

# Domain decomposition methods for nonlinear problems in structural mechanics

Pierre Gosselet

LMT-Cachan – ENS Cachan/CNRS/UPMC/PRES UniverSud Paris

June 6, 2011

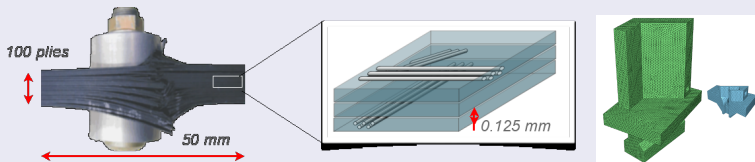
- J. Pebrel, G. Desmeure – Christian Rey
- P. Kerfriden (U. Cardiff), K. Saavedra, L. Gendre – Olivier Allix



# Objectives

Mechanical point of view

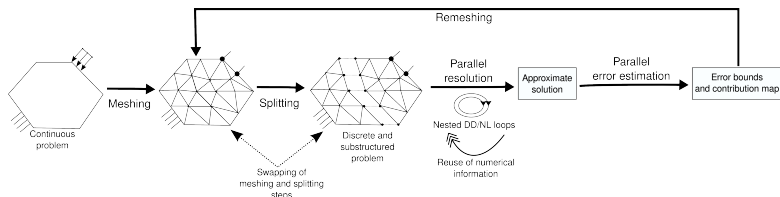
## Problems with large ratio between scales



- Delamination of lamified composites plates
  - Plasticity at the basis of turbine blades
  - Large transformation – hyperelastic materials – buckling
- 
- Large number of dof ( $> 10^6$ ), need to use domain decomposition methods to conduct parallel computations.
  - Local phenomena may have a global influence (stress redistribution, large displacements).

# Objective

## Parallel adaptive computational strategy



### Reuse of numerical information

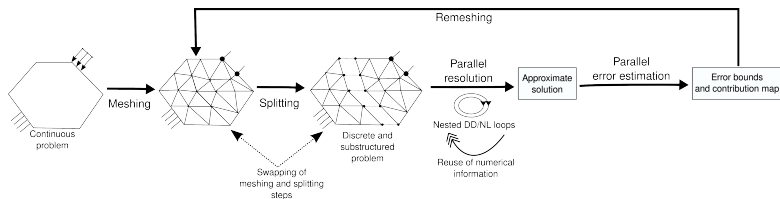
Definition of a reduced model which enables to accelerate computations (PGD [Nouy, 2010], POD-Newton [Ryckelynck, Chinesta, Cueto, and Ammar, 2006; Kerfriden, Gosselet, Adhikari, and Bordas, 2011], Krylov solvers [Gosselet, Rey, and Pebrel, 2011])

### Verification

Expensive step which enables to validate numerical results and provides remeshing maps. (Parallel error estimation in [Parret-Fréaud, Rey, Gosselet, and Feyel, 2010])

# Objective

Parallel adaptive computational strategy



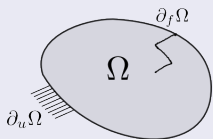
## Nonlinearities

**Nonlinearity:** it is handled by an iterative algorithm (Newton, ANM, Uzawa...) which leads to the resolution of a sequence of global linear systems (and Gauss point problems).

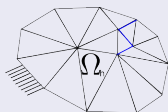
**Domain decomposition:** in order to decouple computations, interface connection conditions between subdomains are reached by the convergence of an iterative process.

What is the best way to nest loops ? first answers: LaTin [Ladevèze, 1985; Ladevèze and Nouy, 2003] Schwarz [Badea, 1991] and others [Cresta, Allix, Rey, and Guinard, 2007; Pebrel, Rey, and Gosselet, 2008].

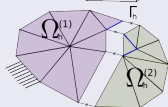
## Last remark



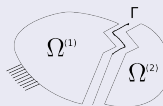
There are two philosophies to use non-overlapping DD to treat a continuous problem (with surface nonlinearities)



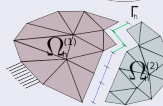
(A) FE discretization (behavior is inside elements)



(B) DD splitting: interfaces connect subdomains (constraints)



(C) DD interfaces have behaviors and own mechanical unknowns



(D) (independent) discretization of subdomains and interfaces

DD enables to model interface nonlinearities. One can compare the quality of the FE solution between A and D, and the convergence speed between B and D.

First part of the talk is related to the left column, second part to the right.

# Outlook

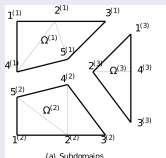
- 1 Brief reminder for linear systems
- 2 DD for nonlinear problems
- 3 The LaTin method
  - Brief presentation
  - Application to delamination
- 4 Conclusion

## Global discrete system



$$\mathbf{K}u = \mathbf{f}$$

## N-subdomain partitioning



- Splitting between boundary dof ( $b$  subscript) and internal ( $i$  subscript)  $\mathbf{u}^{(s)} = \begin{pmatrix} \mathbf{u}_i^{(s)} \\ \mathbf{u}_b^{(s)} \end{pmatrix}$
- Trace operator  $\mathbf{t}^{(s)} = \begin{pmatrix} \mathbf{0}_{bi} & \mathbf{I}_{bb} \end{pmatrix} : \mathbf{t}^{(s)}\mathbf{u}^{(s)} = \mathbf{u}_b^{(s)}$
- Introduction of the nodal reaction  $\lambda_b^{(s)}$  of neighbors on subdomain ( $s$ )

$$\begin{pmatrix} \ddots & \mathbf{0} & \\ \mathbf{0} & \mathbf{K}^{(s)} & \mathbf{0} \\ & \mathbf{0} & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \mathbf{u}^{(s)} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{f}^{(s)} \\ \vdots \end{pmatrix} + \begin{pmatrix} \ddots & & \\ & \mathbf{t}^{(s)T} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \lambda_b^{(s)} \\ \vdots \end{pmatrix}$$

$$\mathbf{K}^\diamond \mathbf{u}^\diamond = \mathbf{f}^\diamond + \mathbf{t}^\diamond{}^T \lambda_b^\diamond$$

# Condensation

## Elimination of internal unknowns

$$\begin{pmatrix} \mathbf{K}_{ij}^{(s)} & \mathbf{K}_{ib}^{(s)} \\ \mathbf{K}_{bi}^{(s)} & \mathbf{K}_{bb}^{(s)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i^{(s)} \\ \mathbf{u}_b^{(s)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i^{(s)} \\ \mathbf{f}_b^{(s)} + \lambda_b^{(s)} \end{pmatrix}$$

$$\begin{cases} \mathbf{u}_i^{(s)} = \mathbf{K}_{ii}^{(s)-1} \left( -\mathbf{K}_{ib}^{(s)} \mathbf{u}_b^{(s)} + \mathbf{f}_i^{(s)} \right) \\ \left( \mathbf{K}_{bb}^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)-1} \mathbf{K}_{ib}^{(s)} \right) \mathbf{u}_b^{(s)} = \mathbf{f}_b^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)-1} \mathbf{f}_i^{(s)} \end{cases}$$

Schur complement  $\mathbf{S}^{(s)} = \left( \mathbf{K}_{bb}^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)-1} \mathbf{K}_{ib}^{(s)} \right)$

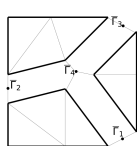
Condensed right hand side  $\mathbf{b}^{(s)} = \mathbf{f}_b^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)-1} \mathbf{f}_i^{(s)}$

## Condensed subdomain equilibrium

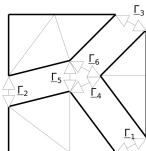
$$\mathbf{S}^\diamond \mathbf{u}_b^\diamond = \mathbf{b}^\diamond + \lambda_b^\diamond$$

(subdomain is a black-box)





(c) Interface nodes



(d) Interface connections

## Description of interfaces

- Set of shared nodes :  $\bar{\Gamma}$ , injection operator  $\bar{\mathbf{A}}^{(s)}$  from  $\Gamma^{(s)}$  to  $\bar{\Gamma}$   
Action-reaction principle writes:

$$\sum_s \bar{\mathbf{A}}^{(s)} \lambda_b^{(s)} = \left( \dots \quad \bar{\mathbf{A}}^{(s)} \quad \dots \right) \begin{pmatrix} \vdots \\ \lambda_b^{(s)} \\ \vdots \end{pmatrix} = \bar{\mathbf{A}}^{\diamond} \lambda_b^{\diamond} = 0$$

- Connection between nodes:  $\underline{\Gamma}$ , Operator  $\underline{\mathbf{A}}^{(s)}$  which satisfies

$$\text{Range}(\underline{\mathbf{A}}^{\diamond T}) = \text{Ker}(\bar{\mathbf{A}}^{\diamond})$$

Continuity of displacement at the interface writes

$$\sum_s \underline{\mathbf{A}}^{(s)} \mathbf{u}_b^{(s)} = \underline{\mathbf{A}}^{\diamond} \mathbf{u}_b^{\diamond} = 0$$

# Few properties of assembling operators

- Two-subdomain case:

$$\overline{\mathbf{A}}^\diamond \lambda_b^\diamond = \lambda_b^{(1)} + \lambda_b^{(2)}$$

$$\underline{\mathbf{A}}^\diamond \mathbf{u}_b^\diamond = \mathbf{u}_b^{(1)} - \mathbf{u}_b^{(2)}$$

- Orthogonality:

$$\overline{\mathbf{A}}^\diamond \underline{\mathbf{A}}^\diamond T = 0$$

- Any boundary vector is the combination of a continuous vector and a balanced vector:

$$\forall \mathbf{x}_b^\diamond, \exists! (\underline{\mathbf{y}}, \overline{\mathbf{z}}) \in \mathbb{R}^{\Gamma} \times \mathbb{R}^{\overline{\Gamma}} / \mathbf{x}_b^\diamond = \underline{\mathbf{A}}^\diamond T \underline{\mathbf{y}} + \overline{\mathbf{A}}^\diamond T \overline{\mathbf{z}}$$

$$\text{indeed } \begin{cases} \overline{\mathbf{z}} = \left( \overline{\mathbf{A}}^\diamond \underline{\mathbf{A}}^\diamond T \right)^{-1} \overline{\mathbf{A}}^\diamond \mathbf{x}_b^\diamond \\ \underline{\mathbf{y}} = \left( \underline{\mathbf{A}}^\diamond \overline{\mathbf{A}}^\diamond T \right)^+ \underline{\mathbf{A}}^\diamond \mathbf{x}_b^\diamond \end{cases}$$

# Few words on pseudo-inverses

- System  $\mathbf{M}\mathbf{x} = \mathbf{b}$  has a solution if  $\mathbf{b} \in \text{Range}(\mathbf{M}) \Leftrightarrow \mathbf{b} \perp \text{Ker}(\mathbf{M}^T)$
- A pseudo-inverse of  $\mathbf{M}$  satisfies

$$\mathbf{M}\mathbf{M}^+\mathbf{b} = \mathbf{b}, \quad \forall \mathbf{b} \in \text{Range}(\mathbf{M})$$

- Pseudo-inverse can be made unique when associated to optimization, for example Moore-Penrose pseudo-inverse  $\mathbf{M}^\dagger$  is the only to satisfy:

$$\mathbf{x} = \mathbf{M}^\dagger \mathbf{b} \Leftrightarrow \begin{cases} \mathbf{x} = \arg \min_{\mathbf{y} \in \mathcal{V}} \|\mathbf{y}\| \\ \mathcal{V} = \arg \min_{\mathbf{z} \in \mathbb{R}^n} \|\mathbf{M}\mathbf{z} - \mathbf{b}\| \end{cases}$$

$\mathbf{M}^\dagger$  gives the minimal norm vector which minimizes the residual norm.

## Back to DD formulation

## Condensed DD problem

$$\text{Find } (\mathbf{u}_b^\diamond, \boldsymbol{\lambda}_b^\diamond) / \begin{cases} \mathbf{S}^\diamond \mathbf{u}_b^\diamond = \mathbf{b}^\diamond + \boldsymbol{\lambda}_b^\diamond \\ \underline{\mathbf{A}}^\diamond \boldsymbol{\lambda}_b^\diamond = 0 \\ \underline{\mathbf{A}}^\diamond \mathbf{u}_b^\diamond = 0 \end{cases}$$

## Primal formulation

$$\mathbf{u}_b^\diamond = \underline{\mathbf{A}}^\diamond{}^T \bar{\mathbf{u}}_b$$

$$\left( \underline{\mathbf{A}}^\diamond \mathbf{S}^\diamond \underline{\mathbf{A}}^\diamond{}^T \right) \bar{\mathbf{u}}_b = \underline{\mathbf{A}}^\diamond \mathbf{b}^\diamond$$

## Dual formulation

$$\boldsymbol{\lambda}_b^\diamond = \underline{\mathbf{A}}^\diamond{}^T \underline{\boldsymbol{\lambda}}_b$$

$$\begin{cases} \left( \underline{\mathbf{A}}^\diamond \mathbf{S}^\diamond + \underline{\mathbf{A}}^\diamond{}^T \right) \underline{\boldsymbol{\lambda}}_b = -\underline{\mathbf{A}}^\diamond \mathbf{S}^\diamond \mathbf{b}^\diamond + \underline{\mathbf{A}}^\diamond \mathbf{R}_b^\diamond \boldsymbol{\alpha}^\diamond \\ \mathbf{R}_b^\diamond{}^T \left( \underline{\mathbf{A}}^\diamond{}^T \underline{\boldsymbol{\lambda}}_b + \mathbf{b}^\diamond \right) = 0 \end{cases}$$

$\mathbf{R}_b^{(s)}$  is basis of the kernel of  $\mathbf{S}^{(s)}$

## Primal solver

One seeks iteratively  $\bar{\mathbf{u}}_b$  defined on  $\bar{\Gamma}$ . At iteration  $k$ , we have

$$\begin{aligned} \mathbf{S}^\diamond \bar{\mathbf{A}}^{\diamond T} \bar{\mathbf{u}}_{bk} - \mathbf{b}^\diamond &= \lambda_{bk}^\diamond \\ \bar{\mathbf{A}}^\diamond \left( \mathbf{S}^\diamond \bar{\mathbf{A}}^{\diamond T} \bar{\mathbf{u}}_{bk} - \mathbf{b}^\diamond \right) &= \bar{\mathbf{r}}_{pk} = \bar{\mathbf{A}}^\diamond \lambda_{bk}^\diamond \end{aligned}$$

Iterative solver residual is the lack of balance of interface tractions

## Neumann-Neumann preconditioner

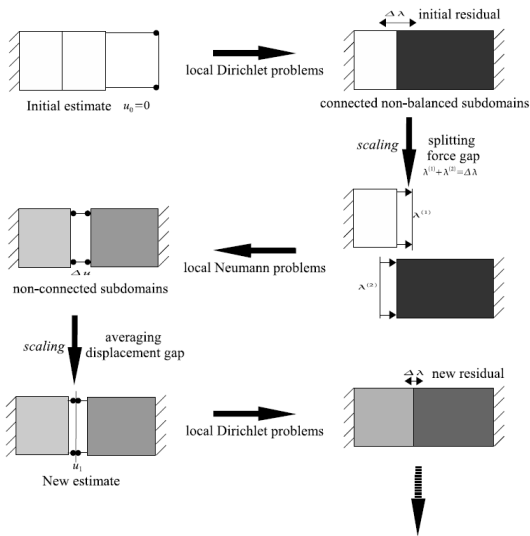
$$\left( \bar{\mathbf{A}}^\diamond \mathbf{S}^\diamond \bar{\mathbf{A}}^{\diamond T} \right)^{-1} \simeq \bar{\mathbf{A}}^{\diamond T+} \mathbf{S}^{\diamond+} \bar{\mathbf{A}}^{\diamond+}$$

## Coarse problem

It ensures that all Neumann problems are well posed.

$$\mathbf{R}_b^{\diamond T} \bar{\mathbf{A}}^{\diamond+} \bar{\mathbf{r}}_{pk} = \mathbf{R}_b^{\diamond T} \bar{\mathbf{A}}^{\diamond+} \bar{\mathbf{A}}^{\diamond+} \lambda_{bk}^\diamond = 0$$

It adds a global component to the preconditioner, it enables to satisfy Saint-Venant's principle and makes the method scalable.



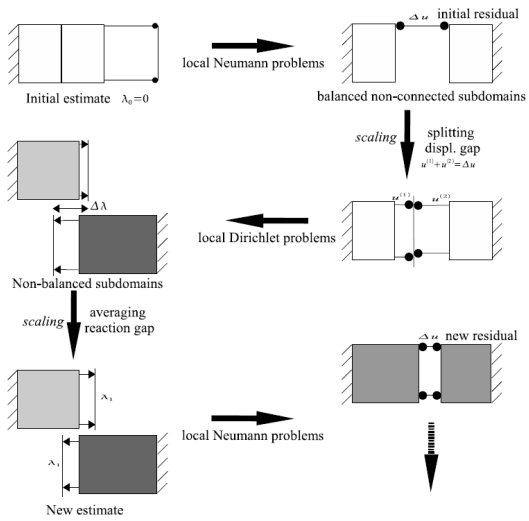
## Dual solver

Same philosophy:

- balanced reaction defined on  $\Gamma$  is iteratively sought
- the unknown satisfies the constraint of equilibrium of subdomains (coarse problem)
- residual is the displacement gap between subdomains

## Dirichlet Preconditioner

$$\left( \underline{\mathbf{A}} \diamond \mathbf{S} \diamond + \underline{\mathbf{A}} \diamond \mathbf{r} \right)^{-1} \simeq \underline{\mathbf{A}} \diamond \mathbf{r} + \mathbf{S} \diamond \underline{\mathbf{A}} \diamond +$$





## Assembling operator pseudo-inverses / scaled assembling

[Klawonn and Widlund, 2001]

## Primal

$$\overline{\mathbf{A}} \diamond \overline{\mathbf{A}} \diamond^+ = \overline{\mathbf{I}}$$

$$\overline{\mathbf{A}} \diamond^+ = \mathbf{X} \diamond \overline{\mathbf{A}} \diamond^T \left( \overline{\mathbf{A}} \diamond \mathbf{X} \diamond \overline{\mathbf{A}} \diamond^T \right)^{-1}$$

## Dual

$$\underline{\mathbf{A}} \diamond \underline{\mathbf{A}} \diamond^+ = \underline{\mathbf{I}}$$

$$\underline{\mathbf{A}} \diamond^+ = \mathbf{Y} \diamond \underline{\mathbf{A}} \diamond^T \left( \underline{\mathbf{A}} \diamond \mathbf{Y} \diamond \underline{\mathbf{A}} \diamond^T \right)^+$$

## Conjugation

if  $\mathbf{Y} \diamond = \mathbf{X} \diamond^{-1}$ 

$$\underline{\mathbf{A}} \diamond^T \underline{\mathbf{A}} \diamond^+ + \overline{\mathbf{A}} \diamond^+ \overline{\mathbf{A}} \diamond = \mathbf{I} \diamond$$

## Usually

 $\mathbf{X} \diamond$  is diagonal :  $\mathbf{X} \diamond = \mathbf{I} \diamond$  or  $\mathbf{X} \diamond = \text{diag}(\mathbf{K} \diamond)$  or better (material stiffness).

- 1 Brief reminder for linear systems
- 2 DD for nonlinear problems
- 3 The LaTin method
  - Brief presentation
  - Application to delamination
- 4 Conclusion

# Discrete nonlinear problem

## Global problem

Small perturbation assumption, for one time increment the problem writes:

$$\mathbf{f}_{int}(\mathbf{u}) + \mathbf{f}_{ext} = 0$$

history (internal variables) is not shown but can be taken into account

## Classical resolution

Use of a Newton solver

$$\frac{\partial \mathbf{f}_{int}}{\partial \mathbf{u}}(\mathbf{u}_k) \delta \mathbf{u} = -(\mathbf{f}_{int}(\mathbf{u}_k) + \mathbf{f}_{ext})$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \delta \mathbf{u}$$

Each iteration is a global linear to solve (with DD). Resolution can be strongly slowed down if an ugly local phenomenon occurs (buckling, damage) [Cresta et al., 2007].

# Discrete nonlinear problem

## Global problem

Small perturbation assumption, for one time increment the problem writes:

$$\mathbf{f}_{int}(\mathbf{u}) + \mathbf{f}_{ext} = 0$$

history (internal variables) is not shown but can be taken into account

## Substructured version

Find  $(\mathbf{u}^\diamond, \boldsymbol{\lambda}_b^\diamond)$  such that

$$\mathbf{f}_{int}^\diamond(\mathbf{u}^\diamond) + \mathbf{f}_{ext}^\diamond + \mathbf{t}^\diamond{}^T \boldsymbol{\lambda}_b^\diamond = 0$$

$$\overline{\mathbf{A}}^\diamond \boldsymbol{\lambda}_b^\diamond = 0$$

$$\underline{\mathbf{A}}^\diamond \mathbf{u}_b^\diamond = 0$$

where the dependency is only local  $\mathbf{f}_{int}^{(s)}(\mathbf{u}^{(s)})$

# Primal condensation

$$\text{Find } \bar{\mathbf{u}}_b \in \mathbb{R}^{\bar{\Gamma}} \text{ such that } \begin{cases} \mathbf{f}_{int}^{\diamond}(\mathbf{u}^{\diamond}) + \mathbf{f}_{ext}^{\diamond} + \mathbf{t}^{\diamond T} \boldsymbol{\lambda}_b^{\diamond} = 0 \\ \mathbf{t}^{\diamond} \mathbf{u}^{\diamond} = \bar{\mathbf{A}}^{\diamond T} \bar{\mathbf{u}}_b \\ \bar{\mathbf{A}}^{\diamond} \boldsymbol{\lambda}_b^{\diamond} = 0 \end{cases}$$

If the local Dirichlet system possesses a unique solution, we can define Operator  $\mathbf{S}_{nl}^{(s)}$

$$\boldsymbol{\lambda}_b^{(s)} = \mathbf{S}_{nl}^{(s)} (\bar{\mathbf{A}}^{(s)T} \bar{\mathbf{u}}_b; \mathbf{f}_{ext}^{(s)})$$

which is a nonlinear version of the Schur complement; it computes the reaction associated to an imposed displacement. It can be computed in the linear case:

$$\mathbf{S}_l^{(s)}(\mathbf{u}_b^{(s)}; \mathbf{f}_{ext}^{(s)}) = \mathbf{S}_t^{(s)} \mathbf{u}^{(s)} - \mathbf{b}^{(s)}$$

The primal nonlinear condensed system writes

$$\text{Find } \bar{\mathbf{u}}_b \in \mathbb{R}^{\bar{\Gamma}} \text{ such that } \bar{\mathbf{A}}^{\diamond} \mathbf{S}_{nl}^{\diamond} (\bar{\mathbf{A}}^{\diamond T} \bar{\mathbf{u}}_b; \mathbf{f}_{ext}^{\diamond}) = 0$$

## Dual condensation

$$\text{Find } \underline{\lambda}_b \in \mathbb{R}^{\Gamma} \text{ such that } \begin{aligned} \mathbf{f}_{int}^{\diamond}(\mathbf{u}^{\diamond}) + \mathbf{f}_{ext}^{\diamond} + \mathbf{t}^{\diamond T} \underline{\mathbf{A}}^{\diamond T} \underline{\lambda}_b &= 0 \\ \underline{\mathbf{A}}^{\diamond} \mathbf{t}^{\diamond} \mathbf{u}^{\diamond} &= 0 \end{aligned}$$

For local Neumann problem to be well-posed, we need to ensure that

$$\mathbf{R}^{\diamond T} (\mathbf{f}_{ext}^{\diamond} + \mathbf{t}^{\diamond T} \underline{\mathbf{A}}^{\diamond T} \underline{\lambda}_b) = 0$$

Then we can define Operator  $\mathbf{D}_{nl}^{(s)}$

$$\mathbf{u}_b^{(s)} = \mathbf{D}_{nl}^{(s)} (\underline{\mathbf{A}}^{(s)T} \underline{\lambda}_b; \mathbf{f}_{ext}^{(s)}) + \mathbf{R}_b^{(s)} \boldsymbol{\alpha}^{(s)}$$

which is a nonlinear version of dual Schur complement, it computes the displacement associated to given reaction (up to a rigid body motion). It can be computed in the linear case:

$$\mathbf{D}_l^{(s)} (\boldsymbol{\lambda}_b^{(s)}; \mathbf{f}_{ext}^{(s)}) = \mathbf{S}_t^{(s)+} (\boldsymbol{\lambda}_b^{(s)} + \mathbf{b}^{(s)})$$

The dual nonlinear condensed system writes

$$\text{Find } \underline{\lambda}_b \in \mathbb{R}^{\Gamma}, \boldsymbol{\alpha}^{\diamond} \text{ such that } \begin{cases} \underline{\mathbf{A}}^{\diamond} \left( \mathbf{D}_{nl}^{\diamond} (\underline{\mathbf{A}}^{\diamond T} \underline{\lambda}_b; \mathbf{f}_{ext}^{\diamond}) + \mathbf{t}^{\diamond} \mathbf{R}^{\diamond} \boldsymbol{\alpha}^{\diamond} \right) = 0 \\ \mathbf{R}^{\diamond T} (\mathbf{f}_{ext}^{\diamond} + \mathbf{t}^{\diamond T} \underline{\mathbf{A}}^{\diamond T} \underline{\lambda}_b) = 0 \end{cases}$$

# Mixed condensation

We can introduce a new interface field  $\boldsymbol{\mu}_b^\diamond = \boldsymbol{\lambda}_b^\diamond + \mathbf{Q}_b^\diamond \mathbf{u}_b^\diamond$  which leads to use Robin boundary conditions on the interface. Interface impedance (virtual stiffness, SPD matrix)  $\mathbf{Q}_b$  is a parameter.

$$\text{Find } \boldsymbol{\mu}_b^\diamond \in \mathbb{R}_b^\diamond \text{ so that } \mathbf{f}_{int}^\diamond(\mathbf{u}^\diamond) - \mathbf{t}^{\diamond T} \mathbf{Q}_b^\diamond \mathbf{u}_b^\diamond + \mathbf{t}^{\diamond T} \boldsymbol{\mu}_b^\diamond + \mathbf{f}_{ext}^\diamond = 0$$

$$\overline{\mathbf{A}}^{\diamond T} \left( \overline{\mathbf{A}}^\diamond \mathbf{Q}_b^\diamond \overline{\mathbf{A}}^{\diamond T} \right)^{-1} \overline{\mathbf{A}}^\diamond \boldsymbol{\mu}_b^\diamond - \mathbf{u}_b^\diamond = 0$$

We introduce a mixed nonlinear Schur complement

$$\mathbf{u}_b^{(s)} = \mathbf{M}_{nl}^{(s)}(\boldsymbol{\mu}_b^{(s)}; \mathbf{f}_{ext}^{(s)}, \mathbf{Q}_b^{(s)})$$

Find  $\boldsymbol{\mu}_b^\diamond \in \mathbb{R}_b^\diamond$  such that

$$\overline{\mathbf{A}}^{\diamond T} \left( \overline{\mathbf{A}}^\diamond \mathbf{Q}_b^\diamond \overline{\mathbf{A}}^{\diamond T} \right)^{-1} \overline{\mathbf{A}}^\diamond \boldsymbol{\mu}_b^\diamond - \mathbf{M}_{nl}^\diamond(\boldsymbol{\mu}_b^\diamond; \mathbf{f}_{ext}^\diamond, \mathbf{Q}_b^\diamond) = 0$$

# Mixed condensation

Alternative mixed condensation

Another formulation is possible, since:

$$\forall \mathbf{x}_b^\diamond \in \mathbb{R}_b^\diamond, \exists! (\underline{\mathbf{y}}, \bar{\mathbf{z}}) \in \mathbb{R}^\Gamma \times \mathbb{R}^{\bar{\Gamma}} / \mathbf{x}_b^\diamond = \mathbf{Q}_b^\diamond \bar{\mathbf{A}}^{\diamond T} \bar{\mathbf{z}} + \underline{\mathbf{A}}^{\diamond T} \underline{\mathbf{y}}$$

$$\text{indeed } \begin{cases} \bar{\mathbf{z}} = \left( \bar{\mathbf{A}}^{\diamond} \mathbf{Q}_b^\diamond \bar{\mathbf{A}}^{\diamond T} \right)^{-1} \bar{\mathbf{A}}^{\diamond} \mathbf{x}_b^\diamond \\ \underline{\mathbf{y}} = \left( \underline{\mathbf{A}}^{\diamond} \mathbf{Q}_b^{\diamond -1} \underline{\mathbf{A}}^{\diamond T} \right)^+ \underline{\mathbf{A}}^{\diamond} \mathbf{Q}_b^{\diamond -1} \mathbf{x}_b^\diamond \end{cases}$$

so that we can rewrite  $\mu_b^\diamond$  in terms of balanced force  $\underline{\lambda}_b$  and continuous displacement fields  $\bar{\mathbf{u}}_b$ :

$$\mu_b^\diamond = \underline{\mathbf{A}}^{\diamond T} \underline{\lambda}_b + \mathbf{Q}_b^\diamond \bar{\mathbf{A}}^{\diamond T} \bar{\mathbf{u}}_b$$



## Resolution of condensed nonlinear system

Primal case

$$\text{Find } \bar{\mathbf{u}}_b \in \mathbb{R}^{\bar{\Gamma}} \text{ such that } \bar{\mathbf{A}} \diamond \mathbf{S}_{nl} \diamond (\bar{\mathbf{A}} \diamond^T \bar{\mathbf{u}}_b; \mathbf{f}_{ext} \diamond) = 0$$

## Newton's solver

$$\left( \bar{\mathbf{A}} \diamond \left( \frac{\partial \mathbf{S}_{nl}}{\partial \bar{\mathbf{u}}_b} \right)_{\bar{\mathbf{u}}_{bk}} \bar{\mathbf{A}} \diamond^T \right) \delta \bar{\mathbf{u}}_b = -\bar{\mathbf{A}} \diamond \mathbf{S}_{nl} \diamond (\bar{\mathbf{A}} \diamond^T \bar{\mathbf{u}}_{bk}; \mathbf{f}_{ext} \diamond)$$

$$\bar{\mathbf{u}}_{bk+1} = \bar{\mathbf{u}}_{bk} + \delta \bar{\mathbf{u}}_b$$

- Right hand side: the computation of the residual is the parallel computation of Dirichlet problems and computation of the interface reaction lack of balance.
- Left hand side: tangent operator is a primal DD operator, it can be computed by the condensation of subdomains' tangent stiffness.

# First outcome

## Algorithm

- Evaluation of NL residual = independent subdomains computations (with various BC's)
- Tangent system = DD-like system (assembling of local condensed operators obtained from tangent stiffness)

Part of the treatment of the nonlinearity is then brought back to the level of the subdomains.

## About BC's

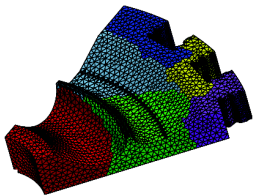
- For linear problems, all formulations are equivalent (tangent systems can be changed from one formulation to another).
- For nonlinear problems, BC's have a strong influence (especially if instabilities are possible), they are not equivalent, performance can be much varying.
- For mixed systems, impedance  $\mathbf{Q}_b$  must be fitted [Gendre, Allix, and Gosselet, 2011].
- For dual formulation, specific (classical) treatment of floating subdomains is necessary.

## Parameters

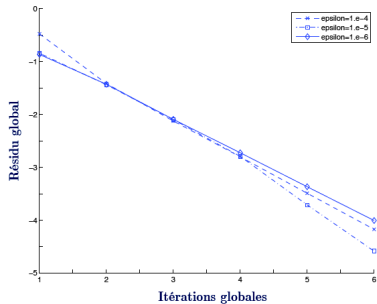
Many thresholds have to be set-up: global and local Newton, Krylov solver.

# First results (primal)

- Pied d'aube de turbine. Comportement endommageable.



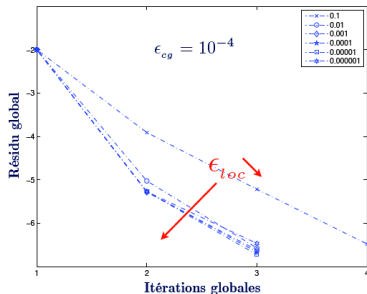
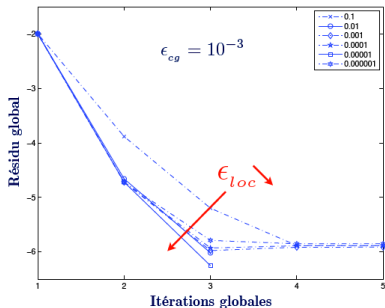
## Influence of initialization



Initialization has almost no influence.

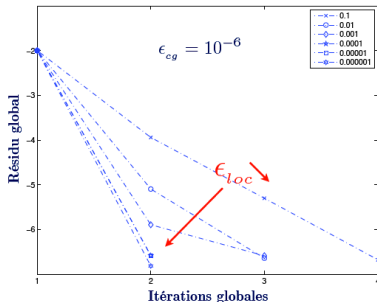
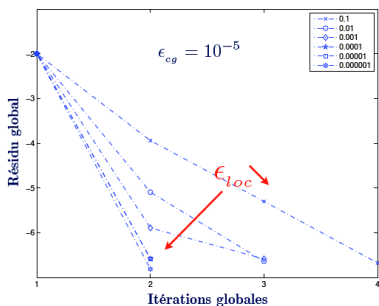
# Influence of the precision of local resolutions

Good local precision diminishes the number of global iterations.



# Influence of the precision of local resolutions

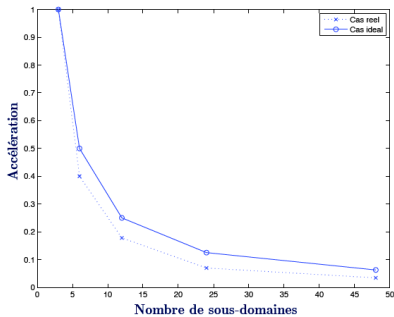
Good local precision diminishes the number of global iterations.



Au delà d'un seuil la précision du solveur de Krylov ne joue plus

# Scalability

CPU time speedup

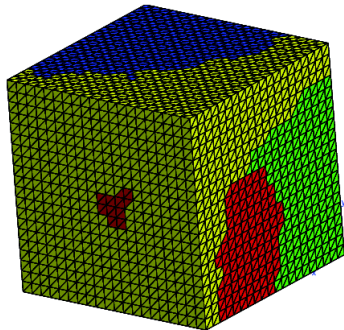


# Results (dual)

- **Problème hyperélastique**

- Matériau hyperélastique de type Mooney Rivlin:

$$\mathcal{W}(I_1, I_2, I_3) = C_1(I_1 - 2I_3 - 1) + C_2(I_2 - 2I_3 - 1) + \frac{C_3}{2}(C_1 + C_2)(I_3 - 1)^2$$



- **Paramètres matériaux**

$$(C_1, C_2, C_3) = (0.0422, -0.0068, 1)$$

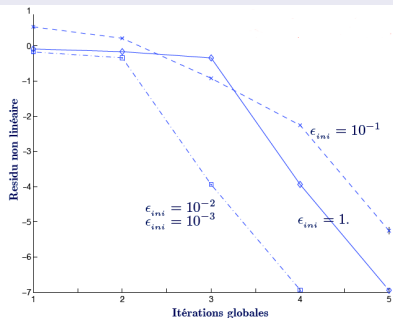
- **Bord gauche encastré.**  
**Bord droit,  $u_d$  imposé**

$$u_d = (0.05, 0.05, 0.05)$$

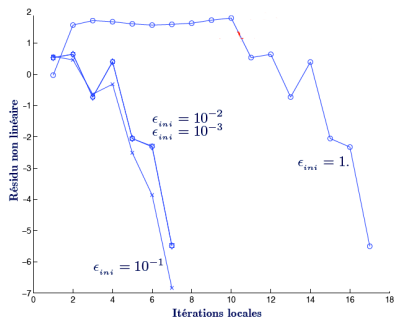
Partir

# Influence of initialization

## Influence on global residual



## Influence on the local residual of first subdomain



Beyond a threshold, the precision of initialization has no more influence.

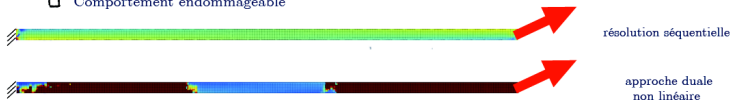


# On the risk of instabilities

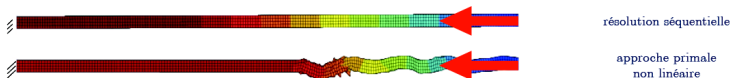
- Deux problèmes décomposés différents: primal et dual. Impliquant la résolution de problèmes non linéaires locaux à conditions de Dirichlet ou Neumann.

- Calculs simples:

- Comportement endommageable



- Comportement hyperélastique



Possibilité d'exciter des instabilités numériques lors de la résolution des problèmes locaux

# Results (mixed)

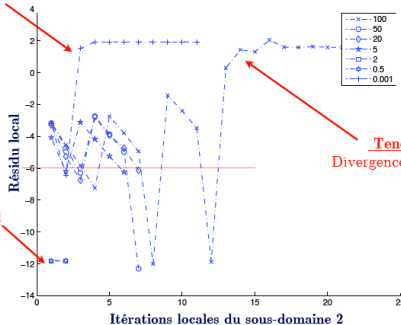
- Poutre endommageable. Sollicitée en traction, deux sous-domaines.



- Influence du rapport T/E sur l'historique de la convergence des problèmes non linéaires locaux

Tend vers le cas dual

Divergence au 2ème problème local



Cas optimal

1 itération globale

1 itération par problème local

Tend vers le cas primal

Divergence au 4ème problème local

# Conclusion on DD for NL problems

- The solution to a nonlinear problem can be split into the solution of parallel nonlinear systems per subdomains and tangent interface systems.
- The interest is that less tangent systems have to be solved (less exchanges).
- The use of different formulations (primal/dual/mixed) leads to different performance.

- Proof of convergence seems easy for material with positive hardening.
- For other cases, the shape of the subdomain has an influence on the stability.
- Extra difficulty for the handling of floating subdomains in large transformations.
- Currently working on the setting up of the respective precisions of solvers.

Other problem: definition of load balancing (maybe using adaptation).

- 1 Brief reminder for linear systems
- 2 DD for nonlinear problems
- 3 The LaTin method
  - Brief presentation
  - Application to delamination
- 4 Conclusion

# The LaTin method

The Large Time Increment method

A method [Ladevèze, 1985] designed since the beginning for nonlinear problems

small transformations, elastoviscoplastic behavior (hardening  $\underline{\mathbf{X}}$ , thermodynamic force  $\underline{\mathbf{Y}}$ ), under normal form  $(\underline{\dot{\underline{\epsilon}}}^P, \underline{\underline{\sigma}}, \underline{\dot{\underline{\mathbf{X}}}}, \underline{\mathbf{Y}})$ :

- Linear equations

$$\text{SA } \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(u^*) = \int_{\Omega} \underline{\mathbf{f}} u^* + \int_{\partial_f \Omega} \underline{\mathbf{F}}_d u^*, \forall u^* \text{ CA0}$$

$$\text{KA } u = u_d \text{ on } \partial_u \Omega, \underline{\underline{\epsilon}}(u) = \underline{\underline{\epsilon}}^e + \underline{\underline{\epsilon}}^P$$

$$\text{State } \underline{\underline{\sigma}} = \mathbf{K} : \underline{\underline{\epsilon}}^e \text{ et } \underline{\mathbf{Y}} = \mathbf{A} : \underline{\mathbf{X}}$$

Initial conditions

- Point-wise equations (in space and time)
  - $(\underline{\dot{\underline{\epsilon}}}^P, \underline{\dot{\underline{\mathbf{X}}}}) \in \mathbf{B}(\underline{\underline{\sigma}}, \underline{\mathbf{Y}})$

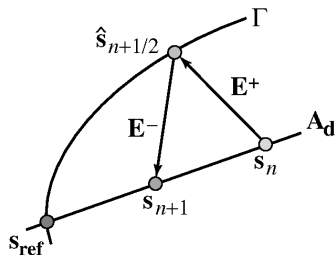
Use of a standard FE method on  $u$ .

The whole time-space solution is iteratively sought.

## Two-step iterative algorithm

- “Local” step
  - Independent per Gauss point and time step
  - Parameter: search direction  $\mathbf{K}^+$  (influence of neighborhood on the point)
  - Nonlinearity handled by a Newton
- Linear step
  - Linear structure-large system at each time step.
  - Parameter: search direction  $\mathbf{K}^-$  (approximate linear behavior)

- Intelligent handling of time by PGD [Ladevèze, Nouy, Passieux,...]
- Convergence proved if  $\mathbf{B}$  max. monotone,  $\mathbf{K}^+ = \mathbf{K}^-$  SPD, convergence observed in many cases
- Cheap iterations (global matrix is constant)



# Domain decomposition

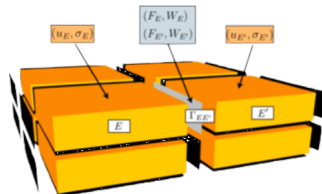
- Interface unknown
  - Traction on the boundary of substructures  $\underline{\mathbf{F}}_E$
  - Interface displacement  $\underline{\mathbf{W}}_E$
- Interface equation
  - Traction balance  $\underline{\mathbf{F}}_E + \underline{\mathbf{F}}_{E'} = 0$  (linear equation)
  - Behavior  $\underline{\mathbf{F}}_E = \mathbf{b}_{EE'}(\underline{\mathbf{W}}_E - \underline{\mathbf{W}}_{E'}, \underline{\mathbf{X}}_{EE'})$

For standard behaviors (perfect, contact, friction), LaTin hypothesis are satisfied  $(\underline{\dot{\epsilon}}^p, \underline{\sigma}_E, \underline{\dot{\mathbf{X}}}_E, \underline{\mathbf{Y}}_E, \underline{\dot{\mathbf{W}}}_E, \underline{\mathbf{F}}_E)$ .

Interface fields  $\underline{\mathbf{W}}_E$  et  $\underline{\mathbf{F}}_E$  are discretized.

- $\underline{\mathbf{W}}_E = \text{trace}(u_E)$  linear
- $\underline{\mathbf{F}}_E = \underline{\underline{\sigma}}_E \cdot \underline{\mathbf{n}}_E$  linear

Rq: Discretizing  $\underline{\mathbf{F}}_E$  is complex, we are currently investigating a strategy inspired from [Bernardi, Rebollo, and Vera, 2008].



## Local step

- Behavior within substructures (parameter  $\mathbf{K}_E^+$ )
- Behavior and balance of interfaces (**communications** between neighboring interfaces) extra parameter  $\mathbf{k}_E^+$

$$(\underline{\mathbf{F}}_E - \widehat{\underline{\mathbf{F}}}_E) + \mathbf{k}_E^+ (\underline{\mathbf{W}}_E - \widehat{\underline{\mathbf{W}}}_E) = 0$$

Nonlinearity handled by Newton solver

## Linear step

- Equilibrium of substructures (fake behavior  $\mathbf{K}_E^-$ )
- Trace relation with interfaces
- Closing...



### Monoscale closing of linear step

Extra parameter = interface search direction  $\underline{\mathbf{k}}_E^-$  (approximate behavior of the rest of the structure)

$$(\underline{\mathbf{F}}_E - \widehat{\underline{\mathbf{F}}}_E) - \underline{\mathbf{k}}_E^- (\underline{\mathbf{W}}_E - \widehat{\underline{\mathbf{W}}}_E) = 0$$

≈ Robin condition on the interface

**No exchange between subdomains**

### Multiscale closing of linear step

Macro spaces for displacement  $\underline{\mathbf{W}}^M$  and tractions  $\underline{\mathbf{F}}^M$

- Verification of macro-balance  $\int_{\Gamma_{EE'}} (\underline{\mathbf{F}}_E + \underline{\mathbf{F}}_{E'}) \cdot \underline{\mathbf{W}}^{M*} = 0$
- If it exist macro behavior  $\underline{\mathbf{F}}_E^M = \mathbf{b}_E^M (\underline{\mathbf{W}}_E^M - \underline{\mathbf{W}}_{E'}^M)$
- macro search direction  $\underline{\mathbf{k}}_E^{-M}$  is verified at best (if no macro behavior exists)
- micro search direction  $\underline{\mathbf{k}}_E^{-m}$

**Small-size exchanges between subdomains, scalability**

# Usual macro spaces

The macro spaces are constituted by the affine part of interface fields. This (large) space includes subdomains' rigid body motions and then enables to satisfy Saint Venant principle. [Ladevèze and Dureisseix, 2000].

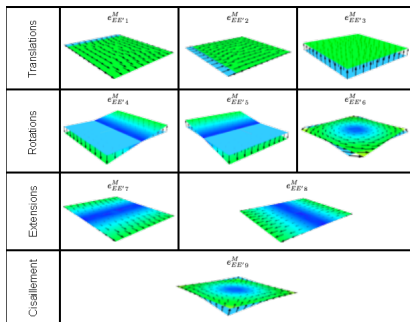


Figure: Base macro usuelle  $\underline{\mathbf{W}}^M$

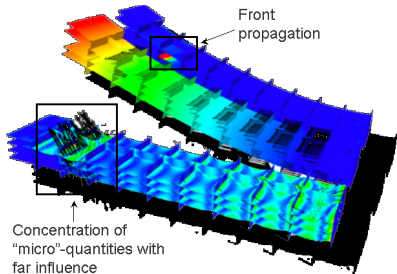
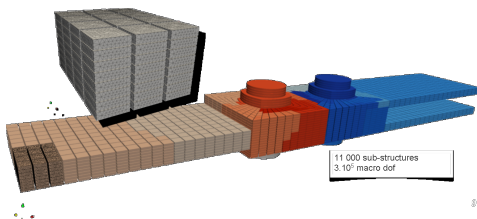
Usually traction and displacement macro basis are chosen identical.

# Summary on LaTin

- Both computation and modeling method (interfaces can bear mechanical behaviors)
- Nonlinearity is treated at the smallest possible scale.
- Sort of modified Newton (constant matrix) with additional local search direction.
- Strongly parallel method (computations independent per Gauss point or per substructures) but convergence may be slow.
- In some cases relaxation is required to prove convergence.
- Many ways to improve the method:
  - search direction
  - macro-displacement space  $\underline{\mathbf{W}}^M$
  - macro-traction space  $\underline{\mathbf{F}}^M$
- difficulties linked to the discretization of interface tractions
- usual macro space is very large

## Pierre Kerfriden's PhD thesis

- Meso-modeling of composites
- Implementation of a cohesive interface [Allix and Ladevèze, 1992]
- Adapted substructuring
- Incremental version



- Many substructures
- Expensive macro problem
- Slow convergence when delamination progresses
- Problems associated to critical points

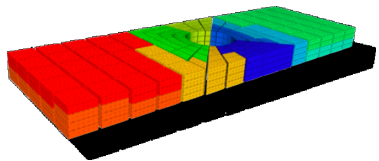
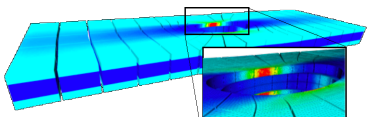
# First difficulties

## Search directions

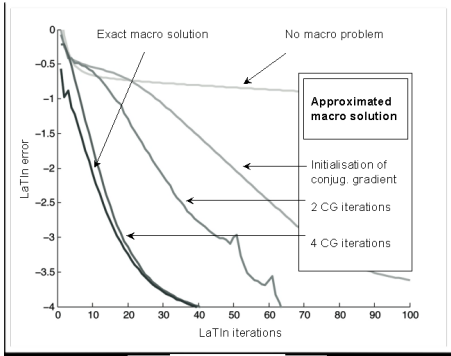
- Setting up of an adaptive strategy for search directions according to interface status (perfect, damaged, ruined, contact).
- When possible (perfect and non-damaged interfaces) verification of the macro behavior.
- frequent update of macro-operators

## Resolution of the macro problem

- the macro problem is discrete, linear and sparse, it corresponds to a mechanical problem with homogenized substructures. We solve it using a BDD approach.
- subdomains are gathered into super-subdomains.
- iterative solution at super-interfaces.
- introduction of a third scale (BDD coarse problem)
- multiple right-hand-side technique for the successive resolutions



- holed plate  $[0/90]_s$ 
  - $3.4 \cdot 10^6$  micro dof
  - $12 \cdot 10^3$  macro dof
  - 12 super-substructures
- Low precision required
- Large grain parallelism



# Pertinence of the macrospace

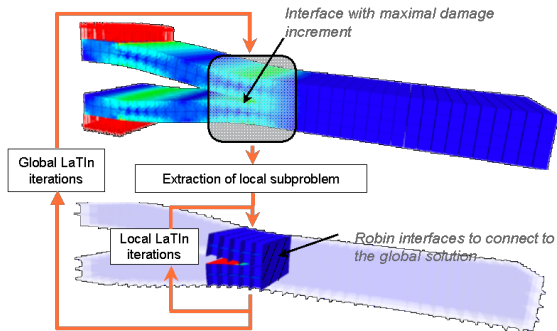
## Problems

- Classical macro basis does not capture long range effects associated to stress concentrations near the delamination front.  
→ **loss of scalability**
- No enrichment of the basis (like [Guidault, Allix, Champaney, and Cornuault, 2008]) seems satisfying.
- Even if an optimal basis was found, it would no more be valid as soon as the front has progressed.

Interface phenomena are too violent to be correctly evaluated at a short distance.

## Sub-iteration box

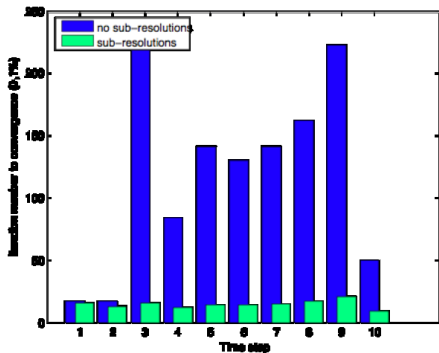
- Solution to a nonlinear problem around the delamination front (using LaTin).
- Robin conditions to connect to the rest of the structure.
- The singularity is filtered so that the classical macro-basis on the boundary of the box is sufficient.





# Restoring scalability

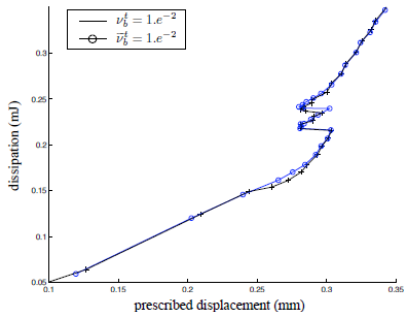
- The number of iterations becomes independent of the delamination state.
- CPU time is significantly decreased.
- The size of the box is a parameter, the optimal size (in order to minimize iterations) depends on the shape of the front (not of the structure).

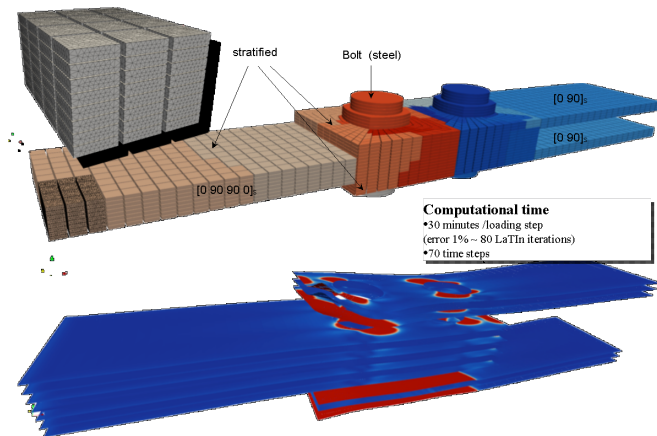


# Beyond instabilities

## Snapbacks and snapthroughs

- Arclength algorithm
- Control of loading by maximal increase of damage
- Auto-adaptation of time step according to the strength of the nonlinearity to follow the path associated to a fine solution.





$12 \cdot 10^6$  dof  $11 \cdot 10^3$  substructures,  $300 \cdot 10^3$  macro dof,  
30 super-substructures, 150 super-macro dof

# Conclusions on the LaTin method

- Due to the slenderness of plates, using DD interfaces as the support of delamination leads to non-natural substructuring and large macro problems.
- The gathering of substructures enables an efficient solving to the macro problem, with convenient level of parallelism.
- The point-wise treatment of nonlinearity leads to difficulties because the scale of the nonlinearity is not the scale of its effects.
- For delamination a more suited scale is the one of the "subiterations" box which enables to filter long-range effects.

# Conclusion

## Scale of treatment of nonlinearities

- Either starting from the LaTin method where we do upscaling or from Newton-Schur where we do downscaling, it is possible to treat nonlinearities at the scale of substructures.
- This local treatment is very interesting to diminish the number of exchanges, to increase the loading increments, to filter too violent singularities.
- Theoretical results are missing if we do not have monotone behaviors (positive hardening), and problematic cases can be imagined (dependence on the shape of the subdomains, branching to bad solutions). Large transformations also set well-posedness issues (wrt rigid body motions).
- To minimize the risks and improve performance, we must provide good representation of the rest of the structure to subdomains, which is done by using Robin bc's (with well chosen impedance) and macro (coarse) problems.

- O. Allix and P. Ladevèze. Interlaminar interface modelling for the prediction of delamination. *Computers and structures*, 22:235–242, 1992.
- Lori Badae. On the schwarz alternating method with more then two subdomains for nonlinear monotone problems. *SIAM J. Numer. Anal.*, 28(1):179–204, 1991.
- C. Bernardi, T. Chacòn Rebollo, and E. Chacòn Vera. A feti method with a mesh independant condition number for the iteration matrix. *Computer Methods in Applied Mechanics and Engineering*, 197:1410–1429, 2008.
- P. Cresta, O. Allix, C. Rey, and S. Guinard. Nonlinear localization strategies for domain decomposition methods in structural mechanics. *Computer Methods in Applied Mechanics and Engineering*, 196:1436–1446, 2007.
- C. Farhat, K. Pierson, and M. Lesoine. The second generation feti methods and their application to the parallel solution of large-scale linear and geometracally non-linear structural analysis problems. *Computer Methods in Applied Mechanics and Engineering*, 184: 333–374, 2000.
- L. Gendre, O. Allix, and P. Gosselet. A two-scale approximation of the schur complement and its use for non-intrusive coupling. *International Journal for Numerical Methods in Engineering*, submitted, 2010.
- L. Gendre, O. Allix, and P. Gosselet. A two-scale approximation of the schur complement and its use for non-intrusive coupling. *International Journal for Numerical Methods in Engineering*, early view online, 2011.
- P. Gosselet, C. Rey, and J. Pebre. Total and selective reuse of krylov subspaces for the solution to a sequence of nonlinear structural problems. *submitted to International Journal for Numerical Methods in Engineering*, 2011.
- P.-A. Guidault, O. Allix, L. Champaney, and S. Cornuault. A multiscale extended finite element method for crack propagation. *Computer Methods in Applied Mechanics and Engineering*, 197(5):381–399, January 2008.
- P. Kerfriden, P. Gosselet, S. Adhikari, and S. Bordas. Bridging proper orthogonal decomposition methods and augmented newton-krylov algorithms: an adaptive model order reduction for highly nonlinear mechanical problems. *Computer Methods in Applied Mechanics and Engineering*, 200(5-8):850–866, 2011.
- A. Klawonn and O.B. Widlund. FETI and Neumann-Neumann iterative substructuring methods: Connections and new results. *Comm. Pure App. Math.*, 54(1):57–90, 2001.
- P. Ladevèze. Sur une famille d’algorithmes en mécanique des structures. *Compte rendu de l’académie des Sciences*, 300(2):41–44, 1985.
- P. Ladevèze and D. Dureisseix. A micro/macro approach for parallel computing of heterogeneous structures. *International Journal for computational Civil and Structural Engineering*, 1:18–28, 2000.
- P. Ladevèze and A. Nouy. On a multiscale computational strategy with time and space homogenization for structural mechanics. *Computer Methods in Applied Mechanics and Engineering*, 192:3061–3087, 2003.
- A. Nouy. A priori model reduction through proper generalized decomposition for solving time-dependent partial differential equations. *Computer Methods in Applied Mechanics and Engineering*, 199(23-24):1603–1626, 2010.
- A. Parret-Fréaud, C. Rey, P. Gosselet, and F. Feyel. Fast estimation of discretization error for fe problems solved by domain decomposition. *Computer Methods in Applied Mechanics and Engineering*, 199(49-52):3315–3323, 2010.
- J. Pebre, C. Rey, and P. Gosselet. A nonlinear dual domain decomposition method: application to structural problems with damage. *International Journal for Multiscale Computational Engineering*, 6(3):251–262, 2008.
- D. Ryckelynck, F. Chinesta, E. Cueto, and A. Ammar. On the "a priori" model reduction: Overview and recent developments. *Archives of Computational Methods in Engineering*, 13:91–128, 2006.

# Discretization of interface forces

Instead of discretizing traction  $\underline{\mathbf{F}}$  on interface  $\Gamma$ , we use Riesz theorem in  $H^{1/2}$

$$\exists \underline{\mathbf{T}} \in H^{1/2}(\Gamma), \int_{\Gamma} \underline{\mathbf{F}} \underline{\mathbf{W}}^* = \langle \underline{\mathbf{T}}, \underline{\mathbf{W}}^* \rangle$$

$$\langle \underline{\mathbf{T}}, \underline{\mathbf{W}}^* \rangle = \int_{\Gamma} \underline{\mathbf{T}} \underline{\mathbf{W}}^* + \iint_{\Gamma \times \Gamma} \frac{(\underline{\mathbf{T}}(x) - \underline{\mathbf{T}}(y)) (\underline{\mathbf{W}}^*(x) - \underline{\mathbf{W}}^*(y))}{|x - y|^d} dx dy$$

$\underline{\mathbf{T}}$  can be discretized like  $\underline{\mathbf{W}}$ .

- This strategy leads to complex computations with dense interface matrices
- We currently work on simpler quadrature for the integration.
- This strategy leads to convergence rate independent of the discretization of the interface (parameter  $h$ ).