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► **To cite this version:**

Fabrizio Durante, Stéphane Girard, Gildas Mazo. Marshall–Olkin type copulas generated by a global shock. *Journal of Computational and Applied Mathematics*, 2016, 296, pp.638–648. 10.1016/j.cam.2015.10.022 . hal-01138228v2

**HAL Id: hal-01138228**

**<https://hal.science/hal-01138228v2>**

Submitted on 14 Oct 2016

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# Marshall–Olkin type copulas generated by a global shock<sup>\*</sup>

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## Abstract

A way to transform a given copula by means of a univariate function is presented. The resulting copula can be interpreted as the result of a global shock affecting all the components of a system modeled by the original copula. The properties of this copula transformation from the perspective of semi–group action are presented, together with some investigations about the related tail behavior. Finally, the whole methodology is applied to model risk assessment.

*Keywords:* Copula; Marshall–Olkin distributions; Shock models.

*2010 Mathematics Subject Classification:* 62H05, 62G32.

## 1 Introduction

Starting with the seminal paper by Marshall and Olkin (1967), Marshall–Olkin distributions (and copulas) have been extensively exploited for modeling multivariate random vectors. **As is known, these distributions arise from an intuitive interpretation in terms of shock models. In fact, a random vector is said to follow a Marshall–Olkin distribution if its components are interpreted as future failure times which are defined as the minimum of independent, exponential arrival times of exogenous shocks.**

Starting with these ideas, different extensions of Marshall–Olkin distributions have been provided in the literature by supposing, for instance, that the shocks follows specific distributions or fail to be independent. See, for instance, Mai and Scherer (2012); Cherubini et al. (2015) and references therein and recent contributions by Li and Pellerey (2011); Kundu et al. (2014); Lin and Li (2014); Ozkut and Bayramoglu (2014). Durante et al. (2015b) calls *Marshall–Olkin machinery* the common stochastic mechanism that drives many of these extensions.

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<sup>\*</sup>This version: September 28, 2015

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Here we are interested in a Marshall–Olkin–type copula generated by a simple mechanism. Given a set of (continuous) random variables with copula  $C$ , we assume that their common behavior is modified by a shock that affects all the variables at the same time. This results in a modification of the copula  $C$  by means of a function  $f$ , which depends on the shock distribution. Despite its simplicity, this modification has several advantages since, for instance, it allows to generate models with various tail dependencies and singularities, as will be clarified in the sequel.

Specifically, in section 2 we present the basic properties of this model and their connections with several results already presented in the literature. Section 3 focuses on the interpretation of this transformation as action of a semigroup of real–valued functions on the class of copulas. The tail behavior induced by the transformation is considered in Section 4. Section 5 illustrates a possible application in the framework of model risk assessment.

## 2 The model and its first properties

In the following, we use standard definitions and properties of copulas, as they can be found, for instance, in (Durante and Sempi, 2015; Joe, 2014; Nelsen, 2006).

Let  $\mathcal{F}$  be the class of increasing and continuous functions  $f: [0, 1] \rightarrow [0, 1]$  such that  $f(1) = 1$  and  $\text{id}/f$  is increasing, where  $\text{id}$  denotes the identity function on  $[0, 1]$ . **The elements of  $\mathcal{F}$  are anti–star–shaped functions (see, e.g., Singpurwalla (2006); Marshall et al. (2011)), i.e. they are characterized by the property  $f(\alpha t) \geq \alpha f(t)$  for every  $\alpha \in [0, 1]$ .** Moreover, if  $f \in \mathcal{F}$ , then for all  $t \in [0, 1]$ ,  $f(t)/t \geq f(1)/1 = 1$ , from which it follows that  $f(t) \geq t$  for all  $t \in [0, 1]$ .

Let  $\mathcal{C}_d$  be the set of all  $d$ -variate copulas.

**Definition 2.1.** For all  $f \in \mathcal{F}$  and  $C \in \mathcal{C}_d$ , the function  $T(f, C): [0, 1]^d \rightarrow [0, 1]$  given, for every  $(u_1, \dots, u_d) \in (0, 1]^d$ , by

$$T(f, C)(u_1, \dots, u_d) = C(f(u_1), \dots, f(u_d)) \frac{\min(u_1, \dots, u_d)}{f(\min(u_1, \dots, u_d))}, \quad (2.1)$$

while  $T(f, C)(u_1, \dots, u_d) = 0$  if  $u_i = 0$  for at least one index  $i \in \{1, \dots, d\}$ , is called *shock transformation of  $C$  via  $f$* .

Since  $T(f, C)$  can be rewritten as

$$T(f, C)(u_1, \dots, u_d) = C(f(u_1), \dots, f(u_d)) M_d(g(u_1), \dots, g(u_d))$$

for all  $(u_1, \dots, u_d) \in [0, 1]^d$  with  $f \cdot g = \text{id}$ , and  $M_d(u_1, \dots, u_d) = \min(u_1, \dots, u_d)$  is the Hoeffding–Fréchet upper bound, it can be interpreted as a particular case of the construction method introduced in (Liebscher, 2008, Theorem 2.1). Thus, the following result easily follows.

**Proposition 2.1.** For all  $f \in \mathcal{F}$  and  $C \in \mathcal{C}_d$ ,  $T(f, C)$  is a copula.

Copula models of type (2.1) extend the bivariate dual extended Marshall–Olkin model by Pinto and Kolev (2015) to the multivariate framework. Moreover, they can be also interpreted as a particular case of the construction principle considered by Durante et al. (2010b). In particular, from the latter reference, the following stochastic interpretation can be derived.

**Proposition 2.2.** *Let  $Y$  be a random variable whose distribution function is  $g$  and let  $(\tilde{X}_1, \dots, \tilde{X}_d)$  be any random vector with copula  $C$  independent of  $Y$ . Denote by  $\tilde{F}_1, \dots, \tilde{F}_d$  be the univariate marginal distributions corresponding to  $\tilde{X}_1, \dots, \tilde{X}_d$  and define  $f = \text{id}/g$ . If  $X_i = f^{-1}(\tilde{F}_i(\tilde{X}_i))$  then  $T(f, C)$  of eq. (2.1) is the distribution function of the random vector  $(Z_1, \dots, Z_d)$  such that  $Z_i = \max\{X_i, Y\}$  for every  $i \in \{1, 2, \dots, d\}$ .*

*Proof.* Since, under independence assumptions, we can write

$$\mathbb{P}(Z_1 \leq u_1, \dots, Z_d \leq u_d) = \mathbb{P}(X_1 \leq u_1, \dots, X_d \leq u_d) \cdot \mathbb{P}(Y \leq \min(u_1, \dots, u_d)),$$

the result follows from the given assumptions.  $\square$

The latter result provides a useful and easy-to-implement algorithm for generating copulas of type (2.1) once an algorithm for generating the starting copula  $C$  is available.

It can be easily proved that, for all  $f \in \mathcal{F}$  and  $C \in \mathcal{C}_d$ :

$$\begin{aligned} T(\text{id}, C) &= C, & T(1, C) &= M_d, \\ T(f, M_d) &= M_d, & T(f, \Pi_d) &= C_f, \end{aligned} \tag{2.2}$$

where  $\Pi_d(u_1, \dots, u_d) = u_1 \dots u_d$  is the independence copula, while  $C_f$  is the copula introduced by Durante et al. (2007) and defined by:

$$C_f(u_1, \dots, u_d) = u_{(1)} \prod_{i=2}^d f(u_{(i)}) \tag{2.3}$$

where  $u_{(1)}, \dots, u_{(d)}$  denotes the order statistics of  $(u_1, \dots, u_d) \in [0, 1]^d$ . See also (Durante and Salvadori, 2010; Mai et al., 2015) and (Durante, 2006; Durante et al., 2008; Durante and Okhrin, 2015) for the bivariate case.

The stochastic mechanism at the basis of eq. (2.1) generally produces a copula  $T(f, C)$  that has a singular component provided that  $f \neq \text{id}$ . In fact, in such a case, the first derivatives  $T(f, C)$  have jumps. The singular component can be easily computed in the bivariate case.

**Proposition 2.3.** *Let  $f$  be in  $\mathcal{F}$  such that both  $f$  and  $g = \text{id}/f$  are absolutely continuous. Let  $C \in \mathcal{C}_2$  be absolutely continuous. Then, the singular component of  $T(f, C)$  is given by*

$$S(x, y) = \int_0^{\min(x, y)} g'(u) C(f(u), f(u)) du,$$

for all  $(x, y) \in [0, 1]^2$ .

*Proof.* The absolutely continuous component of  $T(f, C)$  is given for all  $(x, y) \in [0, 1]^2$  by

$$A(x, y) = \int_0^x \int_0^y \frac{\partial^2 T(f, C)}{\partial x \partial y}(u, v) dudv.$$

Let us assume  $x \leq y$ , the other case being similar, and consider the expansion

$$\begin{aligned} A(x, y) &= \int_0^x \int_0^u \frac{\partial^2 T(f, C)}{\partial x \partial y}(u, v) dv du + \int_0^x \int_u^y \frac{\partial^2 T(f, C)}{\partial x \partial y}(u, v) dv du \\ &=: A_1(x, y) + A_2(x, y). \end{aligned}$$

Let us first focus on  $A_1(x, y)$ . The integration is performed on the half-space  $\{(u, v) \in [0, 1]^2 : u \geq v\}$  where  $T(f, C)(u, v) = C(f(u), f(v))g(v)$  and thus

$$\begin{aligned} A_1(x, y) &= \int_0^x \left[ \frac{\partial C}{\partial x}(f(u), f(v)) f'(u) g(v) \right]_0^u du \\ &= \int_0^x \frac{\partial C}{\partial x}(f(u), f(u)) f'(u) g(u) du - g(0) \int_0^x \frac{\partial C}{\partial x}(f(u), f(0)) f'(u) du \\ &=: A_3(x) + g(0)C(f(0), f(0)) - g(0)C(f(x), f(0)). \end{aligned}$$

Remarking that  $g(0)C(f(x), f(0)) \leq g(0)M(f(x), f(0)) = g(0)f(0) = 0$ , it follows that

$$A_1(x, y) = A_3(x) = \int_0^x \frac{\partial C}{\partial x}(f(u), f(u)) f'(u) g(u) du.$$

Let us now turn to  $A_2(x, y)$ . The integration is performed on the half-space  $\{(u, v) \in [0, 1]^2 : u \leq v\}$  where  $T(f, C)(u, v) = C(f(u), f(v))g(u)$  and thus

$$\begin{aligned} A_2(x, y) &= \int_0^x \left[ \frac{\partial C}{\partial x}(f(u), f(v)) f'(u) g(u) + g'(u) C(f(u), f(v)) \right]_u^y du \\ &= \int_0^x \left[ \frac{\partial C}{\partial x}(f(u), f(y)) f'(u) g(u) + g'(u) C(f(u), f(y)) \right] du \\ &\quad - \int_0^x g'(u) C(f(u), f(u)) du - A_3(x) \\ &= C(f(x), f(y))g(x) - C(f(0), f(y))g(0) - \int_0^x g'(u) C(f(u), f(u)) du - A_3(x) \\ &= T(f, C)(x, y) - \int_0^x g'(u) C(f(u), f(u)) du - A_3(x). \end{aligned}$$

As a conclusion, when  $x \leq y$ ,

$$A(x, y) = T(f, C)(x, y) - \int_0^x g'(u) C(f(u), f(u)) du,$$

and the result is proved.  $\square$

Given  $C \in \mathcal{C}_2$  and  $f \in \mathcal{F}$ , one may wonder whether the popular measures of concordance (see (Nelsen, 2006, Section 5.1) for examples) like Spearman's  $\rho$  and Kendall's  $\tau$  associated with  $C$  increase (or decrease) when applying the transformation  $T(f, C)$ . Actually, as can be seen from next examples, depending on  $C$ , both cases are possible.

**Example 2.1.** Consider the copula  $C_1$  that distributes uniformly the mass on the segments joining  $(0, 0.5)$  and  $(0.5, 0)$ , and  $(1, 0.5)$  and  $(0.5, 1)$ . Let  $T(f, C_1)$  be the copula obtained from eq. (2.1) when  $f(t) = \sqrt{t}$ . Then  $\rho(C_1) = 0.5 < \rho(T(f, C_1)) \approx 0.58$ . See Figure 1.

**Example 2.2.** Consider the copula  $C_2$  that distributes uniformly the mass on the segments joining  $(0, 0)$  and  $(0.5, 0.5)$ , and  $(1, 0.5)$  and  $(0.5, 1)$ . Let  $T(f, C_2)$  be the copula obtained from eq. (2.1) when  $f(t) = \sqrt{t}$ . Then  $\rho(C_2) = 0.75 > \rho(T(f, C_2)) \approx 0.60$ . See Figure 1.

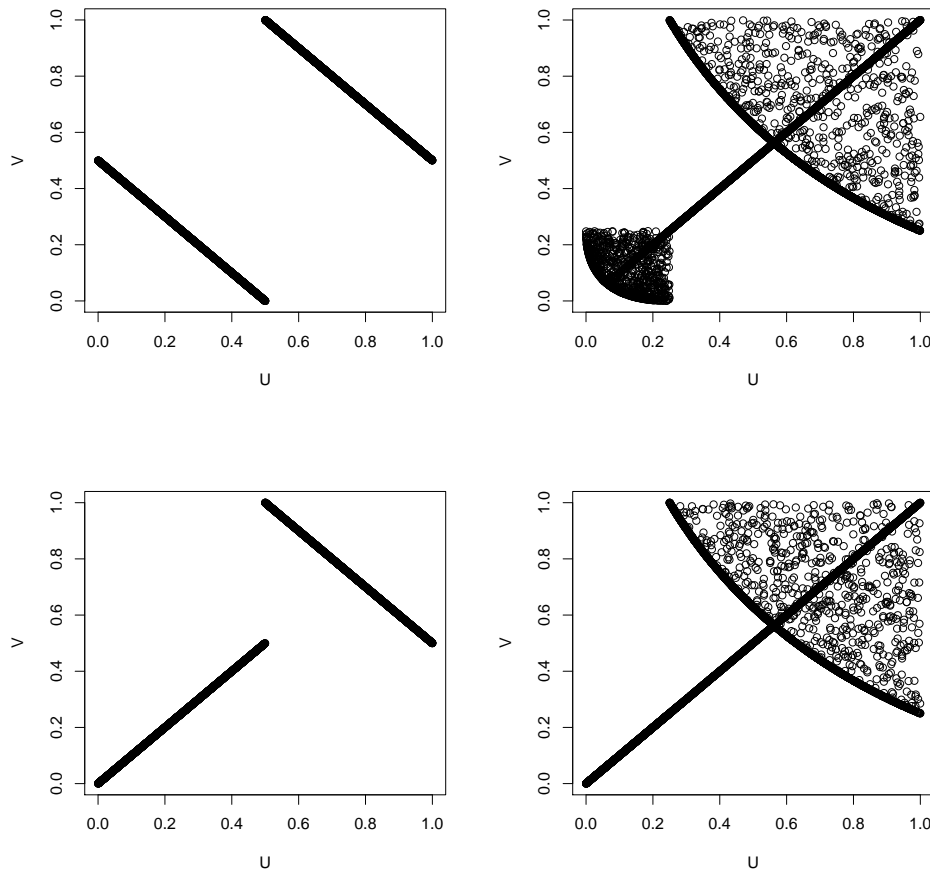


Figure 1: Scatterplot of 5000 points from copulas  $C_1$  (top left),  $T(f, C_1)$  (top right) described in Example 2.1 and  $C_2$  (bottom left),  $T(f, C_2)$  (bottom right) described in Example 2.2.

In order to allow an ordering in the measures of concordance, it is enough to ensure that, given  $C \in \mathcal{C}_2$  and  $f \in \mathcal{F}$ ,  $T(f, C) \succ C$  (or  $T(f, C) \prec C$ ). Recall that  $\succ$  is the symbol for concordance ordering (see, e.g., Durante and Sempi (2015)), i.e.  $C_1 \succ C_2$  means that  $C_1(u, v) \geq C_2(u, v)$  for all  $(u, v) \in [0, 1]^2$ . A first answer is given by the next proposition.

**Proposition 2.4.** *If  $C \in \mathcal{C}_2$  verifies*

$$\frac{C(u_1, v_1)}{M(u_1, v_1)} \leq \frac{C(u_2, v_2)}{M(u_2, v_2)} \text{ for all } u_1 \leq u_2 \text{ and } v_1 \leq v_2, \quad (2.4)$$

*then  $T(f, C) \succ C$  for every  $f \in \mathcal{F}$ .*

*Proof.* Clearly,  $T(f, C)(u, v) \geq C(u, v)$  for all  $(u, v) \in [0, 1]^2$  is equivalent to

$$\frac{C(f(u), f(v))}{f(M(u, v))} \geq \frac{C(u, v)}{M(u, v)} \quad (2.5)$$

for all  $(u, v) \in [0, 1]^2$ . Since  $f(x) \geq x$  for all  $x \in [0, 1]$ , it is readily seen that (2.4) implies (2.5).  $\square$

As a straightforward consequence of Proposition 2.4, we get that (2.4) implies  $\rho(T(f, C)) \geq \rho(C)$  and  $\tau(T(f, C)) \geq \tau(C)$  (see, e.g., Scarsini (1984)).

A second sufficient condition for  $T(f, C) \succ C$  can be deduced from (Liebscher, 2011, Proposition 0.1(b)) and is given by:

$$C(u_1, v_1)C(u_2, v_2) \geq C(u_1u_2, v_1v_2) \text{ for all } (u_1, u_2, v_1, v_2) \in [0, 1]^4. \quad (2.6)$$

The following lemma establishes that (2.4) is a weaker condition than (2.6).

**Lemma 2.1.** *Condition (2.6) implies (2.4) but condition (2.4) does not imply (2.6).*

*Proof.* Suppose that condition (2.6) holds and let  $u_1 \leq u_2, v_1 \leq v_2$ . Let us also assume without loss of generality that  $u_1 \geq v_1$ , the other case being similar. We thus have

$$\begin{aligned} \frac{C(u_1, v_1)}{M(u_1, v_1)} &= \frac{1}{v_1} C(u_1, M(u_2, v_2) v_1 / M(u_2, v_2)) \\ &\leq \frac{1}{v_1} C(u_1, M(u_2, v_2)) C(1, v_1 / M(u_2, v_2)) = \frac{C(u_1, M(u_2, v_2))}{M(u_2, v_2)} \\ &\leq \frac{C(u_1, v_2)}{M(u_2, v_2)} \leq \frac{C(u_2, v_2)}{M(u_2, v_2)} \end{aligned}$$

which corresponds to condition (2.4).

To show that, conversely, condition (2.4) does not imply (2.6), consider the Cuadras-Augé copula:  $C(u, v) = (uv)^\theta M(u, v)^{1-\theta}$  with  $\theta \in [0, 1]$ . Now,  $C(u, v)/M(u, v) = \max(u, v)^\theta$  and thus (2.4) holds. Thus, condition (2.6) is equivalent in this particular case to  $M(u_1, v_1)M(u_2, v_2) \geq M(u_1u_2, v_1v_2)$  which is false in general.  $\square$

Finally, if  $C$  is positive lower orthant dependent (PLOD), i.e.  $C \succ \Pi_d$ , then, for every  $f \in \mathcal{F}$ ,  $T(f, C) \succ C_f \succ \Pi_d$ , where  $C_f$  is given by (2.3).

### 3 Semi-group action interpretation of the model

At a more abstract level, eq. (2.1) defines a mapping  $T: \mathcal{F} \times \mathcal{C}_d \rightarrow \mathcal{C}_d$ . Since the set  $\mathcal{F}$  equipped with function composition is a semi-group with identity and, for all  $f_1, f_2 \in \mathcal{F}$  and  $C \in \mathcal{C}_d$

$$T(f_1 \circ f_2, C) = T(f_1, T(f_2, C)),$$

then  $T$  is the action of  $\mathcal{F}$  over the class of copulas  $\mathcal{C}_d$ . Together with  $T$ , one can also consider the associated mappings  $T(f, \cdot): \mathcal{C}_d \rightarrow \mathcal{C}_d$  and  $T(\cdot, C): \mathcal{F} \rightarrow \mathcal{F}$  for fixed  $f$  and  $C$ . Such mappings are continuous when  $\mathcal{F}$  and  $\mathcal{C}_d$  are equipped with the supremum norm, as the following result shows.

**Proposition 3.1.**

- (i) Let  $(C_1, C_2) \in \mathcal{C}_d^2$ . Then  $\|T(f, C_1) - T(f, C_2)\|_\infty \leq \|C_1 - C_2\|_\infty$  for all  $f \in \mathcal{F}$ .
- (ii) Let  $(f_1, f_2) \in \mathcal{F}^2$ . Then,  $\|T(f_1, C) - T(f_2, C)\|_\infty \leq (d+1)\|f_1 - f_2\|_\infty$  for all  $C \in \mathcal{C}_d$ .

*Proof.*

(i) Consider  $g = \text{id}/f$  with  $f \in \mathcal{F}$ , and  $C_1, C_2 \in \mathcal{C}_d$ . Then

$$\begin{aligned} & \sup_{\mathbf{u} \in [0,1]^d} |T(f, C_1)(u_1, \dots, u_d) - T(f, C_2)(u_1, \dots, u_d)| \\ &= \sup_{\mathbf{u} \in [0,1]^d} |C_1(f(u_1), \dots, f(u_d)) - C_2(f(u_1), \dots, f(u_d))| \min(g(u_1), \dots, g(u_d)) \\ &= \sup_{\mathbf{x} \in [f(0), 1]^d} |C_1(x_1, \dots, x_d) - C_2(x_1, \dots, x_d)| \min\left(\frac{f^{-1}(x_1)}{x_1}, \dots, \frac{f^{-1}(x_d)}{x_d}\right), \end{aligned}$$

where  $f^{-1}$  denotes the quantile inverse of  $f$  (Embrechts and Hofert, 2013). Besides,  $f(u) \geq u$  for all  $u \in [0, 1]$  and thus  $x \geq f^{-1}(x)$  for all  $x \in [f(0), 1]$ . As a consequence,

$$\begin{aligned} & \sup_{\mathbf{u} \in [0,1]^d} |T(f, C_1)(u_1, \dots, u_d) - T(f, C_2)(u_1, \dots, u_d)| \\ & \leq \sup_{\mathbf{x} \in [f(0), 1]^d} |C_1(x_1, \dots, x_d) - C_2(x_1, \dots, x_d)| \\ & \leq \sup_{\mathbf{u} \in [0,1]^d} |C_1(u_1, \dots, u_d) - C_2(u_1, \dots, u_d)|, \end{aligned}$$

and the conclusion follows.

(ii) Let  $f_1, f_2$  belong to  $\mathcal{F}$ ,  $g_1 = \text{id}/f_1$ ,  $g_2 = \text{id}/f_2$ , and  $C \in \mathcal{C}_d$ . For all  $(u_1, \dots, u_d) \in [0, 1]^d$ , the following expansion holds:

$$\begin{aligned} & T(f_1, C)(u_1, \dots, u_d) - T(f_2, C)(u_1, \dots, u_d) \\ &= \min(g_2(u_1), \dots, g_2(u_d))(C(f_1(u_1), \dots, f_1(u_d)) - C(f_2(u_1), \dots, f_2(u_d))) \\ & \quad + C(f_1(u_1), \dots, f_1(u_d))(\min(g_1(u_1), \dots, g_1(u_d)) - \min(g_2(u_1), \dots, g_2(u_d))) \\ &=: \Delta_1(u_1, \dots, u_d) + \Delta_2(u_1, \dots, u_d). \end{aligned}$$



From (Nelsen, 2006, Theorem 2.2.4), we have

$$|C(f_1(u_1), \dots, f_1(u_d)) - C(f_2(u_1), \dots, f_2(u_d))| \leq \sum_{i=1}^d |f_1(u_i) - f_2(u_i)| \leq d \|f_1 - f_2\|_\infty.$$

It follows that  $\|\Delta_1\|_\infty \leq d \|f_1 - f_2\|_\infty$ . Let us now focus on  $\Delta_2$ . Introducing  $u_{(1)} = \min(u_1, \dots, u_d)$ , we have

$$\begin{aligned} \Delta_2(u_1, \dots, u_d) &= C(f_1(u_1), \dots, f_1(u_d))(g_1(u_{(1)}) - g_2(u_{(1)})) \\ &= C(f_1(u_1), \dots, f_1(u_d))u_{(1)} \frac{(f_2(u_{(1)}) - f_1(u_{(1)}))}{f_1(u_{(1)})f_2(u_{(1)})}. \end{aligned}$$

Since  $C \leq M_d$ , it follows that

$$|\Delta_2(u_1, \dots, u_d)| \leq |f_2(u_{(1)}) - f_1(u_{(1)})| \frac{u_{(1)}}{f_2(u_{(1)})} = g_2(u_{(1)}) |f_2(u_{(1)}) - f_1(u_{(1)})|,$$

and therefore  $\|\Delta_2\|_\infty \leq \|f_1 - f_2\|_\infty$ . The result is hence proved.  $\square$

For a fixed  $f$ , one may define recursively the following operation. For a given  $C \in \mathcal{C}_d$  and  $f \in \mathcal{F}$ , we may define recursively the following operation in  $\mathcal{C}$ .

**Definition 3.1.** For a fixed  $f \in \mathcal{F}$ , we define, for every  $C \in \mathcal{C}$ ,

$$T^n(f, C) := T^{n-1}(f, T(f, C))$$

for every  $n \geq 2$ , with  $T^1 := T$ .

In particular, for fixed  $f$  and  $C$ , the set  $\{T^n(f, C) : n \geq 1\}$  describes the orbit of  $C$  under successive applications of the operator  $T(f, \cdot)$ . Due to the definition of  $T$ , it easily follows that  $T^n(f, C) = T(f^n, C)$ , where  $f^n := f \circ \dots \circ f$  denotes the  $n$ -th composition of  $f$  with itself. The limiting behavior (under pointwise convergence) of the orbit  $\{T^n(f, C) : n \in \mathbb{N}\}$  for  $n \rightarrow \infty$  is described below. Here, we remark that pointwise convergence for copulas is equivalent to uniform convergence (see, e.g., Durante and Sempi (2015)).

**Proposition 3.2.** Let  $C$  be in  $\mathcal{C}_d$  and let  $f \in \mathcal{F}$ . Set  $A_f := \{x \in [0, 1], f(x) = x\}$  and  $a_f := \inf A_f$ . Then  $T^n(f, C) \rightarrow T(f^\infty, C)$  pointwisely as  $n \rightarrow \infty$ , where  $f^\infty(x) := \max(x, a_f)$  for all  $x \in [0, 1]$ .

*Proof.* Let  $f \in \mathcal{F}$ . First of all, notice that, since  $f(1) = 1$ , there exists  $x_0 \in [0, 1]$  such that  $f(x_0)/x_0 = 1$ . Then, for all  $x \geq x_0$ ,  $f(x)/x \leq f(x_0)/x_0 = 1$ . It follows that  $f(x) \leq x$  for all  $x \in [x_0, 1]$ . Therefore,  $f(x) = x$  for all  $x \in [x_0, 1]$ .

Moreover, since  $T^n(f, \cdot) = T(f^n, \cdot)$ , it is enough to establish the convergence of the sequence  $(f^n(x))_{n \geq 1}$  for all  $x \in [0, 1]$ . Since  $x \leq f(x) \leq 1$  for all  $x \in [0, 1]$ , the above sequence is non-decreasing and upper bounded by the constant function equal to 1. The sequence  $(f^n(x))_{n \geq 1}$  thus converges to a limit denoted by  $f^\infty(x)$  for all  $x \in [0, 1]$ . Letting  $n \rightarrow \infty$  in  $f^{n+1}(x) = f(f^n(x))$  yields the functional equation  $f^\infty(x) = f(f^\infty(x))$  and therefore  $f^\infty(x) \in A_f$ .

Let us consider two cases:

- If  $x \leq a_f$  then, remarking that  $f^\infty$  is increasing entails  $f^\infty(x) \leq f^\infty(a_f) = f(a_f) = a_f$  from one hand, and  $f^\infty(x) \geq a_f$  from the other hand. It follows that  $f^\infty(x) = a_f$ .
- If  $x > a_f$  then  $f^\infty(x) = f(x) = x$ .

The conclusion hence follows.  $\square$

From the latter result and (2.2), the following particular cases arise.

**Corollary 3.1.** *Let  $C$  be in  $\mathcal{C}_d$  and let  $f \in \mathcal{F}$ . Set  $A_f := \{x \in [0, 1], f(x) = x\}$  and  $a_f := \inf A_f$ . The following properties hold:*

(a) *If  $a_f = 0$ , then  $T(f^\infty, C) = C$ .*

(b) *If  $a_f = 1$ , then  $T(f^\infty, C) = M_d$ .*

(c) *If  $0 < a_f < 1$ , then*

$$T(f^\infty, C)(u_1, \dots, u_d) = \frac{1}{a_f} C(\max(u_1, a_f), \dots, \max(u_d, a_f)) \min(u_1, \dots, u_d, a_f)$$

*for all  $(u_1, \dots, u_d) \in [0, 1]^d$ .*

**Example 3.1.** Let  $C$  be a 2-copula and let  $f \in \mathcal{F}$  be such that  $0 < a_f < 1$ . Then the copula  $T(f^\infty, C)$  is given by

$$T(f^\infty, C) = \begin{cases} \frac{C(a_f, a_f)M_2(u_1, u_2)}{a_f}, & (u_1, u_2) \in [0, a_f]^2, \\ \frac{C(a_f, u_2)u_1}{a_f}, & (u_1, u_2) \in [0, a_f] \times [a_f, 1], \\ \frac{C(u_1, a_f)u_2}{a_f}, & (u_1, u_2) \in [a_f, 1] \times [0, a_f], \\ C(u_1, u_2), & \text{otherwise.} \end{cases}$$

As illustrated in Figure 2, the expression of the copula  $T(f^\infty, C)$  is obtained by splitting the domain  $[0, 1]^2$  in four different regions  $R_i, i = 1, 2, 3, 4$ . Moreover, the probability mass assigned to the region  $R_i$  ( $i = 1, 2, 3, 4$ ) is equal to the  $C$ -volume of  $R_i$ , however the way how the probability is distributed in each region  $R_i$  may be different. Constructions of this type are known as *patchwork copulas*, and have been considered in Durante et al. (2009, 2013).

**Example 3.2.** Let  $f_\alpha(x) = x^\alpha$  or  $f_\alpha(x) = \alpha x + (1 - \alpha)$ , with  $\alpha \in [0, 1]$ . If  $\alpha < 1$  then  $A_{f_\alpha} = \{1\}$ ,  $a_{f_\alpha} = 1$  and therefore  $f_\alpha^\infty(x) = 1$  for all  $x \in [0, 1]$  with  $T(f_\alpha^\infty, C) = M_d$ . If  $\alpha = 1$  then  $A_{f_1} = [0, 1]$ ,  $a_{f_1} = 0$  and therefore  $f_1^\infty(x) = x$  for all  $x \in [0, 1]$  with  $T(f_1^\infty, C) = C$  the original copula.

Note that these results can also be found by direct calculations since, in the considered cases,  $f_\alpha^n = f_{\alpha^n}$ . If  $\alpha < 1$ , then  $f_\alpha^n(x) \rightarrow 1$  as  $n \rightarrow \infty$  for all  $x \in (0, 1]$  and thus  $T^n(f_\alpha, C) = T(f_\alpha^n, C) \rightarrow M$ . If  $\alpha = 1$ , then  $f_1^n(x) = x$  for all  $n > 0$  and for all  $x \in [0, 1]$  and thus  $T^n(f_1, C) = T(f_1^n, C) = T(f_1, C) = C$ .

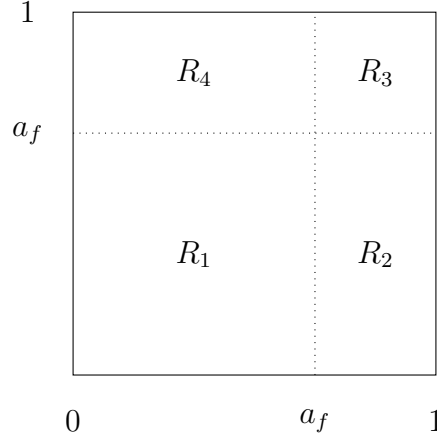


Figure 2: Illustration of Example 3.1.

**Remark 3.1.** Various other constructions of copulas can be represented as an operation of type  $\mathcal{G} \times \mathcal{C}_d \rightarrow \mathcal{C}_d$  for a suitable class of functions  $\mathcal{G}$ . Such examples include, for instance, the distortions of copulas (Durante et al., 2010a; Valdez and Xiao, 2011; Di Bernardino and Rullière, 2013).

## 4 Tail behavior of the model

Let us study now how the transformation (2.1) modifies the tail behavior of a given copula  $C$ . To this end, we use the tail dependence coefficients (see, e.g. Durante et al. (2015a)) and the maximum domain of attraction (see, e.g., Gudendorf and Segers (2010)). **These two ways of measuring the tails have practical impact in risk estimation, since it has been recognized in several studies in quantitative risk management that these measures provide valuable tools to understand the compound risk. See, for instance, Joe (2014); McNeil et al. (2015) and the references therein.**

First, we focus on the bivariate case and consider the tail dependence coefficients. It is easily shown that the lower tail dependence coefficients of  $T(f, C)$  and  $C$  are the same for all  $f \in \mathcal{F}$  and  $C \in \mathcal{C}_d$ . More interestingly, the upper tail dependence coefficient is increased by the  $T(f, \cdot)$  mapping. In fact, the following result holds.

**Proposition 4.1.** *Let  $f \in \mathcal{F}$  and  $C \in \mathcal{C}_d$ .*

(i)  $\lambda_L(T(f, C)) = \lambda_L(C)$ .

(ii) *If  $f$  is continuously differentiable on a left neighbourhood of 1 then*

$$\lambda_U(T(f, C)) = 1 - (1 - \lambda_U(C))f'(1) \geq \lambda_U(C).$$

*Proof.* The proofs of (i) and (ii) are similar. Let us focus on (ii). Let  $u \in (0, 1]$ , recall that  $g = \text{id}/f$  and consider the expansion

$$\frac{T(f, C)(u, u) - 1}{u - 1} = \frac{C(f(u), f(u)) - 1}{f(u) - 1} \frac{f(u) - 1}{u - 1} g(u) + \frac{g(u) - 1}{u - 1}.$$

Letting  $u \rightarrow 1$ , it follows that

$$2 - \lambda_U(T(f, C)) = (2 - \lambda_U(C))f'(1)g(1) + g'(1).$$

Since  $g'(1) = 1 - f'(1)$  and  $g(1) = 1$ , the result follows.  $\square$

In the multivariate case, a more comprehensive view of the tail behavior of a copula  $C$  may be given by calculating its limiting extreme-value copula, defined as

$$C^\sharp(u_1, \dots, u_d) = \lim_{n \uparrow \infty} \left[ C(u_1^{1/n}, \dots, u_d^{1/n}) \right]^n, \quad (u_1, \dots, u_d) \in [0, 1]^d.$$

If  $C$  is a copula belonging to the domain of attraction of the extreme-value copula  $C^\sharp$ , then  $T(f, C)$  belongs to the domain of attraction of an extreme-value copula  $T^\sharp(f, C)$ , which is simply the geometric mean between  $C^\sharp$  and the copula  $M_d$ . This result is obtained below.

**Proposition 4.2.** *Suppose that  $f \in \mathcal{F}$  is continuously differentiable on a left neighbourhood of 1. Then, for all  $C \in \mathcal{C}_d$ :*

$$T^\sharp(f, C) = (C^\sharp)^{f'(1)} M_d^{1-f'(1)}.$$

*Proof.* Let  $u \in (0, 1]$ . A Taylor expansion of  $f$  on a left neighbourhood of 1 yields

$$f(u^{1/n}) = f\left(\exp\left(\frac{\log(u)}{n}\right)\right) = f\left(1 + \frac{\log(u)}{n}(1 + o(1))\right) = 1 + f'(1 + o(1)) \frac{\log(u)}{n}(1 + o(1)),$$

as  $n \rightarrow \infty$  and consequently,

$$\begin{aligned} (f(u^{1/n}))^n &= \left(1 + f'(1 + o(1)) \frac{\log(u)}{n}(1 + o(1))\right)^n \\ &= \exp\left(n \log\left(1 + f'(1 + o(1)) \frac{\log(u)}{n}(1 + o(1))\right)\right) \\ &= \exp(f'(1 + o(1)) \log(u)(1 + o(1))) \\ &= u^{f'(1 + o(1))}(1 + o(1)). \end{aligned}$$

For all copula  $C$  and all  $(u_1, \dots, u_d) \in [0, 1]^d$ , we thus have

$$C^n(f(u_1^{1/n}), \dots, f(u_d^{1/n})) = C^n\left(\left[u_1^{f'(1 + o(1))}(1 + o(1))\right]^{1/n}, \dots, \left[u_d^{f'(1 + o(1))}(1 + o(1))\right]^{1/n}\right).$$

Besides,  $C^n(x_1^{1/n}, \dots, x_d^{1/n}) \rightarrow C^\sharp(x_1, \dots, x_d)$  for all  $(x_1, \dots, x_d) \in [0, 1]^d$  as  $n \rightarrow \infty$ . It follows that

$$C^n(f(u_1^{1/n}), \dots, f(u_d^{1/n})) \rightarrow C^\sharp(u_1^{f'(1)}, \dots, u_d^{f'(1)}) = (C^\sharp(u_1, \dots, u_d))^{f'(1)}, \quad (4.1)$$

in view of the homogeneity property of extreme-value copulas. As a particular case of (4.1), one can obtain that

$$M_d^n(g(u_1^{1/n}), \dots, g(u_d^{1/n})) \rightarrow (M_d^\sharp(u_1, \dots, u_d))^{g'(1)} = (M_d(u_1, \dots, u_d))^{1-f'(1)} \quad (4.2)$$

since  $M_d^\sharp = M_d$  and  $g'(1) = 1 - f'(1)$ . Collecting (4.1) and (4.2) yields

$$\left(T(f, C)(u_1^{1/n}, \dots, u_d^{1/n})\right)^n \rightarrow (C^\sharp(u_1, \dots, u_d))^{f'(1)} (M_d(u_1, \dots, u_d))^{1-f'(1)}$$

as  $n \rightarrow \infty$  and the result is proved.  $\square$

## 5 Illustration: assessing model risk in hydrology

The transformation of copulas described in eq. (2.1) has an intuitive interpretation in model risk assessment.

In fact, suppose that a copula  $C$  has been chosen for some available data. Then, for a suitable  $f$ ,  $T(f, C)$  represents another possible copula that may interpret the random phenomenon of interest when a global shock hits the behavior of the system we are considering. In other words,  $T(f, C)$  can be interpreted as a possible alternative model when a global factor may affect the original system. Thus (see, e.g., Barriou and Scandolo (2015)), starting with a reference model given by the copula  $C$ ,  $T(\cdot, C)$  creates a set of alternative models  $\mathcal{A}$  around the reference one by varying the function  $f$  (or its parameters), that is

$$\mathcal{A} = \{T(f, C) : f \in \mathcal{F}'\},$$

where  $\mathcal{F}'$  is a subset of  $\mathcal{F}$ . Hence, the proposed copula transformation can be employed as a possible way to create a sort of *tolerance set* of models that can be used to assess the model risk when a possible global shock may hit the system.

Specifically, given a distribution-based risk measurement  $\nu$  and a suitable probability distribution function  $F = C(F_1, \dots, F_d)$  for a given phenomenon (fitted to some available data), it could be of interest to compute the set

$$\nu(\mathcal{A}) = \{\nu(F) : F = C(F_1, \dots, F_d), C \in \mathcal{A}\},$$

i.e. to calculate the range of the risk measures when the copula is supposed to vary in a pre-determined set of values, while the marginals  $F_1, \dots, F_d$  are fixed.

**Remark 5.1.** The calculation of ranges of possible risk measurements under copula uncertainty is an issue that has been long considered in the recent years, especially for risk aggregation in a financial and insurance context. See, for instance, Goovaerts et al. (2011); Embrechts et al. (2013); Bernard and Vanduffel (2014) and references therein. Here, the main difference is that the source of uncertainty in the copula model is known and is related to the appearance of a global shock that may hit the system.

| Description                | Parameter | Unit              | Value |
|----------------------------|-----------|-------------------|-------|
| Water density              | $\rho_W$  | kg/m <sup>3</sup> | 1000  |
| Cube density               | $\rho_S$  | kg/m <sup>3</sup> | 2600  |
| N.o of units displaced     | $N_d$     | -                 | 1.5   |
| Gravitational acceleration | $g$       | m/s <sup>2</sup>  | 9.81  |

Table 1: Parameters associated with eq. (5.1)

For the purpose of practical illustration, we consider an application in coastal engineering related to the preliminary design of a rubble mound breakwater, following the framework already outlined in (Salvadori et al., 2015), see also Pappadà et al. (2015).

The target is to compute the quantiles (e.g., value-at-risk) associated to the weight  $W$  of a concrete cube element forming the breakwater structure, assuming that the environmental load is given by the pair of nonindependent variables  $(H, D)$ , where  $H$  represents the significant wave height (in meters), and  $D$  the sea storm duration (in hours). Moreover, we suppose the existence of structure function  $\Psi$ , calibrated for the buoy of Alghero (Sardinia, Italy) previously investigated in (Salvadori et al., 2014, 2015), which allows to express  $W$  via  $(H, D)$  by means of the formula

$$W = \Psi(H, D) = \rho_S \cdot \left[ H \left( \frac{2\pi H}{g [4.597 \cdot H^{0.328}]^2} \right)^{0.1} \right]^3 / \left[ \left( \frac{\rho_S}{\rho_W} - 1 \right) \cdot \left( 1 + \frac{6.7 \cdot N_d^{0.4}}{(3600 D / [4.597 \cdot H^{0.328}])^{0.3}} \right) \right]^3 \quad (5.1)$$

where the values of the equation parameters are as reported in Table 1.

In order to focus on the effects of the dependence in the risk quantification (without considering the marginal effects), we assume in the following that the margins of  $H$  and  $D$  are Generalized Weibull laws, whose parameters values have been fixed accordingly to the results by (Salvadori et al., 2014). Obviously, other possible choices could be done as well.

Let us assume, hence, that the dependence structure of  $(H, D)$  can be modeled by a given copula  $C$ , the idea is to consider how the quantiles of  $W$  may vary when calculated over a tolerance set of models that are associated to  $C$  via the transformation  $T(\cdot, C)$ . To this end, consider the sets of possible models of type

$$F = C(F_H, F_D),$$

where  $F_H, F_D$  are the previously considered marginals applied to  $H$  and  $D$ , respectively, while  $C$  is a copula of type  $T(f, C)$ , where  $f(t) = t^\alpha$  for  $\alpha \in [0, 1]$  and  $C$  belongs to Frank, Gumbel and Clayton family of copulas with different Kendall's tau varying in  $\{0.25, 0.50, 0.75\}$ . Here the global shock should be understood as a large environmental event that may affect the behavior of both  $H$  and  $D$ .

According to Salvadori et al. (2015), the quantiles are calculated by simulating from the reference model (here we apply the algorithm suggested by Proposition 2.2) a large number of data points (here,  $10^7$ ).

The quantile calculations from 0.95 to 0.995 are illustrated in Figure 3. **As it can be seen, the shock transformation tends to increase the quantile values under any considered dependence scenario. Moreover, its relative effect becomes more evident under weak dependence (i.e. smaller Kendall's  $\tau$ ) and in absence of upper tail dependence (i.e. for Clayton and Frank copula models). Thus, the presence of a global shock amplifies the risk measurement especially when the initial model does not present strong (tail) dependency, i.e. it is not conservative from a risk manager perspective.**

## 6 Conclusions

We have introduced a mechanism to modify a given copula  $C$  by means of a univariate function  $f$ . The resulting copula can be interpreted as the result of a global shock affecting all the components of a system modeled by  $C$ . Moreover, we study the properties of this copula transformation from the perspective of semi-group action. In this respect, special attention is devoted to the study of the tail behavior of the resulting copulas.

Finally, the whole methodology is interpreted as a tool for model risk assessment and is applied to a problem arising in environmental engineering.

## Acknowledgments

We would like to thank two anonymous Reviewers for their useful comments. Moreover, we thank Juan Fernández-Sánchez for useful suggestions about a preliminary version of this work.

The first author acknowledges the support of Free University of Bozen-Bolzano, School of Economics and management, via the project FLORIDA.

## References

- Barrieu, P. and Scandolo, G. (2015). Assessing financial model risk. *European J. Oper. Res.*, 242(2):546–556.
- Bernard, C. and Vanduffel, S. (2015). A new approach to assessing model risk in high dimensions. *J. Bank. Fin.*, 58:166–178.
- Cherubini, U., Durante, F., and Mulinacci, S., editors (2015). *Marshall–Olkin Distributions – Advances in Theory and Practice*, Springer Proceedings in Mathematics & Statistics. Springer.
- Di Bernardino, E. and Rullière, D. (2013). On certain transformations of Archimedean copulas: Application to the non-parametric estimation of their generators. *Dependence Modeling*, 1:1–36.
- Durante, F. (2006). A new class of symmetric bivariate copulas. *J. Nonparametr. Stat.*, 18(7-8):499–510, (2007).
- Durante, F., Fernández-Sánchez, J., and Pappadà, R. (2015a). Copulas, diagonals and tail dependence. *Fuzzy Sets and Systems*, 264:22–41.

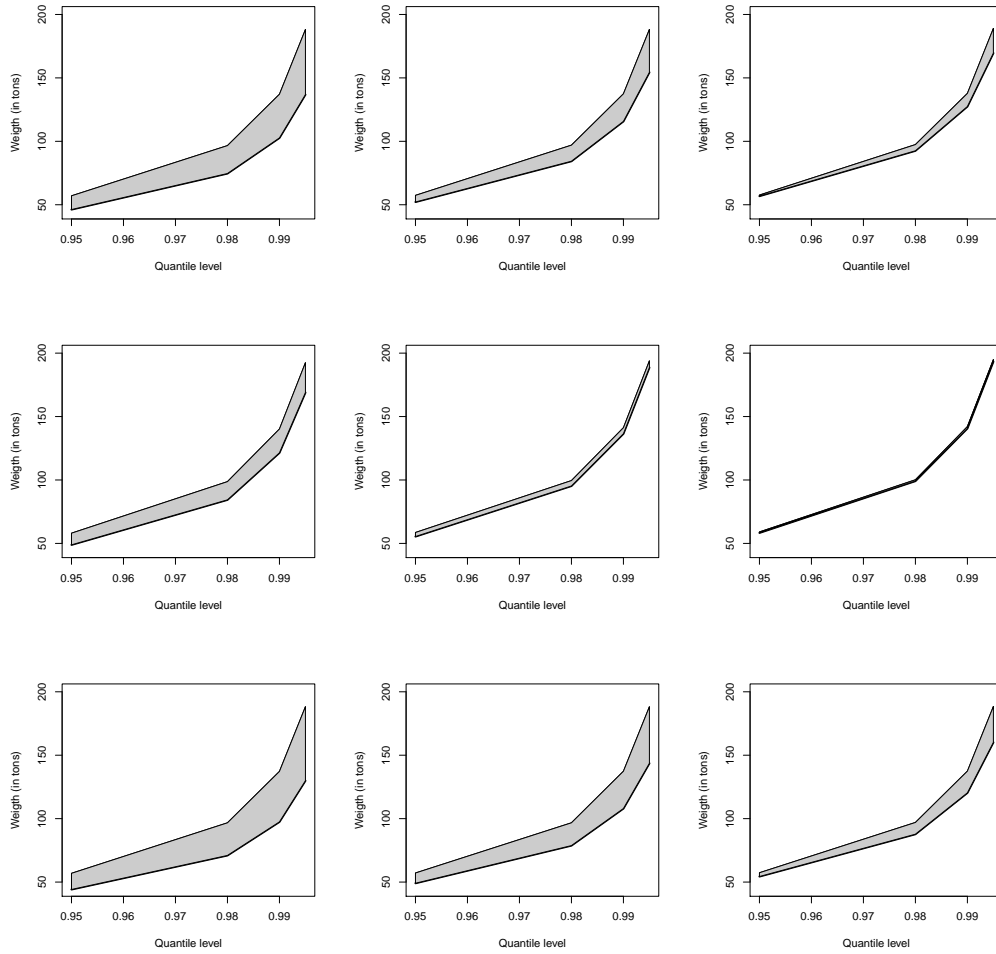


Figure 3: Range of variation of quantiles associated with  $W$  of eq. (5.1) when the copula is of type  $T(t^\alpha, C)$  for  $\alpha \in ]0, 1[$  and  $C$  is Frank copula (first row), Gumbel copula (second row) or Clayton copula (third row) with Kendall's  $\tau$  equal to 0.25 (left), 0.50 (middle), and 0.75 (right). The lower line indicates the quantile associated with the copula  $C$ .



- Durante, F., Fernández-Sánchez, J., and Sempi, C. (2013). Multivariate patchwork copulas: a unified approach with applications to partial comonotonicity. *Insurance Math. Econom.*, 53:897–905.
- Durante, F., Fernández-Sánchez, J., and Trutschnig, W. (2014). Multivariate copulas with hairpin support. *J. Multivariate Anal.*, 130:323–334.
- Durante, F., Foschi, R., and Sarkoci, P. (2010a). Distorted copulas: constructions and tail dependence. *Comm. Statist. Theory Methods*, 39(12):2288–2301.
- Durante, F., Girard, S., and Mazo, G. (2015b). Copulas based on Marshall–Olkin machinery. In Cherubini, U., Durante, F., and Mulinacci, S., editors, *Marshall–Olkin Distributions – Advances in Theory and Practice*, Springer Proceedings in Mathematics & Statistics, pages 15–31. Springer.
- Durante, F., Hofert, M., and Scherer, M. (2010b). Multivariate hierarchical copulas with shocks. *Methodol. Comput. Appl. Probab.*, 12(4):681–694.
- Durante, F., Kolesárová, A., Mesiar, R., and Sempi, C. (2008). Semilinear copulas. *Fuzzy Sets and Systems*, 159(1):63–76.
- Durante, F. and Okhrin, O. (2015). Estimation procedures for exchangeable Marshall copulas with hydrological application. *Stoch. Environ. Res Risk Assess.*, 29(1):205–226.
- Durante, F., Quesada-Molina, J. J., and Úbeda-Flores, M. (2007). On a family of multivariate copulas for aggregation processes. *Inform. Sci.*, 177(24):5715–5724.
- Durante, F. and Salvadori, G. (2010). On the construction of multivariate extreme value models via copulas. *Environmetrics*, 21(2):143–161.
- Durante, F., Saminger-Platz, S., and Sarkoci, P. (2009). Rectangular patchwork for bivariate copulas and tail dependence. *Comm. Statist. Theory Methods*, 38(15):2515–2527.
- Durante, F. and Sempi, C. (2015). *Principles of Copula Theory*. CRC/Chapman & Hall, Boca Raton.
- Embrechts, P. and Hofert, M. (2013). A note on generalized inverses. *Math. Methods Oper. Res.*, 77(3):423–432.
- Embrechts, P., Puccetti, G., and Rüschendorf, L. (2013). Model uncertainty and VaR aggregation. *J. Bank. Financ.*, 37(8):2750–2764.
- Goovaerts, M. J., Kaas, R., and Laeven, R. J. A. (2011). Worst case risk measurement: Back to the future? *Insurance Math. Econom.*, 49(3):380–392.
- Gudendorf, G. and Segers, J. (2010). Extreme-value copulas. In Jaworski, P., Durante, F., Härdle, W. K., and Rychlik, T., editors, *Copula Theory and its Applications*, volume 198 of *Lecture Notes in Statistics - Proceedings*, pages 127–145. Springer, Berlin Heidelberg.
- Joe, H. (2014). *Dependence Modeling with Copulas*. Chapman & Hall/CRC, London.
- Kundu, D., Franco, M., and Vivo, J.-M. (2014). Multivariate distributions with proportional reversed hazard marginals. *Comput. Statist. Data Anal.*, 77:98–112.
- Li, X. and Pellerey, F. (2011). Generalized Marshall–Olkin distributions and related bivariate aging properties. *J. Multivariate Anal.*, 102(10):1399–1409.
- Liebscher, E. (2008). Construction of asymmetric multivariate copulas. *J. Multivariate Anal.*, 99(10):2234–2250.

- Liebscher, E. (2011). Erratum to “Construction of asymmetric multivariate copulas”. *J. Multivariate Anal.*, 102(4):869–870.
- Lin, J. and Li, X. (2014). Multivariate generalized Marshall-Olkin distributions and copulas. *Methodol. Comput. Appl. Probab.*, 16(1):53–78.
- Mai, J.-F., Schenk, S., and Scherer, M. (2015). Exchangeable exogenous shock models. *Bernoulli*, in press.
- Mai, J.-F. and Scherer, M. (2012). *Simulating copulas*. Imperial College Press, London.
- Marshall, A. W. and Olkin, I. (1967). A multivariate exponential distribution. *J. Amer. Statist. Assoc.*, 62:30–44.
- Marshall, A. W., Olkin, I., and Arnold, B. C. (2011). *Inequalities: theory of majorization and its applications*. Springer Series in Statistics. Springer, New York, second edition.
- McNeil, A. J., Frey, R., and Embrechts, P. (2015). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press, Princeton, second edition.
- Nelsen, R. B. (2006). *An introduction to copulas*. Springer Series in Statistics. Springer, New York, second edition.
- Ozkut, M. and Bayramoglu, I. (2014). On Marshall-Olkin type distribution with effect of shock magnitude. *J. Comput. Appl. Math.*, 271:150–162.
- Pappadà, R., Perrone, E., Durante, F., and Salvadori, G. (2015). Spin-off extreme value and Archimedean copulas for estimating the multivariate structural risk. *Stoch. Environ. Res Risk Assess.*, in press.
- Pinto, J. and Kolev, N. (2015). Extended Marshall–Olkin model and its dual version. In Cherubini, U., Durante, F., and Mulinacci, S., editors, *Marshall–Olkin Distributions – Advances in Theory and Practice*, Springer Proceedings in Mathematics & Statistics, pages 87–113. Springer.
- Salvadori, G., Durante, F., Tomasicchio, G. R., and D’Alessandro, F. (2015). Practical guidelines for the multivariate assessment of the structural risk in coastal and off–shore engineering. *Coastal Engineering*, 95:77–83.
- Salvadori, G., Tomasicchio, G. R., and D’Alessandro, F. (2014). Practical guidelines for multivariate analysis and design in coastal and off-shore engineering. *Coastal Engineering*, 88:1–14.
- Scarsini, M. (1984). On measures of concordance. *Stochastica*, 8(3):201–218.
- Singpurwalla, N. D. (2006). *Reliability and risk*. Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester.
- Valdez, E. A. and Xiao, Y. (2011). On the distortion of a copula and its margins. *Scand. Actuar. J.*, 4:292–317.