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Variable selection in multiple regression with random design

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Abstract

We propose a method for variable selection in multiple regression with random predictors. This method is based on a criterion that permits to reduce the variable selection problem to a problem of estimating suitable permutation and dimensionality. Then, estimators for these parameters are proposed and the resulting method for selecting variables is shown to be consistent. A simulation study that permits to gain understanding of the performances of the proposed approach and to compare it with an existing method is given.

Keywords

Variable selection; Multiple linear regression; Random design; Selection criterion; Consistency

1. Introduction

The selection of variables and models is an old and important problem in statistics, and several approaches have been proposed to deal with it for various methods of multivariate statistical analysis. For linear regression, many model selection criteria have been proposed in the literature. Surveys on earlier work in this field may be found in [6, 13, 14], whereas some monographs on this topic are available (e.g., [7, 8]). Most of the methods that have been proposed for variable selection in linear regression deal with the case where the covariates are assumed to be nonrandom; for this case, many selection criteria have been introduced in the literature. These include the FPE criterion ([13, 14, 12, 15]), cross-validation ([16, 11]), AIC and C_p type criteria (e.g., [4]), the prediction error criterion ([5]), and so on. There is just a few works dealing with the case where the covariates are random, although its importance that have been recognized in [2] who argued that that this case typically gives higher prediction errors than the fixed design counterparts and hence more is gained by variable selection. Linear regression with random design were considered in [17, 9] for variable selection, but these works only deal with univariate models, that is models for which the response is a real random variable. A recent work that considered multiple regression model is [1] in which a method based on

applying an adaptative LASSO type penalty and a novel BIC-type selection criterion have been proposed in order to select both predictors and responses.

In this paper we extend the approach introduced in [9] to the case of multiple regression. In Section 2, the multiple regression model that is used is presented as well as a statement of the variable selection problem. Then, the used criterion is introduced and we give a characterization result that permits to reduce the variable selection problem to an estimation problem for two parameters. In Section 3, we propose our method for selecting variables by estimating the two previous parameters, and we prove its consistency. Section 4 is devoted to a simulation study which permits to evaluate finite sample performances of the proposal and to compare it with the method given in [1]. Proofs of lemmas and theorems are given in Section 5.

2. Model and criterion for selection

In this section, the multiple regression model in which we are interested is introduced and a statement of the corresponding variable selection problem is given. It is described as a problem of estimation of a suitable set. A criterion permitting to characterize this set is proposed as well as an estimator of this criterion. Finally, we give a result that permits to obtain asymptotic properties of this estimator.

2.1. Model and statement of the problem

We consider the multiple regression model given by:

$$Y = BX + \varepsilon \tag{1}$$

where X and Y are random vectors valued into \mathbb{R}^p and \mathbb{R}^q respectively with $p \geq 2$ and $q \geq 2$, B is a $q \times p$ matrix of real coefficients, and ε is a random vector valued into \mathbb{R}^q and which is independent of X . Writing

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix}, Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_q \end{pmatrix}, \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_q \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qp} \end{pmatrix}$$

it is easily seen that Model (1) is equivalent to having a set of p univariate regression models given by:

$$Y_i = \sum_{j=1}^p b_{ij} X_j + \varepsilon_i, \quad i = 1, \dots, q, \tag{2}$$

and can also be written as

$$Y = \sum_{j=1}^p X_j \mathbf{b}_{\bullet j} + \varepsilon \tag{3}$$

where

$$\mathbf{b}_{\bullet j} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{qj} \end{pmatrix}.$$

We are interested with the variable selection problem, that is identifying the X_j 's which are not relevant in the previous set of models, on the basis of an i.i.d. sample $(X^{(k)}, Y^{(k)})_{1 \leq k \leq n}$ of (X, Y) . We say that a

variable X_j is not relevant if the corresponding coefficients vector $\mathbf{b}_{\bullet j}$ is null. So, putting $I = \{1, \dots, p\}$ we consider the subset $I_0 = \{j \in I / \|\mathbf{b}_{\bullet j}\|_{\mathbb{R}^q} = 0\}$ which is assumed to be non-empty, and we tackle the variable selection problem in Model (1) as a problem of estimating the set I_0 or, equivalently, the set $I_1 = I - I_0$. In order to simplify the estimation of I_1 we will first characterize it by means of a criterion which introduced below.

2.2. Characterization of I_1

Without loss of generality, we assume that X and Y are centered; thus, that is also the case for ε . Furthermore, denoting by $\|\cdot\|_{\mathbb{R}^k}$ the usual Euclidean norm of \mathbb{R}^k , we assume that $\mathbb{E}(\|X\|_{\mathbb{R}^p}^4) < +\infty$ and $\mathbb{E}(\|Y\|_{\mathbb{R}^q}^4) < +\infty$. Then, it is possible to define the covariance operators

$$V_1 = \mathbb{E}(X \otimes X) \quad \text{and} \quad V_{12} = \mathbb{E}(Y \otimes X), \quad (4)$$

where \otimes denotes the tensor product of vectors defined as follows: when E and F are euclidean spaces and (u, v) is a pair belonging to $E \times F$, the tensor product $u \otimes v$ is the linear map from E to F such that

$$\forall h \in E, \quad (u \otimes v)(h) = \langle u, h \rangle_E v,$$

where $\langle \cdot, \cdot \rangle_E$ denotes the inner product in E .

Remark 1. In all of the paper, we essentially use covariance operators, but the translation into matrix terms is obvious and more details can be found in [3]. Particularly, when u and v are vectors in \mathbb{R}^p and \mathbb{R}^q respectively, the matrix related to the operator $u \otimes v$, relative to canonical bases, is vu^T where u^T is the transpose of u . So, if matricial expressions are preferred to operators, one can identify the operators given in (4) with the matrices $V_1 = \mathbb{E}(XX^T)$ and $V_{12} = \mathbb{E}(XY^T)$.

In all of the paper, the operator V_1 is assumed to be invertible. For any subset K of I , let A_K be the projector

$$x = (x_i)_{i \in I} \in \mathbb{R}^p \mapsto x_K = (x_i)_{i \in K} \in \mathbb{R}^{\text{card}(K)}$$

and put $\Pi_K := A_K^* (A_K V_1 A_K^*)^{-1} A_K$, where A^* denotes the adjoint operator of A . Then, we introduce the criterion

$$\xi_K = \|V_{12} - V_1 \Pi_K V_{12}\| \quad (5)$$

where $\|\cdot\|$ denotes the usual operator norm given by $\|A\| = \sqrt{\text{tr}(A^*A)}$. This criterion permits to give a more explicit expression of I_1 as stated in the following lemma.

Lemma 1. *We have $I_1 \subset K$ if, and only if, $\xi_K = 0$.*

This lemma permits to characterize the fact that an integer i belongs to I_0 . Indeed, since having $i \in I_0$ is equivalent to having $I_1 \subset I - \{i\}$, we deduce from this lemma that one has $i \in I_0$ if, and only if, $\xi_{K_i} = 0$ where $K_i = I - \{i\}$. Then I_1 consists of the elements of I for which ξ_{K_i} does not vanish. Now, let us consider the unique permutation σ of I satisfying:

- (i) $\xi_{K\sigma(1)} \geq \xi_{K\sigma(2)} \geq \dots \geq \xi_{K\sigma(p)}$;
- (ii) $\xi_{K\sigma(i)} = \xi_{K\sigma(j)}$ and $i < j$ imply $\sigma(i) < \sigma(j)$.

Since I_0 is not empty, there exists an integer $s \in I$, that we call the dimensionality, satisfying

$$\xi_{K\sigma(1)} \geq \xi_{K\sigma(2)} \geq \dots \geq \xi_{K\sigma(s)} > 0 = \xi_{K\sigma(s+1)} = \dots = \xi_{K\sigma(s+1)}.$$

Therefore, we obviously have the following characterization of I_1 :

Lemma 2. $I_1 = \{\sigma(k) / 1 \leq k \leq s\}$.

This result shows that estimation of I_1 reduces to that of the two parameters σ and s . So, our method for selecting variables will be based on estimating these parameters; in the next subsection, an estimator of the used criterion will be introduced. That will be the basis of the proposed procedure for variable selection.

2.3. Estimation of the criterion

Recalling that we have an i.i.d. sample $(X^{(k)}, Y^{(k)})_{1 \leq k \leq n}$ of (X, Y) , we consider the sample means

$$\bar{X}^{(n)} = n^{-1} \sum_{k=1}^n X^{(k)}, \quad \bar{Y}^{(n)} = n^{-1} \sum_{k=1}^n Y^{(k)},$$

and the empirical covariance operators

$$\hat{V}_1^{(n)} = n^{-1} \sum_{k=1}^n (X^{(k)} - \bar{X}^{(n)}) \otimes (X^{(k)} - \bar{X}^{(n)}),$$

and

$$\hat{V}_{12}^{(n)} = n^{-1} \sum_{k=1}^n (Y^{(k)} - \bar{Y}^{(n)}) \otimes (X^{(k)} - \bar{X}^{(n)}).$$

Then, for any $K \subset I$, an estimator of ξ_K is given by

$$\hat{\xi}_K^{(n)} = \|\hat{V}_{12}^{(n)} - \hat{V}_1^{(n)} \hat{\Pi}_K^{(n)} \hat{V}_{12}^{(n)}\|$$

where $\hat{\Pi}_K^{(n)} = A_K^* (A_K V_1^{(n)} A_K^*)^{-1} A_K$. The result given below permits to obtain asymptotic properties of this estimator. As usual, when E and F are Euclidean vector spaces, we denote by $\mathcal{L}(E, F)$ the vector space of operators from E to F . When $E = F$, we simply write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$. Each element A of $\mathcal{L}(\mathbb{R}^{p+q})$ can be written as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} \in \mathcal{L}(\mathbb{R}^p)$, $A_{12} \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^p)$, $A_{21} \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ and $A_{22} \in \mathcal{L}(\mathbb{R}^q)$. Then we consider the projectors

$$P_1 : A \in \mathcal{L}(\mathbb{R}^{p+q}) \mapsto A_{11} \in \mathcal{L}(\mathbb{R}^p) \quad \text{and} \quad P_2 : A \in \mathcal{L}(\mathbb{R}^{p+q}) \mapsto A_{12} \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^p),$$

and we have:

Proposition 1. *We have*

$$\sqrt{n} \hat{\xi}_K^{(n)} = \|\hat{\Psi}_K^{(n)}(\hat{H}^{(n)}) + \sqrt{n} \delta_K\|,$$

where $\delta_K = V_{12} - V_1 \Pi_K V_{12}$, $(\hat{\Psi}_K^{(n)})_{n \in \mathbb{N}^*}$ is a sequence of random operators which converges almost surely, as $n \rightarrow +\infty$, to the operator Ψ_K of $\mathcal{L}(\mathcal{L}(\mathbb{R}^{p+q}), \mathcal{L}(\mathbb{R}^q, \mathbb{R}^p))$ given by:

$$\Psi_K(A) = P_2(A) - P_1(A) \Pi_K V_{12} + V_1 \Pi_K P_1(A) \Pi_K V_{12} - V_1 \Pi_K P_2(A),$$

and $(\hat{H}^{(n)})_{n \in \mathbb{N}^*}$ is a sequence of random variables valued into $\mathcal{L}(\mathbb{R}^{p+q})$ which converges in distribution to random variable H having a normal distributon with mean 0 and covariance operator given by:

$$\Gamma = \mathbb{E}((Z \otimes Z - V) \tilde{\otimes} (Z \otimes Z - V)),$$

Z being the \mathbb{R}^{p+q} -valued random variable given by

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$$

and $\tilde{\otimes}$ is the tensor product between elements of $\mathcal{L}(\mathbb{R}^{p+q})$ related to the inner product $\langle A, B \rangle = \text{tr}(A^* B)$.

3. Selection of variables

Lemma 2 shows that estimation of I_1 reduces to that of σ and s . In this section, estimators for these two parameters are proposed and consistency properties are established for them.

3.1. Estimation of σ and s

Let us consider a sequence $(f_n)_{n \in \mathbb{N}^*}$ of functions from I to \mathbb{R}_+ such that there exists a real $\alpha \in]0, 1/2[$ and a strictly decreasing function $f : I \rightarrow \mathbb{R}_+$ satisfying:

$$\forall i \in I, \quad \lim_{n \rightarrow +\infty} (n^\alpha f_n(i)) = f(i).$$

Then, recalling that $K_i = I - \{i\}$, we put

$$\widehat{\phi}_i^{(n)} = \widehat{\xi}_{K_i}^{(n)} + f_n(i) \quad (i \in I)$$

and we take as estimator of σ the random permutation $\widehat{\sigma}^{(n)}$ of I such that

$$\widehat{\phi}_{\widehat{\sigma}^{(n)}(1)}^{(n)} \geq \widehat{\phi}_{\widehat{\sigma}^{(n)}(2)}^{(n)} \geq \dots \geq \widehat{\phi}_{\widehat{\sigma}^{(n)}(p)}^{(n)}$$

and if $\widehat{\phi}_{\widehat{\sigma}^{(n)}(i)}^{(n)} = \widehat{\phi}_{\widehat{\sigma}^{(n)}(j)}^{(n)}$ with $i < j$, then $\widehat{\sigma}^{(n)}(i) < \widehat{\sigma}^{(n)}(j)$. Furthermore, we consider the random set $\widehat{J}_i^{(n)} = \{\widehat{\sigma}^{(n)}(j); 1 \leq j \leq i\}$ and the random variable

$$\widehat{\psi}_i^{(n)} = \widehat{\xi}_{\widehat{J}_i^{(n)}}^{(n)} + g_n(\widehat{\sigma}^{(n)}(i)) \quad (i \in I)$$

where $(g_n)_{n \in \mathbb{N}^*}$ is a sequence of functions from I to \mathbb{R}_+ such that there exist a real $\beta \in]0, 1[$ and a strictly increasing function $g : I \rightarrow \mathbb{R}_+$ satisfying:

$$\forall i \in I, \quad \lim_{n \rightarrow +\infty} (n^\beta g_n(i)) = g(i).$$

Then, we take as estimator of s the random variable

$$\widehat{s}^{(n)} = \min \left\{ i \in I / \widehat{\psi}_i^{(n)} = \min_{j \in I} \left(\widehat{\psi}_j^{(n)} \right) \right\}.$$

The variable selection is achieved by taking the random set

$$\widehat{I}_1^{(n)} = \left\{ \widehat{\sigma}^{(n)}(i) ; 1 \leq i \leq \widehat{s}^{(n)} \right\}$$

as estimator of I_1 .

3.2. Consistency

The following theorem establishes consistency for the preceding estimators :

Theorem 2. *We have:*

- (i) $\lim_{n \rightarrow +\infty} P(\widehat{\sigma}^{(n)} = \sigma) = 1;$
- (ii) $\widehat{s}^{(n)}$ converges in probability to s , as $n \rightarrow +\infty$.

As a consequence of this theorem, we easily obtain: $\lim_{n \rightarrow +\infty} P(\widehat{I}_1^{(n)} = I_1) = 1$. This shows the consistency of our method for selecting variables in the model (1).

4. Simulations

| Sample size | Proposed method | ASCCA |
|-------------|-----------------|----------|
| 50 | 0.00105 | 5.323e-6 |
| 100 | 0.00012 | 0.00052 |
| 500 | 1.009e-6 | 9.075e-6 |
| 800 | 20602e-7 | 5.789e-7 |
| 1000 | 1.243e-7 | 1.308e-7 |
| 2000 | 1.436e-8 | 1.692e-8 |

Table 1: Average of prediction errors over 2000 replications

In this section, we report results of a simulation study which was made in order to check the efficacy of the proposed approach and to compare it with an existing method: the ASCCA method introduced by An et al. (2013). This latter method is based on re-casting the multivariate regression problem as a classical CCA problem for which a least squares type formulation is constructed, and applying an adaptive LASSO type penalty together with a BIC-type selection criterion (see [1] for more details). Our simulated data is based on two independent data sets: training data and test data, each with sample size $n = 50, 100, 500, 800, 1000, 2000$. The training data is used for selecting variables by using both our method, with penalty terms $f_n(i) = n^{-1/4} i^{-1}$ and $g_n(i) = n^{-3/4} i$, and the ASCCA method. The test data is used for computing prediction error given by

$$e = \frac{1}{n} \sum_{k=1}^n \|Y^{(k)} - \hat{Y}^{(k)}\|_{\mathbb{R}^q}^2,$$

where $Y^{(k)}$ is an observed response and $\hat{Y}^{(k)}$ is the usual linear predictor of $Y^{(k)}$ computed by using the variables selected at the previous step, that is $\hat{Y}^{(k)} = (\mathbb{X}^T \mathbb{X})^{-1} \mathbb{X}^T Y^{(k)}$ where \mathbb{X} is a matrix with n rows and columns containing the observations of the X_j 's that have been selected in the previous step. Each data set was generated as follows: $X^{(k)}$ is generated from a multivariate normal distribution in \mathbb{R}^7 with mean 0 and covariance $\text{cov}(X_i^{(k)}, X_j^{(k)}) = 0.5^{|i-j|}$ for any $1 \leq i, j \leq 7$, and the corresponding response $Y^{(k)}$ is generated according to (1) with

$$B = \begin{pmatrix} 3 & 0 & 0 & 1.5 & 0 & 0 & 2 \\ 4 & 0 & 0 & 2.5 & 0 & 0 & -1 \\ 5 & 0 & 0 & 0.5 & 0 & 0 & 3 \\ 6 & 0 & 0 & 3 & 0 & 0 & 1 \\ 7 & 0 & 0 & 6 & 0 & 0 & 4 \end{pmatrix}$$

and the related error term $\varepsilon^{(k)}$ having a multivariate normal distribution in \mathbb{R}^5 with mean 0 and covariance matrix $0.5 I_5$, where I_5 denotes the 5-dimensional identity matrix. The outputs of the numerical experiment are the averages of the aforementioned prediction errors over 2000 independent replications. The results are reported in Table 1. Our method gives the better results for $n \geq 100$ but was outperformed by the ASCCA method for $n = 50$.

5. Proofs

5.1. Proof of Lemma 1

Denoting by (Ω, \mathcal{A}, P) the considered probability space, we consider the operators:

$$L_1 : x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \in \mathbb{R}^p \mapsto \sum_{j=1}^p x_j X_j \in L^2(\Omega, \mathcal{A}, P) \text{ and } L_2 : y = \begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix} \in \mathbb{R}^q \mapsto \sum_{i=1}^q y_i Y_i \in L^2(\Omega, \mathcal{A}, P)$$

with adjoints are respectively given by:

$$L_1^* : Z \in L^2(\Omega, \mathcal{A}, P) \mapsto \mathbf{E}(ZX) \in \mathbb{R}^p, \text{ and } L_2^* : Z \in L^2(\Omega, \mathcal{A}, P) \mapsto \mathbf{E}(ZY) \in \mathbb{R}^q.$$

It is easy to verify that $L_1^* L_1 = V_1$ and $L_1^* L_2 = V_{12}$. Denoting by $R(A)$ the range of the operator A , and from the fact that the orthogonal projector $\Pi_{R(A)}$ onto $R(A)$ is given by $\Pi_{R(A)} = A(A^*A)^{-1}A^*$, we clearly have

$$\xi_K = \|L_1^* L_2 - L_1^* L_1 A_K^* (A_K L_1^* L_1 A_K^*)^{-1} A_K L_1^* L_2\| = \|L_1^* L_2 - L_1^* \Pi_{R(L_1 A_K^*)} L_2\| = \|L_1^* \Pi_{R(L_1 A_K^*)^\perp} L_2\|, \quad (6)$$

where E^\perp denotes the orthogonal space of the vector space E . For any vector $\alpha = (\alpha_1, \dots, \alpha_q)^T$ in \mathbb{R}^q , one has

$$L_2(\alpha) = \sum_{i=1}^q \alpha_i Y_i = \sum_{i=1}^q \alpha_i \left(\sum_{j=1}^p b_{ij} X_j + \varepsilon_i \right) = \sum_{i=1}^q \sum_{j=1}^p \alpha_i b_{ij} X_j + \sum_{i=1}^q \alpha_i \varepsilon_i.$$

Since for any $u = (u_1, \dots, u_p)^T \in \mathbb{R}^p$, we have

$$\langle L_1(u), \alpha_i \varepsilon_i \rangle = \sum_{j=1}^p u_j \langle X_j, \alpha_i \varepsilon_i \rangle = \sum_{j=1}^p u_j \alpha_i \mathbb{E}(X_j \varepsilon_i) = \sum_{j=1}^p u_j \alpha_i \mathbb{E}(X_j) \mathbb{E}(\varepsilon_i) = 0,$$

it follows that $\alpha_i \varepsilon_i \in R(L_1)^\perp$ and, from $R(L_1)^\perp \subset R(L_1 A_K^*)^\perp$, we obtain

$$L_1^* \Pi_{R(L_1 A_K^*)^\perp} \alpha_i \varepsilon_i = L_1^* \alpha_i \varepsilon_i = \mathbb{E}(\alpha_i \varepsilon_i X) = \alpha_i \mathbf{E}(\varepsilon_i) \mathbf{E}(X) = 0.$$

Thus,

$$L_1^* \Pi_{R(L_1 A_K^*)^\perp} L_2(\alpha) = L_1^* \Pi_{R(L_1 A_K^*)^\perp} \sum_{i=1}^q \sum_{j=1}^p \alpha_i b_{ij} X_j = \sum_{i=1}^q \alpha_i L_1^* \Pi_{R(L_1 A_K^*)^\perp} L_1(\mathbf{b}_{i\bullet}), \quad (7)$$

where

$$\mathbf{b}_{i\bullet} = \begin{pmatrix} b_{i1} \\ b_{i2} \\ \vdots \\ b_{ip} \end{pmatrix}.$$

If $\xi_K = 0$, then considering, for $i = 1, \dots, q$, the vector $\alpha = (0, \dots, 0, 1, 0, \dots, 0)$ of \mathbb{R}^q whose coordinates are null except the i -th one which equals 1, we deduce from (7) that $L_1^* \Pi_{R(L_1 A_K^*)^\perp} L_1(\mathbf{b}_{i\bullet}) = 0$. Since, for any operator A , $\ker(A^*A) = \ker(A)$, it follows that we have $\Pi_{R(L_1 A_K^*)^\perp} L_1(\mathbf{b}_{i\bullet}) = 0$, that is

$$L_1(\mathbf{b}_{i\bullet}) \in R(L_1 A_K^*). \quad (8)$$

Denoting by $|K|$ the cardinality of K and putting $K = \{k_1, k_2, \dots, k_{|K|}\}$, we deduce from (8) that there exists a vector $\beta = (\beta_1, \dots, \beta_{|K|})^T \in \mathbb{R}^{|K|}$ such that $L_1(\mathbf{b}_{i\bullet}) = L_1 A_K^* \beta$, that is

$$\sum_{j=1}^p b_{ij} X_j = \sum_{\ell=1}^{|K|} \beta_\ell X_{k_\ell}$$

and, equivalently,

$$\sum_{\ell=1}^{|K|} (b_{ik_\ell} - \beta_\ell) X_{k_\ell} + \sum_{\ell \in I-K} b_{ij} X_j = 0. \quad (9)$$

Since V_1 is invertible we have $\ker(L_1) = \ker(L_1^* L_1) = \ker(V_1) = \{0\}$. Then, X_1, \dots, X_p are linearly independent and, therefore, (9) implies that, for all $j \in I - K$, $b_{ij} = 0$. This property holds for any $i \in \{1, \dots, q\}$, then we deduce that $I - K \subset I_0$ and, equivalently, that $I_1 \subset K$. Reciprocally, we first have

$$\begin{aligned} L_1^* \Pi_{R(L_1 A_K^*)^\perp} L_1(\mathbf{b}_{i\bullet}) &= L_1^* \Pi_{R(L_1 A_K^*)^\perp} \sum_{j=1}^p b_{ij} X_j \\ &= L_1^* \Pi_{R(L_1 A_K^*)^\perp} \left(\sum_{j \in K} b_{ij} X_j + \sum_{j \in I-K} b_{ij} X_j \right) \\ &= L_1^* \Pi_{R(L_1 A_K^*)^\perp} \left(\sum_{\ell=1}^{|K|} b_{ik_\ell} X_{k_\ell} + \sum_{j \in I-K} b_{ij} X_j \right). \end{aligned}$$

If $I_1 \subset K$, then $I - K \subset I_0$ and, consequently, for all $j \in I - K$, $b_{ij} = 0$. Thus

$$L_1^* \Pi_{R(L_1 A_K^*)^\perp} L_1(\mathbf{b}_{i\bullet}) = L_1^* \Pi_{R(L_1 A_K^*)^\perp} \left(\sum_{\ell=1}^{|K|} b_{ik_\ell} X_{k_\ell} \right) = L_1^* \Pi_{R(L_1 A_K^*)^\perp} L_1 A_K^*(\mathbf{b}_{i\bullet}) = 0$$

because $L_1 A_K^*(\mathbf{b}_{i\bullet}) \in R(L_1 A_K^*)$. Then, from (7) and (6), we deduce that $\xi_K = 0$.

5.2. Proof of Proposition 1

We have:

$$\begin{aligned} \sqrt{n} \widehat{\xi}_K^{(n)} &= \|\sqrt{n}(\widehat{V}_{12}^{(n)} - V_{12}) - \sqrt{n}(\widehat{V}_1^{(n)} - V_1) \widehat{\Pi}_K^{(n)} \widehat{V}_{12}^{(n)} - V_1 (\sqrt{n}(\widehat{\Pi}_K^{(n)} - \Pi_K)) \widehat{V}_{12}^{(n)} \\ &\quad - V_1 \Pi_K (\sqrt{n}(\widehat{V}_{12}^{(n)} - V_{12})) + \sqrt{n} \delta_K\|, \end{aligned}$$

and since

$$\begin{aligned} \widehat{\Pi}_K^{(n)} - \Pi_K &= A_K^* \left((A_K \widehat{V}_1^{(n)} A_K^*)^{-1} - (A_K V_1 A_K^*)^{-1} \right) A_K \\ &= A_K^* \left(-(A_K \widehat{V}_1^{(n)} A_K^*)^{-1} \left(A_K \widehat{V}_1^{(n)} A_K^* - A_K V_1 A_K^* \right) (A_K V_1 A_K^*)^{-1} \right) A_K \\ &= -\widehat{\Pi}_K^{(n)} \left(\widehat{V}_1^{(n)} - V_1 \right) \Pi_K, \end{aligned}$$

it follows:

$$\begin{aligned} \sqrt{n} \widehat{\xi}_K^{(n)} &= \|\sqrt{n}(\widehat{V}_{12}^{(n)} - V_{12}) - \sqrt{n}(\widehat{V}_1^{(n)} - V_1) \widehat{\Pi}_K^{(n)} \widehat{V}_{12}^{(n)} \\ &\quad + V_1 \widehat{\Pi}_K^{(n)} \left(\sqrt{n}(\widehat{V}_1^{(n)} - V_1) \right) \Pi_K \widehat{V}_{12}^{(n)} \\ &\quad - V_1 \Pi_K \left(\sqrt{n}(\widehat{V}_{12}^{(n)} - V_{12}) \right) + \sqrt{n} \delta_K\|. \end{aligned} \quad (10)$$

Let us consider the \mathbb{R}^{p+q} -valued random vectors

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad Z^{(k)} = \begin{pmatrix} X^{(k)} \\ Y^{(k)} \end{pmatrix}, \quad k = 1, \dots, n;$$

the covariance operator of Z is given by $V = \mathbb{E}(Z \otimes Z)$ and can be written as

$$V = \begin{pmatrix} V_1 & V_{12} \\ V_{21} & V_2 \end{pmatrix} \quad (11)$$

where $V_2 = \mathbb{E}(Y \otimes Y)$ and $V_{21} = V_{12}^*$. Further, putting

$$\bar{Z}^{(n)} = n^{-1} \sum_{k=1}^n Z^{(k)}, \quad \text{and} \quad \hat{V}^{(n)} = n^{-1} \sum_{k=1}^n (Z^{(k)} - \bar{Z}^{(n)}) \otimes (Z^{(k)} - \bar{Z}^{(n)}),$$

we can write

$$\hat{V}^{(n)} = \begin{pmatrix} \hat{V}_1^{(n)} & \hat{V}_{12}^{(n)} \\ \hat{V}_{21}^{(n)} & \hat{V}_2^{(n)} \end{pmatrix} \quad (12)$$

where $\hat{V}_2^{(n)} = n^{-1} \sum_{k=1}^n (Y^{(k)} - \bar{Y}^{(n)}) \otimes (Y^{(k)} - \bar{Y}^{(n)})$ and $\hat{V}_{21}^{(n)} = (\hat{V}_{12}^{(n)})^*$. Then we deduce from (10), (11) and (12) that $\sqrt{n}\hat{\xi}_K^{(n)} = \|\hat{\Psi}_K^{(n)}(\hat{H}^{(n)}) + \sqrt{n}\delta_K\|$, where $\hat{H}^{(n)} = \sqrt{n}(\hat{V}^{(n)} - V)$ and $\hat{\Psi}_K^{(n)}$ is the random operator from $\mathcal{L}(\mathbb{R}^{p+q})$ to $\mathcal{L}(\mathbb{R}^p)$ defined by

$$\forall A \in \mathcal{L}(\mathbb{R}^{p+q}), \quad \hat{\Psi}_K^{(n)}(A) = P_2(A) - P_1(A)\hat{\Pi}_K^{(n)}\hat{V}_{12}^{(n)} + V_1\hat{\Pi}_K^{(n)}P_1(A)\Pi_A\hat{V}_{12}^{(n)} - V_1\Pi_K P_2(A).$$

Considering the usual operators norm $\|\cdot\|_\infty$ defined in $\mathcal{L}(E, F)$ by $\|A\|_\infty = \sup_{x \in E - \{0\}} \|Ax\|_F / \|x\|_E$ and recalling that, for two operators A and B , one has $\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty$, we obtain

$$\begin{aligned} \|\hat{\Psi}_K^{(n)}(A) - \Psi_K(A)\|_\infty &= \left\| -P_1(A) \left(\hat{\Pi}_K^{(n)} - \Pi_K \right) \hat{V}_{12}^{(n)} - P_1(A) \Pi_K \left(\hat{V}_{12}^{(n)} - V_{12} \right) \right. \\ &\quad \left. + V_1 \left(\hat{\Pi}_K^{(n)} - \Pi_K \right) P_1(A) \Pi_K \hat{V}_{12}^{(n)} + -V_1 \Pi_K P_1(A) \Pi_K \left(\hat{V}_{12}^{(n)} - V_{12} \right) \right\|_\infty \\ &\leq \|P_1(A)\|_\infty \left[\|\hat{\Pi}_K^{(n)} - \Pi_K\|_\infty \|\hat{V}_{12}^{(n)}\|_\infty + \|\Pi_K\|_\infty \|\hat{V}_{12}^{(n)} - V_{12}\|_\infty \right. \\ &\quad \left. + \|V_1\|_\infty \|\Pi_K\|_\infty \|\hat{\Pi}_K^{(n)} - \Pi_K\|_\infty \|\hat{V}_{12}^{(n)}\|_\infty \right. \\ &\quad \left. + \|V_1\|_\infty \|\Pi_K\|_\infty^2 \|\hat{V}_{12}^{(n)} - V_{12}\|_\infty \right] \\ &\leq \left[\|\hat{\Pi}_K^{(n)} - \Pi_K\|_\infty \|\hat{V}_{12}^{(n)}\|_\infty + \|\Pi_K\|_\infty \|\hat{V}_{12}^{(n)} - V_{12}\|_\infty \right. \\ &\quad \left. + \|V_1\|_\infty \|\Pi_K\|_\infty \|\hat{\Pi}_K^{(n)} - \Pi_K\|_\infty \|\hat{V}_{12}^{(n)}\|_\infty \right. \\ &\quad \left. + \|V_1\|_\infty \|\Pi_K\|_\infty^2 \|\hat{V}_{12}^{(n)} - V_{12}\|_\infty \right] \|P_1\|_{\infty, \infty} \|A\|_\infty, \end{aligned}$$

where $\|T\|_{\infty, \infty} := \sup_{A \in \mathcal{L}(\mathbb{R}^{p+q}) - \{0\}} \|T(A)\|_\infty / \|A\|_\infty$. Hence

$$\begin{aligned} \|\hat{\Psi}_K^{(n)} - \Psi_K\|_{\infty, \infty} &\leq \left[\|1 + \|V_1\|_\infty \|\Pi_K\|_\infty\| \|\hat{V}_{12}^{(n)}\|_\infty \|\hat{\Pi}_K^{(n)} - \Pi_K\|_\infty \|P_1\|_{\infty, \infty} \right. \\ &\quad \left. + [1 + \|V_1\|_\infty \|\Pi_K\|_\infty] \|\Pi_K\|_\infty \|\hat{V}_{12}^{(n)} - V_{12}\|_\infty \|P_1\|_{\infty, \infty} \right] \end{aligned} \quad (13)$$

From the strong law of large numbers it is easily seen that $\hat{V}_1^{(n)}$ (resp. $\hat{V}_{12}^{(n)}$) converges almost surely, as $n \rightarrow +\infty$ to V_1 (resp. V_{12}). Therefore, $\hat{\Pi}_K^{(n)}$ converges almost surely, as $n \rightarrow +\infty$ to Π_K , and from (13) we deduce that $\hat{\Psi}_K^{(n)}$ converges almost surely, as $n \rightarrow +\infty$ to Ψ_K . It remains to obtain the asymptotic distribution of $\hat{H}^{(n)}$. We have $\hat{H}^{(n)} = \hat{H}_1^{(n)} - \hat{H}_2^{(n)}$ where

$$\hat{H}_1^{(n)} = \sqrt{n} \left(\frac{1}{n} \sum_{k=1}^n Z_k \otimes Z_k - V \right) \quad \text{and} \quad \hat{H}_2^{(n)} = \frac{1}{\sqrt{n}} \left((\sqrt{n}\bar{Z}^{(n)}) \otimes (\sqrt{n}\bar{Z}^{(n)}) \right).$$

The central limit theorem ensures that $\widehat{H}_1^{(n)}$ (resp. $\sqrt{n}\overline{Z}^{(n)}$) converges in distribution, as $n \rightarrow +\infty$, to a random variable H (resp. U) having a centered normal distribution with covariance operator Γ (resp. Γ') given by

$$\Gamma = \mathbb{E}((Z \otimes Z - V) \otimes (Z \otimes Z - V)) \quad (\text{resp. } \Gamma' = \mathbb{E}(Z \otimes Z)).$$

Hence, $\widehat{H}_2^{(n)}$ converges in probability, as $n \rightarrow +\infty$, to 0 and Slutsky theorem permits to conclude that $\widehat{H}^{(n)}$ converges in distribution, as $n \rightarrow +\infty$, to H .

5.3. Proof of Theorem 2

We just need to prove the lemma which is given below. Then the proof of Theorem 1 is similar than that of Theorem 3.1 in [10]. Let $r \in \mathbb{N}^*$ and $(m_1, \dots, m_r) \in (\mathbb{N}^*)^r$ such that $\sum_{\ell=1}^r m_\ell = p$ and

$$\xi_{K_{\sigma(1)}} = \dots = \xi_{K_{\sigma(m_1)}} > \xi_{K_{\sigma(m_1+1)}} = \dots = \xi_{K_{\sigma(m_1+m_2)}} > \dots > \xi_{K_{\sigma(m_1+m_2+\dots+m_{r-1}+1)}} = \dots = \xi_{K_{\sigma(m_1+m_2+\dots+m_r)}}.$$

Then, putting $E = \{\ell \in \mathbb{N}^* / 1 \leq \ell \leq r, m_\ell \geq 2\}$ and $F_\ell := \left\{ \left(\sum_{k=0}^{\ell-1} m_k \right) + 1, \dots, \left(\sum_{k=0}^{\ell} m_k \right) - 1 \right\}$ with $m_0 = 0$, we have:

Lemma 3. *If $E \neq \emptyset$, then for all $\ell \in E$ and all $i \in F_\ell$, the sequence $n^\alpha \left(\widehat{\xi}_{K_{\sigma(i)}}^{(n)} - \widehat{\xi}_{K_{\sigma(i+1)}}^{(n)} \right)$ converges in probability to 0 as $n \rightarrow +\infty$.*

Proof. Let us put $\gamma_\ell = \xi_{K_{\sigma(i)}} = \xi_{K_{\sigma(i+1)}}$; if $\gamma_\ell = 0$, then

$$\begin{aligned} \left| n^\alpha \left(\widehat{\xi}_{K_{\sigma(i)}}^{(n)} - \widehat{\xi}_{K_{\sigma(i+1)}}^{(n)} \right) \right| &= n^{\alpha-\frac{1}{2}} \left| \|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)})\| - \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)})\| \right| \\ &\leq n^{\alpha-\frac{1}{2}} \left\| \left(\widehat{\Psi}_{K_{\sigma(i)}}^{(n)} - \widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)} \right) (\widehat{H}^{(n)}) \right\| \\ &\leq n^{\alpha-\frac{1}{2}} \|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)} - \widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}\|_\infty \|\widehat{H}^{(n)}\|, \end{aligned}$$

Since $\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}$ and $\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}$ converge almost surely, as $n \rightarrow +\infty$, to $\Psi_{K_{\sigma(i)}}$ and $\Psi_{K_{\sigma(i+1)}}$ respectively, and since $\widehat{H}^{(n)}$ converges in distribution, as $n \rightarrow +\infty$, to H , it follows from the preceding inequality and from $\alpha < 1/2$ that $n^\alpha \left(\widehat{\xi}_{K_{\sigma(i)}}^{(n)} - \widehat{\xi}_{K_{\sigma(i+1)}}^{(n)} \right)$ converges in probability to 0 as $n \rightarrow +\infty$. If $\gamma_\ell \neq 0$, we have

$$\begin{aligned} n^\alpha \left(\widehat{\xi}_{K_{\sigma(i)}}^{(n)} - \widehat{\xi}_{K_{\sigma(i+1)}}^{(n)} \right) &= n^{\alpha-\frac{1}{2}} \left(\|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) + \sqrt{n}\delta_{K_{\sigma(i)}}\| - \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) + \sqrt{n}\delta_{K_{\sigma(i+1)}}\| \right) \\ &= \frac{n^{\alpha-\frac{1}{2}} \left(\|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)})\|^2 - \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)})\|^2 \right)}{\|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) + \sqrt{n}\delta_{K_{\sigma(i)}}\| + \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) + \sqrt{n}\delta_{K_{\sigma(i+1)}}\|} \\ &+ \frac{2n^\alpha \left(\left\langle \delta_{K_{\sigma(i)}}, \widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) \right\rangle - \left\langle \delta_{K_{\sigma(i+1)}}, \widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) \right\rangle \right)}{\|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) + \sqrt{n}\delta_{K_{\sigma(i)}}\| + \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) + \sqrt{n}\delta_{K_{\sigma(i+1)}}\|} \\ &= \frac{n^{\alpha-1} \left(\|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)})\|^2 - \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)})\|^2 \right)}{\|n^{-\frac{1}{2}}\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) + \delta_{K_{\sigma(i)}}\| + \|n^{-\frac{1}{2}}\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) + \delta_{K_{\sigma(i+1)}}\|} \\ &+ \frac{2n^{\alpha-\frac{1}{2}} \left(\left\langle \delta_{K_{\sigma(i)}}, \widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) \right\rangle - \left\langle \delta_{K_{\sigma(i+1)}}, \widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) \right\rangle \right)}{\|n^{-\frac{1}{2}}\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) + \delta_{K_{\sigma(i)}}\| + \|n^{-\frac{1}{2}}\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) + \delta_{K_{\sigma(i+1)}}\|}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product defined by $\langle A, B \rangle = \text{tr}(A^*B)$. First,

$$\begin{aligned}
& \left| n^{\alpha-1} \left(\|\widehat{\Psi}_{\sigma(j)}^{(n)}(\widehat{H}^{(n)})\|^2 - \|\widehat{\Psi}_{\sigma(j+1)}^{(n)}(\widehat{H}^{(n)})\|^2 \right) \right| \\
& \leq n^{\alpha-1} \left(\|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)})\|^2 + \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)})\|^2 \right) \\
& \leq n^{\alpha-1} \left(\|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}\|_{\infty}^2 + \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}\|_{\infty}^2 \right) \|\widehat{H}^{(n)}\|^2
\end{aligned} \tag{14}$$

and, further,

$$\begin{aligned}
& \left| 2n^{\alpha-\frac{1}{2}} \left(\left\langle \delta_{K_{\sigma(i)}}, \widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) \right\rangle - \left\langle \delta_{K_{\sigma(i+1)}}, \widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) \right\rangle \right) \right| \\
& \leq 2n^{\alpha-\frac{1}{2}} \left(\left| \left\langle \delta_{K_{\sigma(i)}}, \widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)}) \right\rangle \right| + \left| \left\langle \delta_{K_{\sigma(i+1)}}, \widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)}) \right\rangle \right| \right) \\
& \leq 2n^{\alpha-\frac{1}{2}} \left(\|\delta_{K_{\sigma(i)}}\| \|\widehat{\Psi}_{K_{\sigma(i)}}^{(n)}(\widehat{H}^{(n)})\| + \|\delta_{K_{\sigma(i+1)}}\| \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}(\widehat{H}^{(n)})\| \right) \\
& \leq 2n^{\alpha-\frac{1}{2}} \gamma_{\ell} \left(\|\widehat{\Psi}^{(n)}\|_{\infty} + \|\widehat{\Psi}_{K_{\sigma(i+1)}}^{(n)}\|_{\infty} \right) \|\widehat{H}^{(n)}\|.
\end{aligned} \tag{15}$$

Equations (14) and (15), and the above recalled convergence properties permit to conclude that the sequence $n^{\alpha} \left(\widehat{\xi}_{K_{\sigma(i)}}^{(n)} - \widehat{\xi}_{K_{\sigma(i+1)}}^{(n)} \right)$ converges in probability to 0, as $n \rightarrow +\infty$. \square

References

- [1] B. An, J. Guo, H. Wang. Multivariate regression shrinkage and selection by canonical correlation analysis. *Comput. Statist. Data Anal.*, 62:93–107, 2013.
- [2] L. Breiman, P. Spector. Submodel selection and evaluation in regression. The X -random case. *Internat. Statist. Rev.*, 60:291–319, 1992.
- [3] J. Dauxois, Y. Romain, S. Viguier. Tensor products and statistics. *Linear Algebra Appl.*, 210:59–88, 1994.
- [4] Y. Fujikoshi, K. Sato. Modified AIC and C_p in multivariate linear regression. *Biometrika*, 84:707–716, 1997.
- [5] Y. Fujikoshi, T. Kan, S. Takahashi, T. Sakurai. Prediction error criterion for selecting variables in a linear regression model. *Ann. Inst. Stat. Math.*, 63:387–403, 2011.
- [6] R. R. Hocking. The analysis and selection in linear regression. *Biometrics*, 32:1–49, 1976.
- [7] H. Linhart, W. Zucchini. *Model selection*. Wiley, New York, 1986.
- [8] A. J. Miller. *Subset selection in regression*. Chapman and Hall, London, 1990.
- [9] G. M. Nkiet. Sélection des variables dans un modèle structurel de régression linéaire. *C. R. Acad. sci. paris I*, 333:1105–1110, 2001.
- [10] G. M. Nkiet. Direct variable selection for discrimination among several groups. *J. Multivariate Anal.*, 105:151–163, 2012.
- [11] J. Shao. Linear model selection by cross-validation. *J. Amer. Statist. Assoc.*, 88:486–494, 1993.
- [12] R. Shibata. Approximate efficiency of a selection procedure for the number of regression variables. *Biometrika*, 71:43–49, 1984.

- [13] M. L. Thomson. Selection of variables in multiple regression. Part I. A review and evaluation. *Internat. Statist. Rev.*, 46:1–19, 1978.
- [14] M. L. Thomson. Selection of variables in multiple regression. Part II. Chosen procedures, computations and examples. *Internat. Statist. Rev.*, 46:129–145, 1978.
- [15] P. Zhang. On the distributional properties of model selection criteria. *J. Amer. Statist. Assoc.*, 87:732–737, 1992.
- [16] P. Zhang. Model selection via multifold cross validation. *Ann. Statist.*, 21:299–313, 1992.
- [17] X. Zheng, W. Y. Loh. A consistent variable selection criterion for linear models with high-dimensional covariates. *Statistica Sinica*, 7:311–325, 1997.