

DGCI conference
Nantes 20-23 April 2016

Digitization of partitions and of tessellations

Jean Serra¹ , B. Ravi Kiran²

1- Université Paris-Est, A3SI-ESIEE LIGM

2 -Ecole Normale Supérieure Paris and Thales-Alenia Space

Purpose

In a partition, the frontiers between classes are not materialized.
...But some operations need them.

2	2	2	8	8	8
2	2	2	8	8	8
2	2	2	8	8	8
8	8	8	2	2	2
8	8	8	2	2	2
8	8	8	2	2	2

- How to combine interiors and frontiers in a unique representation?
- Is that possible with any partition of set E ?
or is there some condition for the classes?

... Here are the problems we try to solve.

Outline

1. **Regular open sets**
2. Tessellations
3. Digital Tessellations by Khalimsky topology
4. Digital Tessellations by Voronoi polyhedra
5. Conclusion

Continuous or Digital Space

One usually separates the questions specific to \mathbf{R}^n or to \mathbf{Z}^n .

But the notions involved in our problem are prior to this distinction...

We prefer to start from **an arbitrary topological space E** .

Notation For $B \in \mathcal{P}(E)$:

- Set $B = \overline{B}$ is *closed*;
- Set $B = B^\circ$ is *open*.

Reminder : regular open sets*

An open set B is regular, or \mathcal{R} -open, when $B = (\overline{B})^\circ$

The family $\mathcal{R}(E)$ of the regular sets B of $\mathcal{P}(E)$ is a **complete boolean lattice** for the inclusion ordering,

The supremum and the infimum are given by

$$\bigvee B_i = (\overline{\bigcup B_i})^\circ \quad ; \quad \bigwedge B_i = (\bigcap B_i)^\circ.$$

and the unique complement of B is

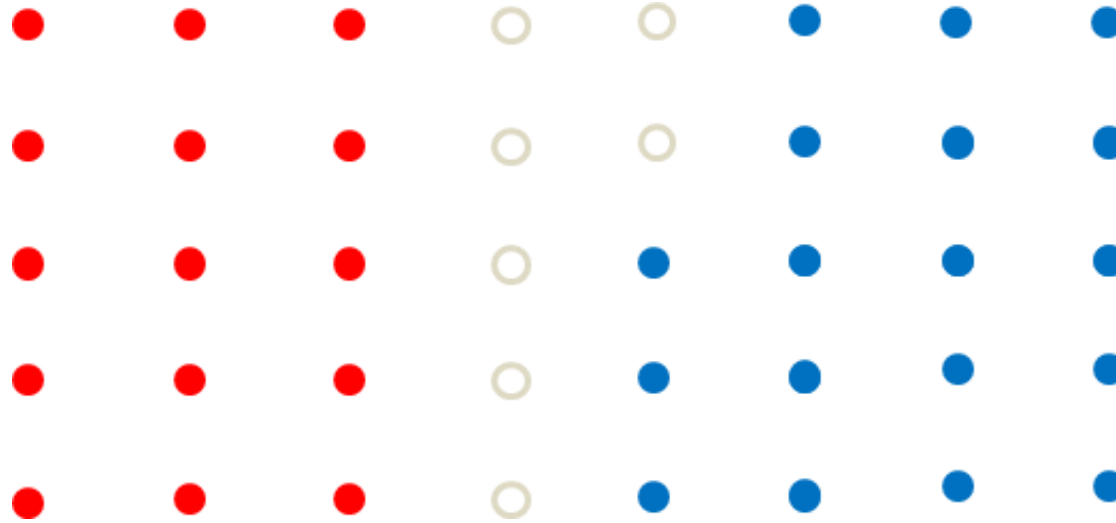
$$(\text{comp} B)^\circ.$$

N.B. *the open sets of $\mathcal{P}(E)$ also form a complete lattice, but it is not complemented.*

Consequence : **all morphological operations** on $\mathcal{P}(E)$ extend to $\mathcal{R}(E)$.

**For regular sets in mathematical morphology, see Matheron (1967,1975,1988,1996)
et Ronse(1990)*

An example of regular open sets



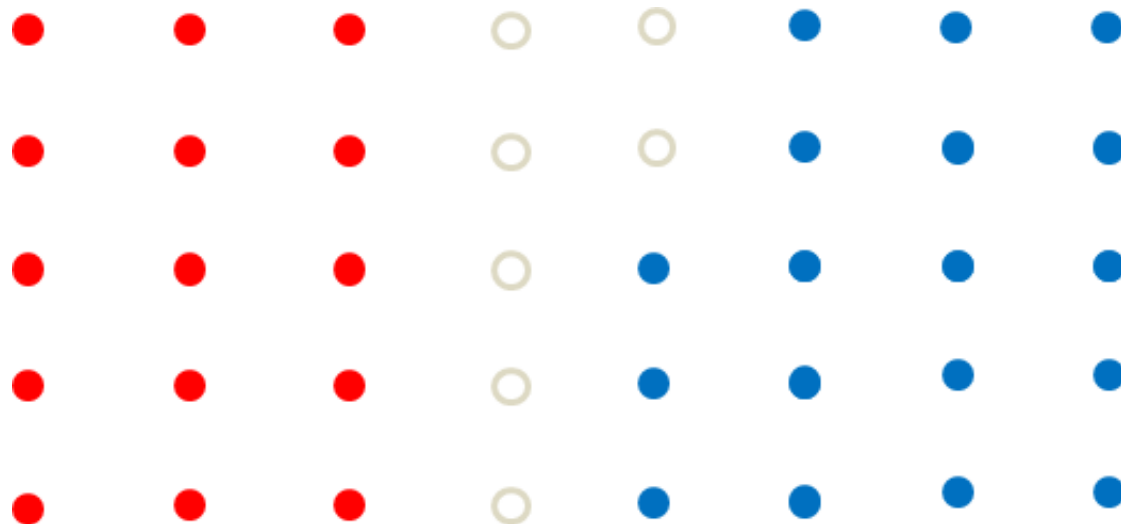
Both blue and red sets are regular open for the topology of the square metric,
... but the white one is not.

Regularisation

Denote by $S = (\overline{B})^\circ$ the \mathcal{R} -open transform of the open set $B \in \mathcal{G}(E)$.

The operation $B \rightarrow S = (\overline{B})^\circ$ is an **algebraic closing** on the open sets of E , and the image of \mathcal{G} is \mathcal{R} .

An example of regularisation

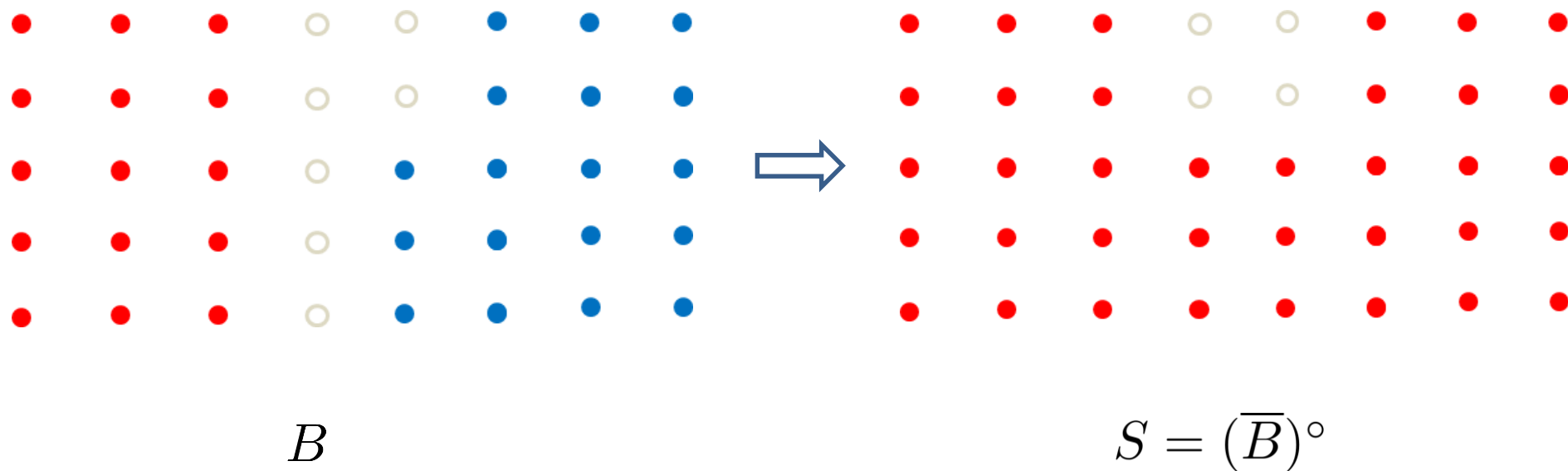


Taken individually, both sets are open and regular for the square metric but their union, still open, is no longer regular.

An example of regularisation

We regularize by the **algebraic closing** $B \rightarrow S = (\overline{B})^\circ$

This closing means that S is the smallest \mathcal{R} -open set that contains B .



Outline

1. Regular open sets
2. **Tessellations**
3. Digital Tessellations by Khalimsky topology
4. Digital Tessellations by Voronoi polyhedra
5. Conclusion

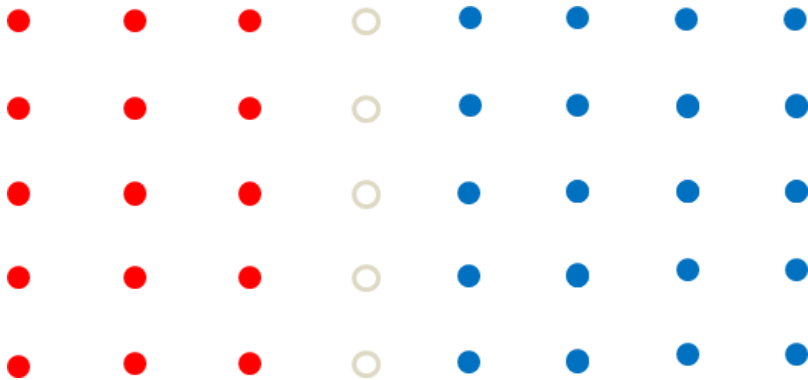
Tessellation

A tessellation τ of E is a family :

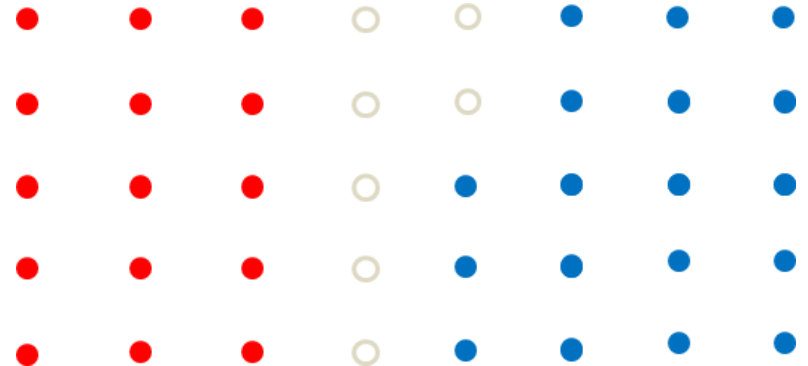
- of disjoint open *classes* $\{B_i, i \in I\}$,

$$\tau = \{B_i, i \in I\} \text{ with } i \neq j \Rightarrow B_i \cap B_j = \emptyset$$

- whose union $\cup B_i$ plus the union of all frontiers $Fr(B_i, B_j), i \neq j$ partitions space E .



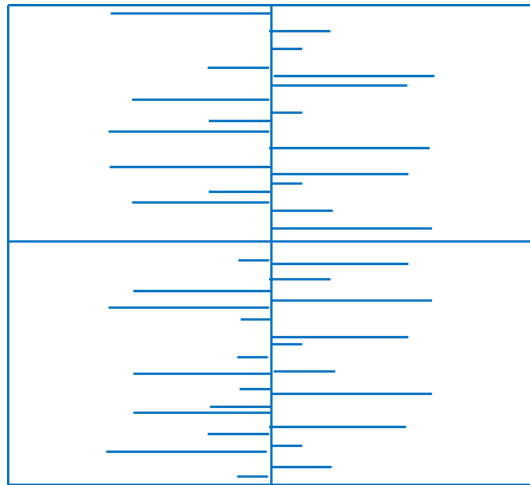
Tessellation



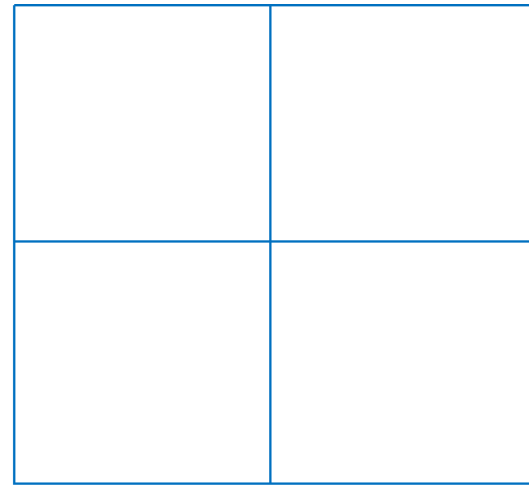
No tessellation

Tessellation and regular sets

Theorem: The family $\tau = \{B_i, i \in I\}$ is a tessellation if and only if all B_i are \mathcal{R} -open.



This is not a tessellation



Tessellation: all classes
have been regularized

Theorem: The set \mathcal{T} of all tessellations $\tau \geq \tau_0$ of E forms a complete lattice for the ordering of the regular sets.

Tessellation and Jordan curves

The theorem reminds us of Jordan's one, though:

- it is true in any topological space,
- it does not focus on the frontiers, but on the classes.
- connectivity is not involved, whereas it is essential in Jordan's one

In \mathbf{R}^2 every Jordan curve induces a tessellation,

... but a tessellation into two open classes, even connected, can have a frontier which is not a Jordan curve.

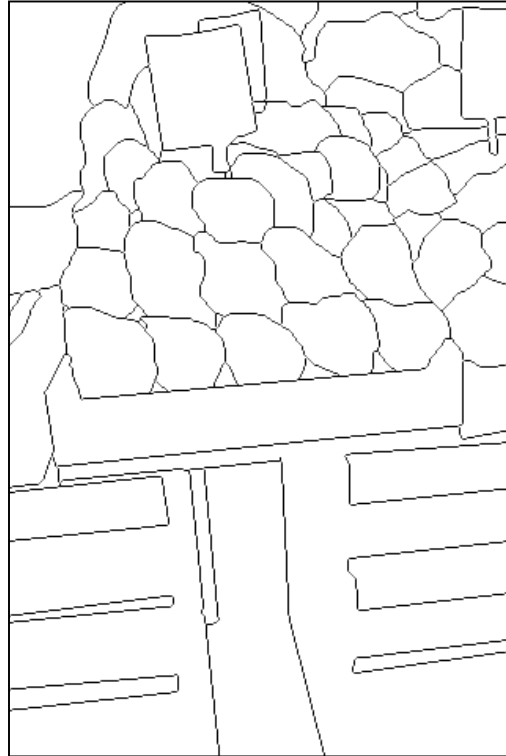
Hierarchies of tessellations

- The tessellations met in image processing are often associated with **hierarchies**, i.e. are elements of totally ordered finite families.
- There is a **minimal tessellation** τ_0 .
- The classes $\{s_i\}$ of the minimal tessellation τ_0 are called **the leaves**, and are supposed in locally finite number.

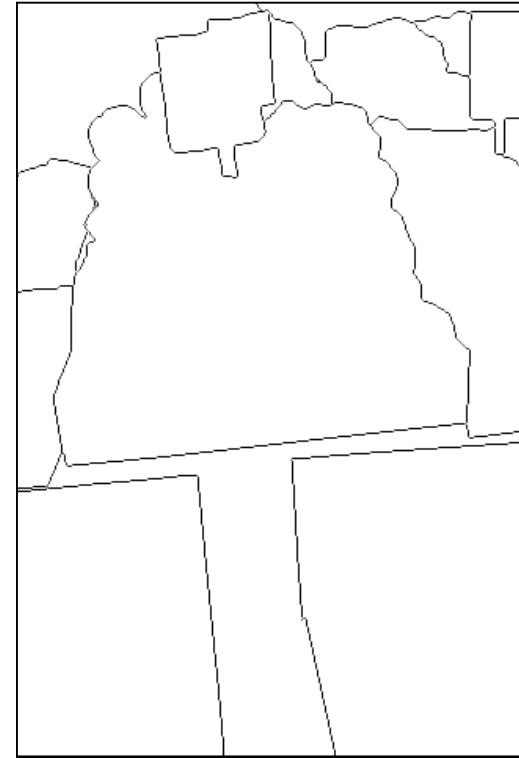
Hierarchies of tessellations



a)



b)



c)

Hierarchies of tessellations

In a hierarchy, the classes of a tessellation τ do not reduce to union of their leaves : the portions of frontiers between adjacent leaves would belong to no classe.

We must find out a law of composition

Let us partition the totality of the leaves into sub-sets

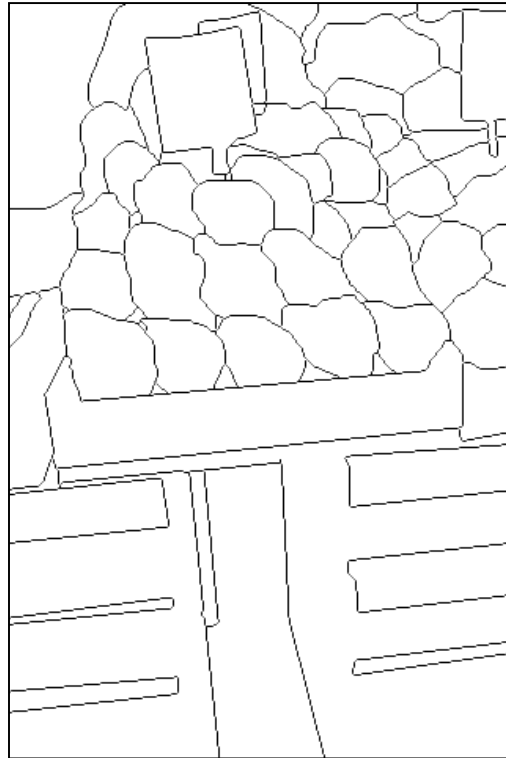
$$B_j = \cup \{s_j, j \in J\}$$

Then the unique tessellation which keeps disjoint the B_j has for classes the \mathcal{R} -open sets $S_j = (\overline{B_j})^\circ$.

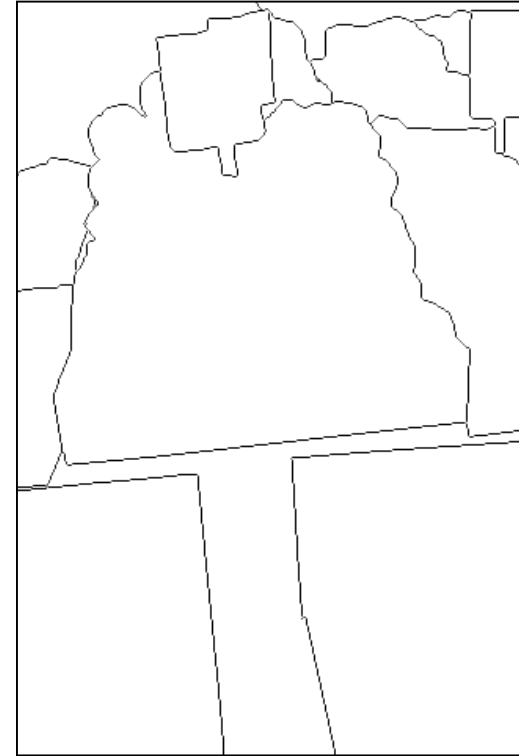
Hierarchies of tessellations



a)



b)



c)

When going from b) to c) the union $B_j = \cup\{s_j, j \in J\}$ is followed by the closing $S_j = (\overline{B_j})^\circ$ which makes it \mathcal{R} -open.

Hierarchies of tessellations

When a portion of frontier disappears, it never reappears upwards.

Therefore, a hierarchy is *characterized* by

1. the level where each portion of frontier disappears,
2. the partition of the leaves into classes.

No connectivity is involved here!

When the classes are connected,

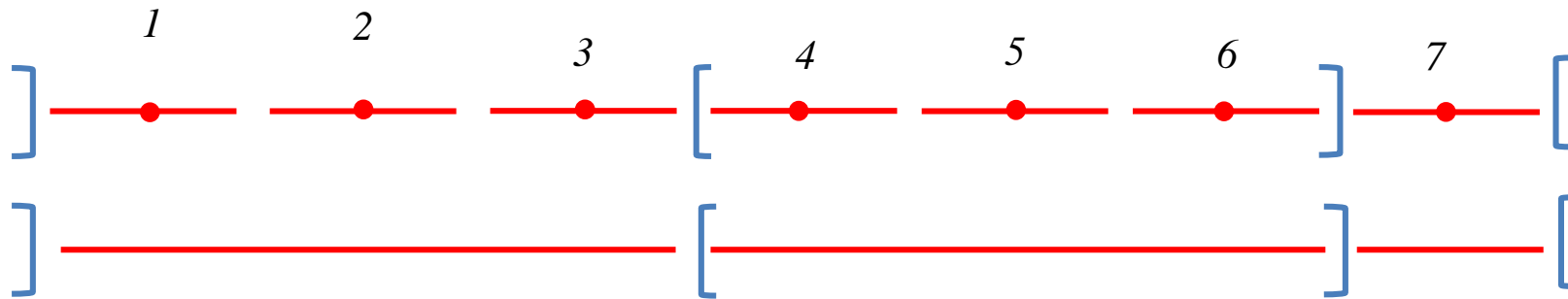
- point 1 comes down to the saliency function,
- point 2 vanishes, but one must control whether the regularization $S_j = (\overline{B_j})^\circ$ is connectivity preserving.

Outline

1. Regular open sets
2. Tessellations
3. **Digital Tessellations by Khalimsky topology**
4. Digital Tessellations by Voronoi polyhedra
5. Conclusion

Reminder on Khalimsky topology

topology: In \mathbf{R}^1 , open (resp. closed) interval when both end labels are odd (resp. even):

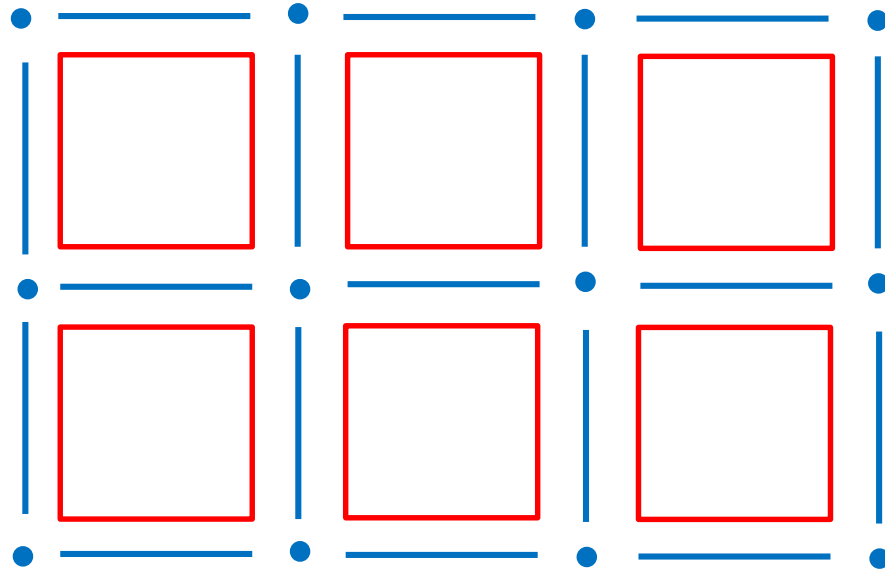


Kovalevski representation :



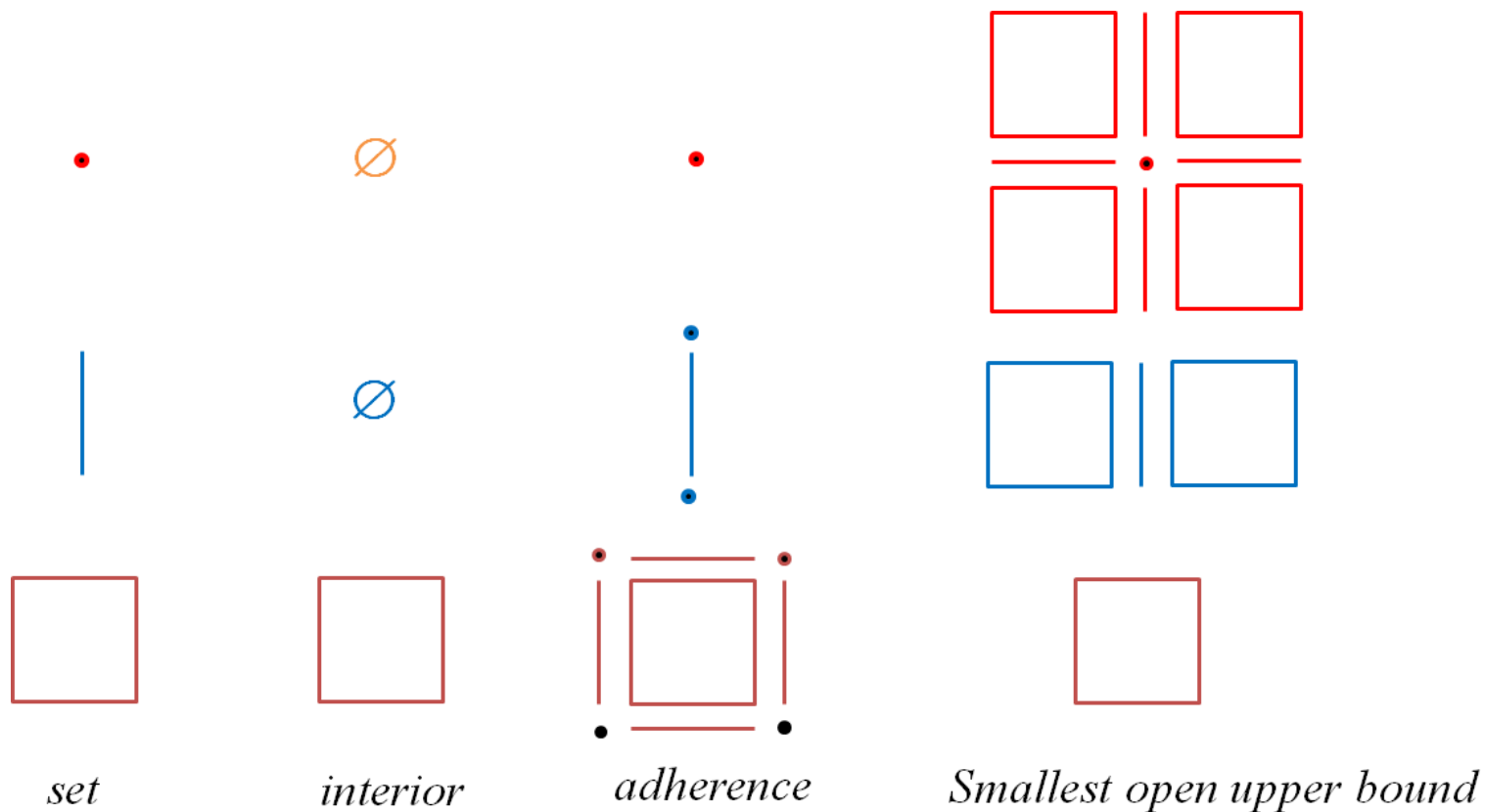
Reminder on Khalimsky topology

The product topology gives in \mathbf{R}^2 :

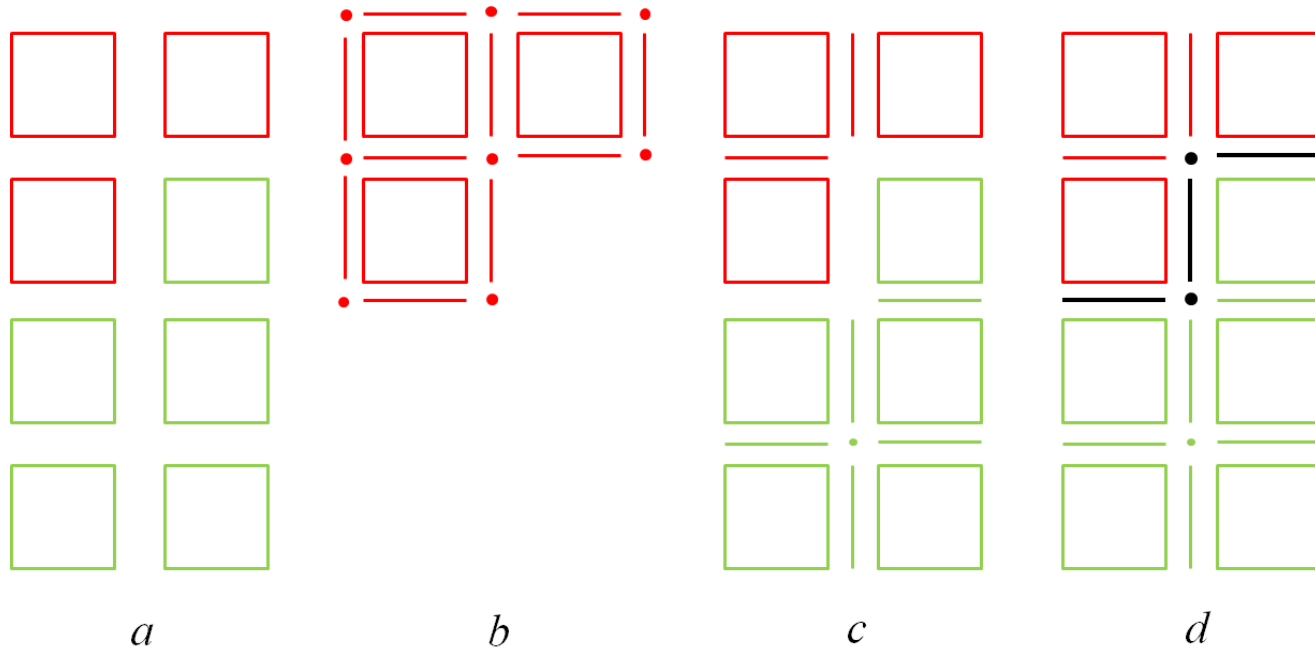


In \mathbf{R}^2 the Kovalevsky cells display Khalimsky topology.
This structure is akin to simplicial simplexes.

Khalimsky Rules



Regularization



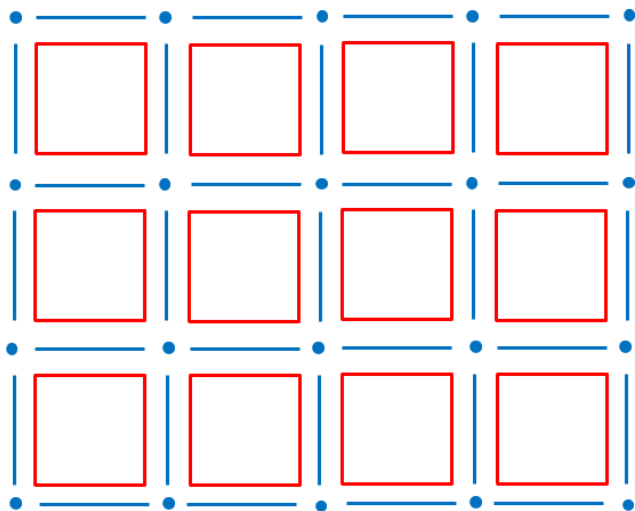
The Figure shows a regularization with :

- (a) eight open elementary cells,
- (b) one elementary close cell
- (c) two R-open sets, and
- (d) their frontier, external to the initial square cells .

Doubling \mathbf{Z}^2

Interpret the points of a set $X \subseteq \mathbf{Z}^2$, as points of odd coordinates in a Khalimsky plane \mathbf{K}^2 which contains twice more points by line and twice more lines.

In the double resolution plane \mathbf{K}^2 , all points of the background have odd coordinates.



Red squares = points of \mathbf{Z}^2 ,

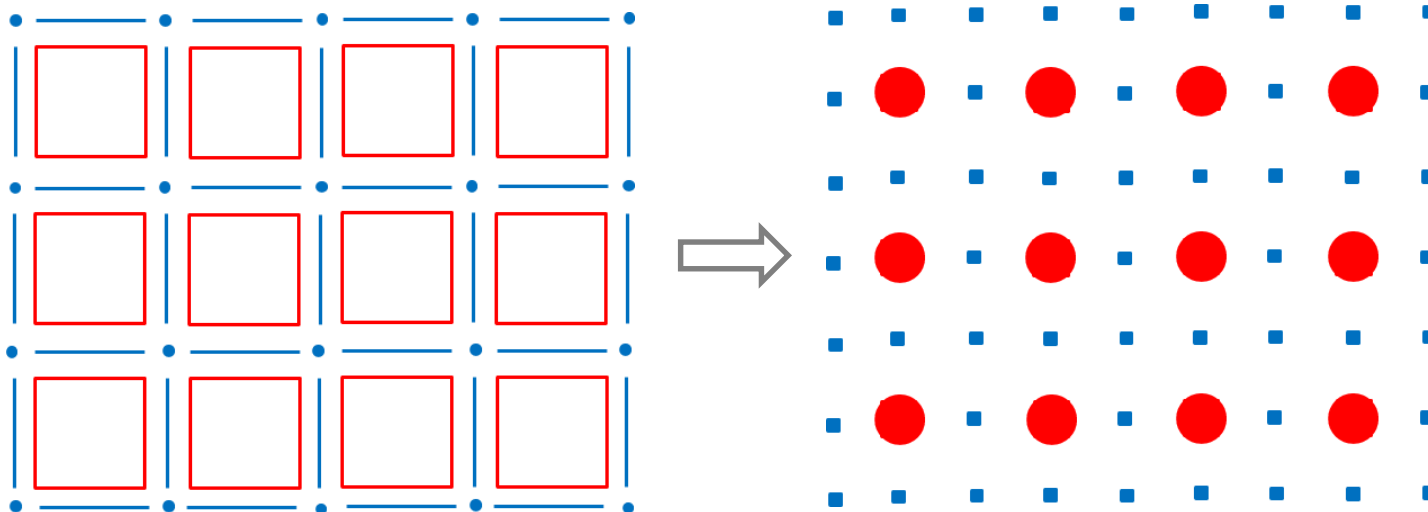
*Blue segments, dots, and red squares
= points of \mathbf{K}^2*

Doubling \mathbb{Z}^2

We meet **the classical rule of the double sampling** :

The fine mesh displays the net of the closed contours which envelop the open classes.

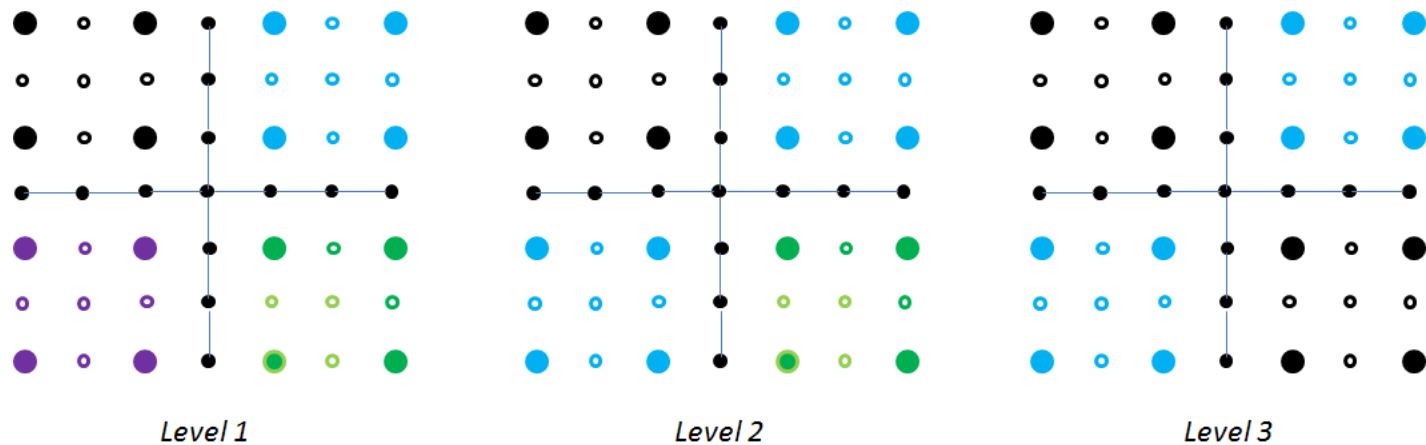
For the sake of simplicity we will indicate the previous Kovalevsky squares by big dots, and the additional segments and points by small dots.



The connectivity trouble

Up to now, no condition on connectivity was introduced.

Hierarchies of tessellations do not need it. But on the other hand, they do not always preserve it:



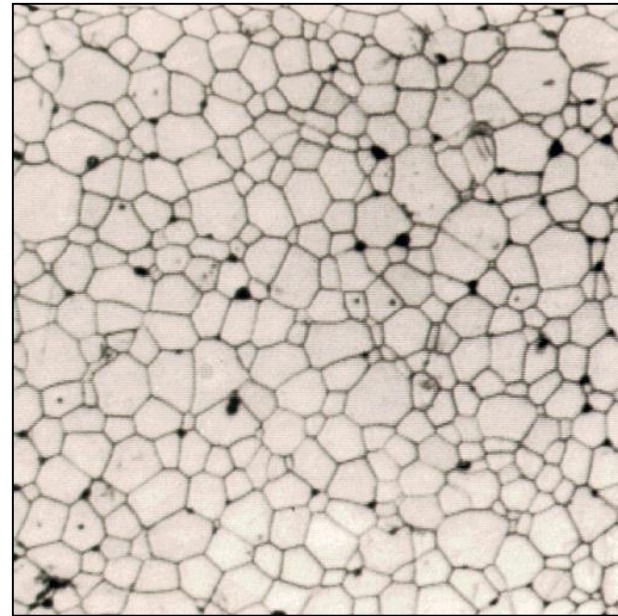
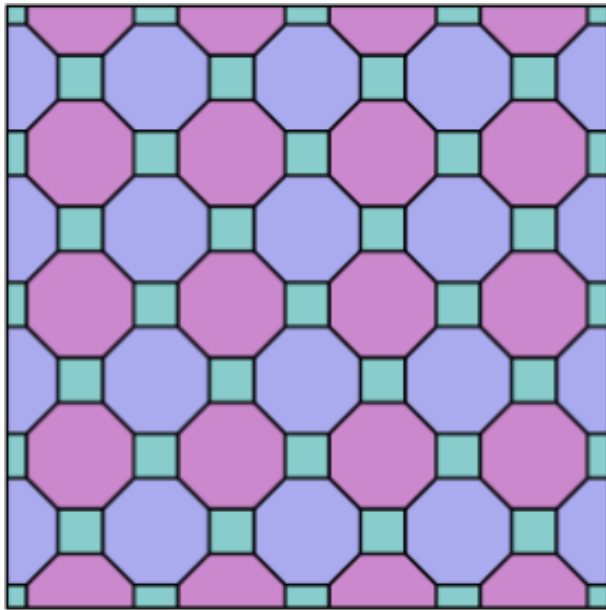
Connected sets will merge into connected sets iff the ambiguous configurations do not exist,

i.e. iff the adherences of the classes never intersect by a point (in 2-D) or a segment (in 3-D).

Strong connectivity

In \mathbf{R}^n , two open connected sets merge into connected sets iff the ambiguous configurations do not exist,

i.e. *iff any two adjacent polyhedra always have a common face of $n-1$ dimension.*



Translation invariance

In addition if translation invariance is required, then the elementary polyhedra in \mathbf{R}^n :

1. *are identical, up to a translation (i.e. regular grid);*
2. *always share a common face of $n-1$ dimension with their neighbors.*

These two constraints lead to:

- the centred squares in \mathbf{R}^2 , yielding Voronoi hexagons,
- the centred cubes in \mathbf{R}^3 , yielding Voronoi truncated octahedra.

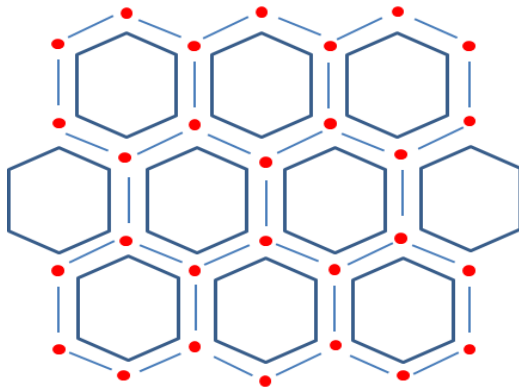
There is no other solution...

Outline

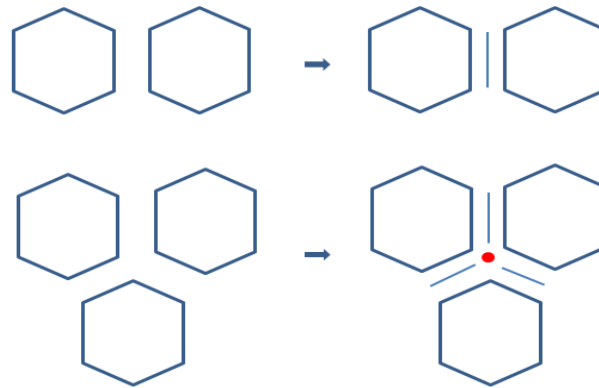
1. Regular open sets
2. Tessellations
3. Digital Tessellations by Khalimsky topology
4. **Digital Tessellations by Voronoi polyhedra**
5. Conclusion

Hexagonal tessellation of \mathbb{R}^2

- The grid is triangular.
- The Voronoi polygons are open hexagons.
- The other open sets are the unions of these hexagons plus the edges adjacent between them,
- and the triple points are closed



a



b

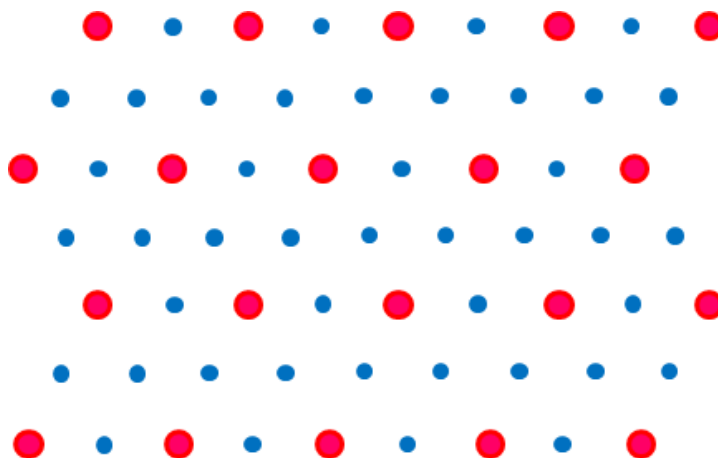
c

Doubling \mathbf{Z}^2

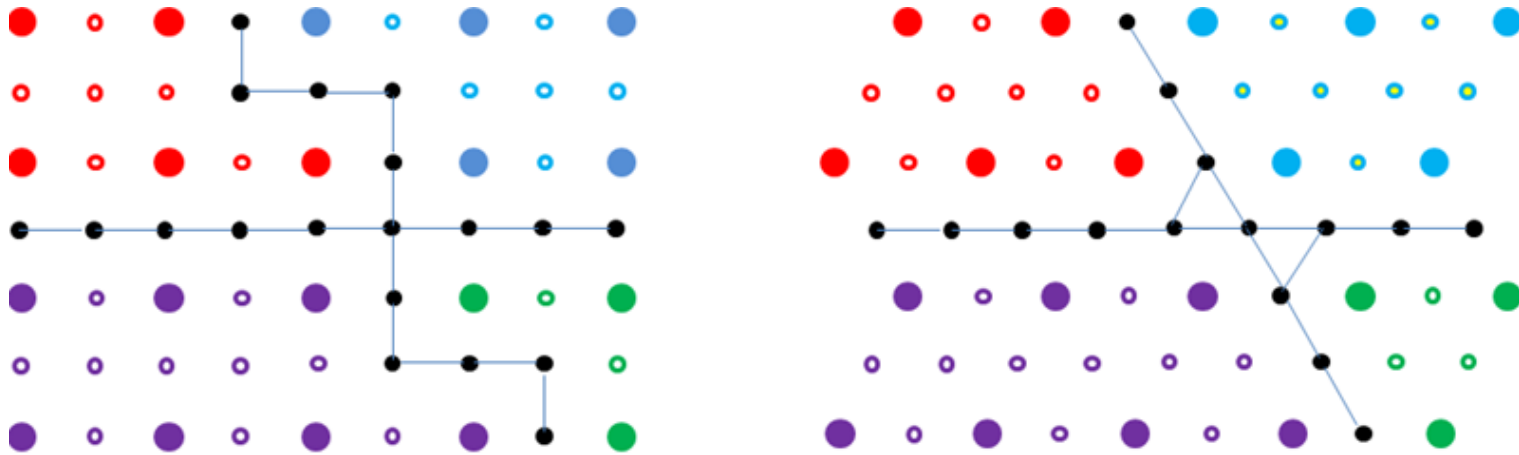
\mathbf{Z}^2 is the triangular grid, and one doubles the pixels along all lines parallel to the three axes \Rightarrow new space \mathbf{H}^2 .

The points of \mathbf{Z}^2 have three odd coordinates in \mathbf{H}^2 , and are identified to the open sets of the hexagonal topology of \mathbf{R}^2 , the other points of \mathbf{H}^2 being closed.

In this new topological space, the \mathcal{R} -open version $(\overline{X})^\circ$ of $X \subseteq \mathbf{Z}^2$ is obtained by adding to X all points comprised between each two open points de X in each of the three directions.



Comparison with Khaminsky topology

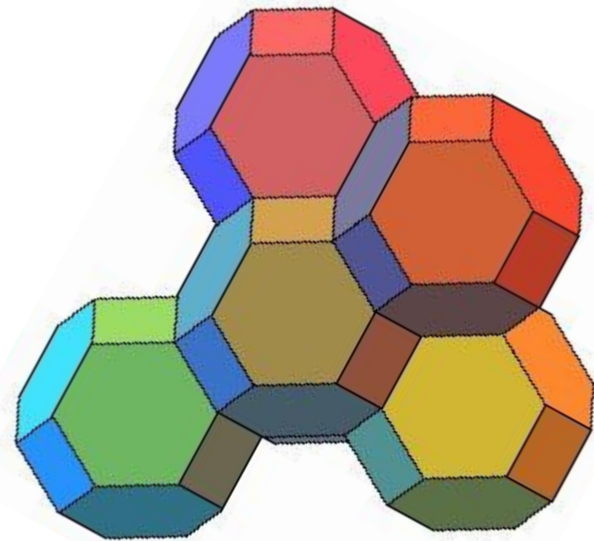
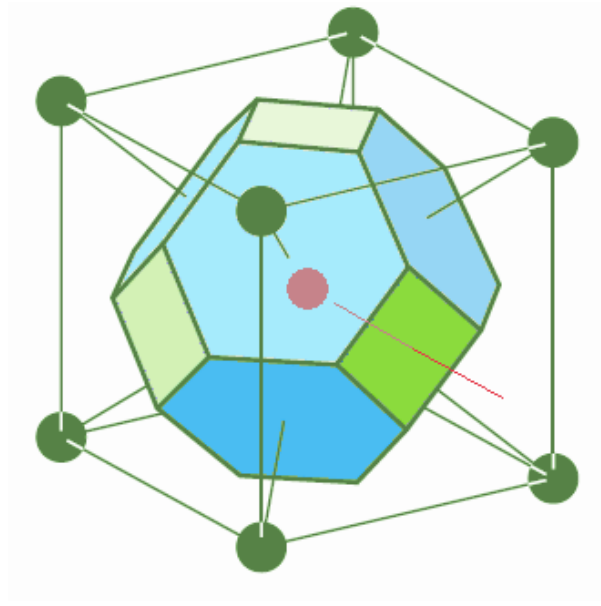


The small black points indicate the net of the frontiers of the tessellation in \mathbf{H}^2 .

Two major differences with Khaminsky square grid.

- the frontiers are no longer simple arcs (clusters of pixels may appear),
- no ambiguous diagonal were removed by suppression of the quadruples points.

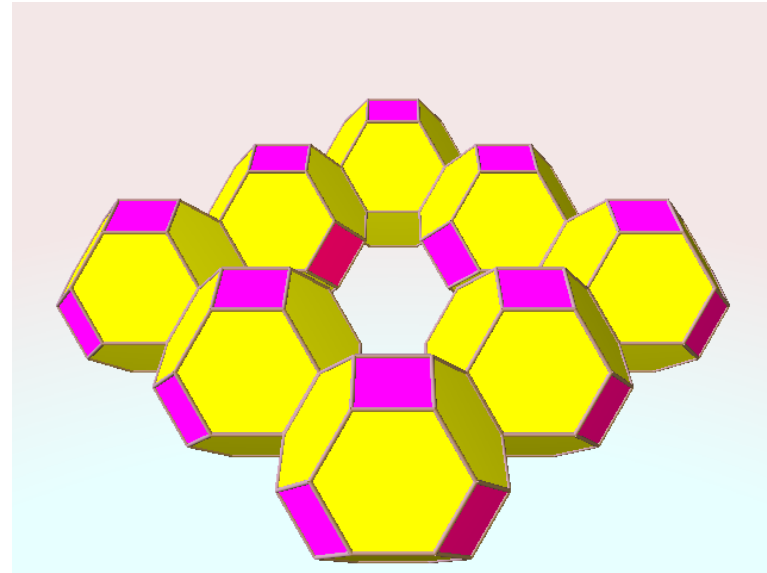
Tessellation of \mathbf{R}^3 by truncated octahedra



The Voronoi polyhedra of the centred cubic grid are the truncated octahedra -or tetrakaidecahedra-

They partition \mathbf{R}^3 in open polyhedra, square and hexagonal faces, triple edges and quadruple vertices. These elements generate a digital topology.

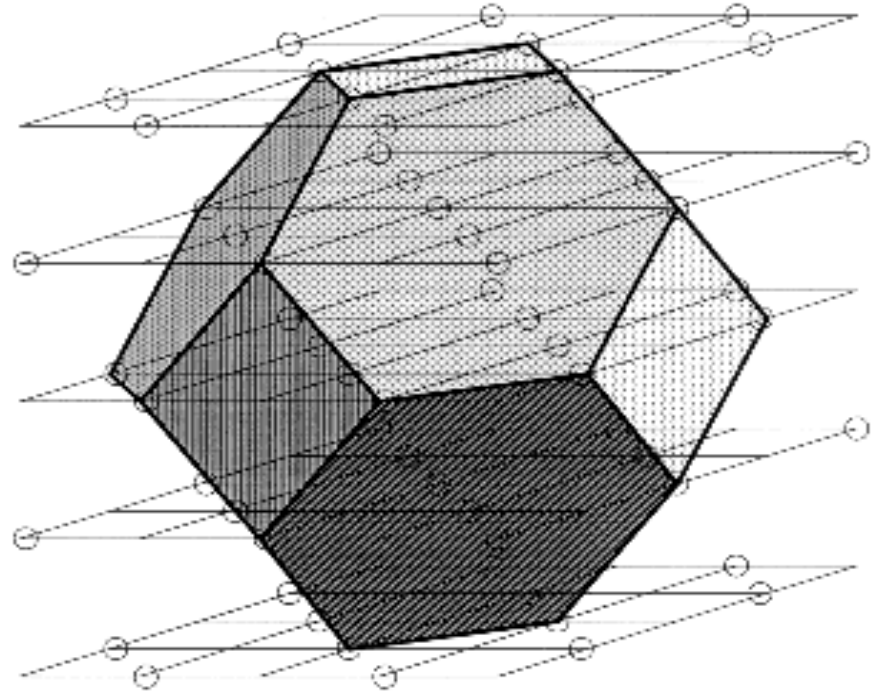
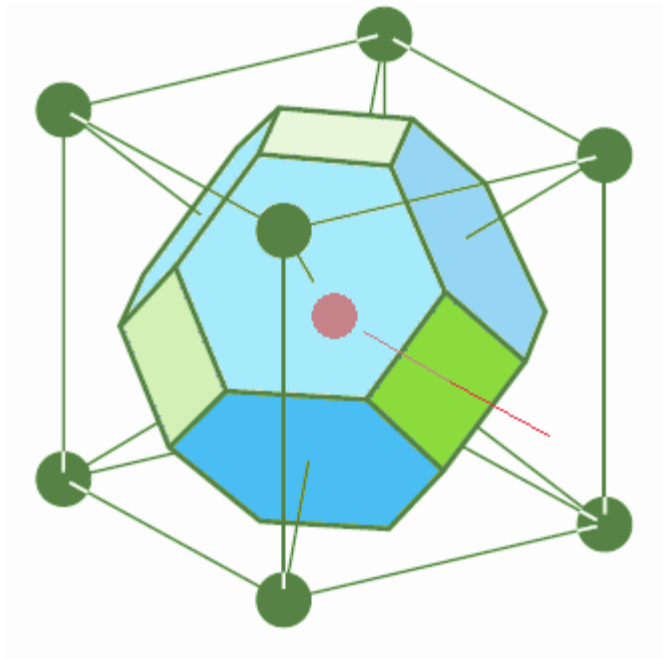
Tessellation of \mathbb{R}^3 by truncated octahedra



The regularization fills up the internal 1-D or 2-D fissures of zero thickness, and the background net is a connected union of faces and edges which completely envelops the classes.

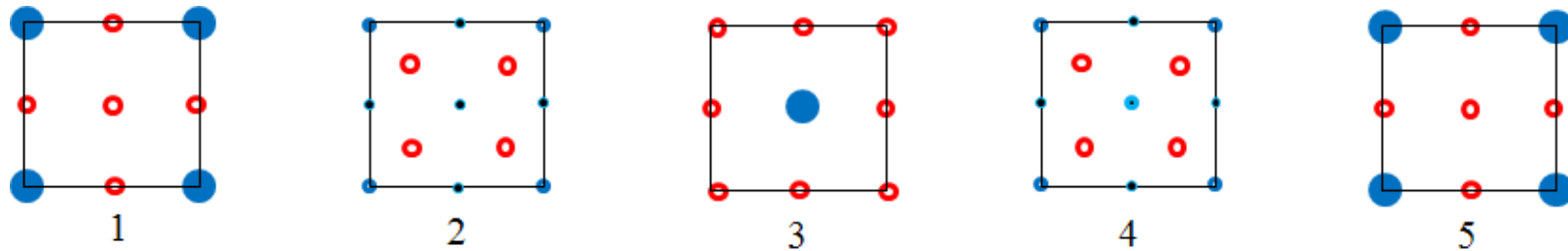
Two adjacent truncated octahedra always share a face (which is not the case with the cubes)

Tessellation of \mathbb{Z}^3 by truncated octahedra



The unit digital truncated octahedron requires five sections.

Tessellation of \mathbf{Z}^3 by truncated octahedra



- *Points of the initial truncated-octahedron of \mathbf{Z}^3*
- *Points added by half spacing in seven directions (sides and diagonals of the cube)*
- *Points added to regularize the truncated-octahedron*

For the rule $\mathbf{Z}^3 \rightarrow \mathbf{H}^3$ one starts from three horizontal planes of the cubic grid containing the vertices (n° 1 and 5) and the centre (n° 3) of the unit cube. The planes n° 2 and 4 are added for generating a centred cubic grid twice finer.

In the three directions of the cube and the four ones of the main diagonals alternate points of \mathbf{Z}^3 with those added for forming \mathbf{H}^3 .

Conclusion

- We proposed a method to combine interiors and frontiers in a unique representation
- The problem was solved by means of regular open sets, and further, tessellations
- In digital cases the passage partition→tessellation involves double resolution
- The ambiguous configurations of the square and the cubic topologies vanish when they are replaced by the hexagonal (\mathbf{Z}^2) and the cube-octahedron topologies (\mathbf{Z}^3).

