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**To link to this article** : DOI :10.1093/jigpal/jzs052  
URL : <http://dx.doi.org/10.1093/jigpal/jzs052>

**To cite this version** : Balbiani, Philippe and Gasquet, Olivier and Schwarzentruher, François *[Agents that look at one another](#)*. (2013) Logic Journal of the IGPL, vol. 21 (n° 3). pp. 438-467. ISSN 1367-0751

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# Agents that look at one another

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## Abstract

Despite the fact that epistemic connectives are sometimes interpreted in concrete structures defined by means of runs and clock time functions, one of the things which strikes one when studying multiagent logics is how abstract their semantics are. Contrasting this fact is the fact that real agents like robots in everyday life and virtual characters in video games have strong links with their spatial environment. In this paper, we introduce multiagent logics which semantics can be defined by means of purely geometrical notions: possible states are defined by means of the positions in  $\mathbb{R}^n$  occupied by agents and the sections of  $\mathbb{R}^n$  seen by agents whereas accessibility relations are defined by means of the ability of agents to imagine possible states compatible with what they currently see.

*Keywords:* Multiagent logics, spatial reasoning, axiomatization/completeness, decidability/complexity.

## 1 Introduction

The field of Artificial Intelligence known as *knowledge representation and reasoning* is concerned with the problem of representing and reasoning about the knowledge of everyday entities called agents. In recent years, much activity in this field has centred on *multiagent logics* — modal languages whose atomic sentences range over sets and whose epistemic connectives represent operations involving those sets. In this respect, see [10], or [11] for an introduction, the most intensively studied epistemic connectives are modal operators of the form  $K_a$  (“ $a$  knows that . . .”). Their semantics use two notions of Kripke models: possible state and accessibility relation. Possible states are states of affairs describing what is true whereas accessibility relations are indistinguishability relations between states of affairs characterizing the ability of agents to determine what states of affairs they can discriminate. And most find these traditional notions like possible state and accessibility relation intuitively transparent. Despite the fact that epistemic connectives are sometimes interpreted in more concrete structures defined by means of runs and clock time functions [12], one of the things which strikes one when studying multiagent logics is how abstract their semantics are.

Contrasting this fact is the fact that real agents like robots in everyday life and virtual characters in video games have strong links with their spatial environment: they occupy positions in it and they see sections of it. Moreover, the knowledge these agents have about their environment mostly depends on the positions they occupy and the sections they see: in the absence of message exchange,  $a$  knows that  $b$  sees  $c$  only if  $a$  sees  $b$ . These features imply the new opinion that geometrical notions like points and sets of points should be integrated into the semantics of multiagent logics. This tension between a traditional abstract semantics and a new spatial semantics disappears when one realizes that possible states and accessibility re-

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lations can be defined by means of purely geometrical notions: possible states can be defined by means of the positions in  $\mathbb{R}^n$  occupied by agents and the sections of  $\mathbb{R}^n$  seen by agents whereas accessibility relations can be defined by means of the ability of agents to imagine possible states compatible with what they currently see. In order to elaborate on this idea, we introduce an epistemic language where sentences like “ $a$  knows that  $b$  sees  $c$ ” can be expressed.

The syntax of our language, in addition to the traditional epistemic connectives of the form  $K_a$  (“ $a$  knows that . . .”) considered in [11, 12], for example, will include atomic formulas of the form  $a \triangleright b$  (“ $a$  sees  $b$ ”), the truth of which will depend on the position in  $\mathbb{R}^n$  occupied by agent  $b$  and the section of  $\mathbb{R}^n$  seen by agent  $a$ . In this setting, the above-mentioned sentence “ $a$  knows that  $b$  sees  $c$  only if  $a$  sees  $b$ ” will be written  $K_a b \triangleright c \rightarrow a \triangleright b$ . Concerning the semantics of our language, in this first attempt at modelling what agents can see and what agents can know, we naturally work with certain simplifications. Firstly, we will require that agents know all the logical consequences of their knowledge. This is the so-called logical omniscience character of agents. For more on the various problems associated with it, see [20, 21]. Secondly, we will require that the visual capacity of agents satisfies the following conditions: agents can see through any agents that may be blocking their view; agents are never faulty, i.e. they always see what they are physically able to see; agents can see infinitely far from their positions; if  $a$  sees  $b$  then  $a$  is perfectly informed about what direction  $b$  is looking in.

The project of relating multiagent logics to space sprang from the fictional two-dimensional world of *Flatland* created by Edwin Abbott in 1884. The third author of the present article developed a logic for studying knowledge of agents along a line where sentences like “ $a$  knows that  $b$  knows that light  $\lambda$  is on” can be expressed [18]. This logic turned out to have a *PSPACE*-complete model checking problem and a *PSPACE*-complete satisfiability problem. The original work was followed up then by further work in collaboration with the first two authors that resulted in an axiomatization of a logic for studying knowledge of agents along a line where sentences like “ $a$  knows that  $b$  sees  $c$ ” can be expressed [2]. In the present article, we generalize these multiagent logics to spaces of greater dimensions. Its section-by-section breakdown is as follows. Section 2 defines the syntax and the semantics of the multiagent logic we will be working with. In Section 3, we study its expressivity. Section 4 gives the axiomatization in dimension 1 of our multiagent logic. In Sections 5 and 6, we investigate the complexity of model checking problems and the complexity of satisfiability problems. Section 7 presents some variants. Some proofs can be found in the annex.

## 2 Syntax and semantics

In this section, we will mostly be concerned with the syntax and the semantics of our multiagent logic. First, we are going to investigate a Cartesian semantics where for some positive integer  $n$ , agents occupy positions in  $\mathbb{R}^n$ . Second, we are going to investigate an abstract semantics in dimension 1 where for some linear order  $\mathcal{T} = (T, <)$  without endpoints, agents occupy positions in  $T$ .

## 2.1 Syntax

Let  $AGT$  be a countable set of agents (with typical members denoted  $a, b$ , etc). The set of all formulas (with typical members denoted  $\phi, \psi$ , etc) is given by the rule

- $\phi ::= a \triangleright b \mid \perp \mid \neg\phi \mid (\phi \vee \psi) \mid K_a\phi$ .

The intended meanings of  $a \triangleright b$  and  $K_a\phi$  are as follows:

- $a \triangleright b$ : “ $a$  sees  $b$ ”,
- $K_a\phi$ : “ $a$  knows that  $\phi$ ”.

### EXAMPLE 2.1

The formula  $K_a b \triangleright c$  can be read “ $a$  knows that  $b$  sees  $c$ ”.

In atomic sentences like  $a \triangleright b$ , we will always assume that  $a, b$  are distinct. This simplification is related to the fact that in Section 2.2, the section seen by an agent is assimilated to an open subset of  $\mathbb{R}^n$  not containing the position occupied by the agent. Let  $\bar{\triangleright}$  be defined by

- $a \bar{\triangleright} b ::= \neg a \triangleright b$ .

Formulas like  $a \triangleright b$  and  $a \bar{\triangleright} b$  are called literals. We adopt the standard definitions for the remaining Boolean connectives. As usual, for all agents  $a$ , we define the epistemic connective  $\hat{K}_a$  as follows:

- $\hat{K}_a\phi ::= \neg K_a \neg\phi$ .

The notion of a subformula is standard. It is usual to omit parentheses if this does not lead to any ambiguity. Considering an enumeration  $(a_1, a_2, \dots)$  of  $AGT$ , let  $k$  be a nonnegative integer. We use  $\phi(a_1, \dots, a_k)$  to denote the fact that  $\phi$  is a formula whose agents form a sublist of  $a_1, \dots, a_k$ . In this case, we shall say that  $\phi$  is a  $k$ -formula. A set  $\Sigma$  of formulas is  $k$ -maximal iff for all  $k$ -formulas  $\phi$ ,  $\phi \in \Sigma$ , or  $\neg\phi \in \Sigma$ .

## 2.2 Cartesian semantics

Let  $n$  be a positive integer. As we mentioned in the introduction, every agent occupies some position in the space and sees some section of the space. Hence, the notion of an  $n$ -scope will be of the utmost interest for us. An  $n$ -scope is a structure of the form  $(x, S)$  where  $x$  is an element of  $\mathbb{R}^n$  and  $S$  is a subset of  $\mathbb{R}^n$ .  $x$  and  $S$  are respectively called the position of the  $n$ -scope  $(x, S)$  and the section of the  $n$ -scope  $(x, S)$ . In a first attempt at modelling what agents can see and what agents can know, one must naturally work with certain simplifications. A natural semantics to look at would be one where the agents' scopes are cones of various angles, with the agents' positions at the basis of the cones. In a first setting, we will consider that agents' scopes are half spaces, with the agents' positions being at the frontier of the halfspaces. In this respect, we shall say that an  $n$ -scope  $(x, S)$  is simple iff

- $S$  is an open half space,
- $x$  is on the frontier of  $S$ .

### EXAMPLE 2.2

If one considers Figure 1 in dimension 2, a 2-scope is defined by the point and the open half plane located on the same side of the line through the point as indicated by the arrow.

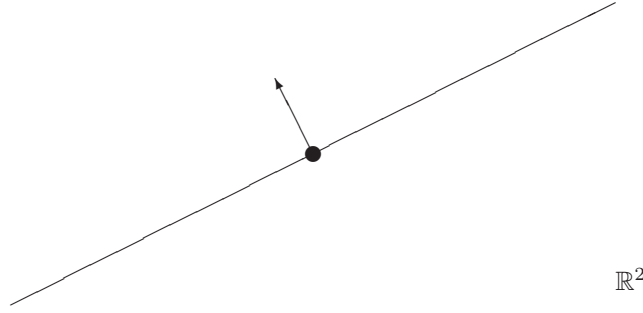


FIG. 1. A simple 2-scope.

We now provide a mechanism for interpreting our formulas in the Cartesian coordinate system  $\mathbb{R}^n$ . An  $n$ -world is a function  $u$  assigning to each agent  $a$  an  $n$ -scope  $u(a)$ . Given an  $n$ -world  $u$  and agents  $a, b$ , if  $u(a) = (x_u(a), S_u(a))$  and  $u(b) = (x_u(b), S_u(b))$  then  $x_u(a)$  and  $x_u(b)$  are respectively the positions in  $\mathbb{R}^n$  occupied by  $a$  and  $b$  in  $u$  and  $S_u(a)$  and  $S_u(b)$  are respectively the sections of  $\mathbb{R}^n$  seen by  $a$  and  $b$  in  $u$ . We will always assume that if  $a \neq b$  then  $x_u(a) \neq x_u(b)$ : distinct agents occupy distinct positions. An  $n$ -world  $u$  is said to be simple iff for all agents  $a$ , the  $n$ -scope  $u(a)$  is simple.

EXAMPLE 2.3

In the simple 2-world represented in Figure 2, the positions and the sections of three distinct agents are depicted.

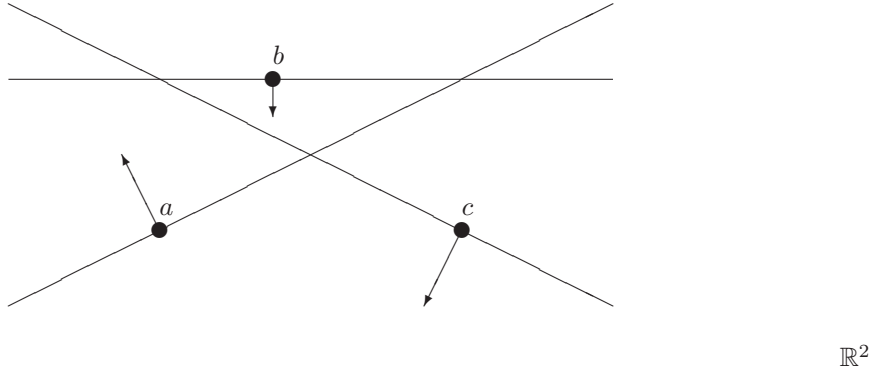


FIG. 2. A simple 2-world.

Given an  $n$ -world  $u$  and agents  $a, b$ , we shall say that  $a$  sees  $b$  in  $u$  iff

- $S_u(a)$  contains  $x_u(b)$ .

Remark that in simple  $n$ -worlds, no agent can see itself.

EXAMPLE 2.4

In the simple 2-world considered in Figure 2,  $a$  sees  $b$ ,  $a$  does not see  $c$ ,  $b$  sees  $a$ ,  $b$  sees  $c$ ,  $c$  sees  $a$  and  $c$  does not see  $b$ .

$n$ -worlds  $u$  and  $v$  are said to be indiscernible for agent  $a$ , in symbols  $u \equiv_a^n v$ , iff  $x_u(a) = x_v(a)$ ,  $S_u(a) = S_v(a)$  and for all agents  $b$ , if  $a \neq b$  then one of the following conditions holds:

- $a$  sees  $b$  in  $u$ ,  $a$  sees  $b$  in  $v$ ,  $x_u(b) = x_v(b)$  and  $S_u(b) = S_v(b)$ ,
- $a$  does not see  $b$  in  $u$  and  $a$  does not see  $b$  in  $v$ .

EXAMPLE 2.5

The simple 2-worlds considered in Figures 2 and 3 are indiscernible for agent  $a$ .

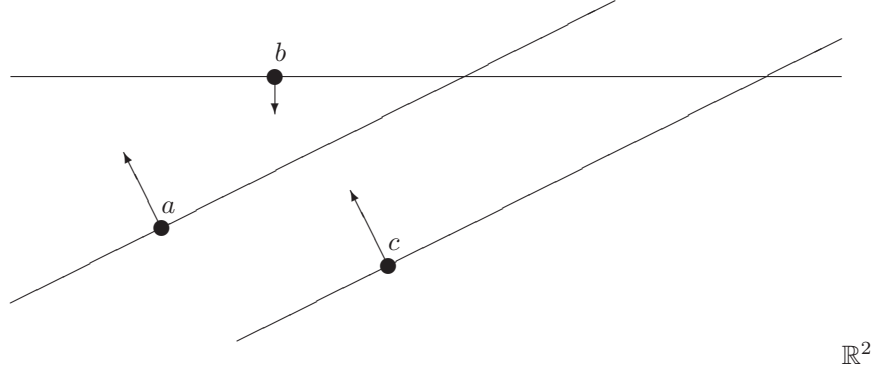


FIG. 3. A simple 2-world.

Remark that  $\equiv_a^n$  is an equivalence relation on the set of all  $n$ -worlds. It will be used to interpret the epistemic connective  $K_a$ .  $n$ -satisfaction is a 3-place relation  $\models^n$  between a nonempty set  $\mathcal{W}$  of  $n$ -worlds, an  $n$ -world  $u$  in  $\mathcal{W}$  and a formula  $\phi$ . It is inductively defined as follows:

- $\mathcal{W}, u \models^n a \triangleright b$  iff  $a$  sees  $b$  in  $u$ ,
- $\mathcal{W}, u \not\models^n \perp$ ,
- $\mathcal{W}, u \models^n \neg \phi$  iff  $\mathcal{W}, u \not\models^n \phi$ ,
- $\mathcal{W}, u \models^n \phi \vee \psi$  iff  $\mathcal{W}, u \models^n \phi$ , or  $\mathcal{W}, u \models^n \psi$ ,
- $\mathcal{W}, u \models^n K_a \phi$  iff for all  $n$ -worlds  $v$  in  $\mathcal{W}$ , if  $u \equiv_a^n v$  then  $\mathcal{W}, v \models^n \phi$ .

As a result,

- $\mathcal{W}, u \models^n a \triangleright b$  iff  $a$  does not see  $b$  in  $u$ ,
- $\mathcal{W}, u \models^n \hat{K}_a \phi$  iff there exists an  $n$ -world  $v$  in  $\mathcal{W}$  such that  $u \equiv_a^n v$  and  $\mathcal{W}, v \models^n \phi$ .

We shall say that a formula  $\phi$  is valid (invalid) in a nonempty set  $\mathcal{W}$  of  $n$ -worlds, in symbols  $\mathcal{W} \text{ val}^n \phi$  ( $\mathcal{W} \text{ inv}^n \phi$ ), iff for all  $n$ -worlds  $u$  in  $\mathcal{W}$ ,  $\mathcal{W}, u \models^n \phi$  ( $\mathcal{W}, u \not\models^n \phi$ ). A formula  $\phi$  is said to be satisfiable (falsifiable) in a nonempty set  $\mathcal{W}$  of  $n$ -worlds, in symbols  $\mathcal{W} \text{ sat}^n \phi$  ( $\mathcal{W} \text{ fal}^n \phi$ ), iff there exists an  $n$ -world  $u$  in  $\mathcal{W}$  such that  $\mathcal{W}, u \models^n \phi$  ( $\mathcal{W}, u \not\models^n \phi$ ). Let  $\mathcal{W}_s^n$  be the set of all simple  $n$ -worlds.

EXAMPLE 2.6

If  $u$  is the simple 2-world depicted in Figure 2 then  $\mathcal{W}_s^2, u \models^2 a \triangleright b$  and  $\mathcal{W}_s^2, u \models^2 K_a b \triangleright a$ . If  $v$  is the simple 2-world depicted in Figure 3 then  $\mathcal{W}_s^2, v \models^2 a \bar{\triangleright} c$  and  $\mathcal{W}_s^2, v \models^2 \hat{K}_a c \bar{\triangleright} a$ .

Unless otherwise stated, we will always assume that all  $n$ -worlds are simple.

### 2.3 Abstract semantics in dimension 1

Let  $\mathcal{T} = (T, <)$  be a linear order without endpoints. As we mentioned in the introduction, every agent occupies some position in the space and sees some section of the space. Hence, the notion of a  $\mathcal{T}$ -scope will be of the utmost interest for us. A  $\mathcal{T}$ -scope is a structure of the form  $(x, S)$  where  $x$  is an element of  $T$  and  $S$  is a subset of  $T$ .  $x$  and  $S$  are respectively called the position of the  $\mathcal{T}$ -scope  $(x, S)$  and the section of the  $\mathcal{T}$ -scope  $(x, S)$ . We shall say that a  $\mathcal{T}$ -scope  $(x, S)$  is simple iff  $S$  is an open semi-interval and  $x$  is on the frontier of  $S$ .

EXAMPLE 2.7

If one considers Figure 4 in  $\mathbb{R}$ , an  $\mathbb{R}$ -scope is defined by the point and the open half line located on the same side of the point as indicated by the arrow.

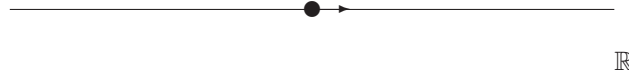


FIG. 4. A simple  $\mathbb{R}$ -scope.

We now provide a mechanism for interpreting our formulas in  $\mathcal{T}$ . A  $\mathcal{T}$ -world is a function  $u$  assigning to each agent  $a$  a  $\mathcal{T}$ -scope  $u(a)$ . Given a  $\mathcal{T}$ -world  $u$  and agents  $a, b$ , if  $u(a) = (x_u(a), S_u(a))$  and  $u(b) = (x_u(b), S_u(b))$  then  $x_u(a)$  and  $x_u(b)$  are respectively the positions in  $\mathcal{T}$  occupied by  $a$  and  $b$  in  $u$  and  $S_u(a)$  and  $S_u(b)$  are respectively the sections of  $\mathcal{T}$  seen by  $a$  and  $b$  in  $u$ . We will always assume that if  $a \neq b$  then  $x_u(a) \neq x_u(b)$ : distinct agents occupy distinct positions. A  $\mathcal{T}$ -world  $u$  is said to be simple iff for all agents  $a$ , the  $\mathcal{T}$ -scope  $u(a)$  is simple.

EXAMPLE 2.8

In the simple  $\mathbb{R}$ -world represented in Figure 5, the positions and the sections of three distinct agents are depicted.

Given a  $\mathcal{T}$ -world  $u$  and agents  $a, b$ , we shall say that  $a$  sees  $b$  in  $u$  iff  $S_u(a)$  contains  $x_u(b)$ . Remark that in simple  $\mathcal{T}$ -worlds, no agent can see itself.

EXAMPLE 2.9

In the simple  $\mathbb{R}$ -world considered in Figure 5,  $a$  sees  $b$ ,  $a$  does not see  $c$ ,  $b$  sees  $a$ ,  $b$  sees  $c$ ,  $c$  does not see  $a$  and  $c$  does not see  $b$ .

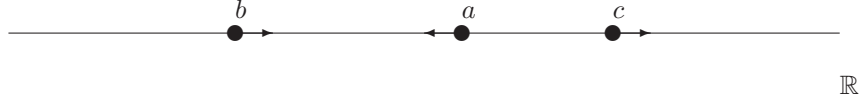


FIG. 5. A simple  $\mathbb{R}$ -world.

$\mathcal{T}$ -worlds  $u$  and  $v$  are said to be indiscernible for agent  $a$ , in symbols  $u \equiv_a^{\mathcal{T}} v$ , iff  $x_u(a) = x_v(a)$ ,  $S_u(a) = S_v(a)$  and for all agents  $b$ , if  $a \neq b$  then one of the following conditions holds:

- $a$  sees  $b$  in  $u$ ,  $a$  sees  $b$  in  $v$ ,  $x_u(b) = x_v(b)$  and  $S_u(b) = S_v(b)$ ,
- $a$  does not see  $b$  in  $u$  and  $a$  does not see  $b$  in  $v$ .

EXAMPLE 2.10

The simple  $\mathbb{R}$ -worlds considered in Figures 5 and 6 are indiscernible for agent  $a$ .

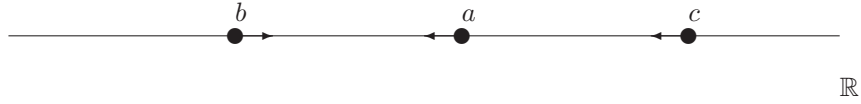


FIG. 6. A simple  $\mathbb{R}$ -world.

Remark that  $\equiv_a^{\mathcal{T}}$  is an equivalence relation on the set of all  $\mathcal{T}$ -worlds. It will be used to interpret the epistemic connective  $K_a$ .  $\mathcal{T}$ -satisfaction is a 3-place relation  $\models^{\mathcal{T}}$  between a nonempty set  $\mathcal{W}$  of  $\mathcal{T}$ -worlds, a  $\mathcal{T}$ -world  $u$  in  $\mathcal{W}$  and a formula  $\phi$ . It is inductively defined as follows:

- $\mathcal{W}, u \models^{\mathcal{T}} a \triangleright b$  iff  $a$  sees  $b$  in  $u$ ,
- $\mathcal{W}, u \not\models^{\mathcal{T}} \perp$ ,
- $\mathcal{W}, u \models^{\mathcal{T}} \neg\phi$  iff  $\mathcal{W}, u \not\models^{\mathcal{T}} \phi$ ,
- $\mathcal{W}, u \models^{\mathcal{T}} \phi \vee \psi$  iff  $\mathcal{W}, u \models^{\mathcal{T}} \phi$ , or  $\mathcal{W}, u \models^{\mathcal{T}} \psi$ ,
- $\mathcal{W}, u \models^{\mathcal{T}} K_a \phi$  iff for all  $\mathcal{T}$ -worlds  $v$  in  $\mathcal{W}$ , if  $u \equiv_a^{\mathcal{T}} v$  then  $\mathcal{W}, v \models^{\mathcal{T}} \phi$ .

As a result,

- $\mathcal{W}, u \models^{\mathcal{T}} a \triangleright b$  iff  $a$  does not see  $b$  in  $u$ ,
- $\mathcal{W}, u \models^{\mathcal{T}} \hat{K}_a \phi$  iff there exists a  $\mathcal{T}$ -world  $v$  in  $\mathcal{W}$  such that  $u \equiv_a^{\mathcal{T}} v$  and  $\mathcal{W}, v \models^{\mathcal{T}} \phi$ .

We shall say that a formula  $\phi$  is valid (invalid) in a nonempty set  $\mathcal{W}$  of  $\mathcal{T}$ -worlds, in symbols  $\mathcal{W} \text{ val}^{\mathcal{T}} \phi$  ( $\mathcal{W} \text{ inv}^{\mathcal{T}} \phi$ ), iff for all  $\mathcal{T}$ -worlds  $u$  in  $\mathcal{W}$ ,  $\mathcal{W}, u \models^{\mathcal{T}} \phi$  ( $\mathcal{W}, u \not\models^{\mathcal{T}} \phi$ ). A formula  $\phi$  is said to be satisfiable (falsifiable) in a nonempty set  $\mathcal{W}$  of  $\mathcal{T}$ -worlds, in symbols  $\mathcal{W} \text{ sat}^{\mathcal{T}} \phi$  ( $\mathcal{W} \text{ fal}^{\mathcal{T}} \phi$ ), iff there exists a  $\mathcal{T}$ -world  $u$  in  $\mathcal{W}$  such that  $\mathcal{W}, u \models^{\mathcal{T}} \phi$  ( $\mathcal{W}, u \not\models^{\mathcal{T}} \phi$ ). Let  $\mathcal{W}_s^{\mathcal{T}}$  be the set of all simple  $\mathcal{T}$ -worlds.

EXAMPLE 2.11

If  $u$  is the simple  $\mathbb{R}$ -world depicted in Figure 5 then  $\mathcal{W}_s^{\mathbb{R}}, u \models^{\mathbb{R}} a \triangleright b$  and  $\mathcal{W}_s^{\mathbb{R}}, u \models^{\mathbb{R}} K_a b \triangleright a$ .

If  $v$  is the simple  $\mathbb{R}$ -world depicted in Figure 6 then  $\mathcal{W}_s^{\mathbb{R}}, v \models^{\mathbb{R}} a \bar{\triangleright} c$  and  $\mathcal{W}_s^{\mathbb{R}}, v \models^{\mathbb{R}} \hat{K}_a c \bar{\triangleright} a$ .

Unless otherwise stated, we will always assume that all  $\mathcal{T}$ -worlds are simple.

### 3 Expressivity

$AGT$  being a countable set of agents, the fact that distinct agents occupy distinct positions imply that our multiagent logic forces its models to be infinite. In this section, we discuss some results concerning the expressive power of our multiagent logic.

#### 3.1 In dimension 1

Let  $\mathcal{T} = (T, <)$  be a linear order without endpoints. Our language cannot distinguish between the notion of satisfiability in  $\mathcal{W}_s^1$  and the notion of satisfiability in  $\mathcal{W}_s^{\mathcal{T}}$ . In order to see why, let us consider an enumeration  $a_1, a_2, \dots$  of  $AGT$ . Given a nonnegative integer  $k$ , we shall say that a binary relation  $Z$  between  $\mathcal{W}_s^1$  and  $\mathcal{W}_s^{\mathcal{T}}$  is a  $k$ -bisimulation between  $\mathcal{W}_s^1$  and  $\mathcal{W}_s^{\mathcal{T}}$  iff for all simple 1-worlds  $u^{(1)}$  and for all simple  $\mathcal{T}$ -worlds  $u^{(\mathcal{T})}$ , if  $u^{(1)} Z u^{(\mathcal{T})}$  then the following conditions are satisfied:

1. For all positive integers  $i, j \leq k$ , if  $a_i$  sees  $a_j$  in  $u^{(1)}$  then  $a_i$  sees  $a_j$  in  $u^{(\mathcal{T})}$ .
2. For all positive integers  $i, j \leq k$ , if  $a_i$  sees  $a_j$  in  $u^{(\mathcal{T})}$  then  $a_i$  sees  $a_j$  in  $u^{(1)}$ .
3. For all positive integers  $i \leq k$  and for all simple 1-worlds  $v^{(1)}$ , if  $u^{(1)} \equiv_{a_i}^1 v^{(1)}$  then there exists a simple  $\mathcal{T}$ -world  $v^{(\mathcal{T})}$  such that  $u^{(\mathcal{T})} \equiv_{a_i}^{\mathcal{T}} v^{(\mathcal{T})}$  and  $v^{(1)} Z v^{(\mathcal{T})}$ .
4. For all positive integers  $i \leq k$  and for all simple  $\mathcal{T}$ -worlds  $v^{(\mathcal{T})}$ , if  $u^{(\mathcal{T})} \equiv_{a_i}^{\mathcal{T}} v^{(\mathcal{T})}$  then there exists a simple 1-world  $v^{(1)}$  such that  $u^{(1)} \equiv_{a_i}^1 v^{(1)}$  and  $v^{(1)} Z v^{(\mathcal{T})}$ .

Interestingly,

LEMMA 3.1

Let  $\phi(a_1, \dots, a_k)$  be a formula. Let  $Z$  be a  $k$ -bisimulation between  $\mathcal{W}_s^1$  and  $\mathcal{W}_s^{\mathcal{T}}$ . For all simple 1-worlds  $u^{(1)}$  and for all simple  $\mathcal{T}$ -worlds  $u^{(\mathcal{T})}$ , if  $u^{(1)} Z u^{(\mathcal{T})}$  then  $\mathcal{W}_s^1, u^{(1)} \models^1 \phi(a_1, \dots, a_k)$  iff  $\mathcal{W}_s^{\mathcal{T}}, u^{(\mathcal{T})} \models^{\mathcal{T}} \phi(a_1, \dots, a_k)$ .

Given a nonnegative integer  $k$ , let  $Z_k$  be the binary relation between  $\mathcal{W}_s^1$  and  $\mathcal{W}_s^{\mathcal{T}}$  such that for all simple 1-worlds  $u^{(1)}$  and for all simple  $\mathcal{T}$ -worlds  $u^{(\mathcal{T})}$ ,  $u^{(1)} Z_k u^{(\mathcal{T})}$  iff for all positive integers  $i, j \leq k$ ,  $a_i$  sees  $a_j$  in  $u^{(1)}$  iff  $a_i$  sees  $a_j$  in  $u^{(\mathcal{T})}$ . Let  $Z_k^*$  be the restriction to  $\mathcal{W}_s^1 \cup \mathcal{W}_s^{\mathcal{T}}$  of the least equivalence relation on  $\mathcal{W}_s^1 \cup \mathcal{W}_s^{\mathcal{T}}$  containing  $Z_k$ .

EXAMPLE 3.2

The simple  $\mathbb{R}$ -worlds considered in Figures 7, 8 and 9 are in the binary relation  $Z_3^*$ .

We have:

LEMMA 3.3

$Z_k$  is a  $k$ -bisimulation between  $\mathcal{W}_s^1$  and  $\mathcal{W}_s^{\mathcal{T}}$ .

Lemmas 3.1 and 3.3 establish the following

PROPOSITION 3.4

Let  $\phi$  be a formula. The following conditions are equivalent:

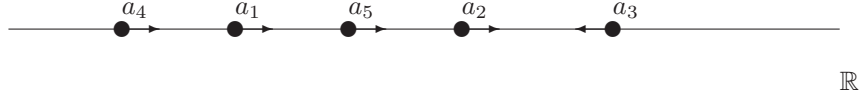


FIG. 7. A simple  $\mathbb{R}$ -world.

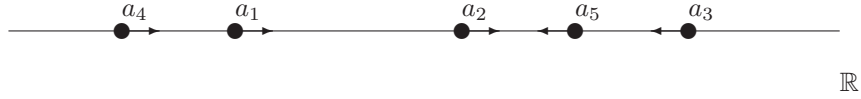


FIG. 8. A simple  $\mathbb{R}$ -world.

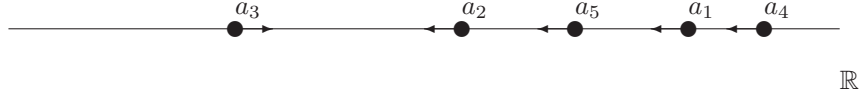


FIG. 9. A simple  $\mathbb{R}$ -world.

1.  $\mathcal{W}_s^1 \text{ sat}^1 \phi$ .
2.  $\mathcal{W}_s^T \text{ sat}^T \phi$ .

PROOF. Let  $k$  be a nonnegative integer such that  $\phi$  is a formula whose agents form a sublist of  $a_1, \dots, a_k$ .

1. $\Rightarrow$ 2. Suppose  $\mathcal{W}_s^1 \text{ sat}^1 \phi(a_1, \dots, a_k)$ . Hence, then there exists a simple 1-world  $u^{(1)}$  such that  $\mathcal{W}_s^1, u^{(1)} \models^1 \phi(a_1, \dots, a_k)$ . Since  $\mathcal{T}$  is a linear order without endpoints, then there exists a simple  $\mathcal{T}$ -world  $u^{(\mathcal{T})}$  such that the temporal relationships between  $a_1, \dots, a_k$  in  $u^{(1)}$  are equal to the temporal relationships between  $a_1, \dots, a_k$  in  $u^{(\mathcal{T})}$ . Obviously,  $u^{(1)} \models^1 Z_k u^{(\mathcal{T})}$ . Since  $\mathcal{W}_s^1, u^{(1)} \models^1 \phi(a_1, \dots, a_k)$ , then by Lemmas 3.1 and 3.3,  $\mathcal{W}_s^T, u^{(\mathcal{T})} \models^T \phi(a_1, \dots, a_k)$ . Thus,  $\mathcal{W}_s^T \text{ sat}^T \phi(a_1, \dots, a_k)$ .

2. $\Rightarrow$ 1. Suppose  $\mathcal{W}_s^T \text{ sat}^T \phi(a_1, \dots, a_k)$ . Hence, then there exists a simple  $\mathcal{T}$ -world  $u^{(\mathcal{T})}$  such that  $\mathcal{W}_s^T, u^{(\mathcal{T})} \models^T \phi(a_1, \dots, a_k)$ . Since  $\mathbb{R}$  is a linear order without endpoints, then there exists a simple 1-world  $u^{(1)}$  such that the temporal relationships between  $a_1, \dots, a_k$  in  $u^{(1)}$  are equal to the temporal relationships between  $a_1, \dots, a_k$  in  $u^{(\mathcal{T})}$ . Obviously,  $u^{(1)} \models^1 Z_k u^{(\mathcal{T})}$ . Since  $\mathcal{W}_s^T, u^{(\mathcal{T})} \models^T \phi(a_1, \dots, a_k)$ , then by Lemmas 3.1 and 3.3,  $\mathcal{W}_s^1, u^{(1)} \models^1 \phi(a_1, \dots, a_k)$ . Thus,  $\mathcal{W}_s^1 \text{ sat}^1 \phi(a_1, \dots, a_k)$ .  $\blacksquare$

Proposition 3.4 has the following immediate consequence.

PROPOSITION 3.5

Let  $\phi$  be a formula. The following conditions are equivalent:

1.  $\mathcal{W}_s^1 \text{ val}^1 \phi$ .
2. There exists a linear order  $\mathcal{T}$  without endpoints such that  $\mathcal{W}_s^{\mathcal{T}} \text{ val}^{\mathcal{T}} \phi$ .
3. For all linear orders  $\mathcal{T}$  without endpoints,  $\mathcal{W}_s^{\mathcal{T}} \text{ val}^{\mathcal{T}} \phi$ .

PROOF.  $2 \Rightarrow 1$ . By Proposition 3.4.

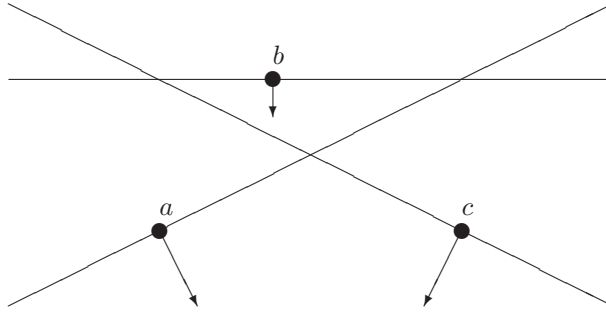
$1 \Rightarrow 3$ . By Proposition 3.4.

$3 \Rightarrow 2$ . Obvious. ■

### 3.2 In dimensions $n \geq 2$

Let  $n \geq 2$ . Our language can distinguish between the notion of satisfiability in  $\mathcal{W}_s^1$  and the notion of satisfiability in  $\mathcal{W}_s^n$ . To illustrate the truth of this, let us consider the four following examples. As a first example, take the formula

- $((a \triangleright b \leftrightarrow b \bar{\triangleright} a) \leftrightarrow (b \triangleright c \leftrightarrow c \bar{\triangleright} b)) \rightarrow (a \triangleright c \leftrightarrow c \bar{\triangleright} a)$ .



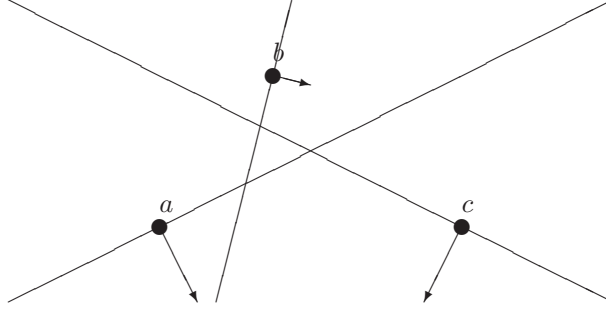
$\mathbb{R}^2$

FIG. 10. A simple 2-world.

Its validity in  $\mathcal{W}_s^1$  follows from the fact that if  $a$  and  $b$  look into the same direction iff  $b$  and  $c$  look into the same direction then  $a$  and  $c$  look into the same direction. Its falsifiability in  $\mathcal{W}_s^2$  is illustrated by Figure 10 where a simple 2-world satisfying  $a \triangleright b \leftrightarrow b \bar{\triangleright} a$  and  $b \triangleright c \leftrightarrow c \bar{\triangleright} b$  — hence, satisfying  $(a \triangleright b \leftrightarrow b \bar{\triangleright} a) \leftrightarrow (b \triangleright c \leftrightarrow c \bar{\triangleright} b)$  — and falsifying  $a \triangleright c \leftrightarrow c \bar{\triangleright} a$  is presented. As a second example, take the formula

- $\neg(a \triangleright b \leftrightarrow a \bar{\triangleright} c) \vee \neg(b \triangleright c \leftrightarrow b \bar{\triangleright} a)$ .

Its validity in  $\mathcal{W}_s^1$  follows from the fact that  $a$  is not between  $b$  and  $c$ , or  $b$  is not between  $c$  and  $a$ . Its falsifiability in  $\mathcal{W}_s^2$  is illustrated by Figure 11 where a simple 2-world satisfying  $a \triangleright b \leftrightarrow a \bar{\triangleright} c$  and  $b \triangleright c \leftrightarrow b \bar{\triangleright} a$  is presented. As a third example, take the formula



$\mathbb{R}^2$

FIG. 11. A simple 2-world.

- $c \bar{\triangleright} a \wedge c \bar{\triangleright} b \rightarrow \hat{K}_c(((a \triangleright b \leftrightarrow b \bar{\triangleright} a) \leftrightarrow (b \triangleright c \leftrightarrow c \bar{\triangleright} b)) \wedge \neg(a \triangleright c \leftrightarrow c \bar{\triangleright} a))$ .

Its falsifiability in  $\mathcal{W}_s^1$  is illustrated by Figure 5 where a simple 1-world satisfying  $c \bar{\triangleright} a \wedge c \bar{\triangleright} b$  and falsifying  $\hat{K}_c(((a \triangleright b \leftrightarrow b \bar{\triangleright} a) \leftrightarrow (b \triangleright c \leftrightarrow c \bar{\triangleright} b)) \wedge \neg(a \triangleright c \leftrightarrow c \bar{\triangleright} a))$  is presented. Its validity

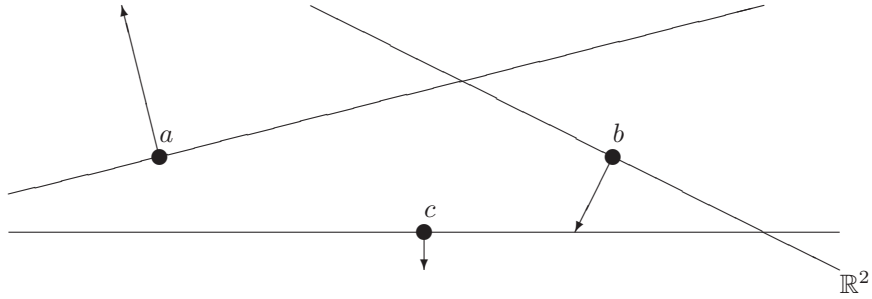


FIG. 12. A simple 2-world.

in  $\mathcal{W}_s^2$  follows from the fact that if  $c$  sees neither  $a$  nor  $b$  then  $c$  can imagine the simple 2-world illustrated by Figure 12 where  $((a \triangleright b \leftrightarrow b \bar{\triangleright} a) \leftrightarrow (b \triangleright c \leftrightarrow c \bar{\triangleright} b)) \wedge \neg(a \triangleright c \leftrightarrow c \bar{\triangleright} a)$  holds. As a fourth example, take the formula

- $c \bar{\triangleright} a \wedge c \bar{\triangleright} b \rightarrow \hat{K}_c((a \triangleright b \leftrightarrow a \bar{\triangleright} c) \wedge (b \triangleright c \leftrightarrow b \bar{\triangleright} a))$ .

Its falsifiability in  $\mathcal{W}_s^1$  is illustrated by Figure 5 where a simple 1-world satisfying  $c \bar{\triangleright} a \wedge c \bar{\triangleright} b$  and falsifying  $\hat{K}_c((a \triangleright b \leftrightarrow a \bar{\triangleright} c) \wedge (b \triangleright c \leftrightarrow b \bar{\triangleright} a))$  is presented. Its validity in  $\mathcal{W}_s^2$  follows

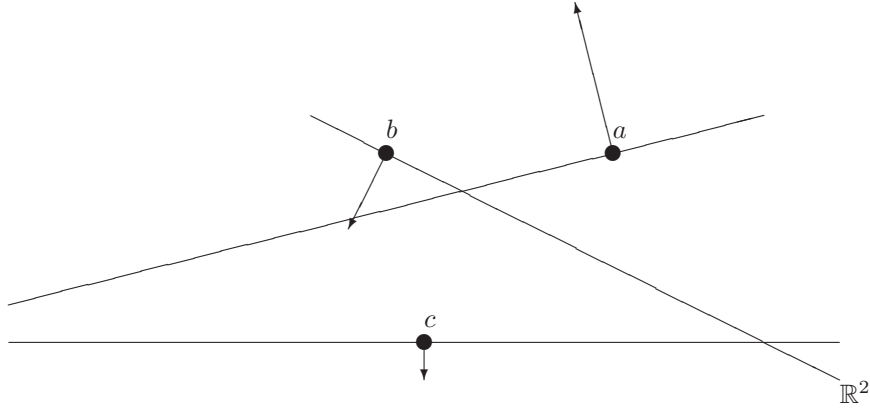


FIG. 13. A simple 2-world.

from the fact that if  $c$  sees neither  $a$  nor  $b$  then  $c$  can imagine the simple 2-world illustrated by Figure 13 where  $(a \triangleright b \leftrightarrow a \bar{\triangleright} c) \wedge (b \triangleright c \leftrightarrow b \bar{\triangleright} a)$  holds. Finally, for all we know, it is still open whether our language can distinguish between the notion of satisfiability in  $\mathcal{W}_s^n$  and the notion of satisfiability in  $\mathcal{W}_s^{n+1}$ .

## 4 Axiomatization and completeness

In this section, our goal is to provide the axiomatization of our multiagent logic in dimension 1. Such an axiomatization will enable us to underline the most representative properties of sentences like “ $a$  knows that  $b$  sees  $c$ ” when interpreted in  $\mathbb{R}$ , or in linear orders without endpoints.

### 4.1 Axiomatization

Let  $G$  be a finite group of agents. A  $G$ -vector is a pair  $\vec{u} = (pos_{\vec{u}}, sec_{\vec{u}})$  where

- $pos_{\vec{u}}$  is a function assigning to each agent  $a \in G$  a positive integer  $pos_{\vec{u}}(a) \leq Card(G)$ ,
- $sec_{\vec{u}}$  is a function assigning to each agent  $a \in G$  an element  $sec_{\vec{u}}(a) \in \{r, l\}$ .

We will always assume that  $pos_{\vec{u}}$  is injective.

#### EXAMPLE 4.1

If  $G = \{a_1, a_2, a_3\}$  then the pair  $\vec{u} = (pos_{\vec{u}}, sec_{\vec{u}})$  defined by  $pos_{\vec{u}}(a_1) = 2$ ,  $pos_{\vec{u}}(a_2) = 1$ ,  $pos_{\vec{u}}(a_3) = 3$ ,  $sec_{\vec{u}}(a_1) = l$ ,  $sec_{\vec{u}}(a_2) = r$  and  $sec_{\vec{u}}(a_3) = l$  is a  $G$ -vector. It corresponds to a situation where  $a_1$  occupies the 2nd position and looks at its left,  $a_2$  occupies the 1st position and looks at its right and  $a_3$  occupies the 3rd position and looks at its left.

Given a  $G$ -vector  $\vec{u}$  and agents  $a, b \in G$ ,  $a$  is said to be seeing  $b$  in  $\vec{u}$  iff one of the following conditions holds:

- $pos_{\vec{u}}(a) < pos_{\vec{u}}(b)$  and  $sec_{\vec{u}}(a) = r$ ,

- $pos_{\vec{u}}(b) < pos_{\vec{u}}(a)$  and  $sec_{\vec{u}}(a) = l$ .

Remark that in  $G$ -vectors, no agent can see itself.

#### EXAMPLE 4.2

In the  $G$ -vector  $\vec{u}$  considered in Example 4.1,  $a_1$  sees  $a_2$ ,  $a_1$  does not see  $a_3$ ,  $a_2$  sees  $a_1$ ,  $a_2$  sees  $a_3$ ,  $a_3$  sees  $a_1$  and  $a_3$  sees  $a_2$ .

We now associate to each  $G$ -vector  $\vec{u}$  the conjunction  $\chi_{\vec{u}}$  of the following literals based on  $G$ :

- for all distinct agents  $a, b \in G$  such that  $a$  sees  $b$  in  $\vec{u}$ , the literal  $a \triangleright b$ ,
- for all distinct agents  $a, b \in G$  such that  $a$  does not see  $b$  in  $\vec{u}$ , the literal  $a \bar{\triangleright} b$ .

#### EXAMPLE 4.3

The conjunction  $\chi_{\vec{u}}$  associated to the  $G$ -vector  $\vec{u}$  considered in Example 4.1 is  $a_1 \triangleright a_2 \wedge a_1 \bar{\triangleright} a_3 \wedge a_2 \triangleright a_1 \wedge a_2 \triangleright a_3 \wedge a_3 \triangleright a_1 \wedge a_3 \triangleright a_2$ .

Let  $a \in G$ . We shall say that the  $G$ -vector  $\vec{v}$  is  $a$ -compatible with the  $G$ -vector  $\vec{u}$ , in symbols  $\vec{u} \equiv_a^G \vec{v}$ , iff for all  $b \in G$ , if  $a \neq b$  then one of the following conditions holds:

- $a$  sees  $b$  in  $\vec{u}$ ,  $a$  sees  $b$  in  $\vec{v}$ ,  $pos_{\vec{u}}(b) = pos_{\vec{v}}(b)$  and  $sec_{\vec{u}}(b) = sec_{\vec{v}}(b)$ ,
- $a$  does not see  $b$  in  $\vec{u}$  and  $a$  does not see  $b$  in  $\vec{v}$ .

#### EXAMPLE 4.4

The  $G$ -vector  $\vec{v} = (pos_{\vec{v}}, sec_{\vec{v}})$  defined by  $pos_{\vec{v}}(a_1) = 2$ ,  $pos_{\vec{v}}(a_2) = 1$ ,  $pos_{\vec{v}}(a_3) = 3$ ,  $sec_{\vec{v}}(a_1) = l$ ,  $sec_{\vec{v}}(a_2) = r$  and  $sec_{\vec{v}}(a_3) = r$  is  $a_1$ -compatible with the  $G$ -vector  $\vec{u}$  considered in Example 4.1.

Remark that  $\equiv_a^G$  is an equivalence relation on the set of all  $G$ -vectors. It will be used to provide one of the proper axioms of our multiagent logic. To continue, another technical lemma is necessary.

#### LEMMA 4.5

The following decision problem is decidable by a deterministic Turing machine in logarithmic space:

**Input:** a finite group  $G$  of agents, an agent  $a \in G$  and  $G$ -vectors  $\vec{u}$  and  $\vec{v}$ ,

**Output:** determine whether  $\vec{u} \equiv_a^G \vec{v}$ .

We shall say that a set  $L$  of formulas in our language is a logic iff  $L$  contains all propositional tautologies and  $L$  is closed under modus ponens (i.e. if  $\phi \in L$  and  $\phi \rightarrow \psi \in L$  then  $\psi \in L$ ). A logic  $L$  is said to be normal iff  $L$  contains all formulas of the form  $K_a(\phi \rightarrow \psi) \rightarrow (K_a\phi \rightarrow K_a\psi)$  and  $L$  is closed under generalization (i.e. if  $\phi \in L$  then  $K_a\phi \in L$ ). Let  $L_{min}$  be the least normal logic in our language that contains the following formulas as proper axioms:

$$Ax_1: ((a \triangleright b \leftrightarrow b \bar{\triangleright} a) \leftrightarrow (b \triangleright c \leftrightarrow c \bar{\triangleright} b)) \rightarrow (a \triangleright c \leftrightarrow c \bar{\triangleright} a),$$

$$Ax_2: \neg(a \triangleright b \leftrightarrow a \bar{\triangleright} c) \vee \neg(b \triangleright c \leftrightarrow b \bar{\triangleright} a),$$

$$Ax_3: K_a a \triangleright b \vee K_a a \bar{\triangleright} b,$$

$$Ax_4: a \triangleright b \rightarrow K_a b \triangleright c \vee K_a b \bar{\triangleright} c,$$

$$Ax_5: \chi_{\vec{u}} \rightarrow \hat{K}_a \chi_{\vec{v}} \text{ where } G \text{ is a finite group of agents, } a \in G \text{ is an agent and } \vec{u} \text{ and } \vec{v} \text{ are } G\text{-vectors such that } \vec{u} \equiv_a^G \vec{v},$$

$Ax_6: K_a\phi \rightarrow \phi$ .

Let us remind that in atomic sentences like  $a \triangleright b$ ,  $a, b$  are distinct. It follows that in dimension 1,  $a \triangleright b \leftrightarrow b \triangleright a$  and  $a \triangleright b \leftrightarrow a \triangleright c$  can be read as “ $a$  and  $b$  look into the same direction” and “ $a$  is between  $b$  and  $c$ ”. There are several points worth making about our proper axioms:  $Ax_1$  says that if  $a$  and  $b$  look into the same direction iff  $b$  and  $c$  look into the same direction then  $a$  and  $c$  look into the same direction;  $Ax_2$  says that  $a$  is not between  $b$  and  $c$ , or  $b$  is not between  $c$  and  $a$ ;  $Ax_3$  says that  $a$  knows whether it looks at  $b$ ;  $Ax_4$  says that if  $a$  looks at  $b$  then  $a$  knows whether  $b$  looks at  $c$ ;  $Ax_5$  says that what is compatible with what  $a$  currently sees is also compatible with what  $a$  currently knows;  $Ax_6$  says that what  $a$  knows is true. Note that we have no need, as proper axioms, of the traditional formulas of positive introspection ( $K_a\phi \rightarrow K_aK_a\phi$ ) and negative introspection ( $\neg K_a\phi \rightarrow K_a\neg K_a\phi$ ). Seeing that these formulas are valid in  $\mathcal{W}_s^1$ , the completeness result described below in Proposition 4.16 implies that they are derivable from the proper axioms.

## 4.2 Completeness

We show first that

PROPOSITION 4.6

Let  $\phi$  be a formula. If  $\phi$  is in  $L_{min}$  then  $\mathcal{W}_s^1 \text{ val}^1 \phi$ .

PROOF. It is readily seen that  $Ax_1$ – $Ax_6$  are valid in  $\mathcal{W}_s^1$ . ■

Slightly less trivial is the following

PROPOSITION 4.7

Let  $\phi$  be a formula. If for all linear orders  $\mathcal{T}$  without endpoints,  $\mathcal{W}_s^{\mathcal{T}} \text{ val}^{\mathcal{T}} \phi$  then  $\phi$  is in  $L_{min}$ .

Proposition 4.7 will be proved by a construction similar to the canonical model construction. Let  $n$  be a positive integer. As usual, any  $L_{min}$ -consistent set of formulas can be extended to an  $n$ -maximal  $L_{min}$ -consistent set of formulas.

LEMMA 4.8

If  $\Sigma$  is an  $L_{min}$ -consistent set of formulas then there exists an  $n$ -maximal  $L_{min}$ -consistent set  $\Delta$  of formulas such that  $\Sigma \subseteq \Delta$ .

Let  $\mathbb{Z}^*$  be the set of all non-zero integers. We will interpret  $n$ -formulas in the linear order  $\mathcal{T} = (T, \prec)$  defined as follows:

- $T = \mathbb{Z}^* \cup \{(0, k) : k \leq n \text{ is a positive integer}\}$ ,
- for all  $\alpha, \beta \in T$ ,  $\alpha \prec \beta$  iff one of the following conditions hold:
  - $\alpha, \beta \in \mathbb{Z}^*$  and  $\alpha < \beta$ ,
  - $\alpha \in \mathbb{Z}^*$ , there exists a positive integer  $j \leq n$  such that  $\beta = (0, j)$  and  $\alpha < 0$ ,
  - there exists a positive integer  $i \leq n$  such that  $\alpha = (0, i)$ ,  $\beta \in \mathbb{Z}^*$  and  $0 < \beta$ ,
  - there exists positive integers  $i, j \leq n$  such that  $\alpha = (0, i)$ ,  $\beta = (0, j)$  and  $i < j$ .

It follows immediately from the definition that

FACT 4.9

$\mathcal{T}$  is a linear order without endpoints.

Let  $\Sigma$  be an  $n$ -maximal  $L_{min}$ -consistent set of formulas. For all positive integers  $k \leq n$ ,  $k$  is said to be  $\Sigma$ -right iff  $k = 1$ , or  $k \neq 1$  and  $(a_1 \triangleright a_k \leftrightarrow a_k \bar{\triangleright} a_1) \in \Sigma$  and  $k$  is said to be  $\Sigma$ -left iff  $k \neq 1$  and  $(a_1 \triangleright a_k \leftrightarrow a_k \triangleright a_1) \in \Sigma$ . The reader may easily verify that *to be  $\Sigma$ -right* and *to be  $\Sigma$ -left* are complementary properties of a positive integer  $k \leq n$ . For all positive integers  $i, j \leq n$ , we shall say that  $i$   $\Sigma$ -precedes  $j$  iff  $i \neq j$  and one of the following conditions holds:

- $i$  is  $\Sigma$ -right and  $a_i \triangleright a_j \in \Sigma$ ,
- $j$  is  $\Sigma$ -left and  $a_j \triangleright a_i \in \Sigma$ ,
- $i$  is  $\Sigma$ -left,  $j$  is  $\Sigma$ -right,  $a_i \bar{\triangleright} a_j \in \Sigma$  and  $a_j \bar{\triangleright} a_i \in \Sigma$ .

It is worth noting at this point the following

FACT 4.10

Let  $i, j, k \leq n$  be positive integers.

1.  $i$  does not  $\Sigma$ -precede  $i$ .
2. If  $i$   $\Sigma$ -precedes  $j$  and  $j$   $\Sigma$ -precedes  $k$  then  $i$   $\Sigma$ -precedes  $k$ .
3. If  $i \neq j$  then  $i$   $\Sigma$ -precedes  $j$ , or  $j$   $\Sigma$ -precedes  $i$ .

As a result, there exists permutations  $\pi_\Sigma^r, \pi_\Sigma^l$  of the set of all positive integers  $k \leq n$  such that for all positive integers  $i, j \leq n$ ,

- $\pi_\Sigma^r(i) < \pi_\Sigma^r(j)$  iff  $i$   $\Sigma$ -precedes  $j$ ,
- $\pi_\Sigma^l(i) < \pi_\Sigma^l(j)$  iff  $j$   $\Sigma$ -precedes  $i$ .

Obviously, for all positive integers  $k \leq n$ ,  $\pi_\Sigma^r(k) + \pi_\Sigma^l(k) = n + 1$ . We now wish to show that there exists a simple  $\mathcal{T}$ -world  $u$  such that for all formulas  $\phi(a_1, \dots, a_n)$ , if  $\phi(a_1, \dots, a_n) \in \Sigma$  then  $\mathcal{W}_s^T, u \models^T \phi(a_1, \dots, a_n)$ . Let  $u_\Sigma^r, u_\Sigma^l$  be simple  $\mathcal{T}$ -worlds such that for all positive integers  $k \leq n$ ,

- $x_{u_\Sigma^r}(a_k) = (0, \pi_\Sigma^r(k))$ ,
- if  $k$  is  $\Sigma$ -right then  $S_{u_\Sigma^r}(a_k) = ](0, \pi_\Sigma^r(k)), +\infty[$ ,
- if  $k$  is  $\Sigma$ -left then  $S_{u_\Sigma^r}(a_k) = ]-\infty, (0, \pi_\Sigma^r(k))]$ ,
- $x_{u_\Sigma^l}(a_k) = (0, \pi_\Sigma^l(k))$ ,
- if  $k$  is  $\Sigma$ -right then  $S_{u_\Sigma^l}(a_k) = ]-\infty, (0, \pi_\Sigma^l(k))]$ ,
- if  $k$  is  $\Sigma$ -left then  $S_{u_\Sigma^l}(a_k) = ](0, \pi_\Sigma^l(k)), +\infty[$ .

Remind that  $Z_n^*$  is the restriction to  $\mathcal{W}_s^T$  of the least equivalence relation on  $\mathcal{W}_s^1 \cup \mathcal{W}_s^T$  containing  $Z_n$ . It follows immediately from the definition that

FACT 4.11

$u_\Sigma^r Z_n^* u_\Sigma^l$ .

Not surprisingly, we have

FACT 4.12

For all positive integers  $i, j \leq n$ , if  $i \neq j$  then the following conditions are equivalent:

1.  $\mathcal{W}_s^T, u_\Sigma^r \models^T a_i \triangleright a_j$ .
2.  $\mathcal{W}_s^T, u_\Sigma^l \models^T a_i \triangleright a_j$ .

3.  $a_i \triangleright a_j \in \Sigma$ .

For all positive integers  $i \leq n$ , let  $\equiv_{a_i}^{\mathcal{T}}$  be the indiscernibility relation between  $\mathcal{T}$ -worlds defined as in Section 2.3. The following fact is basic.

FACT 4.13

Let  $\Delta$  be an  $n$ -maximal consistent set of formulas. For all positive integers  $i \leq n$ , the following conditions are equivalent:

1.  $u_{\Sigma}^r \equiv_{a_i}^{\mathcal{T}} u_{\Delta}^r$ , or  $u_{\Sigma}^r \equiv_{a_i}^{\mathcal{T}} u_{\Delta}^l$ .
2.  $u_{\Sigma}^l \equiv_{a_i}^{\mathcal{T}} u_{\Delta}^r$ , or  $u_{\Sigma}^l \equiv_{a_i}^{\mathcal{T}} u_{\Delta}^l$ .

An important further result is

FACT 4.14

Let  $\Delta$  be an  $n$ -maximal consistent set of formulas. For all positive integers  $i \leq n$ , if  $K_{a_i} \Sigma \subseteq \Delta$  then

- $u_{\Sigma}^r \equiv_{a_i}^{\mathcal{T}} u_{\Delta}^r$ , or  $u_{\Sigma}^r \equiv_{a_i}^{\mathcal{T}} u_{\Delta}^l$ ,
- $u_{\Sigma}^l \equiv_{a_i}^{\mathcal{T}} u_{\Delta}^r$ , or  $u_{\Sigma}^l \equiv_{a_i}^{\mathcal{T}} u_{\Delta}^l$ .

With this established, the rest is easy.

FACT 4.15

Let  $\psi(a_1, \dots, a_n)$  be a formula. The following conditions are equivalent:

1.  $\mathcal{W}_s^{\mathcal{T}}, u_{\Sigma}^r \models^{\mathcal{T}} \psi(a_1, \dots, a_n)$ .
2.  $\mathcal{W}_s^{\mathcal{T}}, u_{\Sigma}^l \models^{\mathcal{T}} \psi(a_1, \dots, a_n)$ .
3.  $\psi(a_1, \dots, a_n) \in \Sigma$ .

The proof of Proposition 4.7 can now be done as follows.

PROOF OF PROPOSITION 4.7. Let  $n$  be a nonnegative integer such that  $\phi$  is a formula whose agents form a sublist of  $a_1, \dots, a_n$ . Suppose  $\phi(a_1, \dots, a_n)$  is an  $L_{min}$ -consistent formula. Hence, by Lemma 4.8, there exists an  $n$ -maximal  $L_{min}$ -consistent set  $\Sigma$  of formulas such that  $\phi(a_1, \dots, a_n) \in \Sigma$ . Thus, by Fact 4.15,  $\mathcal{W}_s^{\mathcal{T}}, u_{\Sigma} \models^{\mathcal{T}} \phi(a_1, \dots, a_n)$ . ■

As a result,

PROPOSITION 4.16

Let  $\phi$  be a formula. The following conditions are equivalent:

1.  $\phi$  is in  $L_{min}$ .
2.  $\mathcal{W}_s^1 \text{ val}^1 \phi$ .
3. There exists a linear order  $\mathcal{T}$  without endpoints such that  $\mathcal{W}_s^{\mathcal{T}} \text{ val}^{\mathcal{T}} \phi$ .
4. For all linear orders  $\mathcal{T}$  without endpoints,  $\mathcal{W}_s^{\mathcal{T}} \text{ val}^{\mathcal{T}} \phi$ .

PROOF. By Proposition 3.5, it suffices to prove that 1. $\Rightarrow$ 2. and 4. $\Rightarrow$ 1.

1. $\Rightarrow$ 2. By Proposition 4.6.

4. $\Rightarrow$ 1. By Proposition 4.7. ■

## 5 Decidability and complexity of model checking problems

In this section, for some positive integers  $k$ , we investigate the decidability and complexity of the model checking problem with respect to  $\mathcal{W}_s^k$ : given a  $k$ -world  $u$  in  $\mathcal{W}_s^k$  and a formula  $\phi$ , determine whether  $\mathcal{W}_s^k, u \models^k \phi$ . The results obtained are summarized as follows: *PSPACE*-complete when  $k = 1$  and *PSPACE*-hard and in *EXSPACE* when  $k = 2$ . Let us remind that in the more traditional epistemic logics considered in [11, 12], for example, model checking problems are usually decidable in deterministic polynomial time.

### 5.1 In dimension 1

First, we prove the following

LEMMA 5.1

The validity problem of quantified Boolean logic is reducible to the model checking problem with respect to  $\mathcal{W}_s^1$ .

As a result,

PROPOSITION 5.2

The model checking problem with respect to  $\mathcal{W}_s^1$  is *PSPACE*-complete.

PROOF. *PSPACE*-hardness follows from Stockmeyer [22] and Lemma 5.1. Membership in *PSPACE* follows from Chandra *et al.* [7], Lemmas 3.1 and 3.3 and the fact that the following alternating algorithm (where without loss of generality, we assume that  $(u(a_1), \dots, u(a_n))$  is coded as a  $G$ -vector with  $G = \{a_1, \dots, a_n\}$ ) decides in polynomial time the model checking problem with respect to  $\mathcal{W}_s^1$ :

**algorithm**  $mc^1((u(a_1), \dots, u(a_n)), \phi(a_1, \dots, a_n))$

**begin**

**case**  $\phi(a_1, \dots, a_n)$  **of**

**begin**

$a_i \triangleright a_j$ :  $(\cdot)$  **if**  $a_i$  sees  $a_j$  in  $(u(a_1), \dots, u(a_n))$  **then succeed else fail**

$\perp$ :  $(\cdot)$  **fail**

$\neg\psi(a_1, \dots, a_n)$ :  $(\cdot)$

**begin**

**call**  $mc^1((u(a_1), \dots, u(a_n)), \psi(a_1, \dots, a_n))$

**if this call succeeds then fail else succeed**

**end**

$\psi_1(a_1, \dots, a_n) \vee \psi_2(a_1, \dots, a_n)$ :  $(\exists)$

**begin**

**choose**  $i$  **in**  $\{1, 2\}$

**call**  $mc^1((u(a_1), \dots, u(a_n)), \psi_i(a_1, \dots, a_n))$

**if this call succeeds then succeed else fail**

**end**

$K_{a_i}\psi(a_1, \dots, a_n)$ :  $(\forall)$

**begin**

**choose a**  $G$ -**vector**  $(v(a_1), \dots, v(a_n))$  **such that**  $(u(a_1), \dots, u(a_n)) \equiv_{a_i}^G (v(a_1), \dots, v(a_n))$

**call**  $mc^1((v(a_1), \dots, v(a_n)), \psi(a_1, \dots, a_n))$

**if this call succeeds then succeed else fail**  
**end**  
**end**  
**end**

Its execution depends primarily on  $\phi(a_1, \dots, a_n)$ , each case being existential, or universal. For example, the case  $((u(a_1), \dots, u(a_n)), \psi_1(a_1, \dots, a_n) \vee \psi_2(a_1, \dots, a_n))$  is existential. It is an accepting case iff for some  $i \in \{1, 2\}$ , the case  $((u(a_1), \dots, u(a_n)), \psi_i(a_1, \dots, a_n))$  is accepting, thus corresponding to the fact that  $\psi_1(a_1, \dots, a_n) \vee \psi_2(a_1, \dots, a_n)$  is true for  $(u(a_1), \dots, u(a_n))$  iff for some  $i \in \{1, 2\}$ ,  $\psi_i(a_1, \dots, a_n)$  is true for  $(u(a_1), \dots, u(a_n))$ . As well, the case  $((u(a_1), \dots, u(a_n)), K_{a_i} \psi(a_1, \dots, a_n))$  is universal. It is an accepting case iff for every  $G$ -vector  $(v(a_1), \dots, v(a_n))$  such that  $(u(a_1), \dots, u(a_n)) \equiv_{a_i}^G (v(a_1), \dots, v(a_n))$ , the case  $((v(a_1), \dots, v(a_n)), \psi(a_1, \dots, a_n))$  is accepting, thus corresponding to the fact that  $K_{a_i} \psi(a_1, \dots, a_n)$  is true for  $(u(a_1), \dots, u(a_n))$  iff for every  $G$ -vector  $(v(a_1), \dots, v(a_n))$  such that  $(u(a_1), \dots, u(a_n)) \equiv_{a_i}^G (v(a_1), \dots, v(a_n))$ ,  $\psi(a_1, \dots, a_n)$  is true for  $(v(a_1), \dots, v(a_n))$ . Cases labelled with  $(\cdot)$  are both existential and universal. All in all, it is clear that  $mc^1((u(a_1), \dots, u(a_n)), \phi(a_1, \dots, a_n))$  correctly solves the model checking problem with respect to  $\mathcal{W}_s^1$  and that it can be implemented in a polynomial time-bounded alternating Turing machine. Hence, the model checking problem with respect to  $\mathcal{W}_s^1$  is in *PSPACE*. ■

## 5.2 In dimension 2

First, we prove the following

### LEMMA 5.3

The validity problem of quantified Boolean logic is reducible to the model checking problem with respect to  $\mathcal{W}_s^2$ .

As a result,

### PROPOSITION 5.4

The model checking problem with respect to  $\mathcal{W}_s^2$  is *PSPACE*-hard and in *EXPSPACE*.

PROOF. *PSPACE*-hardness follows from Stockmeyer [22] and Lemma 5.3. A general strategy for proving a decision problem to be in *EXPSPACE* is to reduce it to a decision problem already known to be in *EXPSPACE*. A suitable decision problem already known to be in *EXPSPACE* is the validity problem of sentences in elementary algebra [4]. The language of elementary algebra is a first-order language with equality. It consists of the constant symbols 0 and 1, the function symbols  $+$  and  $\times$  of arity 2 and the relation symbol  $<$  of arity 2. Suppose that we are given a 2-world  $(u(a_1), \dots, u(a_n))$  in  $\mathcal{W}_s^2$  and a formula  $\phi(a_1, \dots, a_n)$ . We shall construct a sentence  $\varphi((u(a_1), \dots, u(a_n)), \phi(a_1, \dots, a_n))$  in elementary algebra such that  $\phi(a_1, \dots, a_n)$  is true for  $(u(a_1), \dots, u(a_n))$  in  $\mathcal{W}_s^2$  iff  $\varphi((u(a_1), \dots, u(a_n)), \phi(a_1, \dots, a_n))$  is valid. Choose distinct individual variables  $x_k^\alpha, y_k^\alpha$  where  $k \leq n$  is a positive integer and  $\alpha$  is a nonnegative integer. For all positive integers  $k \leq n$ ,  $x_k^0, x_k^1, \dots$  and  $y_k^0, y_k^1, \dots$  will represent the abscissas and the ordinates of the positions in  $\mathbb{R}^2$  occupied by  $a_k$  in such-and-such 2-world. Choose distinct individual variables  $z_k^\alpha, t_k^\alpha$  where  $k \leq n$  is a positive integer and  $\alpha$  is a nonnegative integer. For all positive integers  $k \leq n$ ,  $z_k^0, z_k^1, \dots$  and  $t_k^0, t_k^1, \dots$  will represent the coordinates of the endpoints in  $\mathbb{R}^2$  of the vectors located at the origin corresponding to the sections seen by  $a_k$  in such-and-such 2-world. Without loss

of generality, let us assume that the 2-world  $(u(a_1), \dots, u(a_n))$  in  $\mathcal{W}_s^2$  is given by means of rational numbers  $r_{x_1^0}, r_{y_1^0}, \dots, r_{x_n^0}, r_{y_n^0}, r_{z_1^0}, r_{t_1^0}, \dots, r_{z_n^0}, r_{t_n^0}$  corresponding to the abscissas and the ordinates of the positions in  $\mathbb{R}^2$  occupied by  $a_1, \dots, a_n$  and the coordinates of the endpoints in  $\mathbb{R}^2$  of the vectors located at the origin corresponding to the sections seen by  $a_1, \dots, a_n$ . Rational numbers being easily definable in the language of elementary algebra, there exists a formula  $\varphi_u(x_1^0, y_1^0, \dots, x_n^0, y_n^0, z_1^0, t_1^0, \dots, z_n^0, t_n^0)$  with free individual variables  $x_1^0, y_1^0, \dots, x_n^0, y_n^0, z_1^0, t_1^0, \dots, z_n^0, t_n^0$  in elementary algebra such that the following conditions are equivalent:

- the formula  $\varphi_u(x_1^0, y_1^0, \dots, x_n^0, y_n^0, z_1^0, t_1^0, \dots, z_n^0, t_n^0)$  holds for the real numbers  $v_{x_1^0}, v_{y_1^0}, \dots, v_{x_n^0}, v_{y_n^0}, v_{z_1^0}, v_{t_1^0}, \dots, v_{z_n^0}, v_{t_n^0}$ ,
- $v_{x_1^0} = r_{x_1^0}, v_{y_1^0} = r_{y_1^0}, \dots, v_{x_n^0} = r_{x_n^0}, v_{y_n^0} = r_{y_n^0}, v_{z_1^0} = r_{z_1^0}, v_{t_1^0} = r_{t_1^0}, \dots, v_{z_n^0} = r_{z_n^0}, v_{t_n^0} = r_{t_n^0}$ .

For all positive integers  $i, j \leq n$  and for all nonnegative integers  $\alpha$ , let  $sees(x_i^\alpha, y_i^\alpha, z_i^\alpha, t_i^\alpha, x_j^\alpha, y_j^\alpha)$  be a formula with free individual variables  $x_i^\alpha, y_i^\alpha, z_i^\alpha, t_i^\alpha, x_j^\alpha, y_j^\alpha$  in elementary algebra such that the following conditions are equivalent:

- the formula  $sees(x_i^\alpha, y_i^\alpha, z_i^\alpha, t_i^\alpha, x_j^\alpha, y_j^\alpha)$  holds for the real numbers  $v_{x_i^\alpha}, v_{y_i^\alpha}, v_{z_i^\alpha}, v_{t_i^\alpha}, v_{x_j^\alpha}, v_{y_j^\alpha}$ ,
- an agent occupying the position in  $\mathbb{R}^2$  defined by  $(v_{x_i^\alpha}, v_{y_i^\alpha})$  and seeing the section in  $\mathbb{R}^2$  defined by  $(v_{z_i^\alpha}, v_{t_i^\alpha})$  sees an agent occupying the position in  $\mathbb{R}^2$  defined by  $(v_{x_j^\alpha}, v_{y_j^\alpha})$ .

For all positive integers  $i \leq n$  and for all nonnegative integers  $\alpha$ , let  $equiv_i(x_1^\alpha, y_1^\alpha, \dots, x_n^\alpha, y_n^\alpha, z_1^\alpha, t_1^\alpha, \dots, z_n^\alpha, t_n^\alpha, x_1^{\alpha+1}, y_1^{\alpha+1}, \dots, x_n^{\alpha+1}, y_n^{\alpha+1}, z_1^{\alpha+1}, t_1^{\alpha+1}, \dots, z_n^{\alpha+1}, t_n^{\alpha+1})$  be a formula with free individual variables  $x_1^\alpha, y_1^\alpha, \dots, x_n^\alpha, y_n^\alpha, z_1^\alpha, t_1^\alpha, \dots, z_n^\alpha, t_n^\alpha, x_1^{\alpha+1}, y_1^{\alpha+1}, \dots, x_n^{\alpha+1}, y_n^{\alpha+1}, z_1^{\alpha+1}, t_1^{\alpha+1}, \dots, z_n^{\alpha+1}, t_n^{\alpha+1}$  in elementary algebra such that the following conditions are equivalent:

- the formula  $equiv_i(x_1^\alpha, y_1^\alpha, \dots, x_n^\alpha, y_n^\alpha, z_1^\alpha, t_1^\alpha, \dots, z_n^\alpha, t_n^\alpha, x_1^{\alpha+1}, y_1^{\alpha+1}, \dots, x_n^{\alpha+1}, y_n^{\alpha+1}, z_1^{\alpha+1}, t_1^{\alpha+1}, \dots, z_n^{\alpha+1}, t_n^{\alpha+1})$  holds for the real numbers  $v_{x_1^\alpha}, v_{y_1^\alpha}, \dots, v_{x_n^\alpha}, v_{y_n^\alpha}, v_{z_1^\alpha}, v_{t_1^\alpha}, \dots, v_{z_n^\alpha}, v_{t_n^\alpha}, v_{x_1^{\alpha+1}}, v_{y_1^{\alpha+1}}, \dots, v_{x_n^{\alpha+1}}, v_{y_n^{\alpha+1}}, v_{z_1^{\alpha+1}}, v_{t_1^{\alpha+1}}, \dots, v_{z_n^{\alpha+1}}, v_{t_n^{\alpha+1}}$ ,
- the 2-worlds defined by  $v_{x_1^\alpha}, v_{y_1^\alpha}, \dots, v_{x_n^\alpha}, v_{y_n^\alpha}, v_{z_1^\alpha}, v_{t_1^\alpha}, \dots, v_{z_n^\alpha}, v_{t_n^\alpha}$  and  $v_{x_1^{\alpha+1}}, v_{y_1^{\alpha+1}}, \dots, v_{x_n^{\alpha+1}}, v_{y_n^{\alpha+1}}, v_{z_1^{\alpha+1}}, v_{t_1^{\alpha+1}}, \dots, v_{z_n^{\alpha+1}}, v_{t_n^{\alpha+1}}$  are indiscernible for  $a_i$ .

The translation  $\tau$  taking nonnegative integers and formulas to formulas in elementary algebra is defined as follows:

- $\tau(\alpha, a_i \triangleright a_j) = sees(x_i^\alpha, y_i^\alpha, z_i^\alpha, t_i^\alpha, x_j^\alpha, y_j^\alpha)$ ,
- $\tau(\alpha, \perp) = \perp$ ,
- $\tau(\alpha, \neg \phi) = \neg \tau(\alpha, \phi)$ ,
- $\tau(\alpha, \phi \vee \psi) = \tau(\alpha, \phi) \vee \tau(\alpha, \psi)$ ,
- $\tau(\alpha, K_{a_i} \phi) = \forall x_1^{\alpha+1} \forall y_1^{\alpha+1} \dots \forall x_n^{\alpha+1} \forall y_n^{\alpha+1} \forall z_1^{\alpha+1} \forall t_1^{\alpha+1} \dots \forall z_n^{\alpha+1} \forall t_n^{\alpha+1} (equiv_i(x_1^\alpha, y_1^\alpha, \dots, x_n^\alpha, y_n^\alpha, z_1^\alpha, t_1^\alpha, \dots, z_n^\alpha, t_n^\alpha, x_1^{\alpha+1}, y_1^{\alpha+1}, \dots, x_n^{\alpha+1}, y_n^{\alpha+1}, z_1^{\alpha+1}, t_1^{\alpha+1}, \dots, z_n^{\alpha+1}, t_n^{\alpha+1}) \rightarrow \tau(\alpha + 1, \phi))$ .

Let  $\varphi((u(a_1), \dots, u(a_n)), \phi(a_1, \dots, a_n))$  be  $\forall x_1^0 \forall y_1^0 \dots \forall x_n^0 \forall y_n^0 \forall z_1^0 \forall t_1^0 \dots \forall z_n^0 \forall t_n^0 (\varphi_u(x_1^0, y_1^0, \dots, x_n^0, y_n^0, z_1^0, t_1^0, \dots, z_n^0, t_n^0) \rightarrow \tau(0, \phi(a_1, \dots, a_n)))$ . Remark that  $\varphi((u(a_1), \dots,$

$u(a_n)), \phi(a_1, \dots, a_n))$  can be computed in logarithmic space. Moreover, the reader may easily verify that  $\phi(a_1, \dots, a_n)$  is true for  $(u(a_1), \dots, u(a_n))$  in  $\mathcal{W}_s^2$  iff  $\varphi((u(a_1), \dots, u(a_n)), \phi(a_1, \dots, a_n))$  is valid. Thus, the model checking problem with respect to  $\mathcal{W}_s^2$  is reducible to the validity problem of sentences in elementary algebra. ■

## 6 Decidability and complexity of satisfiability problems

In this section, for some positive integers  $k$ , we investigate the decidability and complexity of the satisfiability problem with respect to  $\mathcal{W}_s^k$ : given a formula  $\phi$ , determine whether  $\mathcal{W}_s^k \text{ sat}^k \phi$ . The results obtained are summarized as follows: *PSPACE*-complete when  $k = 1$  and *PSPACE*-hard and in *EXPSPACE* when  $k = 2$ . Let us remind that in the more traditional epistemic logics considered in [11, 12], for example, satisfiability problems are usually decidable in nondeterministic polynomial time or in polynomial space, according as there exists only one agent or there exists at least two agents.

### 6.1 In dimension 1

First, we prove the following

LEMMA 6.1

The validity problem of quantified Boolean logic is reducible to the satisfiability problem with respect to  $\mathcal{W}_s^1$ .

As a result,

PROPOSITION 6.2

The satisfiability problem with respect to  $\mathcal{W}_s^1$  is *PSPACE*-complete.

PROOF. *PSPACE*-hardness follows from Stockmeyer [22] and Lemma 6.1. Membership in *PSPACE* follows from Savitch [17], Proposition 3.4 and the fact that the following nondeterministic algorithm decides in polynomial space the satisfiability problem with respect to  $\mathcal{W}_s^1$ :

```

algorithm  $\text{sat}^1(\phi(a_1, \dots, a_n))$ 
begin
  choose a G-vector  $(u(a_1), \dots, u(a_n))$ 
  call  $mc_{PSPACE}^1((u(a_1), \dots, u(a_n)), \phi(a_1, \dots, a_n))$ 
  if this call succeeds then succeed else fail
end

```

where  $mc_{PSPACE}^1$  is a deterministic algorithm that decides in polynomial space the model checking problem with respect to  $\mathcal{W}_s^1$  (by Proposition 5.2, we know that there exists such algorithms). It is clear that  $\text{sat}^1$  correctly solves the satisfiability problem with respect to  $\mathcal{W}_s^1$  and that it can be implemented in a polynomial space-bounded nondeterministic Turing machine. Hence, the satisfiability problem with respect to  $\mathcal{W}_s^1$  is in *PSPACE*. ■

### 6.2 In dimension 2

First, we prove the following

### LEMMA 6.3

The validity problem of quantified Boolean logic is reducible to the satisfiability problem with respect to  $\mathcal{W}_s^2$ .

As a result,

### PROPOSITION 6.4

The satisfiability problem with respect to  $\mathcal{W}_s^2$  is *PSPACE*-hard and in *EXSPACE*.

PROOF. The proof is similar to the proof of Proposition 5.4. *PSPACE*-hardness follows from Stockmeyer [22] and Lemma 6.3. Membership in *EXSPACE* follows from Ben-Or *et al.* [4] and the fact that the satisfiability problem with respect to  $\mathcal{W}_s^2$  is reducible to the validity problem of sentences in elementary algebra. Suppose that we are given a formula  $\phi(a_1, \dots, a_n)$ . We shall construct a sentence  $\varphi(\phi(a_1, \dots, a_n))$  in elementary algebra such that  $\phi(a_1, \dots, a_n)$  is satisfiable in  $\mathcal{W}_s^2$  iff  $\varphi(\phi(a_1, \dots, a_n))$  is valid. Choose distinct individual variables  $x_k^\alpha, y_k^\alpha$  where  $k \leq n$  is a positive integer and  $\alpha$  is a nonnegative integer, choose distinct individual variables  $z_k^\alpha, t_k^\alpha$  where  $k \leq n$  is a positive integer and  $\alpha$  is a nonnegative integer and let  $\varphi_u(x_1^0, y_1^0, \dots, x_n^0, y_n^0, z_1^0, t_1^0, \dots, z_n^0, t_n^0)$ ,  $sees(x_i^\alpha, y_i^\alpha, z_i^\alpha, t_i^\alpha, x_j^\alpha, y_j^\alpha)$  and  $equiv_i(x_1^\alpha, y_1^\alpha, \dots, x_n^\alpha, y_n^\alpha, z_1^\alpha, t_1^\alpha, \dots, z_n^\alpha, t_n^\alpha, x_1^{\alpha+1}, y_1^{\alpha+1}, \dots, x_n^{\alpha+1}, y_n^{\alpha+1}, z_1^{\alpha+1}, t_1^{\alpha+1}, \dots, z_n^{\alpha+1}, t_n^{\alpha+1})$  be the formulas in elementary algebra defined in the proof of Proposition 5.4. Let  $\tau$  be the translation taking nonnegative integers and formulas to formulas in elementary algebra defined in the proof of Proposition 5.4. Let  $\varphi(\phi(a_1, \dots, a_n))$  be  $\exists x_1^0 \exists y_1^0 \dots \exists x_n^0 \exists y_n^0 \exists z_1^0 \exists t_1^0 \dots \exists z_n^0 \exists t_n^0 \tau(0, \phi(a_1, \dots, a_n))$ . Remark that  $\varphi(\phi(a_1, \dots, a_n))$  can be computed in logarithmic space. Moreover, the reader may easily verify that  $\phi(a_1, \dots, a_n)$  is satisfiable in  $\mathcal{W}_s^2$  iff  $\varphi(\phi(a_1, \dots, a_n))$  is valid. Thus, the satisfiability problem with respect to  $\mathcal{W}_s^2$  is reducible to the validity problem of sentences in elementary algebra. ■

## 7 Variants

In this section, we discuss several variants not captured by the syntax and the semantics considered in Sections 2–6. For most of them, the axiomatization/completeness issue and the decidability/complexity issue are still unsettled.

### 7.1 Group knowledge

There exists various notions of what may be called *group knowledge*. A well-known example of such a notion of knowledge for a group of agents is the following:  $C_G\phi$  (“ $\phi$  is common knowledge in the group  $G$ ”). The usual semantics definition of common knowledge runs as follows within the context of  $\mathcal{W}_s^n$ :  $\mathcal{W}_s^n, u \models^n C_G\phi$  iff for all  $n$ -worlds  $v$  in  $\mathcal{W}_s^n$ , if  $u \bigcup^+ \{\equiv_a^n : a \in G\} v$  then  $\mathcal{W}_s^n, v \models^n \phi$ ,  $\bigcup^+ \{\equiv_a^n : a \in G\}$  denoting the transitive closure of the union of the indiscernibility relations concerning agents in group  $G$ . In this respect, see [12] for details, the key valid principles of common knowledge are

- $C_G\phi \rightarrow \bigwedge \{K_a(\phi \wedge C_G\phi) : a \in G\}$ ,
- $C_G(\phi \rightarrow \bigwedge \{K_a\phi : a \in G\}) \rightarrow \bigwedge \{K_a(\phi \rightarrow C_G\phi) : a \in G\}$ .

However,  $\mathcal{W}_s^1$  invalidates very specific formulas. For instance, choose distinct agents  $a_1, a_2, a_3, b, c$ . The reader may easily verify that  $\mathcal{W}_s^1 \text{ inv}^1 K_{\{a_1, a_2, a_3\}} b \triangleright c$ . Another interesting example of such a notion of knowledge for a group of agents is the following:  $D_G\phi$  (“ $\phi$

is distributed knowledge in the group  $G$ )). The usual semantics definition of distributed knowledge runs as follows within the context of  $\mathcal{W}_s^n$ :  $\mathcal{W}_s^n, u \models^n D_G \phi$  iff for all  $n$ -worlds  $v$  in  $\mathcal{W}_s^n$ , if  $u \cap \{\equiv_a^n: a \in G\} v$  then  $\mathcal{W}_s^n, v \models^n \phi$ ,  $\bigcap \{\equiv_a^n: a \in G\}$  denoting the intersection of the indiscernibility relations concerning agents in group  $G$ . In this respect, see [12] for details, the key valid principle of distributed knowledge is

- $D_G \phi \rightarrow \bigwedge \{K_a \phi: a \in G\}$ .

Nevertheless,  $\mathcal{W}_s^1$  validates formulas that are not valid in a more general setting. For example, choose a finite group  $G$  of agents. The reader may easily verify that for all formulas  $\phi(G)$ ,  $\mathcal{W}_s^1 \text{ val}^1 D_G \phi(G) \leftrightarrow \phi(G)$ .

## 7.2 Announcements

We have considered in Sections 2–6 that the knowledge our agents have about their environment mostly depends on the positions they occupy and the sections they see. The truth is that knowledge is also affected by the messages our agents exchange. Following the intuition behind the logic of public announcements, see [10] for an introduction, for all formulas  $\varphi$ , let us add modal operators of the form  $[\varphi]$  (“after announcement of  $\varphi$ , it holds that . . .”) to our language. As usual, for all formulas  $\varphi$ , we define the modal operator  $\langle \varphi \rangle$  as follows:

- $\langle \varphi \rangle \phi ::= \neg[\varphi]\neg\phi$ .

Within the context of a nonempty set  $\mathcal{W}^n$  of  $n$ -worlds,  $\mathcal{W}_{|\varphi}^n$  being the set of all  $n$ -worlds  $v$  in  $\mathcal{W}^n$  such that  $\mathcal{W}^n, v \models^n \varphi$ ,  $n$ -satisfaction of  $[\varphi]\phi$  is defined by

- $\mathcal{W}^n, u \models^n [\varphi]\phi$  iff if  $\mathcal{W}^n, u \models^n \varphi$  then  $\mathcal{W}_{|\varphi}^n, \models^n \phi$ .

As a result,

- $\mathcal{W}^n, u \models^n \langle \varphi \rangle \phi$  iff  $\mathcal{W}^n, u \models^n \varphi$  and  $\mathcal{W}_{|\varphi}^n, \models^n \phi$ .

See [19] for a study of this variant. This variant is interesting because it can be considered as a logic of public communications between agents that look at one another, formulas like  $[K_a \varphi]\phi$  and  $\langle K_a \varphi \rangle \phi$  being read “if  $a$  knows that  $\varphi$  then after the announcement of  $\varphi$  by  $a$ ,  $\phi$  holds” and “ $a$  knows that  $\varphi$  and after the announcement of  $\varphi$  by  $a$ ,  $\phi$  holds”.

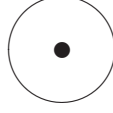
## 7.3 Visual abilities

We have considered in Sections 2–6 that our agents had similar visual abilities. What happens if agents’ sights vary? In Section 2.2, for all positive integers  $n$ , we have defined  $n$ -worlds as functions assigning  $n$ -scopes to agents. If the visual abilities of our agents are similar to those of a radar then the  $n$ -scopes assigned to agents can be classed as open  $n$ -discs of such-and-such diameter. We shall say that an  $n$ -scope  $(x, S)$  is circular iff  $S$  is an open  $n$ -disc and  $x$  is the center of  $S$ .

EXAMPLE 7.1

If one considers Figure 14 in dimension 2, a 2-scope is defined by the point and the open 2-disc delimited by the circle.

An  $n$ -world  $u$  is said to be circular iff for all agents  $a$ , the  $n$ -scope  $u(a)$  is circular. Let  $\mathcal{W}_c^n$  be the set of all circular  $n$ -worlds. Choose distinct agents  $a, b, c$ . The reader may easily



$\mathbb{R}^2$

FIG. 14. A circular 2-scope.

verify that  $\mathcal{W}_s^1 \text{ val}^1 a \triangleright b \wedge b \triangleright a \rightarrow a \triangleright c \vee b \triangleright c$  and  $\mathcal{W}_c^1 \text{ fal}^1 a \triangleright b \wedge b \triangleright a \rightarrow a \triangleright c \vee b \triangleright c$ . Hence, our language can distinguish between the notion of satisfiability in  $\mathcal{W}_s^1$  and the notion of satisfiability in  $\mathcal{W}_c^1$ . We have considered in Sections 2–6 that the knowledge our agents have about their environment mostly depends on the positions they occupy and the sections they see. The truth is that knowledge is also affected by the amount of effort our agents put in. Following the intuition behind the logic of subset spaces, see [15] for an introduction, let us add modal operators of the form  $\Box_a$  (“whatever the effort  $a$  puts in, . . .”) to our language. As usual, for all agents  $a$ , we define the modal operator  $\Diamond_a$  as follows:

- $\Diamond_a \phi ::= \neg \Box_a \neg \phi$ .

We shall say that agent  $a$  has sharpened its range of vision between  $n$ -scopes  $u$  and  $v$  iff the only difference between  $u$  and  $v$  lies in the fact that  $S_u(a) \subsetneq S_v(a)$ . Within the context of  $\mathcal{W}_c^n$ ,  $n$ -satisfaction of  $\Box_a \phi$  is defined by

- $\mathcal{W}_c^n, u \models^n \Box_a \phi$  iff for all  $n$ -worlds  $v$  in  $\mathcal{W}_c^n$ , if  $a$  has sharpened its range of vision between  $u$  and  $v$  then  $\mathcal{W}_c^n, v \models^n \phi$ .

As a result,

- $\mathcal{W}_c^n, u \models^n \Diamond_a \phi$  iff there exists an  $n$ -world  $v$  in  $\mathcal{W}_c^n$  such that  $a$  has sharpened its range of vision between  $u$  and  $v$  and  $\mathcal{W}_c^n, v \models^n \phi$ .

We first observe that  $\mathcal{W}_c^n \text{ val}^n \Box_a \phi \rightarrow \phi$ ,  $\mathcal{W}_c^n \text{ val}^n \Box_a \phi \rightarrow \Box_a \Box_a \phi$  and  $\mathcal{W}_c^n \text{ val}^n \Diamond_a \phi \wedge \Diamond_a \psi \rightarrow \Diamond_a (\phi \wedge \Diamond_a \psi) \vee \Diamond_a (\Diamond_a \phi \wedge \psi)$ . In other respects, for all  $\Box$ -free formulas  $\phi$ ,  $\mathcal{W}_c^n \text{ val}^n \Box_a \hat{K}_a \phi \rightarrow \Diamond_a K_a \phi$ . Let us consider an enumeration  $a_0, a_1, \dots$  of  $AGT$ . Consider a nonnegative integers  $i \leq n$ . Let  $\phi_i^\triangleright$  be the conjunction of the following literals:

- for all nonnegative integers  $j \leq n$ , if  $i \neq j$  then  $a_i \triangleright a_j$ .

Let  $\phi_i^{\bar{\triangleright}}$  be the conjunction of the following literals:

- for all nonnegative integers  $j \leq n$ , if  $i \neq j$  then  $a_i \bar{\triangleright} a_j$ .

Let  $\phi_i$  be  $\phi_i^{\bar{\triangleright}} \wedge \Box_{a_i} \phi_i^\triangleright$ .  $\phi_i$  says that: for all nonnegative integers  $j \leq n$ , if  $i \neq j$  then  $a_i$  does not see  $a_j$ ; whatever the effort  $a_i$  puts in, for all nonnegative integers  $j \leq n$ , if  $i \neq j$  then  $a_i$  sees  $a_j$ . As a result,  $\phi_i$  implies that for all nonnegative integers  $j, k \leq n$ , if  $i \neq j$  and  $i \neq k$

then the distance between the positions of  $a_i$  and  $a_j$  and the distance between the positions of  $a_i$  and  $a_k$  are equal. Let  $\phi$  be  $\phi_0 \wedge \dots \wedge \phi_n$ . The reader may easily verify that  $\mathcal{W}_c^n \text{inv}^n \phi$  and  $\mathcal{W}_c^{n+1} \text{sat}^{n+1} \phi$ . Hence, our language can distinguish between the notion of satisfiability in  $\mathcal{W}_c^n$  and the notion of satisfiability in  $\mathcal{W}_c^{n+1}$ .

## 8 Conclusion

This article considered a logic for studying knowledge of agents where sentences like “ $a$  knows that  $b$  sees  $c$ ” can be expressed. We have studied its expressivity, axiomatized validity in  $\mathcal{W}_s^1$  and investigated the complexity of the model checking and satisfiability problems in  $\mathcal{W}_s^1$  and  $\mathcal{W}_s^2$ . Much remains to be done. Firstly, there is the problem of the complete axiomatization of validity in  $\mathcal{W}_s^2$ . Secondly, there is the question of the precise complexity of the model checking and satisfiability problems in  $\mathcal{W}_s^2$ . Thirdly, there is the issue of the variants considered in Section 7. Of course, one could as well formulate these problems, questions and issues in dimensions  $n \geq 3$ .

## Acknowledgements

We want to thank all our colleagues of the *Institut de recherche en informatique de Toulouse* who made several helpful comments for improving the correctness and the readability of this article. We are also grateful for the detailed comments referees made on a preliminary version of this paper.

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## Annex

PROOF OF LEMMA 3.1. By induction on  $\phi(a_1, \dots, a_k)$ . The argument is similar to that given in Blackburn *et al.* [5, Theorem 2.20], as the reader should check. ■

PROOF OF LEMMA 3.3. Let  $u^{(1)}$  be a simple 1-world and  $u^{(\mathcal{T})}$  be a simple  $\mathcal{T}$ -world such that  $u^{(1)} Z_k u^{(\mathcal{T})}$ .

1. Let  $i, j \leq k$  be positive integers such that  $a_i$  sees  $a_j$  in  $u^{(1)}$ . Since  $u^{(1)} Z_k u^{(\mathcal{T})}$ , then  $a_i$  sees  $a_j$  in  $u^{(\mathcal{T})}$ .

2. Let  $i, j \leq k$  be positive integers such that  $a_i$  sees  $a_j$  in  $u^{(\mathcal{T})}$ . Since  $u^{(1)} Z_k u^{(\mathcal{T})}$ , then  $a_i$  sees  $a_j$  in  $u^{(1)}$ .

3. Let  $i \leq k$  be a positive integer and  $v^{(1)}$  be a simple 1-world such that  $u^{(1)} \equiv_{a_i}^1 v^{(1)}$ . Consider the set of all simple  $\mathcal{T}$ -worlds  $v^{(\mathcal{T})}$  such that  $x_{u^{(\mathcal{T})}}(a_i) = x_{v^{(\mathcal{T})}}(a_i)$ ,  $S_{u^{(\mathcal{T})}}(a_i) = S_{v^{(\mathcal{T})}}(a_i)$  and for all positive integers  $j \leq k$ , if  $a_i$  sees  $a_j$  in  $u^{(\mathcal{T})}$  then  $x_{u^{(\mathcal{T})}}(a_j) = x_{v^{(\mathcal{T})}}(a_j)$  and  $S_{u^{(\mathcal{T})}}(a_j) = S_{v^{(\mathcal{T})}}(a_j)$ . Since  $\mathcal{T}$  is a linear order without endpoints, then this set contains a simple  $\mathcal{T}$ -world  $v^{(\mathcal{T})}$  such that,  $j_1, \dots, j_l$  being a list of all the positive integers  $j \leq k$  such that  $a_i$  does not see  $a_j$  in  $u^{(\mathcal{T})}$ , the temporal relationships between  $a_{j_1}, \dots, a_{j_l}$  in  $v^{(1)}$  are equal to the temporal relationships between  $a_{j_1}, \dots, a_{j_l}$  in  $v^{(\mathcal{T})}$ . Since  $u^{(1)} Z_k u^{(\mathcal{T})}$ , then  $u^{(\mathcal{T})} \equiv_{a_i}^{\mathcal{T}} v^{(\mathcal{T})}$  and  $v^{(1)} Z_k v^{(\mathcal{T})}$ .

4. Let  $i \leq k$  be a positive integer and  $v^{(\mathcal{T})}$  be a simple  $\mathcal{T}$ -world such that  $u^{(\mathcal{T})} \equiv_{a_i}^{\mathcal{T}} v^{(\mathcal{T})}$ . Consider the set of all simple 1-worlds  $v^{(1)}$  such that  $x_{u^{(1)}}(a_i) = x_{v^{(1)}}(a_i)$ ,  $S_{u^{(1)}}(a_i) = S_{v^{(1)}}(a_i)$  and for all positive integers  $j \leq k$ , if  $a_i$  sees  $a_j$  in  $u^{(1)}$  then  $x_{u^{(1)}}(a_j) = x_{v^{(1)}}(a_j)$  and  $S_{u^{(1)}}(a_j) = S_{v^{(1)}}(a_j)$ . Since  $\mathbb{R}$  is a linear order without endpoints, then this set contains a simple 1-world  $v^{(1)}$  such that,  $j_1, \dots, j_l$  being a list of all the positive integers  $j \leq k$  such that  $a_i$  does not see  $a_j$  in  $u^{(1)}$ , the temporal relationships between  $a_{j_1}, \dots, a_{j_l}$  in  $v^{(1)}$  are equal to the temporal relationships between  $a_{j_1}, \dots, a_{j_l}$  in  $v^{(\mathcal{T})}$ . Since  $u^{(1)} Z_k u^{(\mathcal{T})}$ , then  $u^{(1)} \equiv_{a_i}^1 v^{(1)}$  and  $v^{(1)} Z_k v^{(\mathcal{T})}$ . ■

PROOF OF LEMMA 4.5. Obvious. ■

PROOF OF LEMMA 4.8. The argument is similar to that given in Blackburn *et al.* [5, Lemma 4.21], as the reader should check. ■

PROOF OF FACT 4.9. Obvious. ■

PROOF OF FACT 4.10. 1. Obvious.

2. Suppose  $i$   $\Sigma$ -precedes  $j$ ,  $j$   $\Sigma$ -precedes  $k$  and  $i$  does not  $\Sigma$ -precede  $k$ . Since  $i$   $\Sigma$ -precedes  $j$ , then  $i \neq j$  and

- (1)  $i$  is  $\Sigma$ -right and  $a_i \triangleright a_j \in \Sigma$ , or (2)  $j$  is  $\Sigma$ -left and  $a_j \triangleright a_i \in \Sigma$ , or (3)  $i$  is  $\Sigma$ -left,  $j$  is  $\Sigma$ -right,  $a_i \bar{\triangleright} a_j \in \Sigma$  and  $a_j \bar{\triangleright} a_i \in \Sigma$ .

Since  $j$   $\Sigma$ -precedes  $k$ , then  $j \neq k$  and

- (4)  $j$  is  $\Sigma$ -right and  $a_j \triangleright a_k \in \Sigma$ , or (5)  $k$  is  $\Sigma$ -left and  $a_k \triangleright a_j \in \Sigma$ , or (6)  $j$  is  $\Sigma$ -left,  $k$  is  $\Sigma$ -right,  $a_j \bar{\triangleright} a_k \in \Sigma$  and  $a_k \bar{\triangleright} a_j \in \Sigma$ .

Since  $i$  does not  $\Sigma$ -precede  $k$ , then  $i = k$ , or  $i \neq k$  and

- (7)  $i$  is not  $\Sigma$ -right, or (8)  $a_i \triangleright a_k \notin \Sigma$ ,
- (9)  $k$  is not  $\Sigma$ -left, or (10)  $a_k \triangleright a_i \notin \Sigma$ ,
- (11)  $i$  is not  $\Sigma$ -left, or (12)  $k$  is not  $\Sigma$ -right, or (13)  $a_i \bar{\triangleright} a_k \notin \Sigma$ , or (14)  $a_k \bar{\triangleright} a_i \notin \Sigma$ .

Hence, we have to consider eighteen cases.

*Case “(1), (4) and  $i = k$ ”.* Since (1), then  $i$  is  $\Sigma$ -right and  $a_i \triangleright a_j \in \Sigma$ . Since (4), then  $j$  is  $\Sigma$ -right and  $a_j \triangleright a_k \in \Sigma$ . Since  $i = k$ , then  $a_j \triangleright a_i \in \Sigma$ . Since  $i \neq j$ , then  $(a_i \triangleright a_j \leftrightarrow a_j \bar{\triangleright} a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -right and  $j$  is  $\Sigma$ -right). Since  $a_i \triangleright a_j \in \Sigma$ , then  $a_j \bar{\triangleright} a_i \in \Sigma$ : a contradiction.

*Case “(1), (4),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”.* Since (1), then  $i$  is  $\Sigma$ -right and  $a_i \triangleright a_j \in \Sigma$ . Since (4), then  $j$  is  $\Sigma$ -right and  $a_j \triangleright a_k \in \Sigma$ . Since (7), or (8), then  $i$  is not  $\Sigma$ -right, or  $a_i \bar{\triangleright} a_k \in \Sigma$ . Since  $i$  is  $\Sigma$ -right, then  $a_i \bar{\triangleright} a_k \in \Sigma$ . Since  $i \neq j$ , then  $(a_i \triangleright a_j \leftrightarrow a_j \bar{\triangleright} a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -right and  $j$  is  $\Sigma$ -right). Since  $a_i \triangleright a_j \in \Sigma$ , then  $a_j \bar{\triangleright} a_i \in \Sigma$ . Since  $a_i \triangleright a_j \in \Sigma$  and  $a_j \triangleright a_k \in \Sigma$ , then  $a_i \triangleright a_k \in \Sigma$ , or  $a_j \triangleright a_i \in \Sigma$  (use axiom  $Ax_2$ ). Since  $a_j \bar{\triangleright} a_i \in \Sigma$ , then  $a_i \triangleright a_k \in \Sigma$ : a contradiction.

*Case “(1), (5) and  $i = k$ ”.* Since (1), then  $i$  is  $\Sigma$ -right. Since (5), then  $k$  is  $\Sigma$ -left. Since  $i = k$ , then  $i$  is  $\Sigma$ -left: a contradiction.

*Case “(1), (5),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”.* Since (1), then  $i$  is  $\Sigma$ -right and  $a_i \triangleright a_j \in \Sigma$ . Since (5), then  $k$  is  $\Sigma$ -left and  $a_k \triangleright a_j \in \Sigma$ . Since (7), or (8), then  $i$  is not  $\Sigma$ -right, or  $a_i \bar{\triangleright} a_k \in \Sigma$ . Since (9), or (10), then  $k$  is not  $\Sigma$ -left, or  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since  $i$  is  $\Sigma$ -right and  $k$  is  $\Sigma$ -left, then  $a_i \bar{\triangleright} a_k \in \Sigma$  and  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since  $a_i \triangleright a_j \in \Sigma$  and  $a_k \triangleright a_j \in \Sigma$ , then  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_i \in \Sigma$  (use axiom  $Ax_2$ ). Since  $a_i \bar{\triangleright} a_k \in \Sigma$ , then  $a_k \triangleright a_i \in \Sigma$ : a contradiction.

*Case “(1), (6) and  $i = k$ ”.* Since (1), then  $a_i \triangleright a_j \in \Sigma$ . Since (6), then  $a_k \bar{\triangleright} a_j \in \Sigma$ . Since  $i = k$ , then  $a_i \bar{\triangleright} a_j \in \Sigma$ : a contradiction.

Case “(1), (6),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”. Since (1), then  $i$  is  $\Sigma$ -right and  $a_i \triangleright a_j \in \Sigma$ . Since (6), then  $j$  is  $\Sigma$ -left,  $k$  is  $\Sigma$ -right,  $a_j \bar{\triangleright} a_k \in \Sigma$  and  $a_k \bar{\triangleright} a_j \in \Sigma$ . Since (7), or (8), then  $i$  is not  $\Sigma$ -right, or  $a_i \bar{\triangleright} a_k \in \Sigma$ . Since  $i$  is  $\Sigma$ -right, then  $a_i \bar{\triangleright} a_k \in \Sigma$ . Since  $i \neq j$ , then  $(a_i \triangleright a_j \leftrightarrow a_j \triangleright a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -right and  $j$  is  $\Sigma$ -left). Since  $a_i \triangleright a_j \in \Sigma$ , then  $a_j \triangleright a_i \in \Sigma$ . Since  $i \neq k$ , then  $(a_i \triangleright a_k \leftrightarrow a_k \bar{\triangleright} a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -right and  $k$  is  $\Sigma$ -right). Since  $a_i \bar{\triangleright} a_k \in \Sigma$ , then  $a_k \triangleright a_i \in \Sigma$ . Since  $a_i \triangleright a_j \in \Sigma$ , then  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_j \in \Sigma$  (use axiom  $Ax_2$ ). Since  $a_i \bar{\triangleright} a_k \in \Sigma$ , then  $a_k \triangleright a_j \in \Sigma$ : a contradiction.

Case “(2), (4) and  $i = k$ ”. Since (2), then  $j$  is  $\Sigma$ -left. Since (4), then  $j$  is  $\Sigma$ -right: a contradiction.

Case “(2), (4),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”. Since (2), then  $j$  is  $\Sigma$ -left. Since (4), then  $j$  is  $\Sigma$ -right: a contradiction.

Case “(2), (5) and  $i = k$ ”. Since (2), then  $j$  is  $\Sigma$ -left and  $a_j \triangleright a_i \in \Sigma$ . Since (5), then  $k$  is  $\Sigma$ -left and  $a_k \triangleright a_j \in \Sigma$ . Since  $i = k$ , then  $i$  is  $\Sigma$ -left and  $a_i \triangleright a_j \in \Sigma$ . Since  $i \neq j$ , then  $(a_i \triangleright a_j \leftrightarrow a_j \bar{\triangleright} a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -left and  $j$  is  $\Sigma$ -left). Since  $a_j \triangleright a_i \in \Sigma$ , then  $a_i \bar{\triangleright} a_j \in \Sigma$ : a contradiction.

Case “(2), (5),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”. Since (2), then  $j$  is  $\Sigma$ -left and  $a_j \triangleright a_i \in \Sigma$ . Since (5), then  $k$  is  $\Sigma$ -left and  $a_k \triangleright a_j \in \Sigma$ . Since (9), or (10), then  $k$  is not  $\Sigma$ -left, or  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since  $k$  is  $\Sigma$ -left, then  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since  $j \neq k$ , then  $(a_j \triangleright a_k \leftrightarrow a_k \bar{\triangleright} a_j) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $j$  is  $\Sigma$ -left and  $k$  is  $\Sigma$ -left). Since  $a_k \triangleright a_j \in \Sigma$ , then  $a_j \bar{\triangleright} a_k \in \Sigma$ . Since  $a_k \bar{\triangleright} a_i \in \Sigma$ , then  $a_j \bar{\triangleright} a_i \in \Sigma$ , or  $a_k \bar{\triangleright} a_j \in \Sigma$  (use axiom  $Ax_2$ ). Since  $a_j \triangleright a_i \in \Sigma$ , then  $a_k \bar{\triangleright} a_j \in \Sigma$ : a contradiction.

Case “(2), (6) and  $i = k$ ”. Since (2), then  $a_j \triangleright a_i \in \Sigma$ . Since (6), then  $a_j \bar{\triangleright} a_k \in \Sigma$ . Since  $i = k$ , then  $a_j \bar{\triangleright} a_i \in \Sigma$ : a contradiction.

Case “(2), (6),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”. Since (2), then  $j$  is  $\Sigma$ -left and  $a_j \triangleright a_i \in \Sigma$ . Since (6), then  $j$  is  $\Sigma$ -left,  $k$  is  $\Sigma$ -right,  $a_j \bar{\triangleright} a_k \in \Sigma$  and  $a_k \bar{\triangleright} a_j \in \Sigma$ . Since (7), or (8), then  $i$  is not  $\Sigma$ -right, or  $a_i \bar{\triangleright} a_k \in \Sigma$ . Since (11), or (12), or (13), or (14), then  $i$  is not  $\Sigma$ -left, or  $k$  is not  $\Sigma$ -right, or  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_i \in \Sigma$ . Suppose  $i$  is  $\Sigma$ -right. Since  $i$  is not  $\Sigma$ -right, or  $a_i \bar{\triangleright} a_k \in \Sigma$ , then  $a_i \bar{\triangleright} a_k \in \Sigma$ . Since  $i \neq k$ , then  $(a_i \triangleright a_k \leftrightarrow a_k \bar{\triangleright} a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -right and  $k$  is  $\Sigma$ -right). Since  $a_i \bar{\triangleright} a_k \in \Sigma$ , then  $a_k \triangleright a_i \in \Sigma$ . Since  $a_j \triangleright a_i \in \Sigma$ , then  $a_j \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_j \in \Sigma$  (use axiom  $Ax_2$ ). Since  $a_j \bar{\triangleright} a_k \in \Sigma$ , then  $a_k \triangleright a_j \in \Sigma$ : a contradiction. Suppose  $i$  is  $\Sigma$ -left. Since  $i \neq k$ , then  $(a_i \triangleright a_k \leftrightarrow a_k \bar{\triangleright} a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -left and  $k$  is  $\Sigma$ -right). Since  $i$  is  $\Sigma$ -left,  $k$  is  $\Sigma$ -right and  $i$  is not  $\Sigma$ -left, or  $k$  is not  $\Sigma$ -right, or  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_i \in \Sigma$ , then  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_i \in \Sigma$ . Since  $(a_i \triangleright a_k \leftrightarrow a_k \bar{\triangleright} a_i) \in \Sigma$ , then  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since  $a_j \triangleright a_i \in \Sigma$ , then  $a_j \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_j \in \Sigma$  (use axiom  $Ax_2$ ). Since  $a_j \bar{\triangleright} a_k \in \Sigma$ , then  $a_k \triangleright a_j \in \Sigma$ : a contradiction.

Case “(3), (4) and  $i = k$ ”. Since (3), then  $a_j \bar{\triangleright} a_i \in \Sigma$ . Since (4), then  $a_j \triangleright a_k \in \Sigma$ . Since  $i = k$ , then  $a_j \triangleright a_i \in \Sigma$ : a contradiction.

Case “(3), (4),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”. Since (3),

then  $i$  is  $\Sigma$ -left,  $j$  is  $\Sigma$ -right,  $a_i \bar{\triangleright} a_j \in \Sigma$  and  $a_j \bar{\triangleright} a_i \in \Sigma$ . Since (4), then  $j$  is  $\Sigma$ -right and  $a_j \triangleright a_k \in \Sigma$ . Since (9), or (10), then  $k$  is not  $\Sigma$ -left, or  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since (11), or (12), or (13), or (14), then  $i$  is not  $\Sigma$ -left, or  $k$  is not  $\Sigma$ -right, or  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_i \in \Sigma$ . Suppose  $k$  is  $\Sigma$ -right. Since  $i$  is  $\Sigma$ -left and  $i$  is not  $\Sigma$ -left, or  $k$  is not  $\Sigma$ -right, or  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_i \in \Sigma$ , then  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_i \in \Sigma$ . Since  $i \neq k$ , then  $(a_i \triangleright a_k \leftrightarrow a_k \triangleright a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -left and  $k$  is  $\Sigma$ -right). Since  $a_i \triangleright a_k \in \Sigma$ , or  $a_k \triangleright a_i \in \Sigma$ , then  $a_i \triangleright a_k \in \Sigma$ . Since  $a_j \triangleright a_k \in \Sigma$ , then  $a_i \triangleright a_j \in \Sigma$ , or  $a_j \triangleright a_i \in \Sigma$  (use axiom  $Ax_2$ ). Since  $a_i \bar{\triangleright} a_j \in \Sigma$ , then  $a_j \bar{\triangleright} a_i \in \Sigma$ : a contradiction. Suppose  $k$  is  $\Sigma$ -left. Since  $k$  is not  $\Sigma$ -left, or  $a_k \bar{\triangleright} a_i \in \Sigma$ , then  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since  $a_j \bar{\triangleright} a_i \in \Sigma$ , then  $a_j \bar{\triangleright} a_k \in \Sigma$ , or  $a_k \bar{\triangleright} a_j \in \Sigma$  (use axiom  $Ax_2$ ). Since  $j \neq k$ , then  $(a_j \bar{\triangleright} a_k \leftrightarrow a_k \bar{\triangleright} a_j) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $j$  is  $\Sigma$ -right and  $k$  is  $\Sigma$ -left). Since  $a_j \bar{\triangleright} a_k \in \Sigma$ , or  $a_k \bar{\triangleright} a_j \in \Sigma$ , then  $a_j \bar{\triangleright} a_k \in \Sigma$ : a contradiction.

Case “(3), (5) and  $i = k$ ”. Since (3), then  $a_i \bar{\triangleright} a_j \in \Sigma$ . Since (5), then  $a_k \triangleright a_j \in \Sigma$ . Since  $i = k$ , then  $a_i \triangleright a_j \in \Sigma$ : a contradiction.

Case “(3), (5),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”. Since (3), then  $i$  is  $\Sigma$ -left,  $j$  is  $\Sigma$ -right,  $a_i \bar{\triangleright} a_j \in \Sigma$  and  $a_j \bar{\triangleright} a_i \in \Sigma$ . Since (5), then  $k$  is  $\Sigma$ -left and  $a_k \triangleright a_j \in \Sigma$ . Since (9), or (10), then  $k$  is not  $\Sigma$ -left, or  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since  $k$  is  $\Sigma$ -left, then  $a_k \bar{\triangleright} a_i \in \Sigma$ . Since  $i \neq k$ , then  $(a_i \triangleright a_k \leftrightarrow a_k \bar{\triangleright} a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -left and  $k$  is  $\Sigma$ -left). Since  $a_k \bar{\triangleright} a_i \in \Sigma$ , then  $a_i \triangleright a_k \in \Sigma$ . Since  $j \neq k$ , then  $(a_j \triangleright a_k \leftrightarrow a_k \triangleright a_j) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $j$  is  $\Sigma$ -right and  $k$  is  $\Sigma$ -left). Since  $a_k \triangleright a_j \in \Sigma$ , then  $a_j \triangleright a_k \in \Sigma$ . Since  $a_i \triangleright a_k \in \Sigma$ , then  $a_i \triangleright a_j \in \Sigma$ , or  $a_j \triangleright a_i \in \Sigma$  (use axiom  $Ax_2$ ). Since  $a_i \bar{\triangleright} a_j \in \Sigma$ , then  $a_j \triangleright a_i \in \Sigma$ : a contradiction.

Case “(3), (6) and  $i = k$ ”. Since (3), then  $j$  is  $\Sigma$ -right. Since (6), then  $j$  is  $\Sigma$ -left: a contradiction.

Case “(3), (6),  $i \neq k$ , (7), or (8), (9), or (10) and (11), or (12), or (13), or (14)”. Since (3), then  $j$  is  $\Sigma$ -right. Since (6), then  $j$  is  $\Sigma$ -left: a contradiction.

3. Suppose  $i \neq j$ ,  $i$  does not  $\Sigma$ -precede  $j$  and  $j$  does not  $\Sigma$ -precede  $i$ . Since  $i \neq j$  and  $i$  does not  $\Sigma$ -precede  $j$ , then

- (1)  $i$  is not  $\Sigma$ -right, or (2)  $a_i \triangleright a_j \notin \Sigma$ ,
- (3)  $j$  is not  $\Sigma$ -left, or (4)  $a_j \triangleright a_i \notin \Sigma$ ,
- (5)  $i$  is not  $\Sigma$ -left, or (6)  $j$  is not  $\Sigma$ -right, or (7)  $a_i \bar{\triangleright} a_j \notin \Sigma$ , or (8)  $a_j \bar{\triangleright} a_i \notin \Sigma$ .

Since  $i \neq j$  and  $j$  does not  $\Sigma$ -precede  $i$ , then

- (9)  $j$  is not  $\Sigma$ -right, or (10)  $a_j \triangleright a_i \notin \Sigma$ ,
- (11)  $i$  is not  $\Sigma$ -left, or (12)  $a_i \triangleright a_j \notin \Sigma$ ,
- (13)  $j$  is not  $\Sigma$ -left, or (14)  $i$  is not  $\Sigma$ -right, or (15)  $a_j \bar{\triangleright} a_i \notin \Sigma$ , or (16)  $a_i \bar{\triangleright} a_j \notin \Sigma$ .

The 1st and 5th items imply conditions (2) and (12), i.e.  $a_i \bar{\triangleright} a_j \in \Sigma$ . The 2nd and 4th items imply conditions (4) and (10), i.e.  $a_j \bar{\triangleright} a_i \in \Sigma$ . Thus, conditions (7), (8), (15) and (16) does not hold. Therefore, the 3rd and 6th items imply  $i$  is  $\Sigma$ -right and  $j$  is  $\Sigma$ -right, or  $i$  is  $\Sigma$ -left and  $j$  is  $\Sigma$ -left. Since  $i \neq j$ , then  $(a_i \triangleright a_j \leftrightarrow a_j \triangleright a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -right and  $j$  is  $\Sigma$ -right, or  $i$  is  $\Sigma$ -left and  $j$  is  $\Sigma$ -left). Since  $a_i \bar{\triangleright} a_j \in \Sigma$ , then  $a_j \triangleright a_i \in \Sigma$ : a

contradiction. ■

PROOF OF FACT 4.11. It suffices to note that for all positive integers  $i, j \leq n$ ,  $a_i$  sees  $a_j$  in  $u_\Sigma^r$  iff  $a_i$  sees  $a_j$  in  $u_\Sigma^l$ . ■

PROOF OF FACT 4.12. By Fact 4.11,  $\mathcal{W}_s^T, u_\Sigma^r \models^T a_i \triangleright a_j$  iff  $\mathcal{W}_s^T, u_\Sigma^l \models^T a_i \triangleright a_j$ . Hence, it suffices to demonstrate that  $\mathcal{W}_s^T, u_\Sigma^r \models^T a_i \triangleright a_j$  iff  $a_i \triangleright a_j \in \Sigma$ .

Suppose  $\mathcal{W}_s^T, u_\Sigma^r \models^T a_i \triangleright a_j$  and  $a_i \triangleright a_j \notin \Sigma$ . Since  $\mathcal{W}_s^T, u_\Sigma^r \models^T a_i \triangleright a_j$ , then  $a_i$  sees  $a_j$  in  $u_\Sigma^r$ . Hence,  $S_{u_\Sigma^r}(a_i)$  contains  $x_{u_\Sigma^r}(a_j)$ . Thus,  $i$  is  $\Sigma$ -right and  $] (0, \pi_\Sigma^r(i)), +\infty[$  contains  $(0, \pi_\Sigma^r(j))$ , or  $i$  is  $\Sigma$ -left and  $] -\infty, (0, \pi_\Sigma^r(i)) [$  contains  $(0, \pi_\Sigma^r(j))$ . Therefore, we have to consider two cases.

*Case “ $i$  is  $\Sigma$ -right and  $] (0, \pi_\Sigma^r(i)), +\infty[$  contains  $(0, \pi_\Sigma^r(j))$ ”.* Since  $] (0, \pi_\Sigma^r(i)), +\infty[$  contains  $(0, \pi_\Sigma^r(j))$ , then  $(0, \pi_\Sigma^r(i)) \prec (0, \pi_\Sigma^r(j))$ . Hence,  $\pi_\Sigma^r(i) < \pi_\Sigma^r(j)$ . Thus,  $i$   $\Sigma$ -precedes  $j$  and  $i \neq j$ . Since  $i$   $\Sigma$ -precedes  $j$ ,  $i$  is  $\Sigma$ -right and  $a_i \triangleright a_j \notin \Sigma$ , then  $j$  is  $\Sigma$ -left and  $a_j \triangleright a_i \in \Sigma$ . Since  $i \neq j$ , then  $(a_i \triangleright a_j \leftrightarrow a_j \triangleright a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -right and  $j$  is  $\Sigma$ -left). Since  $a_i \triangleright a_j \notin \Sigma$ , then  $a_j \triangleright a_i \notin \Sigma$ : a contradiction.

*Case “ $i$  is  $\Sigma$ -left and  $] -\infty, (0, \pi_\Sigma^r(i)) [$  contains  $(0, \pi_\Sigma^r(j))$ ”.* Since  $] -\infty, (0, \pi_\Sigma^r(i)) [$  contains  $(0, \pi_\Sigma^r(j))$ , then  $(0, \pi_\Sigma^r(j)) \prec (0, \pi_\Sigma^r(i))$ . Hence,  $\pi_\Sigma^r(j) < \pi_\Sigma^r(i)$ . Thus,  $j$   $\Sigma$ -precedes  $i$  and  $i \neq j$ . Since  $j$   $\Sigma$ -precedes  $i$ ,  $i$  is  $\Sigma$ -left and  $a_i \triangleright a_j \notin \Sigma$ , then  $j$  is  $\Sigma$ -right and  $a_j \triangleright a_i \in \Sigma$ . Since  $i \neq j$ , then  $(a_i \triangleright a_j \leftrightarrow a_j \triangleright a_i) \in \Sigma$  (use axiom  $Ax_1$  and the fact that  $i$  is  $\Sigma$ -left and  $j$  is  $\Sigma$ -right). Since  $a_i \triangleright a_j \notin \Sigma$ , then  $a_j \triangleright a_i \notin \Sigma$ : a contradiction.

Suppose  $a_i \triangleright a_j \in \Sigma$  and  $\mathcal{W}_s^T, u_\Sigma^r \not\models^T a_i \triangleright a_j$ . Since  $\mathcal{W}_s^T, u_\Sigma^r \not\models^T a_i \triangleright a_j$ , then  $a_i$  does not see  $a_j$  in  $u_\Sigma^r$ . Hence,  $S_{u_\Sigma^r}(a_i)$  does not contain  $x_{u_\Sigma^r}(a_j)$ . Thus,  $i$  is  $\Sigma$ -right and  $] (0, \pi_\Sigma^r(i)), +\infty[$  does not contain  $(0, \pi_\Sigma^r(j))$ , or  $i$  is  $\Sigma$ -left and  $] -\infty, (0, \pi_\Sigma^r(i)) [$  does not contain  $(0, \pi_\Sigma^r(j))$ . Therefore, we have to consider two cases.

*Case “ $i$  is  $\Sigma$ -right and  $] (0, \pi_\Sigma^r(i)), +\infty[$  does not contain  $(0, \pi_\Sigma^r(j))$ ”.* Since  $] (0, \pi_\Sigma^r(i)), +\infty[$  does not contain  $(0, \pi_\Sigma^r(j))$ , then  $(0, \pi_\Sigma^r(i)) \not\prec (0, \pi_\Sigma^r(j))$ . Hence,  $\pi_\Sigma^r(i) \not< \pi_\Sigma^r(j)$ . Thus,  $i$  does not  $\Sigma$ -precede  $j$ . Since  $i$  is  $\Sigma$ -right, then  $a_i \triangleright a_j \notin \Sigma$ : a contradiction.

*Case “ $i$  is  $\Sigma$ -left and  $] -\infty, (0, \pi_\Sigma^r(i)) [$  does not contain  $(0, \pi_\Sigma^r(j))$ ”.* Since  $] -\infty, (0, \pi_\Sigma^r(i)) [$  does not contain  $(0, \pi_\Sigma^r(j))$ , then  $(0, \pi_\Sigma^r(j)) \not\prec (0, \pi_\Sigma^r(i))$ . Hence,  $\pi_\Sigma^r(j) \not< \pi_\Sigma^r(i)$ . Thus,  $j$  does not  $\Sigma$ -precede  $i$ . Since  $i$  is  $\Sigma$ -left, then  $a_i \triangleright a_j \notin \Sigma$ : a contradiction. ■

PROOF OF FACT 4.13. Obvious. ■

PROOF OF FACT 4.14. Suppose  $K_{a_i} \Sigma \subseteq \Delta$ . By Fact 4.13, it suffices to demonstrate that  $u_\Sigma^r \equiv_{a_i}^T u_\Delta^r$ , or  $u_\Sigma^r \equiv_{a_i}^T u_\Delta^l$ . Let  $j_1^\Sigma, \dots, j_{l^\Sigma}^\Sigma$  being a list of all the positive integers  $j^\Sigma \leq n$  such that  $a_i \triangleright a_{j^\Sigma} \in \Sigma$  and  $j_1^\Delta, \dots, j_{l^\Delta}^\Delta$  being a list of all the positive integers  $j^\Delta \leq n$  such that  $a_i \triangleright a_{j^\Delta} \in \Delta$ . Since  $K_{a_i} \Sigma \subseteq \Delta$ , then these two lists are equal (use axioms  $Ax_3, Ax_4$  and  $Ax_6$ ). Obviously,

- if  $i$  is  $\Sigma$ -right and  $i$  is  $\Delta$ -right, or  $i$  is  $\Sigma$ -left and  $i$  is  $\Delta$ -left then  $u_\Sigma^r \equiv_{a_i}^T u_\Delta^r$  and  $u_\Sigma^l \equiv_{a_i}^T u_\Delta^l$ ,

- if  $i$  is  $\Sigma$ -right and  $i$  is  $\Delta$ -left, or  $i$  is  $\Sigma$ -left and  $i$  is  $\Delta$ -right then  $u_\Sigma^r \equiv_{a_i}^\tau u_\Delta^l$  and  $u_\Sigma^l \equiv_{a_i}^\tau u_\Delta^r$ .

Hence,  $u_\Sigma^r \equiv_{a_i}^\tau u_\Delta^r$ , or  $u_\Sigma^r \equiv_{a_i}^\tau u_\Delta^l$  and  $u_\Sigma^l \equiv_{a_i}^\tau u_\Delta^r$ , or  $u_\Sigma^l \equiv_{a_i}^\tau u_\Delta^l$ . ■

PROOF OF FACT 4.15. By induction on  $\psi(a_1, \dots, a_k)$ . The argument is similar to that given in Blackburn *et al.* [5, Lemma 4.21]. The case for  $a_i \triangleright a_j$  follows from Fact 4.12. The cases for the Boolean connectives follow from the fact that  $\Sigma$  is an  $n$ -maximal  $L_{min}$ -consistent set of formulas. It remains to deal with the epistemic connective  $K_{a_i}$ . By Lemmas 3.1 and 3.3 and Fact 4.11,  $\mathcal{W}_s^T, u_\Sigma^r \models^\tau K_{a_i} \psi(a_1, \dots, a_n)$  iff  $\mathcal{W}_s^T, u_\Sigma^l \models^\tau K_{a_i} \psi(a_1, \dots, a_n)$ . Hence, it suffices to demonstrate that  $\mathcal{W}_s^T, u_\Sigma^r \models^\tau K_{a_i} \psi(a_1, \dots, a_n)$  iff  $K_{a_i} \psi(a_1, \dots, a_n) \in \Sigma$ .

Suppose  $\mathcal{W}_s^T, u_\Sigma^r \models^\tau K_{a_i} \psi(a_1, \dots, a_n)$  and  $K_{a_i} \psi(a_1, \dots, a_n) \notin \Sigma$ . Since  $K_{a_i} \psi(a_1, \dots, a_n) \notin \Sigma$ , then  $K_{a_i} \Sigma \cup \{\neg \psi(a_1, \dots, a_n)\}$  is an  $L_{min}$ -consistent set of formulas. Hence, by Lemma 4.8, there exists an  $n$ -maximal  $L_{min}$ -consistent set  $\Delta$  of formulas such that  $K_{a_i} \Sigma \cup \{\neg \psi(a_1, \dots, a_n)\} \subseteq \Delta$ . Thus,  $K_{a_i} \Sigma \subseteq \Delta$  and  $\neg \psi(a_1, \dots, a_n) \in \Delta$ . Since  $K_{a_i} \Sigma \subseteq \Delta$ , then by Fact 4.14,  $u_\Sigma^r \equiv_{a_i}^\tau u_\Delta^r$ , or  $u_\Sigma^r \equiv_{a_i}^\tau u_\Delta^l$ . Since  $\neg \psi(a_1, \dots, a_n) \in \Delta$ , then by induction hypothesis,  $\mathcal{W}_s^T, u_\Delta^r \not\models^\tau \psi(a_1, \dots, a_n)$  and  $\mathcal{W}_s^T, u_\Delta^l \not\models^\tau \psi(a_1, \dots, a_n)$ . Since  $u_\Sigma^r \equiv_{a_i}^\tau u_\Delta^r$ , or  $u_\Sigma^r \equiv_{a_i}^\tau u_\Delta^l$ , then  $\mathcal{W}_s^T, u_\Sigma^r \not\models^\tau K_{a_i} \psi(a_1, \dots, a_n)$ : a contradiction.

Suppose  $K_{a_i} \psi(a_1, \dots, a_n) \in \Sigma$  and  $\mathcal{W}_s^T, u_\Sigma^r \not\models^\tau K_{a_i} \psi(a_1, \dots, a_n)$ . Since  $\mathcal{W}_s^T, u_\Sigma^r \not\models^\tau K_{a_i} \psi(a_1, \dots, a_n)$ , then there exists a  $\mathcal{T}$ -world  $v$  in  $\mathcal{W}_s^T$  such that  $u_\Sigma^r \equiv_{a_i}^\tau v$  and  $\mathcal{W}_s^T, v \not\models^\tau \psi(a_1, \dots, a_n)$ . Without loss of generality, we can assume that for all positive integers  $k \leq n$ ,  $x_v(a_k) \in \{(0, l) : l \leq n \text{ is a positive integer}\}$ . Let  $G = \{a_1, \dots, a_n\}$  and  $\vec{u}_\Sigma^r, \vec{v}$  be the  $G$ -vectors associated to  $u_\Sigma^r, v$  in the obvious way. Obviously,  $\mathcal{W}_s^T, u_\Sigma^r \models^\tau \chi_{\vec{u}_\Sigma^r}$  and  $\mathcal{W}_s^T, v \models^\tau \chi_{\vec{v}}$ . Since  $\mathcal{W}_s^T, u_\Sigma^r \models^\tau \chi_{\vec{u}_\Sigma^r}$ , then by Fact 4.12,  $\chi_{\vec{u}_\Sigma^r} \in \Sigma$ . Since  $u_\Sigma^r \equiv_{a_i}^\tau v$ , then  $\vec{u}_\Sigma^r \equiv_{a_i}^G \vec{v}$ . Hence,  $\chi_{\vec{u}_\Sigma^r} \rightarrow \hat{K}_{a_i} \chi_{\vec{v}}$  is an instance of axiom  $Ax_5$ . Since  $\chi_{\vec{u}_\Sigma^r} \in \Sigma$ , then  $\hat{K}_{a_i} \chi_{\vec{v}} \in \Sigma$ . Since  $K_{a_i} \psi(a_1, \dots, a_n) \in \Sigma$ , then  $\hat{K}_{a_i} (\psi(a_1, \dots, a_n) \wedge \chi_{\vec{v}}) \in \Sigma$ . Thus,  $K_{a_i} \Sigma \cup \{\psi(a_1, \dots, a_n) \wedge \chi_{\vec{v}}\}$  is an  $L_{min}$ -consistent set of formulas. Hence, by Lemma 4.8, there exists an  $n$ -maximal  $L_{min}$ -consistent set  $\Delta$  of formulas such that  $K_{a_i} \Sigma \cup \{\psi(a_1, \dots, a_n) \wedge \chi_{\vec{v}}\} \subseteq \Delta$ . Thus,  $\psi(a_1, \dots, a_n) \in \Delta$  and  $\chi_{\vec{v}} \in \Delta$ . Since  $\psi(a_1, \dots, a_n) \in \Delta$ , then by induction hypothesis,  $\mathcal{W}_s^T, u_\Delta^r \models^\tau \psi(a_1, \dots, a_n)$ . Since  $\chi_{\vec{v}} \in \Delta$ , then by Fact 4.12,  $\mathcal{W}_s^T, u_\Delta^r \models^\tau \chi_{\vec{v}}$ . Since  $\mathcal{W}_s^T, v \models^\tau \chi_{\vec{v}}$ , then  $u_\Delta^r Z_n v$ . Since  $\mathcal{W}_s^T, u_\Delta^r \models^\tau \psi(a_1, \dots, a_n)$ , then by Lemmas 3.1 and 3.3,  $\mathcal{W}_s^T, v \models^\tau \psi(a_1, \dots, a_n)$ : a contradiction. ■

PROOF OF LEMMA 5.1. Given a quantified Boolean expression  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$  based on the propositional quantifiers  $\sigma_1, \dots, \sigma_n$  and the Boolean variables  $P_1, \dots, P_n$ , we wish to construct a 1-world  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  and a formula  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  such that  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$ , considered as a quantified Boolean expression, is valid in quantified Boolean logic iff  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$ , considered as a formula, is true for  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  in  $\mathcal{W}_s^1$ . Choose distinct agents  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$ . Let  $G = \{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}$  and  $\psi$  be the conjunction of the following literals based on  $G$ :

- for all positive integers  $i, j \leq n$  such that  $i < j$ , the literals  $a_i \bar{\triangleright} a_j$  and  $a_i \bar{\triangleright} b_j$ ,
- the literal  $a_i \bar{\triangleright} b_i$ ,
- for all positive integers  $i, j \leq n$  such that  $i > j$ , the literals  $a_i \triangleright a_j$  and  $a_i \triangleright b_j$ .

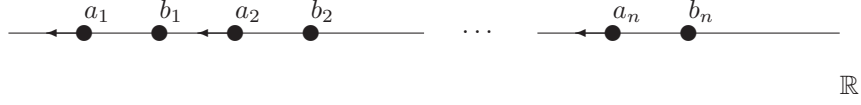


FIG. 15. A simple  $\mathbb{R}$ -world.

Obviously,  $\psi$  is true for a 1-world  $u$  in  $\mathcal{W}_s^1$  iff  $u$  looks like the simple  $\mathbb{R}$ -world depicted in Figure 15. Let  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  be such a 1-world. Now, consider the sequence  $\phi_{n+1}, \phi_n, \dots, \phi_1$  of formulas defined as follows:

- $\phi_{n+1} = \theta(b_1 \triangleright a_1, \dots, b_n \triangleright a_n)$ ,
- for all positive integers  $k \leq n$ ,  $\phi_k = \text{if } \sigma_k = \exists \text{ then } \hat{K}_{a_k}(\psi \wedge \phi_{k+1}) \text{ else } K_{a_k}(\psi \rightarrow \phi_{k+1})$ .

Let  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  be  $\phi_1$ . Remark that  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  and  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  can be computed in logarithmic space. Moreover, the reader may easily verify that  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$ , considered as a quantified Boolean expression, is valid in quantified Boolean logic iff  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$ , considered as a formula, is true for  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  in  $\mathcal{W}_s^1$ . Thus, the validity problem of quantified Boolean logic is reducible to the model checking problem with respect to  $\mathcal{W}_s^1$ .  $\blacksquare$

PROOF OF LEMMA 5.3. The proof is similar to the proof of Lemma 5.1. Given a quantified Boolean expression  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$  based on the propositional quantifiers  $\sigma_1, \dots, \sigma_n$  and the Boolean variables  $P_1, \dots, P_n$ , we wish to construct a 2-world  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  and a formula  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  such that  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$ , considered as a quantified Boolean expression, is valid in quantified Boolean logic iff  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$ , considered as a formula, is true for  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  in  $\mathcal{W}_s^2$ . Choose distinct agents  $a_1, \dots, a_n, a'_1, \dots, a'_n$  and  $b_1, \dots, b_n$ . Let  $G = \{a_1, \dots, a_n, a'_1, \dots, a'_n\} \cup \{b_1, \dots, b_n\}$  and  $\chi$  be the conjunction of the following literals based on  $G$ :

- for all positive integers  $i, j \leq n$  such that  $i < j$ , the literals  $a_i \bar{\triangleright} a_j$ ,  $a_i \bar{\triangleright} a'_j$ ,  $a_i \bar{\triangleright} b_j$ ,  $a'_i \bar{\triangleright} a_j$ ,  $a'_i \bar{\triangleright} a'_j$  and  $a'_i \bar{\triangleright} b_j$ ,
- the literals  $a_i \triangleright a'_i$ ,  $a_i \bar{\triangleright} b_i$ ,  $a'_i \triangleright a_i$  and  $a'_i \bar{\triangleright} b_i$ ,
- for all positive integers  $i, j \leq n$  such that  $i > j$ , the literals  $a_i \triangleright a_j$ ,  $a_i \triangleright a'_j$ ,  $a_i \triangleright b_j$ ,  $a'_i \triangleright a_j$ ,  $a'_i \triangleright a'_j$  and  $a'_i \triangleright b_j$ .

Obviously,  $\chi$  is true for a 2-world  $u$  in  $\mathcal{W}_s^2$  iff  $u$  looks like the simple  $\mathbb{R}^2$ -world depicted in Figure 16. Let  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  be such a 2-world. Now, consider the sequence  $\phi_{n+1}, \phi_n, \dots, \phi_1$  of formulas defined as follows:

- $\phi_{n+1} = \theta(b_1 \triangleright a_1, \dots, b_n \triangleright a_n)$ ,
- for all positive integers  $k \leq n$ ,  $\phi_k = \text{if } \sigma_k = \exists \text{ then } \hat{K}_{a_k}(\chi \wedge \phi_{k+1}) \text{ else } K_{a_k}(\chi \rightarrow \phi_{k+1})$ .

Let  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  be  $\phi_1$ . Remark that  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  and  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  can be computed in logarithmic space. Moreover, the

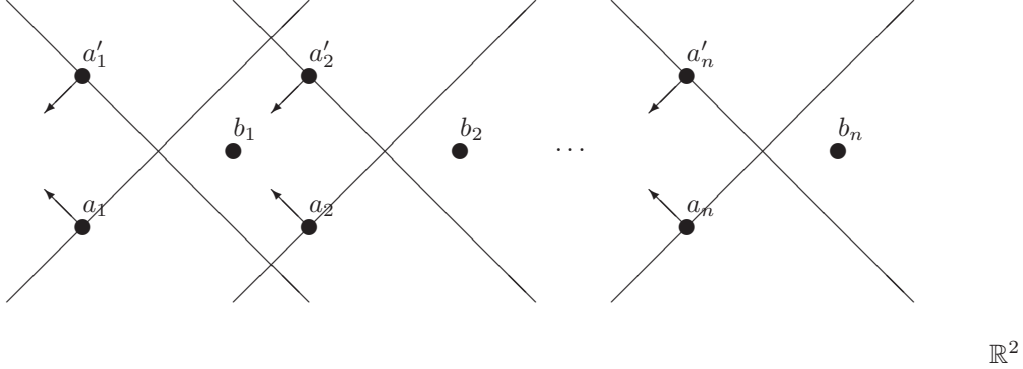


FIG. 16. A simple  $\mathbb{R}^2$ -world.

reader may easily verify that  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$ , considered as a quantified Boolean expression, is valid in quantified Boolean logic iff  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$ , considered as a formula, is true for  $u(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  in  $\mathcal{W}_s^2$ . Thus, the validity problem of quantified Boolean logic is reducible to the model checking problem with respect to  $\mathcal{W}_s^2$ . ■

PROOF OF LEMMA 6.1. Given a quantified Boolean expression  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$  based on the propositional quantifiers  $\sigma_1, \dots, \sigma_n$  and the Boolean variables  $P_1, \dots, P_n$ , we wish to construct a formula  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  such that  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$ , considered as a quantified Boolean expression, is valid in quantified Boolean logic iff  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$ , considered as a formula, is satisfiable in  $\mathcal{W}_s^1$ . Choose distinct agents  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$ . Let  $G = \{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}$  and  $\psi$  be the conjunction of literals based on  $G$  defined in the proof of Lemma 5.1. Let  $\phi_{n+1}, \phi_n, \dots, \phi_1$  be the sequence of formulas defined in the proof of Lemma 5.1. Let  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  be  $\psi \wedge \phi_1$ . Remark that  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  can be computed in logarithmic space. Moreover, the reader may easily verify that  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$ , considered as a quantified Boolean expression, is valid in quantified Boolean logic iff  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$ , considered as a formula, is satisfiable in  $\mathcal{W}_s^1$ . Hence, the validity problem of quantified Boolean logic is reducible to the satisfiability problem with respect to  $\mathcal{W}_s^1$ . ■

PROOF OF LEMMA 6.3. The proof is similar to the proof of Lemma 6.1. Given a quantified Boolean expression  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$  based on the propositional quantifiers  $\sigma_1, \dots, \sigma_n$  and the Boolean variables  $P_1, \dots, P_n$ , we wish to construct a formula  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  such that  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$ , considered as a quantified Boolean expression, is valid in quantified Boolean logic iff  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$ , considered as a formula, is satisfiable in  $\mathcal{W}_s^2$ . Choose distinct agents  $a_1, \dots, a_n, a'_1, \dots, a'_n$  and  $b_1, \dots, b_n$ . Let  $G = \{a_1, \dots, a_n, a'_1, \dots, a'_n\} \cup \{b_1, \dots, b_n\}$  and  $\chi$  be the conjunction of literals based on  $G$  defined in the proof of Lemma 5.3. Let  $\phi_{n+1}, \phi_n, \dots, \phi_1$  be the sequence

of formulas defined in the proof of Lemma 5.3. Let  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  be  $\chi \wedge \phi_1$ . Remark that  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$  can be computed in logarithmic space. Moreover, the reader may easily verify that  $\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n)$ , considered as a quantified Boolean expression, is valid in quantified Boolean logic iff  $\phi(\sigma_1 P_1 \dots \sigma_n P_n \theta(P_1, \dots, P_n))$ , considered as a formula, is satisfiable in  $\mathcal{W}_s^2$ . Hence, the validity problem of quantified Boolean logic is reducible to the satisfiability problem with respect to  $\mathcal{W}_s^2$ . ■