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Holomorphic curves into algebraic varieties intersecting moving hypersurface targets

William Cherry, Gerd Dethloff and Tran Van Tan

Abstract

In [Ann. of Math.169 (2009)], Min Ru proved a second main theorem for algebraically nondegenerate holomorphic curves in complex projective varieties intersecting fixed hypersurface targets. In this paper, by using a different proof method, we generalize this result to moving hypersurface targets.

1 Introduction

During the last century, several Second Main Theorems have been established for linearly nondegenerate holomorphic curves in complex projective spaces intersecting (fixed or moving) hyperplanes, and we now have a satisfactory knowledge about it. Motivated by a paper of Corvaja-Zannier [5] in Diophantine approximation, in 2004 Ru [15] proved a Second Main Theorem for algebraically nondegenerate holomorphic curves in the complex projective space \mathbb{CP}^n intersecting (fixed) hypersurface targets, which settled a long-standing conjecture of Shiffman [17]. In 2011, Dethloff-Tan [6] generalized this result of Ru to moving hypersurface targets (this means where the coefficients of the hypersurfaces are meromorphic functions) in \mathbb{CP}^n . In 2009, Ru [16] generalized his Second Main Theorem to the case of holomorphic curves in smooth complex varieties of dimension n . The main idea in the approach of all the papers mentioned above is to estimate systems of n hypersurfaces in general position by systems of hyperplanes, and then to reduce to the case of hyperplanes. To prove the Second Main Theorem for the case of curves in smooth complex varieties intersecting (fixed) hypersurfaces, in [16], Ru uses

the finite morphism $\phi : V \rightarrow \mathbb{CP}^{q-1}$, $\phi(x) := [Q_1(x) : \cdots : Q_q(x)]$, where the Q_j 's are homogeneous polynomials (with common degree) defining the given hypersurfaces. Thanks to this finite morphism, he can use a generalization of Mumford's identity (the version with explicit estimates obtained by Evertse and Ferretti [9, 10]) for the variety $\text{Im}\phi \subset \mathbb{CP}^{q-1}$. However, for the case of moving hypersurfaces, we do not have such a morphism. So in order to carry out the idea to estimate systems of n hypersurfaces in general position by systems of hyperplanes, and then to reduce to the case of hyperplanes, we have to go back to the filtration method used for the case of curves in the complex projective space by Corvaja-Zannier [5], Ru [15] and Dethloff-Tan [6]. In order to compute the dimensions of the various vector subspaces produced by this filtration method, the following property was used: If homogeneous polynomials Q_0, \dots, Q_n in $\mathbb{C}[x_0, \dots, x_n]$ have no non-trivial common solutions, then $\{Q_0, \dots, Q_n\}$ is a regular sequence (see [6] for the extension to the case of moving hypersurface targets). However, this property is not true for the general case of varieties $V \subset \mathbb{CP}^M$, and is related to whether or not the homogenous coordinate ring of V is Cohen-Macaulay. So by dropping this restriction on the variety V and thereby losing regular sequences, we can no longer exactly calculate the dimensions of these vector subspaces. What we do instead is first to show that by specializing the coefficients of the polynomials corresponding to the moving hypersurfaces in generic points we do not change the dimensions of these vector spaces, then to observe that the Hilbert sequence asymptotics imply that most of the subspaces have the expected dimensions, and finally to prove that we can neglect those rare subspaces where the dimension is not as expected. Another difficulty in the case of moving hypersurface targets is that they are in general position only for generic points. In order to overcome this difficulty, in Dethloff-Tan [6] we used resultants in order to control the locus where the divisors are not in general position. For the more general case of varieties V this technique becomes more complicated since the ideal of the resultants is not a principal ideal in general (unless V is a complete intersection variety). But we will observe (see section 2) that using any nonzero element of this ideal will still be enough for our purpose.

Let f be a holomorphic mapping of \mathbb{C} into \mathbb{CP}^M , with a reduced representation $f := (f_0 : \cdots : f_M)$. The characteristic function $T_f(r)$ of f is defined

by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta, \quad r > 1,$$

where $\|f\| := \max\{|f_0|, \dots, |f_M|\}$.

Let ν be a divisor on \mathbb{C} . The counting function of ν is defined by

$$N_\nu(r) := \int_1^r \log \frac{\sum_{|z|<t} \nu(z)}{t} dt, \quad r > 1.$$

For a non-zero meromorphic function φ , denote by ν_φ the zero divisor of φ , and set $N_\varphi(r) := N_{\nu_\varphi}(r)$. Let Q be a homogeneous polynomial in the variables x_0, \dots, x_M with coefficients which are meromorphic functions. If $Q(f) := Q(f_0, \dots, f_M) \not\equiv 0$, we define $N_f(r, Q) := N_{Q(f)}(r)$. Denote by $Q(z)$ the homogeneous polynomial over \mathbb{C} obtained by evaluating the coefficients of Q at a specific point $z \in \mathbb{C}$ in which all coefficient functions of Q are holomorphic (in particular $Q(z)$ can be the zero polynomial).

We say that a meromorphic function φ on \mathbb{C} is “small” with respect to f if $T_\varphi(r) = o(T_f(r))$ as $r \rightarrow \infty$ (outside a set of finite Lebesgue measure).

Denote by \mathcal{K}_f the set of all “small” (with respect to f) meromorphic functions on \mathbb{C} . Then \mathcal{K}_f is a field.

For a positive integer d , we set

$$\mathcal{T}_d := \{(i_0, \dots, i_M) \in \mathbb{N}_0^{M+1} : i_0 + \dots + i_M = d\}.$$

Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be a set of $q \geq n + 1$ homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_M]$, $\deg Q_j = d_j \geq 1$. We write

$$Q_j = \sum_{I \in \mathcal{T}_{d_j}} a_{jI} x^I \quad (j = 1, \dots, q)$$

where $x^I = x_0^{i_0} \dots x_M^{i_M}$ for $x = (x_0, \dots, x_M)$ and $I = (i_0, \dots, i_M)$. Denote by $\mathcal{K}_{\mathcal{Q}}$ the field over \mathbb{C} of all meromorphic functions on \mathbb{C} generated by $\{a_{jI} : I \in \mathcal{T}_{d_j}, j \in \{1, \dots, q\}\}$. It is clearly a subfield of \mathcal{K}_f .

Let $V \subset \mathbb{CP}^M$ be an arbitrary projective variety of dimension n , generated by the homogeneous polynomials in its ideal $\mathcal{I}(V)$. Assume that f is non-constant and $\text{Im} f \subset V$. Denote by $\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)$ the ideal in $\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V)$. Equivalently $\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)$ is the (infinite-dimensional) $\mathcal{K}_{\mathcal{Q}}$ -subvector space of $\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V)$. We note that $Q(f) \equiv 0$

for every homogeneous polynomial $Q \in \mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)$. We say that f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$ if there is no homogeneous polynomial $Q \in \mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M] \setminus \mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)$ such that $Q(f) \equiv 0$.

The set \mathcal{Q} is said to be V -admissible (or in (weakly) general position (with respect to V)) if there exists $z \in \mathbb{C}$ in which all coefficient functions of all Q_j , $j = 1, \dots, q$ are holomorphic and such that for any $1 \leq j_0 < \dots < j_n \leq q$ the system of equations

$$\begin{cases} Q_{j_i}(z)(x_0, \dots, x_M) = 0 \\ 0 \leq i \leq n \end{cases} \quad (1.1)$$

has no solution (x_0, \dots, x_M) satisfying $(x_0 : \dots : x_M) \in V$. As we will show in section 2, in this case this is true for all $z \in \mathbb{C}$ excluding a discrete subset of \mathbb{C} .

As usual, by the notation “ $\|P$ ” we mean that the assertion P holds for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int_E dr < +\infty$.

Our main result is stated as follows:

Main Theorem. *Let $V \subset \mathbb{CP}^M$ be an irreducible (possibly singular) variety of dimension n , and let f be a non-constant holomorphic map of \mathbb{C} into V . Let $\mathcal{Q} = \{Q_1, \dots, Q_q\}$ be a V -admissible set of homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_M]$ with $\deg Q_j = d_j \geq 1$. Assume that f is algebraically nondegenerate over $\mathcal{K}_{\mathcal{Q}}$. Then for any $\varepsilon > 0$,*

$$\|(q - n - 1 - \varepsilon)T_f(r) \leq \sum_{j=1}^q \frac{1}{d_j} N_f(r, Q_j).$$

In the special case where the coefficients of the polynomials Q_j 's are constant and the variety V is smooth, the above theorem is the Second Main Theorem of Ru in [16]. According to Vojta ([19], p. 183) generalizing this theorem of Ru to singular varieties can be done already by his proof methods without essential changes of the proof (see also Chen-Ru-Yan [3], [4]), so the essential generalization in our main result is the one to moving targets.

We define the defect of f with respect to a homogenous polynomial $Q \in \mathcal{K}_f[x_0, \dots, x_M]$ of degree d with $Q(f) \not\equiv 0$ by

$$\delta_f(Q) := \lim_{r \rightarrow +\infty} \inf \left(1 - \frac{N_f(r, Q)}{d \cdot T_f(r)} \right).$$

As a corollary of the Main Theorem we get the following defect relation.

Corollary 1.1. *Under the assumptions of the Main theorem, we have*

$$\sum_{j=1}^q \delta_f(Q_j) \leq n + 1.$$

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2 Lemmas

Let \mathcal{K} be an arbitrary field over \mathbb{C} generated by a set of meromorphic functions on \mathbb{C} . Let V be a sub-variety in \mathbb{CP}^M of dimension n defined by the homogeneous ideal $\mathcal{I}(V) \subset \mathbb{C}[x_0, \dots, x_M]$. Denote by $\mathcal{I}_{\mathcal{K}}(V)$ the ideal in $\mathcal{K}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V)$.

For each positive integer k and for any (finite or infinite dimensional) \mathbb{C} -vector sub-space W in $\mathbb{C}[x_0, \dots, x_M]$ or for any \mathcal{K} -vector sub-space W in $\mathcal{K}[x_0, \dots, x_M]$, we denote by W_k the vector sub-space consisting of all homogeneous polynomials in W of degree k (and of the zero polynomial; we remark that W_k is necessarily of finite dimension).

The Hilbert polynomial H_V of V is defined by

$$H_V(N) := \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_M]_N}{\mathcal{I}(V)_N}, \quad N \in \mathbb{N}.$$

By the usual theory of Hilbert polynomials (see e.g. [11]), for $N \gg 0$, we have

$$H_V(N) = \deg V \cdot \frac{N^n}{n!} + O(N^{n-1}).$$

Definition 2.1. Let W be a \mathcal{K} -vector sub-space in $\mathcal{K}[x_0, \dots, x_M]$. For each $z \in \mathbb{C}$, we denote

$$W(z) := \{P(z) : P \in W, \text{ all coefficients of } P \text{ are holomorphic at } z\}.$$

It is clear that $W(z)$ is a \mathbb{C} -vector sub-space of $\mathbb{C}[x_0, \dots, x_M]$.

Lemma 2.2. Let W be a \mathcal{K} -vector sub-space in $\mathcal{K}[x_0, \dots, x_M]_N$. Assume that $\{h_j\}_{j=1}^K$ is a basis of W . Then $\{h_j(a)\}_{j=1}^K$ is a basis of $W(a)$ (and in particular $\dim_{\mathcal{K}} W = \dim_{\mathbb{C}} W(a)$) for all $a \in \mathbb{C}$ excluding a discrete subset.

Proof. Let (c_{ij}) be the matrix of coefficients of $\{h_j\}_{j=1}^K$. Since $\{h_j\}_{j=1}^K$ are linearly independent over \mathcal{K} , there exists a square submatrix A of (c_{ij}) of order K and such that $\det A \neq 0$. Let a be an arbitrary point in \mathbb{C} such that $\det A(a) \neq 0$ and such that all coefficients of $\{h_j\}_{j=1}^K$ are holomorphic at a . For each $P \in W$ whose coefficients are all holomorphic at a , we write $P = \sum_{j=1}^K t_j h_j$ with $t_j \in \mathcal{K}$. In fact, there are coefficients b_j ($j = 1, \dots, K$) of P such that (t_1, \dots, t_K) is the unique solution in \mathcal{K}^K of the following system of linear equations:

$$A \cdot \begin{pmatrix} t_1 \\ \vdots \\ t_K \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_K \end{pmatrix}.$$

By our choice of a , so in particular we have $\det A(a) \neq 0$, and since $\{b_j\}_{j=1}^K$ are holomorphic at a , we get that the $\{t_j\}_{j=1}^K$ are holomorphic at a . Therefore, $P(a) = \sum_{j=1}^K t_j(a) h_j(a)$, $t_j(a) \in \mathbb{C}$. On the other hand, still by our choice of a , we have $h_j(a) \in W(a)$ for all $j \in \{1, \dots, K\}$. Hence, $\{h_j(a)\}_{j=1}^K$ is a generating system of $W(a)$. Since $\det A(a) \neq 0$, the matrix $(c_{ij}(a))$ has maximum rank. Therefore, $\{h_j(a)\}_{j=1}^K$ are also linearly independent over \mathbb{C} . \square

Throughout of this section, we consider a V -admissible set of $(n+1)$ homogeneous polynomials Q_0, \dots, Q_n in $\mathcal{K}[x_0, \dots, x_M]$ of common degree d . We write

$$Q_j = \sum_{I \in \mathcal{T}_d} a_{jI} x^I, \quad (j = 0, \dots, n),$$

where $a_{jI} \in \mathcal{K}$ and \mathcal{T}_d is again the set of all $I := (i_0, \dots, i_M) \in \mathbb{N}_0^{M+1}$ with $i_0 + \dots + i_M = d$.

Let $t = (\dots, t_{jI}, \dots)$ be a family of variables. Set

$$\widetilde{Q}_j = \sum_{I \in \mathcal{T}_d} t_{jI} x^I \in \mathbb{C}[t, x], \quad (j = 0, \dots, n).$$

We have

$$\widetilde{Q}_j(\dots, a_{jI}(z), \dots, x_0, \dots, x_M) = Q_j(z)(x_0, \dots, x_M).$$

Assume that the ideal $\mathcal{I}(V)$ is generated by homogeneous polynomials P_1, \dots, P_m . Since $\{Q_0, \dots, Q_n\}$ is a V -admissible set, there exists $z_0 \in \mathbb{C}$ such that the homogeneous polynomials $P_1, \dots, P_m, Q_0(z_0), \dots, Q_n(z_0)$ in $\mathbb{C}[x_0, \dots, x_M]$ have no common non-trivial solutions. Denote by ${}_{\mathbb{C}[t]}(P_1, \dots, P_m, \widetilde{Q}_0, \dots, \widetilde{Q}_n)$ the ideal in the ring of polynomials in x_0, \dots, x_M with coefficients in $\mathbb{C}[t]$ generated by $P_1, \dots, P_m, \widetilde{Q}_0, \dots, \widetilde{Q}_n$. A polynomial \widetilde{R} in $\mathbb{C}[t]$ is called an *inertia form* of the polynomials $P_1, \dots, P_m, \widetilde{Q}_0, \dots, \widetilde{Q}_n$ if it has the following property (see e.g. [20]):

$$x_i^s \cdot \widetilde{R} \in {}_{\mathbb{C}[t]}(P_1, \dots, P_m, \widetilde{Q}_0, \dots, \widetilde{Q}_n)$$

for $i = 0, \dots, M$ and for some non-negative integer s .

It is well known that $(m+n+1)$ homogeneous polynomials $P_i(x_0, \dots, x_M)$, $\widetilde{Q}_j(\dots, t_{jI}, \dots, x_0, \dots, x_M)$, $i \in \{1, \dots, m\}$, $j \in \{0, \dots, n\}$ have no common non-trivial solutions in x_0, \dots, x_M for special values t_{jI}^0 of t_{jI} if and only if there exists an inertia form \widetilde{R} (depending on t_{jI}^0) such that $\widetilde{R}(\dots, t_{jI}^0, \dots) \neq 0$ (see e.g. [20], page 254). Choose such a \widetilde{R} for the special values $t_{jI}^0 = a_{jI}(z_0)$, and put $R(z) := \widetilde{R}(\dots, a_{kI}(z), \dots) \in \mathcal{K}$. Then by construction, $R(z_0) \neq 0$, hence $R \in \mathcal{K} \setminus \{0\}$, so in particular R only vanishes on a discrete subset of \mathbb{C} , and, by the above property of the inertia form \widetilde{R} , outside this discrete subset, $Q_0(z), \dots, Q_n(z)$ have no common solutions in V . Furthermore, by the definition of the inertia forms, there exists a non-negative integer s such that

$$x_i^s \cdot R \in {}_{\mathcal{K}}(P_1, \dots, P_m, Q_0, \dots, Q_n), \text{ for } i = 0, \dots, M, \quad (2.1)$$

where ${}_{\mathcal{K}}(P_1, \dots, P_m, Q_0, \dots, Q_n)$ is the ideal in $\mathcal{K}[x_0, \dots, x_M]$ generated by $P_1, \dots, P_m, Q_0, \dots, Q_n$.

Let f be a nonconstant meromorphic map of \mathbb{C} into \mathbb{CP}^M . Denote by \mathcal{C}_f the set of all non-negative functions $h : \mathbb{C} \rightarrow [0, +\infty] \subset \overline{\mathbb{R}}$, which are of the form

$$\frac{|u_1| + \cdots + |u_k|}{|v_1| + \cdots + |v_\ell|}, \quad (2.2)$$

where $k, \ell \in \mathbb{N}$, $u_i, v_j \in \mathcal{K}_f \setminus \{0\}$.

By the First Main Theorem we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |\phi(re^{i\theta})| d\theta = o(T_f(r)), \quad \text{as } r \rightarrow \infty$$

for $\phi \in \mathcal{K}_f$. Hence, for any $h \in \mathcal{C}_f$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ h(re^{i\theta}) d\theta = o(T_f(r)), \quad \text{as } r \rightarrow \infty.$$

It is easy to see that sums, products and quotients of functions in \mathcal{C}_f are again in \mathcal{C}_f .

By the result on the inertia forms mentioned above, similarly to Lemma 2.2 in [6], we have

Lemma 2.3. *Let $\{Q_j\}_{j=0}^n$ be a V -admissible set of homogeneous polynomials of degree d in $\mathcal{K}[x_0, \dots, x_M]$. If $\mathcal{K} \subset \mathcal{K}_f$, then there exist functions $h_1, h_2 \in \mathcal{C}_f \setminus \{0\}$ such that,*

$$h_2 \cdot \|f\|^d \leq \max_{j \in \{0, \dots, n\}} |Q_j(f_0, \dots, f_M)| \leq h_1 \cdot \|f\|^d.$$

In fact, the second inequality is elementary. In order to obtain the first inequality, we use equation (2.1) in the same way as the corresponding equation in Lemma 2.1 in [6], and we observe that we have $P_i(f_0, \dots, f_M) \equiv 0$ for $i = 1, \dots, m$ since $f(\mathbb{C}) \subset V$, so the maximum only needs to be taken over the $Q_j(f_0, \dots, f_M)$, $j = 0, \dots, n$. The rest of the proof is identically to the one of Lemma 2.2 in [6].

Let N be a positive integer divisible by d . Denote by τ_N the set of all $I := (i_1, \dots, i_n) \in \mathbb{N}_0^n$ with $\|I\| := i_1 + \cdots + i_n \leq \frac{N}{d}$. We use the lexicographic order in τ_N .

Definition 2.4. For each $I = (i_1, \dots, i_n) \in \tau_N$, denote by \mathcal{L}_N^I the set of all $\gamma \in \mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}$ such that for each $E > I$ there exists $\gamma_E \in \mathcal{K}[x_0, \dots, x_M]_{N-d\|E\|}$ satisfying

$$Q_1^{i_1} \cdots Q_n^{i_n} \gamma - \sum_{E=(e_1, \dots, e_n) > I} Q_1^{e_1} \cdots Q_n^{e_n} \gamma_E \in \mathcal{I}_{\mathcal{K}}(V)_N.$$

Remark 2.5. It is easy to see that \mathcal{L}_N^I is a \mathcal{K} -vector sub-space of $\mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}$, and $(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|} \subset \mathcal{L}_N^I$ for all $I \in \tau_N$, where $(\mathcal{I}(V), Q_1, \dots, Q_n)$ is the ideal in $\mathcal{K}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V) \cup \{Q_1, \dots, Q_n\}$.

Set

$$m_N^I := \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}}{\mathcal{L}_N^I}. \quad (2.3)$$

Let $\gamma_{I1}, \dots, \gamma_{Im_N^I} \in \mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}$ such that they form a basis of the \mathcal{K} -vector space $\frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}}{\mathcal{L}_N^I}$.

Lemma 2.6. $\{[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I1}], \dots, [Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{Im_N^I}], I = (i_1, \dots, i_n) \in \tau_N\}$ is a basis of the \mathcal{K} -vector space $\frac{\mathcal{K}[x_0, \dots, x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N}$.

Proof. Firstly, we prove that:

$$\{[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I1}], \dots, [Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{Im_N^I}], I = (i_1, \dots, i_n) \in \tau_N\} \quad (2.4)$$

are linearly independent.

Indeed, let $t_{I\ell} \in \mathcal{K}$, ($I = (i_1, \dots, i_n) \in \tau_N, \ell \in \{1, \dots, m_N^I\}$) such that

$$\sum_{I \in \tau_N} (t_{I1}[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I1}] + \cdots + t_{Im_N^I}[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{Im_N^I}]) = 0.$$

Then

$$\sum_{I \in \tau_N} Q_1^{i_1} \cdots Q_n^{i_n} (t_{I1}\gamma_{I1} + \cdots + t_{Im_N^I}\gamma_{Im_N^I}) \in \mathcal{I}_{\mathcal{K}}(V)_N. \quad (2.5)$$

By the definition of \mathcal{L}_N^I , and by (2.5), we get

$$t_{I^*1}\gamma_{I^*1} + \cdots + t_{I^*m_N^{I^*}}\gamma_{I^*m_N^{I^*}} \in \mathcal{L}_N^{I^*},$$

where I^* is the smallest elements of τ_N .

On the other hand, $\{\gamma_{I^*1}, \dots, \gamma_{I^*m_N^{I^*}}\}$ form a basis of $\frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I^*\|}}{\mathcal{L}_N^{I^*}}$.

Hence,

$$t_{I^*1} = \dots = t_{I^*m_N^{I^*}} = 0. \quad (2.6)$$

Then, by (2.5), we have

$$\sum_{I \in \tau_N \setminus \{I^*\}} Q_1^{i_1} \dots Q_n^{i_n} (t_{I1} \gamma_{I1} + \dots + t_{Im_N^I} \gamma_{Im_N^I}) \in \mathcal{I}_{\mathcal{K}}(V)_N.$$

Then, similarly to (2.6), we have

$$t_{\tilde{I}1} = \dots = t_{\tilde{I}m_N^{\tilde{I}}} = 0,$$

where \tilde{I} is the smallest element of $\tau_N \setminus \{I^*\}$.

Continuing the above process, we get that $t_{I\ell} = 0$ for all $I \in \tau_N$ and $\ell \in \{1, \dots, m_N^I\}$, and hence, we get (2.4).

Denote by \mathcal{L} the \mathcal{K} -vector sub-space in $\mathcal{K}[x_0, \dots, x_M]_N$ generated by

$$\{Q_1^{i_1} \dots Q_n^{i_n} \cdot \gamma_{I1}, \dots, Q_1^{i_1} \dots Q_n^{i_n} \cdot \gamma_{Im_N^I}, I = (i_1, \dots, i_n) \in \tau_N\}.$$

Now we prove that: For any $I = (i_1, \dots, i_n) \in \tau_N$, we have

$$Q_1^{i_1} \dots Q_n^{i_n} \cdot \gamma_I \in \mathcal{L} + \mathcal{I}_{\mathcal{K}}(V)_N \quad (2.7)$$

for all $\gamma_I \in \mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}$.

Set $I' = (i'_1, \dots, i'_n) := \max\{I : I \in \tau_N\}$. Since $\gamma_{I'1}, \dots, \gamma_{I'm_N^{I'}}$ form a basis of $\frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I'\|}}{\mathcal{L}_N^{I'}}$, for any $\gamma_{I'} \in \mathcal{K}[x_0, \dots, x_M]_{N-d\|I'\|}$, we have

$$\gamma_{I'} = \sum_{\ell=1}^{m_N^{I'}} t_{I'\ell} \cdot \gamma_{I'\ell} + h_{I'\ell}, \text{ where } h_{I'\ell} \in \mathcal{L}_N^{I'}, \text{ and } t_{I'\ell} \in \mathcal{K}. \quad (2.8)$$

On the other hand, by the definition of $\mathcal{L}_N^{I'}$, we have $Q_1^{i'_1} \dots Q_n^{i'_n} \cdot h_{I'\ell} \in \mathcal{I}_{\mathcal{K}}(V)_N$ (note that $I' = \max\{I : I \in \tau_N\}$). Hence,

$$Q_1^{i'_1} \dots Q_n^{i'_n} \cdot \gamma_{I'} = \sum_{\ell=1}^{m_N^{I'}} t_{I'\ell} Q_1^{i'_1} \dots Q_n^{i'_n} \cdot \gamma_{I'\ell} + Q_1^{i'_1} \dots Q_n^{i'_n} \cdot h_{I'\ell} \in \mathcal{L} + \mathcal{I}_{\mathcal{K}}(V)_N.$$

We get (2.7) for the case where $I = I'$.

Assume that (2.7) holds for all $I > I^* = (i_1^*, \dots, i_n^*)$. We prove that (2.7) holds also for $I = I^*$.

Indeed, similarly to (2.8), for any $\gamma_{I^*} \in \mathcal{K}[x_0, \dots, x_M]_{N-d\|I^*\|}$, we have

$$\gamma_{I^*} = \sum_{\ell=1}^{m_N^{I^*}} t_{I^*\ell} \cdot \gamma_{I^*\ell} + h_{I^*\ell}, \text{ where } h_{I^*\ell} \in \mathcal{L}_N^{I^*}, \text{ and } t_{I^*\ell} \in \mathcal{K}.$$

Then,

$$Q_1^{i_1^*} \cdots Q_n^{i_n^*} \cdot \gamma_{I^*} = \sum_{\ell=1}^{m_N^{I^*}} t_{I^*\ell} Q_1^{i_1^*} \cdots Q_n^{i_n^*} \cdot \gamma_{I^*\ell} + Q_1^{i_1^*} \cdots Q_n^{i_n^*} \cdot h_{I^*\ell}. \quad (2.9)$$

Since $h_{I^*\ell} \in \mathcal{L}_N^{I^*}$, we have

$$Q_1^{i_1^*} \cdots Q_n^{i_n^*} \cdot h_{I^*\ell} - \sum_{E=(e_1, \dots, e_n) > I^*} Q_1^{e_1} \cdots Q_n^{e_n} \cdot g_E \in \mathcal{I}_{\mathcal{K}}(V)_N,$$

for some $g_E \in \mathcal{K}[x_0, \dots, x_M]_{N-d\|E\|}$.

Therefore, by the induction hypothesis,

$$Q_1^{i_1^*} \cdots Q_n^{i_n^*} \cdot h_{I^*\ell} \in \mathcal{L} + \mathcal{I}_{\mathcal{K}}(V)_N.$$

Then, by (2.9), we have

$$Q_1^{i_1^*} \cdots Q_n^{i_n^*} \cdot \gamma_{I^*} \in \mathcal{L} + \mathcal{I}_{\mathcal{K}}(V)_N.$$

This means that (2.7) holds for $I = I^*$. Hence, by (descending) induction we get (2.7).

For any $Q \in \mathcal{K}[x_0, \dots, x_M]_N$, we write $Q = Q_1^0 \cdots Q_n^0 \cdot Q$. Then by (2.7), we have

$$Q \in \mathcal{L} + \mathcal{I}_{\mathcal{K}}(V)_N.$$

Hence,

$$\{[Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I_1}], \dots, [Q_1^{i_1} \cdots Q_n^{i_n} \cdot \gamma_{I_{m_N^I}}], I = (i_1, \dots, i_n) \in \tau_N\}$$

is a generating system of $\frac{\mathcal{K}[x_0, \dots, x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N}$. Combining with (2.4), we get the conclusion of Lemma 2.6. \square

Lemma 2.7. *For each positive integer N divisible by d , denote by \mathcal{N} the set of all integers $k \in \{0, d, 2d, \dots, N\}$ such that there exists $I \in \tau_N$, satisfying $N - d\|I\| = k$, and $m_N^I \neq \deg V \cdot d^n$. Then, $\#\mathcal{N} = O(1)$ as $N \rightarrow \infty$.*

Proof. Denote by $(\mathcal{I}(V), Q_1, \dots, Q_n)$ the ideal in $\mathcal{K}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V) \cup \{Q_1, \dots, Q_n\}$. For each z in \mathbb{C} such that all coefficients of Q_j ($j = 1, \dots, n$) are holomorphic at z , we denote by $(\mathcal{I}(V), Q(z), \dots, Q_n(z))$ the ideal in $\mathbb{C}[x_0, \dots, x_M]$ generated by $\mathcal{I}(V) \cup \{Q_1(z), \dots, Q_n(z)\}$.

We have

$$(\mathcal{I}(V), Q_1(z), \dots, Q_n(z)) \subset (\mathcal{I}(V), Q_1, \dots, Q_n)(z). \quad (2.10)$$

Indeed, for any $P \in (\mathcal{I}(V), Q_1(z), \dots, Q_n(z))$, we write $P = G + Q_1(z) \cdot P_1 + \dots + Q_n(z) \cdot P_n$, where $G \in \mathcal{I}(V)$, and $P_i \in \mathbb{C}[x_0, \dots, x_M]$. Take $\tilde{P} := G + Q_1 \cdot P_1 + \dots + Q_n \cdot P_n \in (\mathcal{I}(V), Q_1, \dots, Q_n)$, then all coefficients of \tilde{P} are holomorphic at z . It is clear that $\tilde{P}(z) = P$. Hence, we get (2.10).

Let I be an arbitrary element in τ_N . Let $\{h_k := \sum_{j=1}^n Q_j \cdot R_{jk} + \sum_{j=1}^{m_k} \gamma_{jk} \cdot g_{jk}\}_{k=1}^K$ be a basic system of $(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|}$, where $g_{jk} \in \mathcal{I}(V)$, and $R_{jk}, \gamma_{jk} \in \mathcal{K}[x_0, \dots, x_M]$ satisfying $\deg(Q_j \cdot R_{jk}) = \deg(\gamma_{jk} \cdot g_{jk}) = N - d\|I\|$. By Lemma 2.2, and since $\{Q_0, \dots, Q_n\}$ is a V -admissible set, there exists $a \in \mathbb{C}$ such that:

- i) $\{h_k(a)\}_{k=1}^K$ is a basic system of $(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|}(a)$,
- ii) all coefficients of $Q_j, R_{jk}, \gamma_{jk}, g_{jk}$ are holomorphic at a , and
- iii) the homogeneous polynomials $Q_0(a), \dots, Q_n(a) \in \mathbb{C}[x_0, \dots, x_M]$ have no common zero points in V .

On the other hand, it is clear that $h_k(a) \in (\mathcal{I}(V), Q_1(a), \dots, Q_n(a))$, for all $k = 1, \dots, K$. Hence, by (2.10), and by i), we have

$$(\mathcal{I}(V), Q_1(a), \dots, Q_n(a))_{N-d\|I\|} = (\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|}(a).$$

Hence, by Remark 2.5, and by Lemma 2.2, for each $I \in \tau_N$, we have (we recall that $a \in \mathbb{C}$ was chosen to satisfy Lemma 2.2)

$$\begin{aligned} \dim_{\mathcal{K}} \mathcal{L}_N^I &\geq \dim_{\mathcal{K}} (\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|} \\ &= \dim_{\mathbb{C}} (\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|}(a) \\ &= \dim_{\mathbb{C}} (\mathcal{I}(V), Q_1(a), \dots, Q_n(a))_{N-d\|I\|}. \end{aligned}$$

This implies that

$$m_N^I = \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}}{\mathcal{L}_N^I} \leq \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_M]_{N-d\|I\|}}{(\mathcal{I}(V), Q_1(a), \dots, Q_n(a))_{N-d\|I\|}}. \quad (2.11)$$

By (2.11), and by the usual result of intersection theory (see [11], Theorem 7.7, page 53), there exists a positive integer n_0 such that

$$m_N^I \leq \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_M]_{N-d\|I\|}}{(\mathcal{I}(V), Q_1(a), \dots, Q_n(a))_{N-d\|I\|}} = \deg V \cdot d^n \quad (2.12)$$

for all $N, I \in \tau_N$ satisfying $N - d \cdot \|I\| \geq n_0$.

Moreover, by Remark 2.5, and by the argument above, for $N, I \in \tau_N$ satisfying $N - d \cdot \|I\| \geq n_0$, we have that

$$m_N^I \neq \deg V \cdot d^n \quad (2.13)$$

if and only if $(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|}$ is a proper subspace of \mathcal{L}_N^I .

Claim 1: For any positive integer N , we have $\dim_{\mathcal{K}} \mathcal{I}_{\mathcal{K}}(V)_N = \dim_{\mathbb{C}} \mathcal{I}(V)_N$.

Indeed, let $\{P_1, \dots, P_s\}$ be a basis of the \mathbb{C} -vector space $\mathcal{I}(V)_N$. It is clear that $\mathcal{I}_{\mathcal{K}}(V)_N$ is a vector space over \mathcal{K} generated by $\mathcal{I}(V)_N$, therefore $\{P_1, \dots, P_s\}$ is also a generating system of $\mathcal{I}_{\mathcal{K}}(V)_N$. Then, for the claim, it suffices to prove that if $t_1, \dots, t_s \in \mathcal{K}$ satisfy

$$t_1 \cdot P_1 + \dots + t_s \cdot P_s \equiv 0, \quad (2.14)$$

then $t_1 = \dots = t_s \equiv 0$. We rewrite (2.14) in the following form

$$A \cdot \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_s \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix},$$

where $A \in \text{Mat}(\binom{M+N}{N} \times s, \mathcal{K})$.

If the above system of linear equations has non-trivial solutions, then $\text{rank}_{\mathcal{K}} A < s$. Then $\text{rank}_{\mathbb{C}} A(z) < s$ for all $z \in \mathbb{C}$ excluding a discrete set. Take $a \in \mathbb{C}$ such that $\text{rank}_{\mathbb{C}} A(a) < s$. Then the following system of linear equations

$$A(a) \cdot \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_s \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix},$$

has some non-trivial solution $(t_1, \dots, t_s) = (\alpha_1, \dots, \alpha_s) \in \mathbb{C}^s \setminus \{0\}$. Then $\alpha_1 \cdot P_1 + \dots + \alpha_s \cdot P_s \equiv 0$, this is a contradiction. Hence, we get Claim 1.

By Lemma 2.6 and by Claim 1, we have

$$\begin{aligned} \sum_{I \in \tau_N} m_N^I &= \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_M]_N}{\mathcal{I}(V)_N} \\ &= \deg V \cdot \frac{N^n}{n!} + O(N^{n-1}), \end{aligned} \quad (2.15)$$

for all N large enough.

On the other hand

$$\begin{aligned} \#\tau_N &= \binom{\frac{N}{d} + n}{n} = \frac{1}{d^n} \cdot \frac{N^n}{n!} + O(N^{n-1}), \text{ and} \\ \#\{I \in \tau_N : N - d \cdot \|I\| \leq n_0\} &= O(N^{n-1}). \end{aligned}$$

Therefore, by (2.12) and (2.15), for $N \gg 0$, divisible by d , we have:

$$\#\{I \in \tau_N : \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}}{\mathcal{L}_N^I} = \deg V \cdot d^n\} = \frac{1}{d^n} \cdot \frac{N^n}{n!} + O(N^{n-1}) \quad (2.16)$$

and

$$\#\{I \in \tau_N : \dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_{N-d\|I\|}}{\mathcal{L}_N^I} \neq \deg V \cdot d^n\} = O(N^{n-1}). \quad (2.17)$$

Denote by $\eta(N)$ the number of integers $k \in \{0, d, 2d, \dots, N\}$ such that there exists $I \in \tau_N$, $N - d \cdot \|I\| = k$ satisfying

$$m_N^I \neq \deg V \cdot d^n. \quad (2.18)$$

For any $I = (i_1, \dots, i_n) \in \tau_N$, with $N - d \cdot \|I\| = k \geq n_0$, by (2.13), we have that (2.18) is equivalent to the existence of a polynomial $\gamma_0 \in \mathcal{K}_{N-d\|I\|}[x_0, \dots, x_M]$ such that

$$\left\{ \begin{array}{l} Q_1^{i_1} \cdots Q_n^{i_n} \gamma_0 = \sum_{E_s = (e_{1s}, \dots, e_{ns}) > I} Q_1^{e_{1s}} \cdots Q_n^{e_{ns}} \gamma_s, \text{ for some } \gamma_s \in \mathcal{K}_{N-d\|E_s\|} \\ \gamma_0 \notin (\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|}. \end{array} \right. \quad (2.19)$$

For any I and γ_0 satisfying the condition above, we will construct a sequence of $\binom{\frac{N}{d}+n-1}{n-1}$ elements $T \in \tau_{2N}$ such that $2N - d \cdot \|T\| = N - d \cdot \|I\| \geq n_0$ and

$$\gamma_0 \in \mathcal{L}_{2N}^T \setminus (\mathcal{I}(V), Q_1, \dots, Q_n)_{2N-d\|T\|}. \quad (2.20)$$

In order to do this, for each $J = (\ell_1, \dots, \ell_n)$ with $\|J\| = \frac{N}{d}$ (there are $\binom{\frac{N}{d}+n-1}{n-1}$ elements J), we take $T := I + J$. Then $2N - d \cdot \|T\| = N - d \cdot \|I\|$, and from (2.19) we get

$$Q_1^{i_1+\ell_1} \dots Q_n^{i_n+\ell_n} \gamma_0 = \sum_{E_s=(e_{1s}, \dots, e_{ns}) > I} Q_1^{e_{1s}+\ell_n} \dots Q_n^{i_{ns}+\ell_n} \gamma_s.$$

On the other hand since $(e_{1s}, \dots, e_{ns}) > I$, we have $(e_{1s} + \ell_1, \dots, e_{ns} + \ell_n) > (i_1 + \ell_1, \dots, i_n + \ell_n)$. Hence, $\gamma_0 \in \mathcal{L}_{2N}^T$. However,

$$\gamma_0 \notin (\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|} = (\mathcal{I}(V), Q_1, \dots, Q_n)_{2N-d\|T\|}.$$

Therefore, we get (2.20).

From the argument above we get that for each $I \in \tau_N$ satisfying $N - d \cdot \|I\| = k \geq n_0$, and

$$(\mathcal{I}(V), Q_1, \dots, Q_n)_{N-d\|I\|} \subsetneq \mathcal{L}_N^I,$$

there is a set \mathcal{A}_I of $\binom{\frac{N}{d}+n-1}{n-1}$ elements $T \in \tau_{2N}$ such that $2N - d\|T\| = N - d\|I\| = k \geq n_0$, and $(\mathcal{I}(V), Q_1, \dots, Q_n)_{2N-d\|T\|} \subsetneq \mathcal{L}_{2N}^T$. It is clear that $\mathcal{A}_I \cap \mathcal{A}_{I'} = \emptyset$, if $\|I\| \neq \|I'\|$. So by this fact, if, for each positive integer t , we still denote $\beta(t)$ the number of $I \in \tau_t$ satisfying $m_t^I \neq \deg V \cdot d^n$, we get

$$\beta(2N) \geq \binom{\frac{N}{d}+n-1}{n-1} \cdot (\eta(N) - O(1)) = O(N^{n-1}) \cdot (\eta(N) - O(1)).$$

On the other hand, by (2.17), we have

$$\beta(2N) \leq O(N^{n-1}).$$

Hence, $\eta(N) = O(1)$. This completes the proof of Lemma 2.7. \square

Lemma 2.8. *For each $s \in \{1, \dots, n\}$, and for $N \gg 0$, divisible by d , we have:*

$$\sum_{I=(i_1, \dots, i_n) \in \tau_N} m_N^I \cdot i_s \geq \frac{\deg V}{d \cdot (n+1)!} N^{n+1} - O(N^n).$$

Proof. By Lemma 2.7, for each $k \in \mathcal{N}^* := \{0, d, 2d, \dots, N\} \setminus \mathcal{N}$, and for each $I := (i_1, \dots, i_n) \in \tau_N$ with $N - d \cdot \|I\| = k$, then $m_N^{I'} = \deg V \cdot d^n$, for all symmetry $I' = (i_{\sigma(1)}, \dots, i_{\sigma(n)})$ of (i_1, \dots, i_n) . Hence,

$$\begin{aligned}
\sum_{I=(i_1, \dots, i_n): N-d\|I\| \in \mathcal{N}^*} m_N^I \cdot i_1 &= \dots = \sum_{I=(i_1, \dots, i_n): N-d\|I\| \in \mathcal{N}^*} m_N^I \cdot i_n \\
&= \deg V \cdot d^n \cdot \sum_{I=(i_1, \dots, i_n): N-d\|I\| \in \mathcal{N}^*} \frac{\|I\|}{n} \\
&= \deg V \cdot d^n \cdot \left(\sum_{I=(i_1, \dots, i_n) \in \tau_N} \frac{\|I\|}{n} - \sum_{I=(i_1, \dots, i_n): N-d\|I\| \in \mathcal{N}} \frac{\|I\|}{n} \right) \\
&\geq \deg V \cdot d^n \left(\sum_{k=0}^{\frac{N}{d}} \frac{k}{n} \cdot \binom{k+n-1}{n-1} - O(1) \binom{\frac{N}{d}+n-1}{n-1} \cdot \frac{N}{nd} \right) \\
&= \deg V \cdot d^n \sum_{k=1}^{\frac{N}{d}} \binom{k+n-1}{n} - O(N^n) \\
&= \deg V \cdot d^n \binom{\frac{N}{d}+n}{n+1} - O(N^n) \\
&\geq \frac{\deg V}{d \cdot (n+1)!} N^{n+1} - O(N^n).
\end{aligned}$$

Hence, for each $i \in \{1, \dots, n\}$

$$\begin{aligned}
\sum_{I=(i_1, \dots, i_n) \in \tau_N} m_N^I \cdot i_s &\geq \sum_{I=(i_1, \dots, i_n): N-d\|I\| \in \mathcal{N}^*} m_N^I \cdot i_s \\
&\geq \frac{\deg V}{d \cdot (n+1)!} N^{n+1} - O(N^n).
\end{aligned}$$

□

We recall that by (2.15), for $N \gg 0$, we have

$$\dim_{\mathcal{K}} \frac{\mathcal{K}[x_0, \dots, x_M]_N}{\mathcal{I}_{\mathcal{K}}(V)_N} = H_V(N) = \deg V \cdot \frac{N^n}{n!} + O(N^{n-1}).$$

Therefore, from Lemmas 2.6, 2.8 we get immediately the following result.

Lemma 2.9. *For all $N \gg 0$ divisible by d , there are homogeneous polynomials $\phi_1, \dots, \phi_{H_V(N)}$ in $\mathcal{K}[x_0, \dots, x_M]_N$ such that they form a basis of the \mathcal{K} -vector space $\frac{\mathcal{K}_{[x_0, \dots, x_M]_N}}{\mathcal{I}_{\mathcal{K}}(V)_N}$, and*

$$\prod_{j=1}^{H_V(N)} \phi_j - (Q_1 \cdots Q_n)^{\frac{\deg V \cdot N^{n+1}}{d \cdot (n+1)!} - u(N)} \cdot P \in \mathcal{I}_{\mathcal{K}}(V)_{N \cdot H_V(N)},$$

where $u(N)$ is a function in N satisfying $u(N) \leq O(N^n)$, P is a homogeneous polynomial of degree

$$N \cdot H_V(N) - \frac{n \cdot \deg V \cdot N^{n+1}}{(n+1)!} + u(N) = \frac{\deg V \cdot N^{n+1}}{(n+1)!} + O(N^n).$$

Lemma 2.10 (see [14]). *Let f be a non-constant holomorphic map of \mathbb{C} into \mathbb{CP}^M . Let $H_j = a_{j0}x_0 + \cdots + a_{jM}x_M$, $j \in \{1, \dots, q\}$ be q linear homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_M]$. Denote by $\mathcal{K}_{\{H_j\}_{j=1}^q}$ the field over \mathbb{C} of all meromorphic functions on \mathbb{C} generated by $\{a_{ji}, i = 0, \dots, M; j = 1, \dots, q\}$. Assume that f is linearly non-degenerate over $\mathcal{K}_{\{H_j\}_{j=1}^q}$. Then for each $\epsilon > 0$, we have*

$$\left\| \frac{1}{2\pi} \int_0^{2\pi} \max_K \log \prod_{k \in K} \left(\frac{\|f\| \cdot \max_{i=0, \dots, M} |a_{ki}|}{|H_k(f)|} (re^{i\theta}) \right) d\theta \right\| \leq (n+1+\epsilon)T_f(r), \quad (2.21)$$

where \max_K is taken over all subsets $K \subset \{1, \dots, q\}$ such that the polynomials H_j , $j \in K$ are linearly independent over $\mathcal{K}_{\{H_j\}_{j=1}^q}$.

Remark 2.11. *Since the coefficients of H_j 's are small functions (with respect to f), by the First Main Theorem, and by (2.21), for each $\epsilon > 0$, we have*

$$\left\| \frac{1}{2\pi} \int_0^{2\pi} \max_K \log \prod_{k \in K} \left(\frac{\|f\|}{|H_k(f)|} (re^{i\theta}) \right) d\theta \right\| \leq (n+1+\epsilon)T_f(r).$$

3 Proof of the Main Theorem

Replacing Q_j by $Q_j^{\frac{d}{d_j}}$, where d is the l.c.m of the Q_j 's, we may assume that the polynomials Q_1, \dots, Q_q have the same degree d . Let $N \gg 0$ be an

integer divisible by d . For each $J := \{j_1, \dots, j_n\} \subset \{1, \dots, q\}$, by Lemma 2.9 (for $\mathcal{K} := \mathcal{K}_{\mathcal{Q}}$), there are homogeneous polynomials $\phi_1^J, \dots, \phi_{H_V(N)}^J$ (depending on J) in $\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]$ and there are functions (common for all J) $u(N), v(N) \leq O(N^n)$ such that they form a basis of the $\mathcal{K}_{\mathcal{Q}}$ -vector space $\frac{\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]}{\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_N}$, and

$$\prod_{\ell=1}^{H_V(N)} \phi_{\ell}^J - (Q_{j_1} \cdots Q_{j_n})^{\frac{\deg V \cdot N^{n+1}}{d \cdot (n+1)!} - u(N)} \cdot P_J \in \mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_{N \cdot H_V(N)},$$

where P_J is a homogeneous polynomial of degree $\frac{\deg V \cdot N^{n+1}}{(n+1)!} + v(N)$. On the other hand, for any $Q \in \mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_{N \cdot H_V(N)}$, we have $Q(f) \equiv 0$. Therefore

$$\prod_{\ell=1}^{H_V(N)} \phi_{\ell}^J(f) = (Q_{j_1}(f) \cdots Q_{j_n}(f))^{\frac{\deg V \cdot N^{n+1}}{d \cdot (n+1)!} - u(N)} \cdot P_J(f).$$

On the other hand, since the coefficients of P_J are small functions (with respect to f), it is easy to see that there exist $h_J \in \mathcal{C}_f$ such that

$$|P_J(f)| \leq \|f\|^{\deg P_J} \cdot h_J = \|f\|^{\frac{\deg V \cdot N^{n+1}}{(n+1)!} + v(N)} \cdot h_J.$$

Hence,

$$\begin{aligned} \log \left(\prod_{\ell=1}^{H_V(N)} |\phi_{\ell}^J(f)| \right) &\leq \left(\frac{\deg V \cdot N^{n+1}}{d \cdot (n+1)!} - u(N) \right) \cdot \log |Q_{j_1}(f) \cdots Q_{j_n}(f)| + \log^+ h_J \\ &\quad + \left(\frac{\deg V \cdot N^{n+1}}{(n+1)!} + v(N) \right) \cdot \log \|f\|. \end{aligned}$$

This implies that there are functions $\omega_1(N), \omega_2(N) \leq O(\frac{1}{N})$ such that

$$\begin{aligned} \log (|Q_{j_1}(f)| \cdots |Q_{j_n}(f)|) &\geq \left(\frac{d \cdot (n+1)!}{\deg V \cdot N^{n+1}} - \frac{\omega_1(N)}{N^{n+1}} \right) \cdot \log \left(\prod_{\ell=1}^{H_V(N)} |\phi_{\ell}^J(f)| \right) \\ &\quad - \frac{1}{N^{n+1}} \log^+ \tilde{h}_J - (d + \omega_2(N)) \cdot \log \|f\|, \quad (3.1) \end{aligned}$$

for some $\tilde{h}_J \in \mathcal{C}_f$.

We fix homogeneous polynomials $\Phi_1, \dots, \Phi_{H_V(N)} \in \mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]_N$ such that they form a basis of the $\mathcal{K}_{\mathcal{Q}}$ -vector space $\frac{\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]_N}{\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_N}$. Then for each subset $J := \{j_1, \dots, j_n\} \in \{1, \dots, q\}$, there exist homogeneous linear polynomials $L_1^J, \dots, L_{H_V(N)}^J \in \mathcal{K}_{\mathcal{Q}}[y_1, \dots, y_{H_V(N)}]$ such that they are linearly independent over $\mathcal{K}_{\mathcal{Q}}$ and

$$\phi_\ell^J - L_\ell^J(\Phi_1, \dots, \Phi_{H_V(N)}) \in \mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_N, \text{ for all } \ell \in \{1, \dots, H_V(N)\}. \quad (3.2)$$

It is easy to see that there exists a meromorphic function φ such that $N_\varphi(r) = o(T_f(r))$, $N_{\frac{1}{\varphi}}(r) = o(T_f(r))$ and $\frac{\Phi_1(f)}{\varphi}, \dots, \frac{\Phi_{H_V(N)}(f)}{\varphi}$ are holomorphic and have no common zeros (note that all coefficients of Φ_ℓ are in $\mathcal{K}_{\mathcal{Q}} \subset \mathcal{K}_f$). Let $F : \mathbb{C} \rightarrow \mathbb{C}\mathbb{P}^{H_V(N)-1}$ be the holomorphic map with the reduced representation $F := \left(\frac{\Phi_1(f)}{\varphi} : \dots : \frac{\Phi_{H_V(N)}(f)}{\varphi} \right)$. Since f is algebraically non-degenerate over $\mathcal{K}_{\mathcal{Q}}$, and since the polynomials $\Phi_1, \dots, \Phi_{H_V(N)}$ form a basis of $\frac{\mathcal{K}_{\mathcal{Q}}[x_0, \dots, x_M]_N}{\mathcal{I}_{\mathcal{K}_{\mathcal{Q}}}(V)_N}$, we get that F is linearly non-degenerate over $\mathcal{K}_{\mathcal{Q}}$. As a corollary, F is linearly non-degenerate over the field over \mathbb{C} generated by all coefficients of L_ℓ 's.

It is easy to see that

$$T_F(r) \leq N \cdot T_f(r) + o(T_f(r)). \quad (3.3)$$

In order to simplify the writing of the following series of inequalities, put $A(N) := \left(\frac{d \cdot (n+1)!}{\deg V \cdot N^{n+1}} - \frac{\omega_1(N)}{N^{n+1}} \right)$. By (3.2), for all $\ell \in \{1, \dots, H_V(N)\}$ we have

$$\log |\phi_\ell^J(f)| = \log |L_\ell^J(F)| + \log |\varphi|.$$

Hence, by (3.1), and by taking $\tilde{h} \in \mathcal{C}_f$ such that $\log^+ \tilde{h}_J \leq \log^+ \tilde{h}$ for all J , we get

$$\begin{aligned} \log (|Q_{j_1}(f)| \cdots |Q_{j_n}(f)|) &\geq A(N) \cdot \left(H_V(N) \cdot \log |\varphi| + \log \left(\prod_{\ell=1}^{H_V(N)} |L_\ell^J(F)| \right) \right) \\ &\quad - \frac{1}{N^{n+1}} \log^+ \tilde{h}_J - (d + \omega_2(N)) \log \|f\| \\ &\geq A(N) \cdot \log \left(\prod_{\ell=1}^{H_V(N)} |L_\ell^J(F)| \right) + A(N) \cdot H_V(N) \cdot \log |\varphi| \\ &\quad - \log^+ \tilde{h} - (d + \omega_2(N)) \log \|f\|. \end{aligned} \quad (3.4)$$

Then, by Lemma 2.3, and by increasing $\tilde{h} \in \mathcal{C}_f$ if necessary, we get

$$\begin{aligned}
\log \prod_{j=1}^q |Q_j(f)| &= \max_{\{\beta_1, \dots, \beta_{q-n}\} \subset \{1, \dots, q\}} \log |Q_{\beta_1}(f) \cdots Q_{\beta_{q-n}}(f)| \\
&\quad + \min_{J=\{j_1, \dots, j_n\} \subset \{1, \dots, q\}} \log |Q_{j_1}(f) \cdots Q_{j_n}(f)| \\
&\geq (q-n)d \cdot \log \|f\| + \min_{J \subset \{1, \dots, q\}, \#J=n} A(N) \cdot \log \left(\prod_{\ell=1}^{H_V(N)} |L_\ell^J(F)| \right) \\
&\quad - (d + \omega_2(N)) \log \|f\| + A(N) \cdot H_V(N) \cdot \log |\varphi| - \log^+ \tilde{h} \\
&= (q-n-1)d \cdot \log \|f\| + \min_{J \subset \{1, \dots, q\}, \#J=n} A(N) \cdot \log \left(\prod_{\ell=1}^{H_V(N)} |L_\ell^J(F)| \right) \\
&\quad - \omega_2(N) \cdot \log \|f\| + A(N) \cdot H_V(N) \cdot \log |\varphi| - \log^+ \tilde{h}
\end{aligned} \tag{3.5}$$

Now for given $\epsilon > 0$ we fix $N = N(\epsilon)$ big enough such that

$$\omega_2(N) \leq \frac{\epsilon}{3} \text{ and } A(N) < 1. \tag{3.6}$$

By using Remark 2.11 to the holomorphic map $F : \mathbb{C} \rightarrow \mathbb{CP}^{H_V(N)-1}$, the error constant $\frac{\epsilon}{2N} > 0$ and the system of linear polynomials $L_1^J, \dots, L_{H_V(N)}^J \in \mathcal{K}_{\mathcal{Q}}[y_1, \dots, y_{H_V(N)}]$, where J runs over all subsets $J := \{j_1, \dots, j_n\} \in \{1, \dots, q\}$, we get:

$$\begin{aligned}
&\left\| \frac{1}{2\pi} \int_0^{2\pi} \max_{J \subset \{1, \dots, q\}, \#J=n} \log \left(\prod_{\ell=1}^{H_V(N)} \frac{\|F\|}{|L_\ell^J(F)|} (re^{i\theta}) \right) d\theta \right. \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \max_K \log \prod_{k \in K} \left(\frac{\|F\|}{|L_k^J(F)|} (re^{i\theta}) \right) d\theta \leq (H_V(N) + \frac{\epsilon}{2N}) T_F(r), \tag{3.7}
\end{aligned}$$

where \max_K is taken over all subsets of the system of linear polynomials $L_1^J, \dots, L_{H_V(N)}^J \in \mathcal{K}_{\mathcal{Q}}[y_1, \dots, y_{H_V(N)}]$, where J runs over all subsets $J := \{j_1, \dots, j_n\} \in \{1, \dots, q\}$, such that these linear polynomials are linearly independent over $\mathcal{K}_{\mathcal{Q}}$.

So, by integrating (3.5) and combining with (3.6) and (3.7) we have (using that $N_\varphi(r) = o(T_f(r))$, $N_{\frac{1}{\varphi}}(r) = o(T_f(r))$, that $A(N) \cdot H_V(N) \leq O(\frac{1}{N})$ and

that $\tilde{h} \in \mathcal{C}_f$)

$$\begin{aligned}
\left\| \sum_{j=1}^q N_f(r, Q_j) \right\| &\geq d(q-n-1)T_f(r) - \frac{\epsilon}{3}T_f(r) + A(N) \cdot H_V(N) \cdot \left(N_\varphi(r) - N_{\frac{1}{\varphi}}(r) \right) - \frac{\epsilon}{12}T_f(r) \\
&\quad + A(N) \cdot \frac{1}{2\pi} \int_0^{2\pi} \min_{J \subset \{1, \dots, q\}, \#J=n} \log \left(\prod_{\ell=1}^{H_V(N)} |L_\ell^J(F)|(re^{i\theta}) \right) d\theta \\
&\geq d(q-n-1)T_f(r) - \frac{\epsilon}{3}T_f(r) - \frac{\epsilon}{12}T_f(r) - \frac{\epsilon}{12}T_f(r) \\
&\quad + A(N) \cdot \frac{1}{2\pi} \int_0^{2\pi} \min_{J \subset \{1, \dots, q\}, \#J=n} \log \left(\prod_{\ell=1}^{H_V(N)} |L_\ell^J(F)|(re^{i\theta}) \right) d\theta \\
&= d(q-n-1)T_f(r) - \frac{\epsilon}{2}T_f(r) \\
&\quad - A(N) \cdot \frac{1}{2\pi} \int_0^{2\pi} \max_{J \subset \{1, \dots, q\}, \#J=n} \log \left(\prod_{\ell=1}^{H_V(N)} \frac{\|F\|}{|L_\ell^J(F)|}(re^{i\theta}) \right) d\theta \\
&\quad + A(N) \cdot H_V(N) \cdot T_F(r) \\
&\geq d(q-n-1)T_f(r) - A(N) \left(H_V(N) + \frac{\epsilon}{2N} \right) T_F(r) \\
&\quad + A(N) \cdot H_V(N) \cdot T_F(r) - \frac{\epsilon}{2}T_f(r) \\
&\geq d(q-n-1)T_f(r) - \frac{\epsilon}{2N}T_F(r) - \frac{\epsilon}{2}T_f(r) \\
&\geq d(q-n-1-\epsilon)T_f(r).
\end{aligned}$$

This completes the proof of the Main Theorem. \square

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