & \text{in } \Omega_-^* \\ (eq.2) & \Delta w_+^\varepsilon(x, y) = h_+^\varepsilon(x, y) & \text{in } \Omega_+^{*\varepsilon} \\ (t.c.) & w_-^\varepsilon = w_+^\varepsilon, \quad \mu_- \frac{\partial w_-^\varepsilon}{\partial n} = \mu_+ \frac{\partial w_+^\varepsilon}{\partial n} & \text{on } \Gamma^* \\ (g.c.) & \int_{\Gamma^*} w_-^\varepsilon(\sigma) d\sigma = \int_{\Gamma^*} w_+^\varepsilon(\sigma) d\sigma = 0 \\ (b.c.) & \frac{\partial w_+^\varepsilon}{\partial n} = l_+^\varepsilon & \text{on } \Gamma_+^{*\varepsilon}, \end{cases}$$

under the compatibility condition

$$(CC_{x,y}^\varepsilon) \begin{cases} \int_{\Omega_-^*} (\mu_- h_-)(x, y) dx dy + \int_{\Omega_+^{*\varepsilon}} (\mu_+ h_+^\varepsilon)(x, y) dx dy \\ + \int_{\Gamma_+^{*\varepsilon}} \mu_+ l_+^\varepsilon(\sigma) d\sigma = 0. \end{cases}$$

Several authors were interested in the study of transmission problems in different spaces with different boundary conditions in Hilbert spaces (see, for

instance, Caloz *et al.* [3] and Nicaise [16]). In [8], Favini *et al.* have considered some transmission problems set on a bounded domain for a second member in an interpolation space. They have used the Da Prato-Grisvard sum theory of linear operators.

A very interesting work is given in Dore *et al.* [5], where the authors have considered some transmission problems on cylindrical domains for second members only in  $L^p$ ,  $p > 1$ .

Many other authors have worked on this subject in many concrete biological situations. We cite, for example, Fear and Stuchly [9], [10], where the cytoplasm is considered an homogeneous material. In Poignard [18], the method of asymptotic expansions is used to model this kind of problems.

In this first work our approach is quite different and uses the theory of operational differential equations set in  $L^p$ - spaces and the celebrated Dore-Venni Theorem.

Note that the small thickness of the membrane surrounding the cytoplasm leads to many numerical difficulties. Therefore, the important question is how to handle them and take into account the effect of this thin layer. So, our aim consists to solve  $(P^\varepsilon)$  for every small  $\varepsilon > 0$  and then, in a forthcoming work, to let  $\varepsilon \rightarrow 0$  in order to find the limiting problem well-posed in the cytoplasm  $\Omega_-^*$  with a good impedance boundary condition on  $\Gamma^*$ . This impedance condition will describe exactly the limiting effect of the thin membrane and will answer our question.

In order to give our main result in this work, let us recall the definition of the following Besov space for  $0 < s < 1$ :

$$B_{p,p}^s(\Gamma_+^{*\varepsilon}) = B_p^s(\Gamma_+^{*\varepsilon}) = W^{s,p}(\Gamma_+^{*\varepsilon}) \\ := \left\{ \psi \in L^p(\Gamma_+^{*\varepsilon}) : \int_{\Gamma_+^{*\varepsilon}} \int_{\Gamma_+^{*\varepsilon}} \frac{|\psi(\theta_1) - \psi(\theta_2)|^p}{|\theta_1 - \theta_2|^{sp+1}} d\theta_1 d\theta_2 < \infty \right\},$$

see Grisvard [11], p. 680 (here the dimension of  $\Gamma_+^{*\varepsilon}$  is one).

**Theorem 1.** *Let  $l_+^\varepsilon \in B_p^{1-\frac{1}{p}}(\Gamma_+^{*\varepsilon})$  and  $h_- \in L^p(\Omega_-^*)$ ,  $h_+^\varepsilon \in L^p(\Omega_+^{*\varepsilon})$ ,  $1 < p < \infty$  with  $p \neq 2$ , satisfying the above compatibility condition  $(CC_{x,y}^\varepsilon)$ . Then, problem  $(P_{x,y}^\varepsilon)$  has a unique solution*

$$w^\varepsilon = \begin{cases} w_-^\varepsilon & \text{in } \Omega_-^* \\ w_+^\varepsilon & \text{in } \Omega_+^{*\varepsilon} \end{cases}$$

such that

$$w_-^\varepsilon \in W^{2,p}(\Omega_-^*), \quad w_+^\varepsilon \in W^{2,p}(\Omega_+^{*\varepsilon}). \quad (1)$$

This maximal  $L^p$ -regularity is very important to solve many quasilinear parabolic evolution equations corresponding to  $(P^\varepsilon)$ . In fact, the use of the fixed point theorem to solve these nonlinear problems requires necessarily optimal regularities such (1).

The organization of the paper is as follows.

In the next section, we show that our model problem can be transformed by natural changes of variables into a particular abstract second order differential problem set in an unbounded cylindrical body. We then give an important

theorem (see Theorem 7) on this problem. In section 3, we collect some useful basic lemmas. In section 4, we solve two auxiliary problems in order to get the representation formula of the abstract solution. We then prove Theorem 7. Finally, in section 5, we go back to our first problem in the biological cell in order to prove Theorem 1.

## 2 Operational formulation and main results

From the polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$ , we consider the changes

$$\begin{cases} v_-^\varepsilon(r, \theta) = w_-^\varepsilon(r \cos \theta, r \sin \theta) \text{ and } v_+^\varepsilon(r, \theta) = w_+^\varepsilon(r \cos \theta, r \sin \theta) \\ k_-^\varepsilon(r, \theta) = h_-^\varepsilon(r \cos \theta, r \sin \theta) \text{ and } k_+^\varepsilon(r, \theta) = h_+^\varepsilon(r \cos \theta, r \sin \theta), \end{cases}$$

and for any  $\sigma \in \Gamma_+^{*\varepsilon}$ , we write  $\sigma = (1 + \varepsilon)(\cos \theta, \sin \theta)$ ,  $\theta \in [-\pi, +\pi[$ , and

$$l_+^\varepsilon(\sigma) = l_+^\varepsilon((1 + \varepsilon)(\cos \theta, \sin \theta)) := L_+^\varepsilon(\theta), \quad (2)$$

then, we have

$$\begin{aligned} \int_{\Gamma_+^{*\varepsilon}} l_+^\varepsilon(\sigma) d\sigma &= \int_{-\pi}^{+\pi} l_+^\varepsilon((1 + \varepsilon)(\cos \theta, \sin \theta)) \sqrt{(1 + \varepsilon)^2} d\theta \\ &= (1 + \varepsilon) \int_{-\pi}^{+\pi} L_+^\varepsilon(\theta) d\theta, \end{aligned} \quad (3)$$

$$v_\pm^\varepsilon(r, -\pi) = w_\pm^\varepsilon(-r, 0), \quad v_\pm^\varepsilon(r, \pi) = w_\pm^\varepsilon(-r, 0),$$

and

$$\frac{\partial v_\pm^\varepsilon}{\partial \theta}(r, -\pi) = -r \frac{\partial w_\pm^\varepsilon}{\partial y}(-r, 0), \quad \frac{\partial v_\pm^\varepsilon}{\partial \theta}(r, \pi) = -r \frac{\partial w_\pm^\varepsilon}{\partial y}(-r, 0).$$

It follows that

$$(PBC_r^\varepsilon) \begin{cases} v_-^\varepsilon(r, -\pi) = v_-^\varepsilon(r, \pi), & \frac{\partial v_-^\varepsilon}{\partial \theta}(r, -\pi) = \frac{\partial v_-^\varepsilon}{\partial \theta}(r, \pi), \quad r \in (0, 1) \\ v_+^\varepsilon(r, -\pi) = v_+^\varepsilon(r, \pi), & \frac{\partial v_+^\varepsilon}{\partial \theta}(r, -\pi) = \frac{\partial v_+^\varepsilon}{\partial \theta}(r, \pi), \quad r \in (1, 1 + \varepsilon). \end{cases}$$

Therefore, problem  $(P_{x,y}^\varepsilon)$  becomes

$$(P_{r,\theta}^\varepsilon) \begin{cases} (eq.1) & (r\partial_r)^2 v_-^\varepsilon(r, \theta) + \partial_\theta^2 v_-^\varepsilon(r, \theta) = r^2 k_-^\varepsilon(r, \theta) \text{ in } \widehat{\Omega}_- \\ (eq.2) & (r\partial_r)^2 v_+^\varepsilon(r, \theta) + \partial_\theta^2 v_+^\varepsilon(r, \theta) = r^2 k_+^\varepsilon(r, \theta) \text{ in } \widehat{\Omega}_+ \\ (t.c.) & \begin{cases} v_-^\varepsilon(1, \theta) = v_+^\varepsilon(1, \theta), \quad \theta \in (-\pi, \pi) \\ \mu_- \frac{\partial v_-^\varepsilon}{\partial r}(1, \theta) = \mu_+ \frac{\partial v_+^\varepsilon}{\partial r}(1, \theta), \quad \theta \in (-\pi, \pi) \end{cases} \\ (g.c.) & \int_{-\pi}^{\pi} v_-^\varepsilon(1, \theta) d\theta = \int_{-\pi}^{\pi} v_+^\varepsilon(1, \theta) d\theta = 0 \\ (b.c.) & \frac{\partial v_+^\varepsilon}{\partial r}(1 + \varepsilon, \theta) = L_+^\varepsilon(\theta), \quad \theta \in (-\pi, \pi), \end{cases}$$

with the periodic boundary conditions ( $PBC_r^\varepsilon$ ) and the compatibility condition

$$(CC_{r,\theta}^\varepsilon) \begin{cases} \int_{\widehat{\Omega}_-} (\mu_- k_-)(r, \theta) r dr d\theta + \int_{\widehat{\Omega}_+^\varepsilon} (\mu_+ k_+^\varepsilon)(r, \theta) r dr d\theta \\ + (1 + \varepsilon) \int_{-\pi}^{+\pi} \mu_+ L_+^\varepsilon(\theta) d\theta = 0; \end{cases}$$

here

$$\widehat{\Omega}_- := (0, 1) \times (-\pi, \pi); \quad \widehat{\Omega}_+^\varepsilon := (1, 1 + \varepsilon) \times (-\pi, \pi).$$

Using the following change of variables

$$\Phi: \begin{array}{ccc} (-\infty, \delta) \times (-\pi, \pi) & \rightarrow & (0, 1 + \varepsilon) \times (-\pi, \pi) \\ (t, \theta) & \mapsto & \Phi(t, \theta) = (e^t, \theta) = (r, \theta), \end{array}$$

where  $\delta = \ln(1 + \varepsilon)$ , and the functions

$$\begin{cases} u_-^\delta(t, \theta) = v_-^\varepsilon(r, \theta), & u_+^\delta(t, \theta) = v_+^\varepsilon(r, \theta) \\ g_-(t, \theta) = r^2 k_-(r, \theta), & g_+^\delta(t, \theta) = r^2 k_+^\varepsilon(r, \theta), \end{cases}$$

our problem ( $P_{r,\theta}^\varepsilon$ ) becomes

$$(P_{t,\theta}^\delta) \begin{cases} (eq.1) & \Delta u_-^\delta(t, \theta) = g_-(t, \theta) \text{ in } \Omega_- \\ (eq.2) & \Delta u_+^\delta(t, \theta) = g_+^\delta(t, \theta) \text{ in } \Omega_+^\delta \\ (t.c.) & \begin{cases} u_-^\delta(0, \theta) = u_+^\delta(0, \theta), \theta \in (-\pi, \pi) \\ \mu_- \frac{\partial u_-^\delta}{\partial t}(0, \theta) = \mu_+ \frac{\partial u_+^\delta}{\partial t}(0, \theta), \theta \in (-\pi, \pi) \end{cases} \\ (g.c.) & \int_{-\pi}^{\pi} u_-^\delta(0, \theta) d\theta = \int_{-\pi}^{\pi} u_+^\delta(0, \theta) d\theta = 0 \\ (b.c.) & \frac{\partial u_+^\delta}{\partial t}(\delta, \theta) = f_+^\delta(\theta), \theta \in (-\pi, \pi), \end{cases}$$

with the periodic boundary conditions

$$(PBC_t^\delta) \begin{cases} u_-^\delta(t, -\pi) = u_-^\delta(t, \pi), & \frac{\partial u_-^\delta}{\partial \theta}(t, -\pi) = \frac{\partial u_-^\delta}{\partial \theta}(t, \pi), t \in (-\infty, 0) \\ u_+^\delta(t, -\pi) = u_+^\delta(t, \pi), & \frac{\partial u_+^\delta}{\partial \theta}(t, -\pi) = \frac{\partial u_+^\delta}{\partial \theta}(t, \pi), t \in (0, \delta), \end{cases}$$

and the compatibility condition

$$(CC_{t,\theta}^\delta) \int_{\Omega_-} (\mu_- g_-)(t, \theta) dt d\theta + \int_{\Omega_+^\delta} (\mu_+ g_+^\delta)(t, \theta) dt d\theta + \int_{-\pi}^{+\pi} \mu_+ f_+^\delta(\theta) d\theta = 0;$$

here

$$f_+^\delta(\theta) := e^\delta L_+^\varepsilon(\theta), \quad \Omega_- := (-\infty, 0) \times (-\pi, \pi), \quad \Omega_+^\delta := (0, \delta) \times (-\pi, \pi).$$

Recall that we want to find  $w_\pm^\varepsilon$  in  $W^{2,p}(\Omega_\pm^{*\varepsilon}, dx dy)$  for problem ( $P_{x,y}^\varepsilon$ ) with  $h_\pm^\varepsilon \in L^p(\Omega_\pm^{*\varepsilon}, dx dy)$ . This last assumption implies that  $g_\pm^\delta$  belong to weighted

spaces. Indeed we have

$$\begin{aligned} \int_{\Omega_{\pm}^{*\varepsilon}} |h_{\pm}^{\varepsilon}(x, y)|^p dx dy &= \int_{\widehat{\Omega}_{\pm}^{\varepsilon}} |k_{\pm}^{\varepsilon}(r, \theta)|^p r dr d\theta \\ &= \int_{\Omega_{\pm}^{\delta}} |k_{\pm}^{\varepsilon}(e^t, \theta)|^p e^{2t} dt d\theta \\ &= \int_{\Omega_{\pm}^{\delta}} |g_{\pm}^{\delta}(t, \theta)|^p e^{2(1-p)t} dt d\theta. \end{aligned}$$

In other words the good assumption is the following

$$g_{\pm} \in L^p_{e^{(-2+2/p)t}}(\Omega_{\pm}^{\delta}, dt d\theta).$$

Let us translate the regularities of  $w_{\pm}^{\varepsilon}$  on  $u_{\pm}^{\delta}$ . We have

$$w_{\pm}^{\varepsilon} \in L^p(\Omega_{\pm}^{*\varepsilon}, dx dy) \Leftrightarrow u_{\pm}^{\delta} \in L^p_{e^{(2/p)t}}(\Omega_{\pm}^{\delta}, dt d\theta).$$

Then, using the fact that

$$\begin{aligned} \frac{\partial w_{\pm}^{\varepsilon}}{\partial x}(x, y) &= \cos \theta \frac{\partial v_{\pm}^{\varepsilon}}{\partial r}(r, \theta) - \frac{1}{r} \sin \theta \frac{\partial v_{\pm}^{\varepsilon}}{\partial \theta}(r, \theta) \\ \frac{\partial w_{\pm}^{\varepsilon}}{\partial y}(x, y) &= \sin \theta \frac{\partial v_{\pm}^{\varepsilon}}{\partial r}(r, \theta) + \frac{1}{r} \cos \theta \frac{\partial v_{\pm}^{\varepsilon}}{\partial \theta}(r, \theta), \end{aligned}$$

and after multiplying by  $\cos \theta$  and  $\sin \theta$ , we deduce that

$$\frac{\partial w_{\pm}^{\varepsilon}}{\partial x}, \frac{\partial w_{\pm}^{\varepsilon}}{\partial y} \in L^p(\Omega_{\pm}^{*\varepsilon}, dx dy),$$

if and only if

$$\frac{\partial u_{\pm}^{\delta}}{\partial t}, \frac{\partial u_{\pm}^{\delta}}{\partial \theta} \in L^p_{e^{(-1+2/p)t}}(\Omega_{\pm}^{\delta}, dt d\theta).$$

By the same way we get

$$\frac{\partial^2 w_{\pm}^{\varepsilon}}{\partial x^2}, \frac{\partial^2 w_{\pm}^{\varepsilon}}{\partial y \partial x}, \frac{\partial^2 w_{\pm}^{\varepsilon}}{\partial y^2} \in L^p(\Omega_{\pm}^{*\varepsilon}, dx dy),$$

if and only if

$$\frac{\partial^2 u_{\pm}^{\delta}}{\partial t^2}, \frac{\partial^2 u_{\pm}^{\delta}}{\partial t \partial \theta}, \frac{\partial^2 u_{\pm}^{\delta}}{\partial \theta^2} \in L^p_{e^{(-2+2/p)t}}(\Omega_{\pm}^{\delta}, dt d\theta).$$

Summarizing, the natural space for  $u_{\pm}^{\delta}$  is

$$E^{2,p}(\Omega_{\pm}^{\delta}, dt d\theta) = \left\{ u_{\pm}^{\delta} : e^{(-|\alpha|+2/p)t} \partial^{\alpha} u_{\pm}^{\delta} \in L^p(\Omega_{\pm}^{\delta}, dt d\theta) \text{ for } |\alpha| \leq 2 \right\}.$$

**Remark 2.** Note that the exponent

$$\varpi := -2 + 2/p$$

is precisely the opposite of the Sobolev exponent of the space  $W^{2,p}$  in two variables.

It would be difficult to use the weighted spaces mentioned above. This is why we define new unknown functions

$$U_{\pm}^{\delta}(t, \theta) = e^{\varpi t} u_{\pm}^{\delta}(t, \theta),$$

and new right hand sides

$$G_{\pm}^{\delta}(t, \theta) = e^{\varpi t} g_{\pm}^{\delta}(t, \theta).$$

**Remark 3.** We observe that, if  $U_{\pm}^{\delta}$  is in the space  $W^{2,p}(\Omega_{\pm}^{\delta}, dt d\theta)$  then  $u_{\pm}^{\delta}$  belongs to  $E^{2,p}(\Omega_{\pm}^{\delta}, dt d\theta)$ . Moreover,  $G_{\pm}^{\delta} \in L^p(\Omega_{\pm}^{\delta}, dt d\theta)$  if and only if  $g_{\pm}^{\delta} \in L^p_{e^{\varpi t}}(\Omega_{\pm}^{\delta}, dt d\theta)$ .

Then the equations verified by  $U_{\pm}^{\delta}$  are

$$(UP_{t,\theta}^{\delta}) \left\{ \begin{array}{l} \text{(eq.1)} \quad \partial_t^2 U_-^{\delta}(t, \theta) - 2\varpi \partial_t U_-^{\delta}(t, \theta) + \varpi^2 U_-^{\delta}(t, \theta) + \partial_{\theta}^2 U_-^{\delta}(t, \theta) \\ \quad = G_-(t, \theta) \text{ in } \Omega_- \\ \text{(eq.2)} \quad \partial_t^2 U_+^{\delta}(t, \theta) - 2\varpi \partial_t U_+^{\delta}(t, \theta) + \varpi^2 U_+^{\delta}(t, \theta) + \partial_{\theta}^2 U_+^{\delta}(t, \theta) \\ \quad = G_+^{\delta}(t, \theta) \text{ in } \Omega_+^{\delta} \\ \text{(t.c.)} \quad \left\{ \begin{array}{l} U_-^{\delta}(0, \theta) = U_+^{\delta}(0, \theta), \theta \in (-\pi, \pi) \\ \mu_- \frac{\partial U_-^{\delta}}{\partial t}(0, \theta) - \mu_+ \frac{\partial U_+^{\delta}}{\partial t}(0, \theta) \\ \quad = \varpi(\mu_- - \mu_+) U_{\pm}^{\delta}(0, \theta), \theta \in (-\pi, \pi) \end{array} \right. \\ \text{(g.c.)} \quad \int_{-\pi}^{\pi} U_-^{\delta}(0, \theta) d\theta = \int_{-\pi}^{\pi} U_+^{\delta}(0, \theta) d\theta = 0 \\ \text{(b.c.)} \quad \frac{\partial U_{\pm}^{\delta}}{\partial t}(\delta, \theta) - \varpi U_{\pm}^{\delta}(\delta, \theta) = F_{\pm}^{\delta}(\theta), \theta \in (-\pi, \pi), \end{array} \right.$$

with the periodic boundary conditions

$$(UPBC_t^{\delta}) \left\{ \begin{array}{l} U_-^{\delta}(t, -\pi) = U_-^{\delta}(t, \pi), \quad \frac{\partial U_-^{\delta}}{\partial \theta}(t, -\pi) = \frac{\partial U_-^{\delta}}{\partial \theta}(t, \pi), \quad t \in (-\infty, 0) \\ U_+^{\delta}(t, -\pi) = U_+^{\delta}(t, \pi), \quad \frac{\partial U_+^{\delta}}{\partial \theta}(t, -\pi) = \frac{\partial U_+^{\delta}}{\partial \theta}(t, \pi), \quad t \in (0, \delta), \end{array} \right.$$

and the compatibility condition

$$(UCC_{t,\theta}^{\delta}) \left\{ \begin{array}{l} \mu_- \int_{\Omega_-} e^{-\varpi t} G_-(t, \theta) dt d\theta + \mu_+ \int_{\Omega_+^{\delta}} e^{-\varpi t} G_+^{\delta}(t, \theta) dt d\theta \\ + \mu_+ e^{-\varpi \delta} \int_{-\pi}^{+\pi} F_+^{\delta}(\theta) d\theta = 0, \end{array} \right.$$

where  $F_+^{\delta}(\theta) := e^{\varpi \delta} f_+^{\delta}(\theta)$ .

Recall that  $E := L^p_{\#}(-\pi, \pi)$  is the space of  $2\pi$ -periodic measurable functions on  $\mathbb{R}$  with locally summable  $p$ -th power, with norm

$$\|\psi\|_E = \left( \int_{-\pi}^{\pi} |\psi(\theta)|^p d\theta \right)^{1/p}.$$



Let  $P$  be the projection operator defined by

$$\begin{aligned} P : L_{\#}^p(-\pi, \pi) &\rightarrow \mathbb{R} \\ \psi &\mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\theta) d\theta. \end{aligned}$$

Now, set

$$E_0 := L_{\#,0}^p(-\pi, \pi) := \left\{ \psi \in L_{\#}^p(-\pi, \pi); P\psi = 0 \right\} = \text{Ker}(P),$$

then

$$E = L_{\#}^p(-\pi, \pi) = E_0 \oplus E_1,$$

where

$$E_1 := \mathbb{R} \cdot f_1 = \{ \lambda f_1, \lambda \in \mathbb{R} \},$$

$f_1(\theta) = 1$  for  $\theta \in [-\pi, \pi]$ .

It is known that every closed subspaces of UMD Banach space is UMD; thus  $E_0$  is UMD (we recall that a Banach space  $X$  is a UMD space if and only if for some  $p > 1$  the Hilbert transform is continuous from  $L^p(\mathbb{R}; X)$  into itself, see Bourgain [1] and Burkholder [2]).

We define the following operators

$$\begin{cases} D(A) = \{ \psi \in E : \psi' \in E, \psi'' \in E \} := W_{\#}^{2,p}(-\pi, \pi) \\ (A\psi)(\theta) = \psi''(\theta), \psi \in D(A), \end{cases}$$

$$\begin{cases} D(A_0) = \{ \psi \in E_0 : \psi' \in E_0, \psi'' \in E_0 \} := W_{\#,0}^{2,p}(-\pi, \pi) \\ (A_0\psi)(\theta) = \psi''(\theta), \psi \in D(A_0). \end{cases} \quad (4)$$

Operator  $A$  is not injective in  $E$  but  $A_0$  is bijective in  $E_0$ .

We have

$$A(I - P)\psi = A_0\psi, \quad AP\psi = 0.$$

Setting

$$U_{\pm}^{\delta} = (I - P)U_{\pm}^{\delta} \oplus PU_{\pm}^{\delta} := W_{\pm}^{\delta} + V_{\pm}^{\delta},$$

and

$$H_{-} := (I - P)G_{-}, \quad H_{+}^{\delta} := (I - P)G_{+}^{\delta}, \quad M_{+}^{\delta} := (I - P)F_{+}^{\delta},$$

problem  $(UP_{t,\theta}^{\delta})$  with the periodic boundary conditions  $(UPBC_t^{\delta})$  splits into the two following problems (respectively in  $E_0$  for the first and in  $E_1$  for the second)

$$(WP_t^{\delta}) \left\{ \begin{array}{l} (eq.1) \quad \begin{cases} (W_{-}^{\delta})''(t) - 2\varpi(W_{-}^{\delta})'(t) + \varpi^2 W_{-}^{\delta}(t) + A_0 W_{-}^{\delta}(t) \\ = H_{-}(t) \text{ on } (-\infty, 0) \end{cases} \\ (eq.2) \quad \begin{cases} (W_{+}^{\delta})''(t) - 2\varpi(W_{+}^{\delta})'(t) + \varpi^2 W_{+}^{\delta}(t) + A_0 W_{+}^{\delta}(t) \\ = H_{+}^{\delta}(t) \text{ on } (0, \delta) \end{cases} \\ (t.c.) \quad \begin{cases} W_{-}^{\delta}(0) = W_{+}^{\delta}(0) \\ \mu_{-}(W_{-}^{\delta})'(0) - \mu_{+}(W_{+}^{\delta})'(0) = \varpi(\mu_{-} - \mu_{+})W_{\pm}^{\delta}(0) \end{cases} \\ (b.c.) \quad (W_{+}^{\delta})'(\delta) - \varpi W_{+}^{\delta}(\delta) = M_{+}^{\delta}, \end{array} \right.$$

and

$$(VP_t^\delta) \begin{cases} (eq.1) & \begin{cases} (V_-^\delta)''(t) - 2\varpi (V_-^\delta)'(t) + \varpi^2 V_-^\delta(t) = PG_-(t) \\ \text{on } (-\infty, 0) \end{cases} \\ (eq.2) & \begin{cases} (V_+^\delta)''(t) - 2\varpi (V_+^\delta)'(t) + \varpi^2 V_+^\delta(t) = PG_+^\delta(t) \\ \text{on } (0, \delta) \end{cases} \\ (t.c.) & \begin{cases} V_-^\delta(0) = V_+^\delta(0) \\ \mu_- (V_-^\delta)'(0) - \mu_+ (V_+^\delta)'(0) = \varpi (\mu_- - \mu_+) V_\pm^\delta(0) \end{cases} \\ (g.c.) & V_-^\delta(0) = V_+^\delta(0) = 0 \\ (b.c.) & (V_+^\delta)'(\delta) - \varpi V_+^\delta(\delta) = PF_+^\delta, \end{cases}$$

under the compatibility condition

$$(VCC_t^\delta) \mu_- \int_{-\infty}^0 e^{-\varpi t} PG_-(t) dt + \mu_+ \int_0^\delta e^{-\varpi t} PG_+^\delta(t) dt - \mu_+ e^{-\varpi \delta} F_+^\delta = 0,$$

with  $H_- \in L^p(-\infty, 0; E_0)$ ,  $H_+^\delta \in L^p(0, \delta; E_0)$ ,  $1 < p < \infty$ ,  $M_+^\delta$  is some given element in  $E_0$ .

For problem  $(VP_t^\delta)$ , we have the following proposition.

**Proposition 4.** *Let  $F_+^\delta \in E$  and  $G_- \in L^p(-\infty, 0; E)$ ,  $G_+^\delta \in L^p(0, \delta; E)$ ,  $1 < p < \infty$ , satisfying  $(VCC_t^\delta)$ . Then, problem  $(VP_t^\delta)$  has a unique solution*

$$V^\delta = \begin{cases} V_-^\delta & \text{in } ]-\infty, 0[ \\ V_+^\delta & \text{in } ]0, \delta[ \end{cases}$$

such that

$$V_-^\delta \in W^{2,p}(-\infty, 0; E_1), \quad V_+^\delta \in W^{2,p}(0, \delta; E_1).$$

*Proof.* The solution is given by

$$V_-^\delta(t) = \int_{-\infty}^0 s e^{\varpi(t-s)} PG_-(s) ds + \int_{-\infty}^t (t-s) e^{\varpi(t-s)} PG_-(s) ds,$$

$$V_+^\delta(t) = t e^{\varpi(t-\delta)} PF_+^\delta + \int_0^\delta s e^{\varpi(t-s)} PG_+^\delta(s) ds - \int_t^\delta (t-s) e^{\varpi(t-s)} PG_+^\delta(s) ds.$$

□

Now, let us specify the essential spectral properties of operator  $(A_0, D(A_0))$ .

**Proposition 5.** *Operator  $A_0$  defined by (4) is linear closed densely defined and boundedly invertible in  $E_0$  with the following properties*

$$\sigma(A_0) = \{-n^2 : n \in \mathbb{N} \setminus \{0\}\},$$

$$\exists C > 0, \forall \lambda \notin ]-\infty, 0[, \|(A_0 - \lambda I)^{-1}\|_{L(E_0)} \leq \frac{C}{1 + |\lambda|},$$

$$\begin{cases} \forall s \in \mathbb{R}, (-A_0)^{is} \in L(E_0) \text{ and} \\ \exists C > 1, \exists \alpha \in ]0, \pi[, \forall s \in \mathbb{R}, \|(-A_0)^{is}\|_{L(E_0)} \leq C e^{\alpha|s|}. \end{cases}$$

*Proof.* First, it is not difficult to have the invertibility of  $A_0$  with the following explicit representation of its inverse in  $E_0$

$$A_0^{-1}(\psi)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [(\theta + \pi)t - t^2/2] \psi(t) dt + \int_{-\pi}^{\theta} (\theta - t) \psi(t) dt, \quad \theta \in (-\pi, \pi),$$

which gives obviously the continuity of  $A_0^{-1}$  on  $E_0$  and thus  $A_0$  is closed. Since the inclusion  $D(A_0) \subset W^{2,p}(-\pi, \pi)$  is compact we deduce by composition that  $A_0^{-1}$  is compact, therefore the spectrum  $\sigma(A_0)$  consists entirely of isolated eigenvalues with finite multiplicities; see Kato [13], Theorem 6.29, p. 187. Now, the equation

$$A_0\psi = \lambda\psi,$$

means

$$\begin{cases} \psi'' = \lambda\psi \\ \psi(-\pi) = \psi(\pi) \\ \psi'(-\pi) = \psi'(\pi); \end{cases}$$

then, necessarily  $\psi \in C^2[-\pi, \pi]$  and

$$\psi(\theta) = C_1 \sinh \sqrt{\lambda}\theta + C_2 \cosh \sqrt{\lambda}\theta.$$

We have two cases: the first one is  $C_1 = C_2 = 0$  which gives  $\psi \equiv 0$ , then no eigenvectors; the second one is

$$\sinh \sqrt{\lambda}\pi = 0 = \sin i\sqrt{\lambda}\pi,$$

that is  $\sqrt{\lambda} = -in$  and so  $\lambda = -n^2$ ,  $n \in \mathbb{N}$ ; but the case  $n = 0$  must be omitted since it corresponds to  $\psi = C_2$  and the condition  $P\psi = 0$  leads to  $\psi \equiv 0$  which cannot be an eigenvector. Finally

$$\sigma(A_0) = \{-n^2 : n \in \mathbb{N} \setminus \{0\}\}.$$

Now an explicit calculus for the spectral equation  $A_0\varphi - \lambda\varphi = \psi$  in  $E_0$ , with  $\lambda \notin ]-\infty, 0]$ , gives

$$\varphi(\theta) = \{(A_0 - \lambda I)^{-1}\psi\}(\theta) = \int_{-\pi}^{\pi} K_{\sqrt{\lambda}}(s, \theta) \psi(s) ds,$$

where

$$K_{\sqrt{\lambda}}(s, \theta) = - \begin{cases} \frac{\cosh \sqrt{\lambda}(\pi - s + \theta)}{2\sqrt{\lambda} \sinh \sqrt{\lambda}\pi} & \text{if } \theta \leq s \leq \pi \\ \frac{\cosh \sqrt{\lambda}(\pi - \theta + s)}{2\sqrt{\lambda} \sinh \sqrt{\lambda}\pi} & \text{if } -\pi \leq s \leq \theta. \end{cases}$$

It is not difficult to verify that  $\varphi \in D(A_0)$  and for any  $\lambda \notin ]-\infty, 0]$ ,  $\psi \in E_0$ , we have, by the well known Schur Lemma

$$\|(A_0 - \lambda I)^{-1}\psi\| \leq \left( \sup_{\theta \in [-\pi, \pi]} \int_{-\pi}^{\pi} |K_{\sqrt{\lambda}}(s, \theta)| ds \right) \|\psi\|,$$

and

$$\begin{aligned}
\int_{-\pi}^{\pi} |K_{\sqrt{\lambda}}(s, \theta)| ds &\leq \frac{1}{2|\lambda|^{1/2} |\sinh \sqrt{\lambda}\pi|} \int_{-\pi}^{\theta} \cosh \operatorname{Re} \sqrt{\lambda}(\pi + s - \theta) ds \\
&\quad + \frac{1}{2|\lambda|^{1/2} |\sinh \sqrt{\lambda}\pi|} \int_{\theta}^{\pi} \cosh \operatorname{Re} \sqrt{\lambda}(\pi - s + \theta) ds \\
&\leq \frac{\sinh \operatorname{Re} \sqrt{\lambda}\pi - \sinh \operatorname{Re} \sqrt{\lambda}\theta}{|\lambda|^{1/2} \operatorname{Re} \sqrt{\lambda} |\sinh \sqrt{\lambda}\pi|} \\
&\leq \frac{\sinh \operatorname{Re} \sqrt{\lambda}\pi}{|\lambda|^{1/2} \operatorname{Re} \sqrt{\lambda} |\sinh \sqrt{\lambda}\pi|} \\
&\leq \frac{1}{|\lambda| \cos(\frac{\arg(\lambda)}{2})}.
\end{aligned}$$

Since  $E_0$  is a UMD space, we deduce from the previous inequality that  $D(A_0)$  is dense in  $E_0$ ; see Haase [12], Proposition 2.1.1, pp. 18-19.

The property of bounded imaginary powers of  $A_0$  is due to Prüss-Sohr [19], Theorem C, p. 166.  $\square$

Therefore, it is well known that

$$B := -(-A_0)^{1/2}$$

generates an analytic semigroup,  $(e^{tB})_{t>0}$ , and in virtue of the spectral mapping theorem (see Haase [12], p. 56), we deduce that  $\sigma(B) = -\mathbb{N} \setminus \{0\}$ . On the other hand, there exist two positive constants  $a, M$  such that, for all  $t > 0$ , for all  $m \in \mathbb{N} \setminus \{0\}$ , we have

$$\|e^{tB}\|_{L(E_0)} \leq M e^{-at}, \quad \|B^m e^{tB}\|_{L(E_0)} \leq M t^{-m} e^{-at}; \quad (5)$$

see Pazy [17], Theorem 6.13, p. 74.

Moreover, it is well known that operator  $(I + e^{2\delta B})^{-1} \in L(E_0)$ ; see Lunardi [15], Proposition 2.3.6, p. 60.

We will consider the two following natural operators

$$B_{\varpi}^+ := B + \varpi I, \quad B_{\varpi}^- := B - \varpi I.$$

Note that,  $B_{\varpi}^+$  with domain  $D(B_{\varpi}^+) = D(B)$  generates an analytic semigroup; see Engel-Nagel [7], Proposition 1.12, p. 164. It is boundedly invertible since for all  $p \in ]1, \infty[$ ,  $-\varpi$  is never equal to  $-n$ ,  $n \in \mathbb{N} \setminus \{0\}$ .

It is not the same for the second operator  $B_{\varpi}^-$ ; since for the unique particular case  $p = 2$ ,  $\varpi$  coincides with the spectral value

$$\varpi = -2 + \frac{2}{p} = -1.$$

This is why we will assume in all this work that

$$p \in ]1, \infty[ \text{ and } p \neq 2.$$

**Remark 6.** Note that the situation  $p = 2$  corresponds to hilbertian case which can be treated by the usual variational method.

We will study existence and uniqueness for a natural solution of  $(WP_t^\delta)$ , that is a function

$$W^\delta = \begin{cases} W_-^\delta & \text{in } ]-\infty, 0[ \\ W_+^\delta & \text{in } ]0, \delta[, \end{cases}$$

such that

$$\begin{cases} W_-^\delta \in W^{2,p}(-\infty, 0; E_0) \cap L^p(-\infty, 0; D(A_0)), \\ W_+^\delta \in W^{2,p}(0, \delta; E_0) \cap L^p(0, \delta; D(A_0)), \end{cases} \quad (6)$$

and verifying  $(WP_t^\delta)$ . Then for our problem  $(WP_t^\delta)$ , we have the following.

**Theorem 7.** Let  $H_- \in L^p(-\infty, 0; E_0)$ ,  $H_+^\delta \in L^p(0, \delta; E_0)$ ,  $1 < p < \infty$  with  $p \neq 2$ . Then, problem  $(WP_t^\delta)$  has a unique solution  $W^\delta$  satisfying (6) if and only if  $M_+^\delta \in (D(A_0), E_0)_{\frac{1}{2p} + \frac{1}{2}, p}$ .

We recall that for all  $\alpha \in ]0, 1[$  and  $q \in [1, \infty]$ , the space

$$(E_0, D(A_0))_{\alpha, q} = (D(A_0), E_0)_{1-\alpha, q}$$

is a real interpolation space between  $D(A_0)$  and  $E_0$ , defined for instance in Triebel [20], p. 96.

This theorem is proved in section 5. Our techniques are essentially based on the Dunford operational calculus, the analytic semigroup theory and the celebrated Dore-Venni Theorem.

**Remark 8.** In this work, we have considered problem  $(WP_t^\delta)$  with  $A_0$  defined by (4). However, we can also study it when  $A_0$  is some closed linear operator of domain  $D(A_0)$  in some complex Banach space  $E_0$  under the following assumptions

$E_0$  is a UMD space,

$$\rho(-A_0) = \{\lambda_j\}_{j \geq 1} \text{ with } 0 < \lambda_1 < \lambda_2 < \lambda_3 \dots,$$

$$\forall j \geq 1, 2 - \frac{2}{p} \neq \sqrt{\lambda_j},$$

$$\exists C > 0, \forall \lambda \in [0, +\infty[, \|(A_0 - \lambda I)^{-1}\|_{L(E_0)} \leq \frac{C}{1 + |\lambda|},$$

$$\begin{cases} \forall s \in \mathbb{R}, (-A_0)^{is} \in L(E_0) \text{ and} \\ \exists C > 1, \exists \alpha \in ]0, \pi[, \forall s \in \mathbb{R}, \|(-A_0)^{is}\|_{L(E_0)} \leq C e^{\alpha|s|}. \end{cases}$$

**Remark 9.** If we have considered  $\Omega_-^*$  as the unit ball of  $\mathbb{R}^m$ , then by analogous calculus we obtain the equations

$$(WP_t^{\delta, m}) \begin{cases} (W_-^\delta)''(t) - (m - 2 - 2\varpi)(W_-^\delta)'(t) + \varpi(\varpi - m + 2)W_-^\delta(t) \\ + A_0 W_-^\delta(t) = H_-(t) \text{ on } (-\infty, 0) \\ (W_+^\delta)''(t) - (m - 2 - 2\varpi)(W_+^\delta)'(t) + \varpi(\varpi - m + 2)W_+^\delta(t) \\ + A_0 W_+^\delta(t) = H_+^\delta(t) \text{ on } (0, \delta) \\ W_-^\delta(0) = W_+^\delta(0) \\ \mu_- (W_-^\delta)'(0) - \mu_+ (W_+^\delta)'(0) = (\mu_- - \mu_+) \varpi W_\pm^\delta(0), \\ (W_+^\delta)'(\delta) - \varpi W_+^\delta(\delta) = M_+^\delta, \end{cases}$$

where  $\varpi = -2 + m/p$  and  $A_0$  is the Laplace-Beltrami operator on the unit sphere  $S^{m-1}$ . Here, we must assume for all  $j \geq 1$  that

$$\varpi(\varpi - m + 2) = \left(\frac{m}{p} - 2\right) \left(\frac{m}{p} - m\right) \neq \lambda_j. \quad (7)$$

Consider in the Banach space  $X = L^p(-\infty, \delta; E)$ , the following operators

$$\begin{cases} D(\Lambda_1) = \{W^\delta = (W_-^\delta, W_+^\delta) \in L^p(-\infty, 0; E) \times L^p(0, \delta; E) : \\ \quad W_-^\delta(0) = W_+^\delta(0), \mu_- (W_-^\delta)'(0) - \mu_+ (W_+^\delta)'(0) = (\mu_- - \mu_+) \varpi W_\pm^\delta(0) \\ \quad \text{and } (W_+^\delta)'(\delta) - \varpi W_+^\delta(\delta) = 0\} \\ \Lambda_1 W^\delta = (W^\delta)'' - (m - 2 - 2\varpi) (W^\delta)' + \varpi(\varpi - m + 2)W^\delta, \\ \\ \begin{cases} D(\Lambda_2) = \{W^\delta = (W_-^\delta, W_+^\delta) \in L^p(-\infty, 0; E) \times L^p(0, \delta; E) : \\ \quad W^\delta(t) \in D(A_0) \text{ a.e. } t \in (-\infty, \delta)\} \\ \Lambda_2 W^\delta(t) = A_0 W^\delta(t). \end{cases} \end{cases}$$

We can verify that operators  $-\Lambda_1$  and  $-\Lambda_2$  are sectorial and that their resolvents commute. We then know that if the spectra  $\sigma(\Lambda_1)$  and  $\sigma(-\Lambda_2)$  do not intersect, then  $\Lambda_1 + \Lambda_2$  is closable. This last condition means that (7) is satisfied. Therefore, we can conjecture that in the critical cases (that is when (7) is not satisfied), the sum  $\Lambda_1 + \Lambda_2$  is not closable.

### 3 Technical lemmas

Set, for all  $f \in L^p(a, b; E_0)$ , with  $a < b$ , and for a.e.  $t \in (a, b)$

$$F(f)(t) := \int_a^b \frac{f(s)}{s+t} ds.$$

Since  $E_0$  is a UMD space, we have

$$F \in L(L^p(a, b; E_0)). \quad (8)$$

In fact, we have for  $f \in L^p(a, b; E_0)$  and for a.e.  $t \in (a, b)$

$$F(f)(t) = \int_a^b \frac{f(s)}{s+t} ds = \int_{\mathbb{R}} \frac{g(s)}{s+t} ds = \int_{\mathbb{R}} \frac{g(\tau-t)}{\tau} d\tau,$$

where  $g = \mathbb{1}_{[a,b]} f$  ( $\mathbb{1}$  is a characteristic function). On the other hand, we have

$$F(f)(t) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon \leq |\tau| \leq \frac{1}{\varepsilon}} \frac{g(\tau-t)}{\tau} d\tau = V_p \left( \frac{1}{\tau} \right) * g(-\cdot) = H(g(-\cdot))(t),$$

where  $V_p$  is the Cauchy principal value and  $H$  is the Hilbert transform. The desired result follows from the fact that  $E_0$  is a UMD space.

We need the following lemmas.

**Lemma 10.** Let  $Q \in \{B, B_{\infty}^-, B_{\infty}^+\}$  and  $f \in L^p(0, \delta; E_0)$ ,  $1 < p < \infty$  with  $p \neq 2$ . Then, the following applications

$$\begin{aligned} t &\mapsto Q \int_0^t e^{(t-s)Q} f(s) ds, \\ t &\mapsto Q \int_t^{\delta} e^{(s-t)Q} f(s) ds, \\ t &\mapsto Q \int_0^{\delta} e^{(t+s)Q} f(s) ds, \end{aligned}$$

are well defined for a.e.  $t \in (0, \delta)$  and belong to  $L^p(0, \delta; E_0)$ .

*Proof.* For the first and the second applications, it is a consequence of the Dore-Venni Theorem [6]. For the third, it suffices to write for a.e.  $t \in (0, \delta)$

$$\begin{aligned} Q \int_0^{\delta} e^{(t+s)Q} f(s) ds &= Q \int_0^t e^{(t+s)Q} f(s) ds + Q \int_t^{\delta} e^{(t+s)Q} f(s) ds \\ &= Q \int_0^t e^{(t-s)Q} (e^{2sQ} f(s)) ds + e^{2tQ} Q \int_t^{\delta} e^{(s-t)Q} f(s) ds. \end{aligned}$$

□

**Lemma 11.** Let  $Q \in \{B, B_{\infty}^-, B_{\infty}^+\}$  and  $f \in L^p(-\infty, 0; E_0)$ ,  $1 < p < \infty$  with  $p \neq 2$ . Then, the following applications

$$\begin{aligned} t &\mapsto Q \int_t^0 e^{(s-t)Q} f(s) ds, \\ t &\mapsto Q \int_{-\infty}^t e^{(t-s)Q} f(s) ds, \\ t &\mapsto Q \int_{-\infty}^0 e^{-(t+s)Q} f(s) ds, \end{aligned}$$

are well defined for a.e.  $t \in (-\infty, 0)$  and belong to  $L^p(-\infty, 0; E_0)$ .

*Proof.* Write, for a.e.  $t \in (-\infty, 0)$ ,

$$Q \int_t^0 e^{(s-t)Q} f(s) ds = Q \int_0^{-t} e^{(-t-\tau)Q} f(-\tau) d\tau$$

and consider the change  $-t = x$  with  $x \in (0, +\infty)$ . Then

$$Q \int_t^0 e^{(s-t)Q} f(s) ds = Q \int_0^x e^{(x-\tau)Q} f(-\tau) d\tau,$$

where  $t \mapsto f(-t) \in L^p(0, +\infty; E_0)$ . Then, it is clear that

$$t \mapsto Q \int_t^0 e^{(s-t)Q} f(s) ds \in L^p(-\infty, 0; E_0),$$

if and only if

$$t \mapsto Q \int_0^t e^{(t-\tau)Q} f(-\tau) d\tau \in L^p(0, \infty; E_0).$$

In virtue of the Dore-Venni Theorem [6], we have

$$t \mapsto Q \int_0^t e^{(t-\tau)Q} f(-\tau) d\tau \in L^p(0, T; E_0),$$

for all  $T > 0$  and by Theorem 2.4, p. 28 in Dore [4], we get

$$t \mapsto Q \int_0^t e^{(t-\tau)Q} f(-\tau) d\tau \in L^p(0, \infty; E_0).$$

For the second application, the proof is not obvious, the idea follows from the method used in Dore [4], pp. 28-29. We have for a.e.  $t \in (-\infty, 0)$ ,

$$Q \int_{-\infty}^t e^{(t-s)Q} f(s) ds = Q \int_{-\infty}^{t-1} e^{(t-s)Q} f(s) ds + Q \int_{t-1}^t e^{(t-s)Q} f(s) ds.$$

For the first integral, we use (5) to get

$$\begin{aligned} & \int_{-\infty}^0 \left\| \int_{-\infty}^{t-1} Q e^{(t-s)Q} f(s) ds \right\|^p dt \\ & \leq M^p \int_{-\infty}^0 \left( \int_{-\infty}^{t-1} \frac{1}{t-s} e^{-(t-s)a} \|f(s)\| ds \right)^p dt \\ & \leq M^p \int_{-\infty}^0 \left( \int_{-\infty}^{t-1} e^{-(t-s)a} \|f(s)\| ds \right)^p dt. \end{aligned}$$

Now, we can write

$$\int_{-\infty}^{t-1} e^{-(t-s)a} \|f(s)\| ds = (h * \Psi)(t),$$

where

$$h(\xi) = \begin{cases} 0 & \text{if } \xi \leq 1 \\ e^{-\xi a} & \text{if } \xi > 1, \end{cases}$$

$$\Psi(\omega) = \begin{cases} \|f(\omega)\| & \text{if } \omega \leq 0 \\ 0 & \text{if } \omega > 0, \end{cases}$$

it is clear that  $\Psi \in L^p(\mathbb{R})$  and  $h \in L^1(\mathbb{R})$ . Then the Young's inequality implies  $h * \Psi \in L^p(\mathbb{R})$ , that is

$$\int_{-\infty}^0 \left\| Q \int_{-\infty}^{t-1} e^{(t-s)Q} f(s) ds \right\|^p dt < \infty.$$

It remains to estimate

$$\int_{-\infty}^0 \left\| Q \int_{t-1}^t e^{(t-s)Q} f(s) ds \right\|^p dt.$$

Consider, for  $j \in \mathbb{N}$ , the truncated functions defined by

$$f_{-j} = \mathbb{I}_{[-j-1, -j]} f,$$



where  $\mathbb{I}_{[-j-1, -j]}$  denotes the characteristic function of the set  $[-j-1, -j]$ . Then, we have

$$\begin{aligned}
& \int_{-\infty}^0 \left\| Q \int_{t-1}^t e^{(t-s)Q} f(s) ds \right\|^p dt \\
&= \sum_{j=0}^{\infty} \int_{-j-1}^{-j} \left\| Q \int_{t-1}^{-j-1} e^{(t-s)Q} f_{-j-1}(s) ds + Q \int_{-j-1}^t e^{(t-s)Q} f_{-j}(s) ds \right\|^p dt \\
&\leq 2^{p-1} \sum_{j=0}^{\infty} \int_{-j-1}^{-j} \left\| Q \int_{t-1}^{-j-1} e^{(t-s)Q} f_{-j-1}(s) ds \right\|^p dt \\
&\quad + 2^{p-1} \sum_{j=0}^{\infty} \int_{-j-1}^{-j} \left\| Q \int_{-j-1}^t e^{(t-s)Q} f_{-j}(s) ds \right\|^p dt.
\end{aligned}$$

Set

$$\begin{cases} I_j = \int_{-j-1}^{-j} \left\| Q \int_{-j-1}^t e^{(t-s)Q} f_{-j}(s) ds \right\|^p dt, \\ J_j = \int_{-j-1}^{-j} \left\| Q \int_{t-1}^{-j-1} e^{(t-s)Q} f_{-j-1}(s) ds \right\|^p dt. \end{cases}$$

Then, by the changes of variables  $\tau = -1 - t - j$  and  $\sigma = -s - j - 1$ , we obtain

$$\begin{aligned}
I_j &= \int_{-1}^0 \left\| Q \int_{-1-j}^{-j-1-\tau} e^{(-1-j-\tau-s)Q} f_{-j}(s) ds \right\|^p d\tau \\
&= \int_{-1}^0 \left\| Q \int_{\tau}^0 e^{(\sigma-\tau)Q} f_{-j}(-\sigma-j-1) d\sigma \right\|^p d\tau,
\end{aligned}$$

therefore, in virtue of Dore-Venni [6], there exists  $C_1 > 0$  such that

$$I_j \leq C_1^p \|f_{-j}(-1-j-\cdot)\|_{L^p(-1,0;E_0)}^p \leq C_1^p \|f_{-j}\|_{L^p(-j-1,-j;E_0)}^p. \quad (9)$$

For  $J_j$ , we use the changes of variables  $\tau = -1 - t - j$  and  $y = s + j + 1$  to obtain

$$\begin{aligned}
J_j &= \int_{-1}^0 \left\| Q \int_{-2-\tau-j}^{-j-1} e^{(-1-\tau-j-s)Q} f_{-j-1}(s) ds \right\|^p d\tau \\
&= \int_{-1}^0 \left\| Q \int_{-1-\tau}^0 e^{(-\tau-y)Q} f_{-j-1}(y-j-1) dy \right\|^p d\tau \\
&\leq \int_{-1}^0 \left( \int_{-1}^0 \left\| Q e^{(-\tau-y)Q} f_{-j-1}(y-j-1) \right\| dy \right)^p d\tau \\
&\leq M^p \int_{-1}^0 \left( \int_{-1}^0 \frac{-1}{\tau+y} \|f_{-j-1}(y-j-1)\| dy \right)^p d\tau.
\end{aligned}$$

Now, by (8) the kernel  $\frac{-1}{\tau+y}$  defines a bounded operator on  $L^p(-1,0;\mathbb{R})$ , then

$$\begin{aligned}
J_j &\leq C_2 M^p \|f_{-j-1}(\cdot - j - 1)\|_{L^p(-1,0;E_0)}^p \\
&\leq C_2 M^p \|f_{-j-1}\|_{L^p(-j-2,-j-1;E_0)}^p.
\end{aligned} \quad (10)$$

Using (9) and (10), we conclude that

$$\begin{aligned} \int_{-\infty}^0 \left\| Q \int_{t-1}^t e^{(t-s)Q} f(s) ds \right\|^p dt &\leq C_2 2^{p-1} M^p \sum_{j=0}^{\infty} \|f_{-j-1}\|_{L^p(-j-2, -j-1; E_0)}^p \\ &\quad + 2^{p-1} C_1^p \sum_{j=0}^{\infty} \|f_{-j}\|_{L^p(-j-1, -j; E_0)}^p. \end{aligned}$$

Since

$$\begin{cases} \sum_{j=0}^{\infty} \|f_{-j}\|_{L^p(-j-1, -j; E_0)}^p = \|f\|_{L^p(-\infty, 0; E_0)}^p, \\ \sum_{j=0}^{\infty} \|f_{-j-1}\|_{L^p(-j-2, -j-1; E_0)}^p \leq \|f\|_{L^p(-\infty, 0; E_0)}^p, \end{cases}$$

we get

$$\int_{-\infty}^0 \left\| Q \int_{t-1}^t e^{(t-s)Q} f(s) ds \right\|^p dt \leq 2^p \max(C_1^p, C_2 M^p) \|f\|_{L^p(-\infty, 0; E_0)}^p.$$

For the third application, we write for a.e.  $t \in (-\infty, 0)$

$$\begin{aligned} &Q \int_{-\infty}^0 e^{-(t+s)Q} f(s) ds \\ &= e^{-2tQ} Q \int_{-\infty}^t e^{(t-s)Q} f(s) ds + Q \int_t^0 e^{(s-t)Q} (e^{-2sQ} f(s)) ds, \end{aligned}$$

and apply the previous results.  $\square$

**Lemma 12.** *Let  $Q \in \{B, B_{\overline{\omega}}^-, B_{\overline{\omega}}^+\}$  and  $f \in L^p(0, \delta; E_0)$ ,  $1 < p < \infty$  with  $p \neq 2$ . Then*

$$t \mapsto Q \int_{-\infty}^0 e^{(t-s)Q} f(s) ds \in L^p(0, \delta; E_0).$$

*Proof.* We have by the change of variables  $t = -x$

$$\begin{aligned} \int_0^{\delta} \left\| Q \int_{-\infty}^0 e^{(t-s)Q} f(s) ds \right\|^p dt &= \int_{-\delta}^0 \left\| Q \int_{-\infty}^0 e^{(-x-s)Q} f(s) ds \right\|^p dx \\ &\leq \int_{-\infty}^0 \left\| Q \int_{-\infty}^0 e^{-(x+s)Q} f(s) ds \right\|^p dx < \infty, \end{aligned}$$

see Lemma 11.  $\square$

**Lemma 13.** *Let  $Q \in \{B, B_{\overline{\omega}}^-, B_{\overline{\omega}}^+\}$  and  $f \in L^p(-\infty, 0; E_0)$ ,  $1 < p < \infty$  with  $p \neq 2$ . Then*

$$t \mapsto Q \int_0^{\delta} e^{(s-t)Q} f(s) ds \in L^p(-\infty, 0; E_0).$$

*Proof.* We have by the change of variables  $t = -x$

$$\begin{aligned} \int_{-\infty}^0 \left\| Q \int_0^\delta e^{(s-t)Q} f(s) ds \right\|^p dt &= \int_0^\infty \left\| Q \int_0^\delta e^{(s+x)Q} f(s) ds \right\|^p dx \\ &= \int_0^\delta \left\| Q \int_0^\delta e^{(s+x)Q} f(s) ds \right\|^p dx \\ &\quad + \int_\delta^\infty \left\| Q \int_0^\delta e^{(s+x)Q} f(s) ds \right\|^p dx, \end{aligned}$$

the first integral is finite since  $t \mapsto Q \int_0^\delta e^{(s+x)Q} f(s) ds \in L^p(0, \delta; E_0)$ , see Lemma 10; for the second integral we use the semigroup properties and the Hölder inequality

$$\begin{aligned} &\int_\delta^\infty \left\| Q \int_0^\delta e^{(s+x)Q} f(s) ds \right\|^p dx \\ &\leq M^p \int_\delta^\infty \left( \int_0^\delta \frac{e^{-a(s+x)}}{s+x} \|f(s)\| ds \right)^p dx \\ &\leq \frac{M^p}{\delta^p} \int_\delta^\infty \left( \int_0^\delta e^{-qa(s+x)} ds \right)^{\frac{p}{q}} dx \|f\|_{L^p(0, \delta; E_0)}^p \\ &\leq K \|f\|_{L^p(0, \delta; E_0)}^p, \end{aligned}$$

with  $1/p + 1/q = 1$ . □

**Lemma 14.** Let  $Q \in \{B, B_{\overline{\omega}}^-, B_{\overline{\omega}}^+\}$  and  $p \in ]1, \infty[$  with  $p \neq 2$ . Then

1.  $\varphi \in (D(A_0), E_0)_{\frac{1}{2p} + \frac{1}{2}, p}$  if and only if  $t \mapsto Qe^{tQ}\varphi \in L^p(0, \delta; E_0)$ .
2.  $\psi \in (D(A_0), E_0)_{\frac{1}{2p}, p}$  if and only if  $t \mapsto Q^2e^{tQ}\psi \in L^p(0, \delta; E_0)$ .

*Proof.* Since  $Q$  generates an analytic semigroup, then, for any  $\theta \in ]0, 1[$ ,  $m \in \mathbb{N} \setminus \{0\}$ , we have

$$(D(Q^m), E_0)_{\theta, p} = \left\{ \varphi \in E_0 : \int_0^\infty \|t^{m\theta} Q^m e^{tQ} \varphi\|^p \frac{dt}{t} < \infty \right\},$$

(see Triebel [20], p. 96). Let  $\varphi \in (D(Q^m), E_0)_{\frac{1}{mp}, p}$ , then

$$\begin{aligned} \int_0^\delta \|Q^m e^{tQ} \varphi\|^p dt &\leq \int_0^\infty \|t^{\frac{1}{p}} Q^m e^{tQ} \varphi\|^p \frac{dt}{t} \\ &\leq \int_0^\infty \|t^{m(\frac{1}{mp})} Q^m e^{tQ} \varphi\|^p \frac{dt}{t} \\ &\leq \|\varphi\|_{(D(Q^m); E_0)_{\frac{1}{mp}, p}}^p. \end{aligned}$$

Conversely, if  $t \mapsto Q^m e^{tQ} \varphi \in L^p(0, \delta; E_0)$ , then

$$\begin{aligned} \int_0^\infty \left\| t^{\frac{1}{m}} Q^m e^{tQ} \varphi \right\|^p \frac{dt}{t} &= \int_0^\infty \left\| t^{\frac{1}{p}} Q^m e^{tQ} \varphi \right\|^p \frac{dt}{t} \\ &= \int_0^\delta \left\| Q^m e^{tQ} \varphi \right\|^p dt + \int_\delta^\infty \left\| Q^m e^{tQ} \varphi \right\|^p dt \\ &= \left\| Q^m e^{tQ} \varphi \right\|_{L^p(0, \delta; E_0)}^p + \int_\delta^\infty \left\| Q^m e^{tQ} \varphi \right\|^p dt. \end{aligned}$$

On the other hand, in virtue of (5), we get

$$\int_\delta^\infty \left\| Q^m e^{tQ} \varphi \right\|^p dt \leq M \int_\delta^\infty t^{-pm} e^{-tpa} dt \|\varphi\|^p \leq K(\delta) \|\varphi\|^p,$$

from which we conclude that

$$\varphi \in (D(Q^m), E_0)_{\frac{1}{mp}, p} \text{ if and only if } t \mapsto Q^m e^{tQ} \varphi \in L^p(0, \delta; E_0).$$

Applying this property for  $m = 1$ , we obtain

$$\varphi \in (D(Q), E_0)_{\frac{1}{p}, p} \text{ if and only if } t \mapsto Q e^{tQ} \varphi \in L^p(0, \delta; E_0),$$

and due to the reiteration property (see Lions-Peetre [14]), we have

$$(D(Q), E_0)_{\frac{1}{p}, p} = (D(Q^2), E_0)_{\frac{1}{2} + \frac{1}{2p}, p} = (D(A_0), E_0)_{\frac{1}{2} + \frac{1}{2p}, p}.$$

□

**Lemma 15.** *Let  $Q \in \{B, B_{\varpi}^-, B_{\varpi}^+\}$  and  $p \in ]1, \infty[$  with  $p \neq 2$ . Then*

$$\varphi \in (D(A_0), E_0)_{\frac{1}{2p} + \frac{1}{2}, p} \text{ if and only if } t \mapsto Q e^{tQ} \varphi \in L^p(-\infty, 0; E_0).$$

*Proof.* The proof is similar to the one used in Lemma 14. □

The following lemma is obtained as in Lunardi [15], p. 59.

**Lemma 16.** *Let  $\nu \geq 0$ . Then, the operator defined by*

$$D_\nu := (1 + \nu) \left( I + \frac{1 - \nu}{1 + \nu} e^{2\delta B} \right) = I + e^{2\delta B} + \nu (I - e^{2\delta B}),$$

*is boundedly invertible.*

**Proposition 17.** *Let  $A_0$  defined by (4) and  $p \in ]1, \infty[$  with  $p \neq 2$ . Then*

1. *if  $1 < p < 2$ ,*

$$(D(A_0); E_0)_{\frac{1}{2p} + \frac{1}{2}, p} = \left\{ \psi \in B_p^{1 - \frac{1}{p}}(-\pi, \pi) : \int_{-\pi}^{\pi} \psi(\theta) d\theta = 0 \right\};$$

2. *if  $p > 2$ ,*

$$\begin{aligned} &(D(A_0); E_0)_{\frac{1}{2p} + \frac{1}{2}, p} \\ &= \left\{ \psi \in B_p^{1 - \frac{1}{p}}(-\pi, \pi) : \psi(-\pi) = \psi(\pi) \text{ and } \int_{-\pi}^{\pi} \psi(\theta) d\theta = 0 \right\}. \end{aligned}$$

*Proof.* We have

$$(D(A_0); E_0)_{\frac{1}{2p} + \frac{1}{2}, p} = (W_{\#,0}^{2,p}(-\pi, \pi); L_{\#,0}^p(-\pi, \pi))_{\frac{1}{2p} + \frac{1}{2}, p},$$

where  $W_{\#,0}^{2,p}(-\pi, \pi)$  is defined by (4). Then, it is well known that

$$W_{\#,0}^{2,p}(-\pi, \pi) \subset \left( W_{\#,0}^{2,p}(-\pi, \pi), L_{\#,0}^p(-\pi, \pi) \right)_{\frac{1}{2p} + \frac{1}{2}, p} \subset L_{\#,0}^p(-\pi, \pi).$$

Moreover, in Grisvard [11], Propositione 3, p. 683, we have

$$(W^{2,p}(-\pi, \pi), L^p(-\pi, \pi))_{\frac{1}{2p} + \frac{1}{2}, p} = B_p^{1-\frac{1}{p}}(-\pi, \pi),$$

and as in p. 708, we get

$$\begin{aligned} & (W_{\#,0}^{2,p}(-\pi, \pi); L_{\#,0}^p(-\pi, \pi))_{\frac{1}{2p} + \frac{1}{2}, p} \\ = & \begin{cases} \left\{ \psi \in B_p^{1-\frac{1}{p}}(-\pi, \pi) : \int_{-\pi}^{\pi} \psi(\theta) d\theta = 0 \right\} & \text{if } 1 < p < 2 \\ \left\{ \psi \in B_p^{1-\frac{1}{p}}(-\pi, \pi) : \psi(-\pi) = \psi(\pi) \text{ and } \int_{-\pi}^{\pi} \psi(\theta) d\theta = 0 \right\} & \text{if } p > 2. \end{cases} \end{aligned}$$

□

## 4 Representation of the solution

### 4.1 Auxiliary problems

Concerning problem  $(WP_t^\delta)$ , our approach consists to solve the following auxiliary problems, the first is set on  $(-\infty, 0)$

$$(AuxP_-^\delta) \begin{cases} (X_-^\delta)''(t) - 2\varpi (X_-^\delta)'(t) + \varpi^2 X_-^\delta(t) + A_0 X_-^\delta(t) = H_-(t) \\ (X_-^\delta)'(0) = \phi, \end{cases}$$

and the second is set on the thin layer  $(0, \delta)$

$$(AuxP_+^\delta) \begin{cases} (X_+^\delta)''(t) - 2\varpi (X_+^\delta)'(t) + \varpi^2 X_+^\delta(t) + A_0 X_+^\delta(t) = H_+^\delta(t) \\ X_+^\delta(0) = \psi \\ (X_+^\delta)'(\delta) - \varpi X_+^\delta(\delta) = M_+^\delta, \end{cases}$$

where  $H_- \in L^p(-\infty, 0; E_0)$ ,  $H_+^\delta \in L^p(0, \delta; E_0)$  and  $M_+^\delta$ ,  $\psi$ ,  $\phi$  are some given elements in  $E_0$ . We are interested in classical solutions for  $(AuxP_-^\delta)$  and  $(AuxP_+^\delta)$ , that is, functions  $X_-^\delta$ ,  $X_+^\delta$  such that

$$X_-^\delta \in W^{2,p}(-\infty, 0; E_0) \cap L^p(-\infty, 0; D(A_0)), \quad (11)$$

$$X_+^\delta \in W^{2,p}(0, \delta; E_0) \cap L^p(0, \delta; D(A_0)), \quad (12)$$

and satisfying respectively  $(AuxP_-^\delta)$  and  $(AuxP_+^\delta)$ . Observe that, from

$$t \mapsto X_-^\delta(-t) \in W^{2,p}(0, +\infty; E_0) \cap L^p(0, +\infty; D(A_0)),$$

and using the notations of Lions-Peetre [14], chapter VI, we obtain

$$t \mapsto X_-^\delta(-t) \in V_2(p, 0, D(A_0); p, 0, E_0),$$

then

$$\phi = (X_-^\delta)'(0) \in T_1^2(p, 0, D(A_0); p, 0, E_0);$$

on the other hand, we have

$$\begin{aligned} & T_1^2(p, 0, D(A_0); p, 0, E_0) \\ &= S\left(p, \frac{1}{p} + 1, D(A_0); p, \frac{1}{p} - 1, E_0\right) \\ &= S\left(p, \frac{1}{2p} + \frac{1}{2}, D(A_0); p, \frac{1}{2p} - \frac{1}{2}, E_0\right) \\ &= T_0^1\left(p, -\frac{1}{2p} + \frac{1}{2}, D(A_0); p, -\frac{1}{2p} + \frac{1}{2}, E_0\right) \\ &= \left\{u(0) : t^{\frac{1}{2p} + \frac{1}{2}}u \in L^p(0, \infty; D(A_0)), t^{\frac{1}{2p} + \frac{1}{2}}u' \in L^p(0, \infty; E_0)\right\} \\ &= (D(A_0); E_0)_{\frac{1}{2p} + \frac{1}{2}, p}, \end{aligned}$$

thus

$$\phi \in (D(A_0); E_0)_{\frac{1}{2p} + \frac{1}{2}, p}.$$

Similarly we have necessarily

$$\psi \in (D(A_0); E_0)_{\frac{1}{2p}, p} \text{ and } M_+^\delta \in (D(A_0); E_0)_{\frac{1}{2p} + \frac{1}{2}, p},$$

by using the natural extension

$$\widetilde{X_+^\delta}(t) = \begin{cases} X_+^\delta(t) & \text{for a.e. } t \in [0, \delta] \\ 0 & \text{for a.e. } t \in ]\delta, +\infty[. \end{cases}$$

Now, classical considerations give the following representation for a.e.  $t \in (0, \delta)$

$$\begin{aligned} X_+^\delta(t) &= e^{tB_\infty^+} \xi_0 + e^{(\delta-t)B_\infty^-} \xi_1 + \frac{1}{2} B^{-1} \int_0^t e^{(t-s)B_\infty^+} H_+^\delta(s) ds \\ &\quad + \frac{1}{2} B^{-1} \int_t^\delta e^{(s-t)B_\infty^-} H_+^\delta(s) ds. \end{aligned}$$

Using the boundary conditions, we obtain

$$\begin{aligned} \xi_0 &= (I + e^{2\delta B})^{-1} \psi + B^{-1} (I + e^{2\delta B})^{-1} e^{\delta B_\infty^-} M_+^\delta \\ &\quad - \frac{1}{2} B^{-1} (I + e^{2\delta B})^{-1} \int_0^\delta e^{sB_\infty^-} H_+^\delta(s) ds \\ &\quad - \frac{1}{2} B^{-1} (I + e^{2\delta B})^{-1} e^{\delta B_\infty^-} \int_0^\delta e^{(\delta-s)B_\infty^+} H_+^\delta(s) ds, \end{aligned}$$

and

$$\begin{aligned} \xi_1 &= (I + e^{2\delta B})^{-1} e^{\delta B_\infty^+} \psi - B^{-1} (I + e^{2\delta B})^{-1} M_+^\delta \\ &\quad + \frac{1}{2} B^{-1} (I + e^{2\delta B})^{-1} \int_0^\delta e^{(\delta-s)B_\infty^+} H_+^\delta(s) ds \\ &\quad - \frac{1}{2} B^{-1} (I + e^{2\delta B})^{-1} e^{\delta B_\infty^+} \int_0^\delta e^{sB_\infty^-} H_+^\delta(s) ds. \end{aligned}$$

We then have the following proposition.

**Proposition 18.** *Let  $H_+^\delta \in L^p(0, \delta; E_0)$ ,  $1 < p < \infty$  with  $p \neq 2$ . Then, problem  $(AuxP_+^\delta)$  has a unique solution  $X_+^\delta$  satisfying (12) if and only if  $M_+^\delta \in (D(A_0), E_0)_{\frac{1}{2p} + \frac{1}{2}, p}$  and  $\psi \in (D(A_0), E_0)_{\frac{1}{2p}, p}$ .*

*Proof.* We can write for a.e.  $t \in (0, \delta)$

$$\begin{aligned} X_+^\delta(t) &= e^{tB_\varpi^+} \xi_0 + e^{(\delta-t)B_\varpi^-} \xi_1 + \frac{1}{2} B^{-1} (B_\varpi^+)^{-1} B_\varpi^+ \int_0^t e^{(t-s)B_\varpi^+} H_+^\delta(s) ds \\ &\quad + \frac{1}{2} B^{-1} (B_\varpi^-)^{-1} B_\varpi^- \int_t^\delta e^{(s-t)B_\varpi^-} H_+^\delta(s) ds, \end{aligned}$$

where

$$\begin{aligned} e^{tB_\varpi^+} \xi_0 &= (I + e^{2\delta B})^{-1} e^{tB_\varpi^+} \psi + B^{-1} (I + e^{2\delta B})^{-1} e^{\delta B_\varpi^-} e^{tB_\varpi^+} M_+^\delta \\ &\quad - \frac{1}{2} B^{-1} (I + e^{2\delta B})^{-1} (B_\varpi^+)^{-1} B_\varpi^+ \int_0^\delta e^{(t+s)B_\varpi^+} (e^{-2s\varpi} H_+^\delta(s)) ds \\ &\quad - \frac{1}{2} B^{-1} (I + e^{2\delta B})^{-1} e^{\delta B_\varpi^-} (B_\varpi^+)^{-1} B_\varpi^+ \int_0^\delta e^{(t+s)B_\varpi^+} H_+^\delta(\delta - s) ds, \end{aligned}$$

and

$$\begin{aligned} &e^{(\delta-t)B_\varpi^-} \xi_1 \\ &= (I + e^{2\delta B})^{-1} e^{\delta B_\varpi^+} e^{(\delta-t)B_\varpi^-} \psi - B^{-1} (I + e^{2\delta B})^{-1} e^{(\delta-t)B_\varpi^-} M_+^\delta \\ &\quad + \frac{1}{2} B^{-1} (I + e^{2\delta B})^{-1} (B_\varpi^-)^{-1} B_\varpi^- \int_0^\delta e^{(\delta-t+s)B_\varpi^-} (e^{2s\varpi} H_+^\delta(\delta - s)) ds \\ &\quad - \frac{1}{2} B^{-1} (I + e^{2\delta B})^{-1} e^{\delta B_\varpi^+} (B_\varpi^-)^{-1} B_\varpi^- \int_0^\delta e^{(\delta-t+s)B_\varpi^-} H_+^\delta(s) ds. \end{aligned}$$

So  $X_+^\delta \in L^p(0, \delta; E_0)$  by Lemma 10. Then, by applying  $A_0 = -B^2$ , we get

$$\begin{aligned} &A_0 X_+^\delta(t) \\ &= -B^2 e^{tB_\varpi^+} \xi_0 - B^2 e^{(\delta-t)B_\varpi^-} \xi_1 - \frac{1}{2} B (B_\varpi^+)^{-1} B_\varpi^+ \int_0^t e^{(t-s)B_\varpi^+} H_+^\delta(s) ds \\ &\quad - \frac{1}{2} B (B_\varpi^-)^{-1} B_\varpi^- \int_t^\delta e^{(s-t)B_\varpi^-} H_+^\delta(s) ds, \end{aligned}$$

where

$$\begin{aligned} B^2 e^{tB_\varpi^+} \xi_0 &= (I + e^{2\delta B})^{-1} B^2 e^{tB_\varpi^+} \psi + (I + e^{2\delta B})^{-1} e^{\delta B_\varpi^-} B e^{tB_\varpi^+} M_+^\delta \\ &\quad - \frac{1}{2} (I + e^{2\delta B})^{-1} B (B_\varpi^+)^{-1} B_\varpi^+ \int_0^\delta e^{(t+s)B_\varpi^+} (e^{-2s\varpi} H_+^\delta(s)) ds \\ &\quad - \frac{1}{2} (I + e^{2\delta B})^{-1} e^{\delta B_\varpi^-} B (B_\varpi^+)^{-1} B_\varpi^+ \int_0^\delta e^{(t+s)B_\varpi^+} H_+^\delta(\delta - s) ds, \end{aligned}$$

and

$$\begin{aligned}
& B^2 e^{(\delta-t)B_{\varpi}^-} \xi_1 \\
&= (I + e^{2\delta B})^{-1} e^{\delta B_{\varpi}^+} B^2 e^{(\delta-t)B_{\varpi}^-} \psi - (I + e^{2\delta B})^{-1} B e^{(\delta-t)B_{\varpi}^-} M_+^{\delta} \\
&+ \frac{1}{2} (I + e^{2\delta B})^{-1} B (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_0^{\delta} e^{(\delta-t+s)B_{\varpi}^-} (e^{2s\varpi} H_+^{\delta}(\delta-s)) ds \\
&- \frac{1}{2} (I + e^{2\delta B})^{-1} e^{\delta B_{\varpi}^+} B (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_0^{\delta} e^{(\delta-t+s)B_{\varpi}^-} H_+^{\delta}(s) ds.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
(X_+^{\delta})'(t) &= B_{\varpi}^+ e^{tB_{\varpi}^+} \xi_0 - B_{\varpi}^- e^{(\delta-t)B_{\varpi}^-} \xi_1 + \frac{1}{2} B^{-1} B_{\varpi}^+ \int_0^t e^{(t-s)B_{\varpi}^+} H_+^{\delta}(s) ds \\
&- \frac{1}{2} B^{-1} B_{\varpi}^- \int_t^{\delta} e^{(s-t)B_{\varpi}^-} H_+^{\delta}(s) ds,
\end{aligned}$$

and

$$\begin{aligned}
(X_+^{\delta})''(t) &= (B_{\varpi}^+)^2 e^{tB_{\varpi}^+} \xi_0 + (B_{\varpi}^-)^2 e^{(\delta-t)B_{\varpi}^-} \xi_1 \\
&+ \frac{1}{2} B_{\varpi}^+ B^{-1} B_{\varpi}^+ \int_0^t e^{(t-s)B_{\varpi}^+} H_+^{\delta}(s) ds \\
&+ \frac{1}{2} B_{\varpi}^- B^{-1} B_{\varpi}^- \int_t^{\delta} e^{(s-t)B_{\varpi}^-} H_+^{\delta}(s) ds + H_+^{\delta}(t).
\end{aligned}$$

Since

$$B (B_{\varpi}^{\pm})^{-1}, B_{\varpi}^{\pm} B^{-1}, (I + e^{2\delta B})^{-1} \in L(E_0),$$

we can deduce, thanks to Lemmas 10 and 14, that

$$A_0 X_+^{\delta}, (X_+^{\delta})'' \in L^p(0, \delta; E_0),$$

if and only if

$$M_+^{\delta} \in (D(A_0), E_0)_{\frac{1}{2p} + \frac{1}{2}, p} \text{ and } \psi \in (D(A_0), E_0)_{\frac{1}{2p}, p}.$$

□

By the same way, the solution of problem  $(AuxP_-^{\delta})$  can be represented, for a.e.  $t \in (-\infty, 0)$ , by

$$\begin{aligned}
X_-^{\delta}(t) &= e^{-tB_{\varpi}^-} \xi_2 + \frac{1}{2} B^{-1} \int_t^0 e^{(s-t)B_{\varpi}^-} H_-(s) ds \\
&+ \frac{1}{2} B^{-1} \int_{-\infty}^t e^{(t-s)B_{\varpi}^+} H_-(s) ds,
\end{aligned}$$

where

$$\xi_2 = -(B_{\varpi}^-)^{-1} \phi + \frac{1}{2} B_{\varpi}^+ (B_{\varpi}^-)^{-1} B^{-1} \int_{-\infty}^0 e^{-sB_{\varpi}^+} H_-(s) ds.$$



By the Hölder's inequality and the negative exponential decreasing of the analytic semigroup  $(e^{xB_{\overline{\omega}}^{\pm}})_{x>0}$ , we have the absolute convergence of the integrals in  $X_-^{\delta}$  and  $\xi_2$ . For example, for a.e.  $t \in (-\infty, 0)$

$$\begin{aligned} \left\| \int_{-\infty}^t e^{(t-s)B_{\overline{\omega}}^+} H_-(s) ds \right\| &\leq M \left( \int_{-\infty}^t e^{-qa(t-s)} ds \right)^{\frac{1}{q}} \|H_-\|_{L^p(-\infty, 0; E_0)} \\ &\leq C \|H_-\|_{L^p(-\infty, 0; E_0)}, \end{aligned}$$

with  $1/p + 1/q = 1$ .

**Proposition 19.** *Let  $H_- \in L^p(-\infty, 0; E_0)$ ,  $1 < p < \infty$  with  $p \neq 2$ . Then, problem  $(AuxP^{\delta})$  has a unique solution  $X_-^{\delta}$  satisfying (11) if and only if  $\phi \in (D(A_0), E_0)_{\frac{1}{2p} + \frac{1}{2}, p}$ .*

*Proof.* For a.e.  $t \in (-\infty, 0)$ , we can write that

$$\begin{aligned} X_-^{\delta}(t) &= e^{-tB_{\overline{\omega}}^-} \xi_2 + \frac{1}{2} B^{-1} (B_{\overline{\omega}}^-)^{-1} B_{\overline{\omega}}^- \int_t^0 e^{(s-t)B_{\overline{\omega}}^-} H_-(s) ds \\ &\quad + \frac{1}{2} B^{-1} (B_{\overline{\omega}}^+)^{-1} B_{\overline{\omega}}^+ \int_{-\infty}^t e^{(t-s)B_{\overline{\omega}}^+} H_-(s) ds, \end{aligned}$$

where

$$\begin{aligned} e^{-tB_{\overline{\omega}}^-} \xi_2 &= - (B_{\overline{\omega}}^-)^{-1} e^{-tB_{\overline{\omega}}^-} \phi \\ &\quad + \frac{1}{2} B_{\overline{\omega}}^+ (B_{\overline{\omega}}^-)^{-2} B^{-1} B_{\overline{\omega}}^- \int_{-\infty}^0 e^{-(t+s)B_{\overline{\omega}}^-} (e^{-2s\overline{\omega}} H_-(s)) ds. \end{aligned}$$

Thus, by applying  $A_0 = -B^2$ , we get

$$\begin{aligned} A_0 X_-^{\delta}(t) &= -B^2 e^{-tB_{\overline{\omega}}^-} \xi_2 - \frac{1}{2} B (B_{\overline{\omega}}^-)^{-1} B_{\overline{\omega}}^- \int_t^0 e^{(s-t)B_{\overline{\omega}}^-} H_-(s) ds \\ &\quad - \frac{1}{2} B (B_{\overline{\omega}}^+)^{-1} B_{\overline{\omega}}^+ \int_{-\infty}^t e^{(t-s)B_{\overline{\omega}}^+} H_-(s) ds, \end{aligned}$$

where

$$\begin{aligned} B^2 e^{-tB_{\overline{\omega}}^-} \xi_2 &= -B (B_{\overline{\omega}}^-)^{-1} B e^{-tB_{\overline{\omega}}^-} \phi \\ &\quad + \frac{1}{2} B_{\overline{\omega}}^+ B (B_{\overline{\omega}}^-)^{-2} B_{\overline{\omega}}^- \int_{-\infty}^0 e^{-(t+s)B_{\overline{\omega}}^-} (e^{-2s\overline{\omega}} H_-(s)) ds. \end{aligned}$$

In addition, we have

$$\begin{aligned} (X_-^{\delta})'(t) &= -B_{\overline{\omega}}^- e^{-tB_{\overline{\omega}}^-} \xi_2 - \frac{1}{2} B^{-1} B_{\overline{\omega}}^- \int_t^0 e^{(s-t)B_{\overline{\omega}}^-} H_-(s) ds \\ &\quad + \frac{1}{2} B^{-1} B_{\overline{\omega}}^+ \int_{-\infty}^t e^{(t-s)B_{\overline{\omega}}^+} H_-(s) ds, \end{aligned}$$

and

$$\begin{aligned} (X_-^{\delta})''(t) &= (B_{\overline{\omega}}^-)^2 e^{-tB_{\overline{\omega}}^-} \xi_2 + \frac{1}{2} B_{\overline{\omega}}^- B^{-1} B_{\overline{\omega}}^- \int_t^0 e^{(s-t)B_{\overline{\omega}}^-} H_-(s) ds \\ &\quad + \frac{1}{2} B_{\overline{\omega}}^+ B^{-1} B_{\overline{\omega}}^+ \int_{-\infty}^t e^{(t-s)B_{\overline{\omega}}^+} H_-(s) ds + H_-(t). \end{aligned}$$

Hence, by Lemmas 11 and 15, we obtain the desired result.  $\square$

## 4.2 Formula of the solution of $(WP_t^\delta)$

Recall our problem

$$(WP_t^\delta) \begin{cases} (W_-^\delta)''(t) - 2\varpi (W_-^\delta)'(t) + \varpi^2 W_-^\delta(t) + A_0 W_-^\delta(t) = H_-(t) \\ \text{on } (-\infty, 0) \\ (W_+^\delta)''(t) - 2\varpi (W_+^\delta)'(t) + \varpi^2 W_+^\delta(t) + A_0 W_+^\delta(t) = H_+^\delta(t) \\ \text{on } (0, \delta) \\ (t.c.) \begin{cases} W_-^\delta(0) = W_+^\delta(0) \\ (W_-^\delta)'(0) - \nu (W_+^\delta)'(0) = (1 - \nu) \varpi W_\pm^\delta(0) \end{cases} \\ (W_+^\delta)'(\delta) - \varpi W_+^\delta(\delta) = M_+^\delta, \end{cases}$$

where  $\nu = \frac{\mu_+}{\mu_-}$ .

Using the representations of  $X_-^\delta$ ,  $X_+^\delta$  and combining with the transmission conditions (t.c.), we obtain the following system of unknowns  $\psi, \phi$

$$\begin{cases} B_\varpi^- \psi + \phi = b_1 \\ -\nu (I + e^{2\delta B})^{-1} \left( B_\varpi^+ - B_\varpi^- e^{2\delta B} + \left( \frac{1}{\nu} - 1 \right) \varpi (I + e^{2\delta B}) \right) \psi + \phi = b_2, \end{cases}$$

where

$$\begin{aligned} b_1 &= \int_{-\infty}^0 e^{-sB_\varpi^+} H_-(s) ds, \\ b_2 &= 2\nu (I + e^{2\delta B})^{-1} e^{\delta B_\varpi^-} M_+^\delta - \nu (I + e^{2\delta B})^{-1} \int_0^\delta e^{sB_\varpi^-} H_+^\delta(s) ds \\ &\quad - \nu (I + e^{2\delta B})^{-1} e^{\delta B_\varpi^-} \int_0^\delta e^{(\delta-s)B_\varpi^+} H_+^\delta(s) ds. \end{aligned}$$

The formal determinant of this abstract system is

$$\begin{aligned} \Delta_\varpi &:= B_\varpi^- + \nu (I + e^{2\delta B})^{-1} \left( B_\varpi^+ - B_\varpi^- e^{2\delta B} + \left( \frac{1}{\nu} - 1 \right) \varpi (I + e^{2\delta B}) \right) \\ &= (I + e^{2\delta B})^{-1} (B_\varpi^- (I + e^{2\delta B}) + \nu B_\varpi^+ - \nu B_\varpi^- e^{2\delta B} + (1 - \nu) \varpi (I + e^{2\delta B})) \\ &= (I + e^{2\delta B})^{-1} (B + B e^{2\delta B} + \nu B - \nu B e^{2\delta B}) \\ &= (1 + \nu) B (I + e^{2\delta B})^{-1} \left( I + \frac{1 - \nu}{1 + \nu} e^{2\delta B} \right) \\ &= B (I + e^{2\delta B})^{-1} D_\nu, \end{aligned}$$

$D_\nu$  is boundedly invertible (see Lemma 16), therefore  $\Delta_\varpi^{-1} \in L(E_0, D(B))$ . Thus

$$\begin{aligned} \psi &= -2\nu B^{-1} D_\nu^{-1} e^{\delta B_\varpi^-} M_+^\delta + B^{-1} D_\nu^{-1} (I + e^{2\delta B}) \int_{-\infty}^0 e^{-sB_\varpi^+} H_-(s) ds \\ &\quad + \nu B^{-1} D_\nu^{-1} \int_0^\delta e^{sB_\varpi^-} H_+^\delta(s) ds + \nu B^{-1} D_\nu^{-1} e^{\delta B_\varpi^-} \int_0^\delta e^{(\delta-s)B_\varpi^+} H_+^\delta(s) ds, \end{aligned}$$

and

$$\begin{aligned}
\phi &= 2\nu B_{\varpi}^{-} B^{-1} D_{\nu}^{-1} e^{\delta B_{\varpi}^{-}} M_{+}^{\delta} \\
&+ (I - B_{\varpi}^{-} B^{-1} D_{\nu}^{-1} (I + e^{2\delta B})) \int_{-\infty}^0 e^{-s B_{\varpi}^{+}} H_{-}(s) ds \\
&- \nu B_{\varpi}^{-} B^{-1} D_{\nu}^{-1} \int_0^{\delta} e^{s B_{\varpi}^{-}} H_{+}^{\delta}(s) ds \\
&- \nu B_{\varpi}^{-} B^{-1} D_{\nu}^{-1} e^{\delta B_{\varpi}^{-}} \int_0^{\delta} e^{(\delta-s) B_{\varpi}^{+}} H_{+}^{\delta}(s) ds.
\end{aligned}$$

Using these two expressions in  $\xi_0$ ,  $\xi_1$ ,  $\xi_2$ , and after some simplifications we obtain, for a.e.  $t \in (0, \delta)$ ,

$$\begin{aligned}
W_{+}^{\delta}(t) &= e^{t B_{\varpi}^{+}} \xi_0 + e^{(\delta-t) B_{\varpi}^{-}} \xi_1 + \frac{1}{2} B^{-1} \int_0^t e^{(t-s) B_{\varpi}^{+}} H_{+}^{\delta}(s) ds \\
&+ \frac{1}{2} B^{-1} \int_t^{\delta} e^{(s-t) B_{\varpi}^{-}} H_{+}^{\delta}(s) ds,
\end{aligned}$$

and for a.e.  $t \in (-\infty, 0)$

$$\begin{aligned}
W_{-}^{\delta}(t) &= e^{-t B_{\varpi}^{-}} \xi_2 + \frac{1}{2} B^{-1} \int_t^0 e^{(s-t) B_{\varpi}^{-}} H_{-}(s) ds \\
&+ \frac{1}{2} B^{-1} \int_{-\infty}^t e^{(t-s) B_{\varpi}^{+}} H_{-}(s) ds,
\end{aligned}$$

where

$$\begin{aligned}
\xi_0 &= (1 - \nu) B^{-1} D_{\nu}^{-1} e^{\delta B_{\varpi}^{-}} M_{+}^{\delta} + B^{-1} D_{\nu}^{-1} \int_{-\infty}^0 e^{-s B_{\varpi}^{+}} H_{-}(s) ds \\
&- \frac{1 - \nu}{2} B^{-1} D_{\nu}^{-1} \int_0^{\delta} e^{s B_{\varpi}^{-}} H_{+}^{\delta}(s) ds \\
&- \frac{1 - \nu}{2} B^{-1} D_{\nu}^{-1} e^{\delta B_{\varpi}^{-}} \int_0^{\delta} e^{(\delta-s) B_{\varpi}^{+}} H_{+}^{\delta}(s) ds,
\end{aligned}$$

$$\begin{aligned}
\xi_1 &= -(1 + \nu) B^{-1} D_{\nu}^{-1} M_{+}^{\delta} + B^{-1} D_{\nu}^{-1} e^{\delta B_{\varpi}^{+}} \int_{-\infty}^0 e^{-s B_{\varpi}^{+}} H_{-}(s) ds \\
&- \frac{1 - \nu}{2} B^{-1} D_{\nu}^{-1} e^{\delta B_{\varpi}^{+}} \int_0^{\delta} e^{s B_{\varpi}^{-}} H_{+}^{\delta}(s) ds \\
&+ \frac{1 + \nu}{2} B^{-1} D_{\nu}^{-1} \int_0^{\delta} e^{(\delta-s) B_{\varpi}^{+}} H_{+}^{\delta}(s) ds,
\end{aligned}$$

and

$$\begin{aligned}
\xi_2 &= -2\nu B^{-1} D_\nu^{-1} e^{\delta B_\varpi^-} M_+^\delta \\
&\quad + B^{-1} \left( -\frac{1}{2} I + D_\nu^{-1} (I + e^{2\delta B}) \right) \int_{-\infty}^0 e^{-s B_\varpi^+} H_-(s) ds \\
&\quad + \nu B^{-1} D_\nu^{-1} \int_0^\delta e^{s B_\varpi^-} H_+^\delta(s) ds \\
&\quad + \nu B^{-1} D_\nu^{-1} e^{\delta B_\varpi^-} \int_0^\delta e^{(\delta-s) B_\varpi^+} H_+^\delta(s) ds.
\end{aligned}$$

### 4.3 Proof of Theorem 7

We can write for a.e.  $t \in (0, \delta)$

$$\begin{aligned}
W_+^\delta(t) &= e^{t B_\varpi^+} \xi_0 + e^{(\delta-t) B_\varpi^-} \xi_1 + \frac{1}{2} B^{-1} (B_\varpi^+)^{-1} B_\varpi^+ \int_0^t e^{(t-s) B_\varpi^+} H_+^\delta(s) ds \\
&\quad + \frac{1}{2} B^{-1} (B_\varpi^-)^{-1} B_\varpi^- \int_t^\delta e^{(s-t) B_\varpi^-} H_+^\delta(s) ds,
\end{aligned}$$

where

$$\begin{aligned}
e^{t B_\varpi^+} \xi_0 &= (1-\nu) B^{-1} D_\nu^{-1} e^{\delta B_\varpi^-} e^{t B_\varpi^+} M_+^\delta \\
&\quad + B^{-1} D_\nu^{-1} (B_\varpi^+)^{-1} B_\varpi^+ \int_{-\infty}^0 e^{(t-s) B_\varpi^+} H_-(s) ds \\
&\quad - \frac{1-\nu}{2} B^{-1} D_\nu^{-1} (B_\varpi^+)^{-1} B_\varpi^+ \int_0^\delta e^{(t+s) B_\varpi^+} (e^{-2s\varpi} H_+^\delta(s)) ds \\
&\quad - \frac{1-\nu}{2} B^{-1} D_\nu^{-1} e^{\delta B_\varpi^-} (B_\varpi^+)^{-1} B_\varpi^+ \int_0^\delta e^{(t+s) B_\varpi^+} H_+^\delta(\delta-s) ds,
\end{aligned}$$

and

$$\begin{aligned}
&e^{(\delta-t) B_\varpi^-} \xi_1 \\
&= -(1+\nu) B^{-1} D_\nu^{-1} e^{(\delta-t) B_\varpi^-} M_+^\delta \\
&\quad + B^{-1} D_\nu^{-1} e^{\delta B_\varpi^+} (B_\varpi^-)^{-1} B_\varpi^- \int_{-\infty}^0 e^{(\delta-t-s) B_\varpi^+} (e^{(\delta-s)\varpi} H_-(s)) ds \\
&\quad - \frac{1-\nu}{2} B^{-1} D_\nu^{-1} e^{\delta B_\varpi^+} (B_\varpi^-)^{-1} B_\varpi^- \int_0^\delta e^{(\delta-t+s) B_\varpi^-} H_+^\delta(s) ds \\
&\quad + \frac{1+\nu}{2} B^{-1} D_\nu^{-1} (B_\varpi^-)^{-1} B_\varpi^- \int_0^\delta e^{(\delta-t+s) B_\varpi^-} (e^{2s\varpi} H_+^\delta(\delta-s)) ds.
\end{aligned}$$

Therefore, by Lemmas 10 and 12, we deduce that  $W_+^\delta \in L^p(0, \delta; E_0)$ . Then, applying  $A_0 = -B^2$  we obtain

$$\begin{aligned}
&A_0 W_+^\delta(t) \\
&= -B^2 e^{t B_\varpi^+} \xi_0 - B^2 e^{(\delta-t) B_\varpi^-} \xi_1 - \frac{1}{2} B (B_\varpi^+)^{-1} B_\varpi^+ \int_0^t e^{(t-s) B_\varpi^+} H_+^\delta(s) ds \\
&\quad - \frac{1}{2} B (B_\varpi^-)^{-1} B_\varpi^- \int_t^\delta e^{(s-t) B_\varpi^-} H_+^\delta(s) ds,
\end{aligned}$$

where

$$\begin{aligned}
B^2 e^{tB_{\varpi}^+} \xi_0 &= (1 - \nu) D_{\nu}^{-1} e^{\delta B_{\varpi}^-} B e^{tB_{\varpi}^+} M_+^{\delta} \\
&\quad + D_{\nu}^{-1} B (B_{\varpi}^+)^{-1} B_{\varpi}^+ \int_{-\infty}^0 e^{(t-s)B_{\varpi}^+} H_-(s) ds \\
&\quad - \frac{1 - \nu}{2} D_{\nu}^{-1} B (B_{\varpi}^+)^{-1} B_{\varpi}^+ \int_0^{\delta} e^{(t+s)B_{\varpi}^+} (e^{-2s\varpi} H_+^{\delta}(s)) ds \\
&\quad - \frac{1 - \nu}{2} D_{\nu}^{-1} e^{\delta B_{\varpi}^-} B (B_{\varpi}^+)^{-1} B_{\varpi}^+ \int_0^{\delta} e^{(t+s)B_{\varpi}^+} H_+^{\delta}(\delta - s) ds,
\end{aligned}$$

and

$$\begin{aligned}
&B^2 e^{(\delta-t)B_{\varpi}^-} \xi_1 \\
&= - (1 + \nu) D_{\nu}^{-1} B e^{(\delta-t)B_{\varpi}^-} M_+^{\delta} \\
&\quad + D_{\nu}^{-1} e^{\delta B_{\varpi}^+} B (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_{-\infty}^0 e^{(\delta-t-s)B_{\varpi}^-} (e^{(\delta-s)\varpi} H_-(s)) ds \\
&\quad - \frac{1 - \nu}{2} D_{\nu}^{-1} e^{\delta B_{\varpi}^+} B (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_0^{\delta} e^{(\delta-t+s)B_{\varpi}^-} H_+^{\delta}(s) ds \\
&\quad + \frac{1 + \nu}{2} D_{\nu}^{-1} B (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_0^{\delta} e^{(\delta-t+s)B_{\varpi}^-} (e^{2s\varpi} H_+^{\delta}(\delta - s)) ds.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(W_+^{\delta})'(t) &= B_{\varpi}^+ e^{tB_{\varpi}^+} \xi_0 - B_{\varpi}^- e^{(\delta-t)B_{\varpi}^-} \xi_1 + \frac{1}{2} B^{-1} B_{\varpi}^+ \int_0^t e^{(t-s)B_{\varpi}^+} H_+^{\delta}(s) ds \\
&\quad - \frac{1}{2} B^{-1} B_{\varpi}^- \int_t^{\delta} e^{(s-t)B_{\varpi}^-} H_+^{\delta}(s) ds,
\end{aligned}$$

and

$$\begin{aligned}
(W_+^{\delta})''(t) &= (B_{\varpi}^+)^2 e^{tB_{\varpi}^+} \xi_0 + (B_{\varpi}^-)^2 e^{(\delta-t)B_{\varpi}^-} \xi_1 \\
&\quad + \frac{1}{2} B_{\varpi}^+ B^{-1} B_{\varpi}^+ \int_0^t e^{(t-s)B_{\varpi}^+} H_+^{\delta}(s) ds \\
&\quad + \frac{1}{2} B_{\varpi}^- B^{-1} B_{\varpi}^- \int_t^{\delta} e^{(s-t)B_{\varpi}^-} H_+^{\delta}(s) ds + H_+^{\delta}(t).
\end{aligned}$$

Therefore, by Lemmas 10, 12, 14 and 16, we get

$$W_+^{\delta} \in W^{2,p}(0, \delta; E_0) \cap L^p(0, \delta; D(A_0)),$$

if and only if

$$M_+^{\delta} \in (D(A_0); E_0)_{\frac{1}{2p} + \frac{1}{2}, p}.$$

For  $W_-^{\delta}$ , we have

$$\begin{aligned}
W_-^{\delta}(t) &= e^{-tB_{\varpi}^-} \xi_2 + \frac{1}{2} B^{-1} (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_t^0 e^{(s-t)B_{\varpi}^-} H_-(s) ds \\
&\quad + \frac{1}{2} B^{-1} (B_{\varpi}^+)^{-1} B_{\varpi}^+ \int_{-\infty}^t e^{(t-s)B_{\varpi}^+} H_-(s) ds,
\end{aligned}$$

where

$$\begin{aligned}
& e^{-tB_{\varpi}^-} \xi_2 \\
&= -2\nu B^{-1} D_{\nu}^{-1} e^{(\delta-t)B_{\varpi}^-} M_+^{\delta} \\
&\quad + B^{-1} \left( -\frac{1}{2}I + D_{\nu}^{-1} (I + e^{2\delta B}) \right) \int_{-\infty}^0 e^{-(t+s)B_{\varpi}^-} (e^{-2s\varpi} H_-(s)) ds \\
&\quad + \nu B^{-1} D_{\nu}^{-1} (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_0^{\delta} e^{(s-t)B_{\varpi}^-} H_+^{\delta}(s) ds \\
&\quad + \nu B^{-1} D_{\nu}^{-1} e^{\delta B_{\varpi}^-} (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_0^{\delta} e^{(s-t)B_{\varpi}^-} (e^{2s\varpi} H_+^{\delta}(\delta-s)) ds.
\end{aligned}$$

Applying  $A_0 = -B^2$  we obtain

$$\begin{aligned}
A_0 W_-^{\delta}(t) &= -B^2 e^{-tB_{\varpi}^-} \xi_2 - \frac{1}{2} B (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_t^0 e^{(s-t)B_{\varpi}^-} H_-(s) ds \\
&\quad - \frac{1}{2} B (B_{\varpi}^+)^{-1} B_{\varpi}^+ \int_{-\infty}^t e^{(t-s)B_{\varpi}^+} H_-(s) ds,
\end{aligned}$$

where

$$\begin{aligned}
& B^2 e^{-tB_{\varpi}^-} \xi_2 \\
&= -2\nu D_{\nu}^{-1} B e^{(\delta-t)B_{\varpi}^-} M_+^{\delta} \\
&\quad + B^{-1} \left( -\frac{1}{2}I + D_{\nu}^{-1} (I + e^{2\delta B}) \right) \int_{-\infty}^0 e^{-(t+s)B_{\varpi}^-} (e^{-2s\varpi} H_-(s)) ds \\
&\quad + \nu D_{\nu}^{-1} B (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_0^{\delta} e^{(s-t)B_{\varpi}^-} H_+^{\delta}(s) ds \\
&\quad + \nu D_{\nu}^{-1} e^{\delta B_{\varpi}^-} B (B_{\varpi}^-)^{-1} B_{\varpi}^- \int_0^{\delta} e^{(s-t)B_{\varpi}^-} (e^{2s\varpi} H_+^{\delta}(\delta-s)) ds.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(W_-^{\delta})'(t) &= -B_{\varpi}^- e^{-tB_{\varpi}^-} \xi_2 - \frac{1}{2} B^{-1} B_{\varpi}^- \int_t^0 e^{(s-t)B_{\varpi}^-} H_-(s) ds \\
&\quad + \frac{1}{2} B^{-1} B_{\varpi}^+ \int_{-\infty}^t e^{(t-s)B_{\varpi}^+} H_-(s) ds,
\end{aligned}$$

and

$$\begin{aligned}
(W_-^{\delta})''(t) &= (B_{\varpi}^-)^2 e^{-tB_{\varpi}^-} \xi_2 + \frac{1}{2} B_{\varpi}^- B^{-1} B_{\varpi}^- \int_t^0 e^{(s-t)B_{\varpi}^-} H_-(s) ds \\
&\quad + \frac{1}{2} B_{\varpi}^+ B^{-1} B_{\varpi}^+ \int_{-\infty}^t e^{(t-s)B_{\varpi}^+} H_-(s) ds.
\end{aligned}$$

Therefore, by Lemmas 11, 13, 15 and 16, we deduce that

$$W_-^{\delta} \in W^{2,p}(-\infty, 0; E_0) \cap L^p(-\infty, 0; D(A_0)),$$

if and only if

$$M_+^{\delta} \in (D(A_0); E_0)_{\frac{1}{2p} + \frac{1}{2}, p}.$$

## 5 Going back to the problem in the biological cell

Let us now go back to our first problem  $(P_{x,y}^\varepsilon)$ . Analyzing  $(UP_{t,\theta}^\delta)$ , we get the two following propositions as immediate consequences of Proposition 4 and Theorem 7 and the fact that operator  $I - P$  is continuous from  $E$  to  $E_0$  and from  $D(A)$  to  $D(A_0)$  and that it transforms elements of  $(D(A), E)_{\frac{1}{2p} + \frac{1}{2}, p}$  into elements of  $(D(A_0), E_0)_{\frac{1}{2p} + \frac{1}{2}, p}$ .

**Proposition 20.** *Let  $G_- \in L^p(\Omega_-)$ ,  $G_+^\delta \in L^p(\Omega_+^\delta)$ ,  $1 < p < 2$ , satisfying  $(UCC_{t,\theta}^\delta)$ . Then, the two following assertions are equivalent.*

1. *Problem  $(UP_{t,\theta}^\delta)$  with the periodic boundary conditions  $(UPBC_t^\delta)$  has a unique solution*

$$U^\delta = \begin{cases} U_-^\delta & \text{in } \Omega_- \\ U_+^\delta & \text{in } \Omega_+^\delta \end{cases}$$

such that

$$U_-^\delta \in W^{2,p}(\Omega_-), \quad U_+^\delta \in W^{2,p}(\Omega_+^\delta).$$

2.  $F_+^\delta \in B_p^{1-\frac{1}{p}}(-\pi, \pi)$ .

**Proposition 21.** *Let  $G_- \in L^p(\Omega_-)$ ,  $G_+^\delta \in L^p(\Omega_+^\delta)$ ,  $p > 2$ , satisfying  $(UCC_{t,\theta}^\delta)$ . Then, the two following assertions are equivalent.*

1. *Problem  $(UP_{t,\theta}^\delta)$  with the periodic boundary conditions  $(UPBC_t^\delta)$  has a unique solution*

$$U^\delta = \begin{cases} U_-^\delta & \text{in } \Omega_- \\ U_+^\delta & \text{in } \Omega_+^\delta \end{cases}$$

such that

$$U_-^\delta \in W^{2,p}(\Omega_-), \quad U_+^\delta \in W^{2,p}(\Omega_+^\delta).$$

2.  $F_+^\delta \in B_p^{1-\frac{1}{p}}(-\pi, \pi)$  and  $F_+^\delta(-\pi) = F_+^\delta(\pi)$ .

Remark 3 and the fact that

$$\begin{aligned} F_+^\delta \in B_p^{1-\frac{1}{p}}(-\pi, \pi) &\Leftrightarrow f_+^\delta \in B_p^{1-\frac{1}{p}}(-\pi, \pi), \\ F_+^\delta(-\pi) = F_+^\delta(\pi) &\Leftrightarrow f_+^\delta(-\pi) = f_+^\delta(\pi), \end{aligned}$$

allow us to establish the following propositions for  $(P_{t,\theta}^\delta)$ .

**Proposition 22.** *Let  $f_+^\delta \in B_p^{1-\frac{1}{p}}(-\pi, \pi)$  and  $g_- \in L_{e^{\varpi t}}^p(\Omega_-, dtd\theta)$ ,  $g_+^\delta \in L_{e^{\varpi t}}^p(\Omega_+^\delta, dtd\theta)$ ,  $1 < p < 2$ , satisfying  $(CC_{t,\theta}^\delta)$ . Then, problem  $(P_{t,\theta}^\delta)$  with the periodic boundary conditions  $(PBC_t^\delta)$  has a unique solution*

$$u^\delta = \begin{cases} u_-^\delta & \text{in } \Omega_- \\ u_+^\delta & \text{in } \Omega_+^\delta \end{cases}$$

such that

$$u_-^\delta \in E^{2,p}(\Omega_-, dtd\theta), \quad u_+^\delta \in E^{2,p}(\Omega_+^\delta, dtd\theta).$$

**Proposition 23.** Let  $f_+^\delta \in B_p^{1-\frac{1}{p}}(-\pi, \pi)$  such that  $f_+^\delta(-\pi) = f_+^\delta(\pi)$ , and  $g_- \in L_{e^{\varpi t}}^p(\Omega_-, dt d\theta)$ ,  $g_+^\delta \in L_{e^{\varpi t}}^p(\Omega_+^\delta, dt d\theta)$ ,  $p > 2$ , satisfying  $(CC_{t,\theta}^\delta)$ . Then, problem  $(P_{t,\theta}^\delta)$  with the periodic boundary conditions  $(PBC_t^\delta)$  has a unique solution

$$u^\delta = \begin{cases} u_-^\delta & \text{in } \Omega_- \\ u_+^\delta & \text{in } \Omega_+^\delta \end{cases}$$

such that

$$u_-^\delta \in E^{2,p}(\Omega_-, dt d\theta), \quad u_+^\delta \in E^{2,p}(\Omega_+^\delta, dt d\theta).$$

We can now prove our main result.

*Proof of Theorem 1.* As we have seen in Section 2, we have

$$\begin{aligned} g_\pm^\delta \in L_{e^{\varpi t}}^p(\Omega_\pm^\delta, dt d\theta) &\Leftrightarrow h_\pm^\varepsilon \in L^p(\Omega_\pm^{*\varepsilon}, dx dy), \\ u_\pm^\delta \in E^{2,p}(\Omega_\pm^\delta, dt d\theta) &\Leftrightarrow w_\pm^\varepsilon \in W^{2,p}(\Omega_\pm^{*\varepsilon}, dx dy). \end{aligned}$$

In addition, thanks to (2) and (3), we get

$$f_+^\delta \in B_p^{1-\frac{1}{p}}(-\pi, \pi) \Leftrightarrow L_+^\varepsilon \in B_p^{1-\frac{1}{p}}(-\pi, \pi) \Leftrightarrow l_+^\varepsilon \in B_p^{1-\frac{1}{p}}(\Gamma_+^{*\varepsilon}).$$

and since, for  $p > 2$ , we have

$$\begin{aligned} f_+^\delta(-\pi) = f_+^\delta(\pi) &\Leftrightarrow L_+^\varepsilon(-\pi) = L_+^\varepsilon(\pi) \\ &\Leftrightarrow l_+^\varepsilon((1+\varepsilon)(\cos(-\pi), \sin(-\pi))) = l_+^\varepsilon((1+\varepsilon)(\cos(\pi), \sin(\pi))), \end{aligned}$$

then  $l_+^\varepsilon$  is only in  $B_p^{1-\frac{1}{p}}(\Gamma_+^{*\varepsilon})$ .  $\square$

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