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A short proof of the existence of supercuspidal representations for all reductive p -adic groups

Raphaël Beuzart-Plessis *

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Résumé

Let G be a reductive p -adic group. It can be important for certain global arguments on the trace formula to know that G admits supercuspidal complex representations. We prove that it is always the case. This result has already been established by A. Kret in [K]. Our argument is of a different nature and is based on the Harish-Chandra theory of cusp forms. It ultimately relies on the existence of elliptic maximal tori in G .

Let p be a prime number and let F be a p -adic field (i.e. a finite extension of \mathbb{Q}_p). We denote by \mathcal{O} the ring of integers of F and we fix a uniformizer $\varpi \in \mathcal{O}$. We also denote by $val : F^\times \rightarrow \mathbb{Z}$ the normalized valuation. Let G be a connected reductive group defined over F . We will denote by \mathfrak{g} the Lie algebra of G . A sentence like "Let $P = MN$ be a parabolic subgroup of G " will mean as usual that P is a parabolic subgroup of G defined over F , that N is its unipotent radical and that M is a Levi component of P also defined over F . Also, all subgroups of G that we consider will be implicitly assumed to be defined over F .

Recall that a smooth representation of $G(F)$ is a pair (π, V_π) where V_π is a complex vector space (usually infinite dimensional) and π is a morphism $G(F) \rightarrow GL(V_\pi)$ such that for all vector $v \in V_\pi$ the stabilizer $Stab_{G(F)}(v)$ of v in $G(F)$ is an open subgroup. Let (π, V_π) be a smooth representation of $G(F)$ and let $P = MN$ be a parabolic subgroup of G . The Jacquet module of (π, V_π) with respect to P is the space

$$V_{\pi, N} = V_\pi / V_\pi(N)$$

where $V_\pi(N)$ is the subspace of V_π generated by the elements $v - \pi(n)v$ for all $v \in V_\pi$ and all $n \in N(F)$. It is also the biggest quotient of V_π on which $N(F)$ acts trivially. There is a natural linear action π_N of $M(F)$ on $V_{\pi, N}$ and $(\pi_N, V_{\pi, N})$ is a smooth representation of $M(F)$. The functor $V_\pi \mapsto V_{\pi, N}$ is an exact functor from the category of smooth representations of $G(F)$ to the category of smooth representations of $M(F)$. Indeed, this follows from the following fact (cf proposition III.2.9 of [Re])

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- (1) Let $(N(F)_k)_{k \geq 0}$ be an increasing sequence of compact-open subgroups of $N(F)$ such that $N(F) = \bigcup_{k \geq 0} N(F)_k$ (such sequence always exists). Then a vector $v \in V_\pi$ belongs to $V_{\pi, N}$ if and only if there exists $k \geq 0$ such that

$$\int_{N(F)_k} \pi(n)v dn = 0$$

Let (π, V_π) be an irreducible smooth representation of $G(F)$ (irreducible means that V_π is nonzero and that it has no non-trivial $G(F)$ -invariant subspace). We say that (π, V_π) is supercuspidal if for all proper parabolic subgroup $P = MN$ of G , the Jacquet module $V_{\pi, N}$ is zero. An equivalent conditions is that the coefficients of (π, V_π) are compactly supported modulo the center (cf theorem VI.2.1 of [Re]).

The purpose of this short article is to show the following

Theorem 1 *$G(F)$ admits irreducible supercuspidal representations.*

This theorem has already been proved by A.Kret ([K]). We propose here a different proof. Namely, we will deduce theorem 1 from the existence of nonzero compactly supported cusp forms, in the sense of Harish-Chandra, for the group $G(F)$. Before stating this existence result, we need to introduce some more definitions and notations. We will denote by $C_c^\infty(G(F))$ the space of complex-valued functions on $G(F)$ that are smooth, i.e. locally constant, and compactly supported. We say that a function $f \in C_c^\infty(G(F))$ is a cusp form if for all proper parabolic subgroup $P = MN$ of G we have

$$\int_{N(F)} f(xn)dn = 0, \quad \forall x \in G(F)$$

(these functions are called supercusp forms in [H-C]). We denote by $C_{c, cusp}^\infty(G(F)) \subseteq C_c^\infty(G(F))$ the subspace of cusp forms. As we said, theorem 1 will follows from the following proposition.

Proposition 1 *We have $C_{c, cusp}^\infty(G(F)) \neq 0$.*

Proof that proposition 1 implies theorem 1 : Let us denote by ρ the action of $G(F)$ on $C_c^\infty(G(F))$ given by right translation. Then, $(\rho, C_c^\infty(G(F)))$ is a smooth representation of $G(F)$. Moreover, it is easy to see that the subspace $C_{c, cusp}^\infty(G(F)) \subseteq C_c^\infty(G(F))$ is $G(F)$ -invariant. We claim the following

- (2) For all proper parabolic subgroup $P = MN$ of G , we have

$$C_{c, cusp}^\infty(G(F))_N = 0$$

Let $P = MN$ be a proper parabolic subgroup of G and let us fix an increasing sequence $(N(F)_k)_{k \geq 0}$ of compact-open subgroups of $N(F)$ such that $N(F) = \bigcup_{k \geq 0} N(F)_k$. Let $f \in C_{c,cusp}^\infty(G(F))$. By (1), it suffices to show the existence of an integer $k \geq 0$ such that

$$\int_{N(F)_k} \rho(n) f dn = 0$$

or what amounts to the same

$$(3) \quad \int_{N(F)_k} f(xn) dn = 0, \quad \forall x \in G(F)$$

Since $Supp(f)$ (the support of the function f) is compact, there exists $k \geq 0$ such that

$$(4) \quad Supp(f) \cap Supp(f)(N(F) \setminus N(F)_k) = \emptyset$$

We now show that (3) is satisfied for such a k . Let $x \in G(F)$. If $x \notin Supp(f)N(F)_k$, the term inside the integral (3) is always zero and there is nothing to prove. Assume now that $x \in Supp(f)N(F)_k$. Up to translating x by an element of $N(F)_k$, we may as well assume that $x \in Supp(f)$. Then, by (4) we have $xn \notin Supp(f)$ for all $n \in N(F) \setminus N(F)_k$. It follows that

$$\int_{N(F)_k} f(xn) dn = \int_{N(F)} f(xn) dn$$

But by definition of $C_{c,cusp}^\infty(G(F))$, this last integral is equal to zero. This proves (3) and ends the proof of (2).

We now show how to deduce from (2) that proposition 1 implies theorem 1. Assume that proposition 1 is satisfied. Then, we can find $f \in C_{c,cusp}^\infty(G(F))$ which is nonzero. Denote by V_f the $G(F)$ -invariant subspace of $C_{c,cusp}^\infty(G(F))$ generated by f and let $V \subseteq V_f$ be a maximal $G(F)$ -invariant subspace among those not containing f (Zorn's lemma). Then, V_f/V is a smooth irreducible representation of $G(F)$. We claim that it is supercuspidal. Indeed, let $P = MN$ be a proper parabolic subgroup of G . By (2) and since the Jacquet module's functor is left exact, we have $V_{f,N} = 0$. Hence, since the Jacquet module's functor is also right exact, we have $(V_f/V)_N = 0$. Thus, V_f/V is indeed a supercuspidal representation and this proves theorem 1. ■

Because of the above, we are now left with proving proposition 1. The strategy is to prove first an analog result on the Lie algebra and then lift it to the group by mean of the exponential map. Let $C_c^\infty(\mathfrak{g}(F))$ be the space of complex-valued smooth and compactly supported functions on $\mathfrak{g}(F)$. We say that a function $\varphi \in C_c^\infty(\mathfrak{g}(F))$ is a cusp form if for all proper parabolic subgroup $P = MN$ of G we have

$$\int_{\mathfrak{n}(F)} \varphi(X + N) dN = 0, \quad \forall X \in \mathfrak{g}(F)$$

We denote by $C_{c,cusp}^\infty(\mathfrak{g}(F)) \subseteq C_c^\infty(\mathfrak{g}(F))$ the subspace of cusp forms. The analog of proposition 1 for the Lie algebra is the following lemma

Lemma 1 *We have $C_{c,cusp}^\infty(\mathfrak{g}(F)) \neq 0$.*

Before proving this lemma, we first show how it implies proposition 1.

Proof that lemma 1 implies proposition 1 : Assume that lemma 1 holds. Then, we can find a nonzero function $\varphi \in C_{c,cusp}^\infty(\mathfrak{g}(F))$. The idea is to lift φ to a function on $G(F)$ using the exponential map. Of course, the exponential map is not necessarily defined on the support of φ . Hence, we need first to scale the function φ so that its support becomes small. Let us fix an element $\lambda \in F^\times$ in all what follows. We define the function φ_λ by

$$\varphi_\lambda(X) = \varphi(\lambda^{-1}X), \quad X \in \mathfrak{g}(F)$$

We easily check that φ_λ is still a cusp form. Recall that there exists an open neighborhood $\omega \subseteq \mathfrak{g}(F)$ of 0 on which the exponential map \exp is defined and such that it realizes an F -analytic isomorphism

$$\exp : \omega \simeq \Omega$$

where $\Omega = \exp(\omega)$. Since $Supp(\varphi_\lambda) = \lambda Supp(\varphi)$, for λ sufficiently small, we have

$$Supp(\varphi_\lambda) \subseteq \omega$$

We henceforth assume that λ is that sufficiently small. This allows us to define a function f_λ on $G(F)$ by setting

$$f_\lambda(g) = \begin{cases} \varphi_\lambda(X) & \text{if } g = \exp(X) \text{ for some } X \in \omega \\ 0 & \text{otherwise} \end{cases}$$

for all $g \in G(F)$. Note that we have $f_\lambda \in C_c^\infty(G(F))$ and obviously the function f_λ is nonzero. Hence, we will be done if we can prove the following

(5) If λ is sufficiently small, the function f_λ is a cusp form.

Let us denote by $\log : \Omega \rightarrow \omega$ the inverse of the exponential map. Then, by the Campbell-Hausdorff formula, it is easy to see that we can find a lattice $L \subseteq \omega$ that satisfies the following condition

$$(6) \quad \log(e^X e^Y) \in X + Y + \varpi^{val_L(X)+val_L(Y)} L$$

for all $X, Y \in L$ and where we have set $val_L(X) = \sup\{k \in \mathbb{Z}; X \in \varpi^k L\}$ for all $X \in \mathfrak{g}(F)$. For all integer $n \geq 0$, we set $K_n = \exp(\varpi^n L)$. It is easy to infer from (6) that K_n is an open-compact subgroup of $G(F)$ for all $n \geq 0$. Since φ is smooth and compactly supported, there exists $n_0 \geq 0$ such that translation by $\varpi^{n_0} L$ leaves φ invariant. Also, since φ is compactly

supported, there exists $n_1 \geq 0$ such that $\text{Supp}(\varphi) \subseteq \varpi^{-n_1}L$. We will show that (5) is true provided $\text{val}(\lambda) \geq 2n_1 + n_0$. Assume this is so and set $n = \text{val}(\lambda) - n_1$. Then, we have

$$(7) \quad \text{Supp}(\varphi_\lambda) = \lambda \text{Supp}(\varphi) \subseteq \lambda \varpi^{-n_1}L = \varpi^n L$$

Hence, it follows that

$$(8) \quad \text{Supp}(f_\lambda) \subseteq K_n$$

Let $P = MN$ be a proper parabolic subgroup of G and let $x \in G(F)$. Consider the integral

$$(9) \quad \int_{N(F)} f_\lambda(xn) dn$$

If $xN(F) \cap K_n = \emptyset$, then by (8) the term inside the integral above is identically zero and it follows that the integral is itself equal to zero. Assume now that $xK_n \cap N(F) \neq \emptyset$. Up to translating x by an element of $N(F)$, we may assume that $x \in K_n$. Then, we may write $x = e^X$ for some $X \in \varpi^n L$. Using again (8), and since K_n is a subgroup of $G(F)$, we see that the integral (9) is supported on $K_n \cap N(F)$. Thus, we have

$$(10) \quad \int_{N(F)} f_\lambda(xn) dn = \int_{K_n \cap N(F)} f_\lambda(e^X n) dn$$

Set $L_N = L \cap \mathfrak{n}(F)$. Then, if we normalize measures correctly, the exponential map induces a measure preserving isomorphism $\varpi^n L_N \simeq K_n \cap N(F)$ so that we have

$$(11) \quad \int_{K_n \cap N(F)} f_\lambda(e^X n) dn = \int_{\varpi^n L_N} f_\lambda(e^X e^N) dN = \int_{\varpi^n L} \varphi_\lambda(\log(e^X e^N)) dN$$

By (6), for all $N \in \varpi^n L_N$ we have

$$(12) \quad \log(e^X e^N) \in X + N + \varpi^{2n}L$$

Moreover, since φ is invariant by translation by $\varpi^{n_0}L$, the function φ_λ is invariant by translation by $\lambda \varpi^{n_0}L = \varpi^{n+n_1+n_0}L$ (recall that $n = \text{val}(\lambda) - n_1$). As $\text{val}(\lambda) \geq 2n_1 + n_0$, we also have $n \geq n_1 + n_0$. So finally, the function φ_λ is invariant by translation by $\varpi^{2n}L$. Thus, by (12), we have

$$\varphi_\lambda(\log(e^X e^N)) = \varphi_\lambda(X + N)$$

for all $N \in \varpi^n L_N$. By (10) and (11), it follows that

$$(13) \quad \int_{N(F)} f_\lambda(xn) dn = \int_{\varpi^n L} \varphi_\lambda(X + N) dN$$

By (7) and since $X \in \varpi^n L$, the function $N \in \mathfrak{n}(F) \mapsto \varphi_\lambda(X + N)$ is supported on $\varpi^n L_N$. Hence, we have

$$\int_{\varpi^n L} \varphi_\lambda(X + N) dN = \int_{\mathfrak{n}(F)} \varphi_\lambda(X + N) dN$$

Since φ_λ is a cusp form, this last integral is zero. Hence, by (13) the integral (9) is also zero. Since it is true for all $x \in G(F)$ and all proper parabolic subgroup $P = MN$ of G , this shows that f_λ is a cusp form. Hence, (5) is indeed satisfied as soon as $\text{val}(\lambda) \geq 2n_1 + n_0$ and this ends the proof that lemma 1 implies proposition 1. ■

It now only remains to prove lemma 1. Recall that a maximal torus T in G is said to be elliptic if $A_T = A_G$, where A_T and A_G denotes the maximal split subtorus in T and the center of G respectively. The proof of lemma 1 will ultimately rely on the following existence result (cf Theorem 6.21 of [PR]) :

Theorem 2 *G admits an elliptic maximal torus.*

Proof of lemma 1 : Let us fix a symmetric non-degenerate bilinear form B on $\mathfrak{g}(F)$ which is $G(F)$ -invariant. Such a bilinear form is easy to construct. On $\mathfrak{g}_{der}(F)$, the derived subalgebra of $\mathfrak{g}(F)$, we have the Killing form B_{Kil} which is symmetric $G(F)$ -invariant and non-degenerate. Hence, we may take $B = B_{\mathfrak{z}} \oplus B_{Kil}$ where $B_{\mathfrak{z}}$ is any non-degenerate symmetric bilinear form on $\mathfrak{z}_G(F)$, the center of $\mathfrak{g}(F)$. Let us also fix a non-trivial continuous additive character $\psi : F \rightarrow \mathbb{C}^\times$. Using those, we can define the Fourier transform on $C_c^\infty(\mathfrak{g}(F))$ by

$$\widehat{\varphi}(Y) = \int_{\mathfrak{g}(F)} \varphi(X) \psi(B(X, Y)) dX, \quad \varphi \in C_c^\infty(\mathfrak{g}(F)), Y \in \mathfrak{g}(F)$$

Of course, this Fourier transform also depends on the choice of a Haar measure on $\mathfrak{g}(F)$. More generally, if V is a subspace of $\mathfrak{g}(F)$ and V^\perp denotes the orthogonal of V with respect to B , we can also define a Fourier transform $C_c^\infty(V) \rightarrow C_c^\infty(\mathfrak{g}(F)/V^\perp)$, $\varphi \mapsto \widehat{\varphi}$, by setting

$$\widehat{\varphi}(Y) = \int_V \varphi(Y) \psi(B(X, Y)) dY, \quad X \in \mathfrak{g}(F)/V^\perp$$

where again we need to choose a Haar measure on V . It is easy to check that for compatible choices of Haar measures, the following diagram commutes

$$\begin{array}{ccc} C_c^\infty(\mathfrak{g}(F)) & \xrightarrow{FT} & C_c^\infty(\mathfrak{g}(F)) \\ \downarrow \text{res}_V & & \downarrow \int_{V^\perp} \\ C_c^\infty(V) & \xrightarrow{FT} & C_c^\infty(\mathfrak{g}(F)/V^\perp) \end{array}$$

where the horizontal arrows are Fourier transforms, the left vertical arrow is given by restriction to V and the right vertical arrow is given by integration over the cosets of V^\perp . For $P = MN$ a parabolic subgroup of G , we have $\mathfrak{p}(F)^\perp = \mathfrak{n}(F)$. The commutation of the above diagram in this particular case gives us (for some compatible choices of Haar measures) the following formula

$$(14) \quad \int_{\mathfrak{n}(F)} \widehat{\varphi}(X + N) dN = \int_{\mathfrak{p}(F)} \varphi(Y) \psi(B(X, Y)) dY$$

for all $\varphi \in C_c^\infty(\mathfrak{g}(F))$ and all $X \in \mathfrak{g}(F)$.

Let T_{ell} be an elliptic maximal torus of G whose existence is insured by theorem 2. Let \mathfrak{t}_{ell} be its Lie algebra and $\mathfrak{t}_{ell,reg} = \mathfrak{t}_{ell} \cap \mathfrak{g}_{reg}$ be the subset of G -regular elements in \mathfrak{t}_{ell} . Denote by $\mathfrak{t}_{ell,reg}(F)^G$ the subset of elements in $\mathfrak{g}_{reg}(F)$ that are $G(F)$ -conjugated to an element of $\mathfrak{t}_{ell,reg}(F)$. Then, $\mathfrak{t}_{ell,reg}(F)^G$ is an open subset of $\mathfrak{g}(F)$ (since the map $T_{ell}(F) \backslash G(F) \times \mathfrak{t}_{ell,reg}(F) \rightarrow \mathfrak{g}(F)$, $(g, X) \mapsto g^{-1}Xg$, is everywhere submersive). In particular, we can certainly find a non-zero function $\varphi \in C_c^\infty(\mathfrak{g}(F))$ whose support is contained in $\mathfrak{t}_{ell,reg}(F)^G$. Let us fix such a function φ . We claim the following

(15) The function $\widehat{\varphi}$ is a cusp form.

Indeed, let $P = MN$ be a proper parabolic subgroup of G and let $X \in \mathfrak{g}(F)$. Then, we need to see that the following integral

$$\int_{\mathfrak{n}(F)} \widehat{\varphi}(X + N) dN$$

is zero. By (14), this integral is equal to

$$\int_{\mathfrak{p}(F)} \varphi(Y) \psi(B(X, Y)) dY$$

Hence, we only need to show that $Supp(\varphi) \cap \mathfrak{p}(F) = \emptyset$. By definition of φ , it even suffices to see that $\mathfrak{t}_{ell,reg}(F)^G \cap \mathfrak{p}(F) = \emptyset$. But this follows immediately from the fact that P being proper, it doesn't contain any elliptic maximal torus of G . ■

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