

On compact bicrossed products

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Abstract

We study compact bicrossed product arising from a matched pair of a discrete group Γ and a compact group G . We exhibit an automatic regularity property of the matched pair and describe the representation theory and the fusion rules of such bicrossed product. We characterize the relative Kazhdan property (T) and the relative Haagerup property of the pair given by G and the bicrossed product in terms of the action of Γ on G . We also provide an explicit example of a non-trivial discrete quantum group with the Kazhdan property (T). Finally, we study all the properties mentioned above for the crossed product quantum group given by an action by quantum automorphisms of a discrete group on a compact quantum group, and also establish the permanence of rapid decay and weak amenability and some several explicit examples.

1 Introduction

In the eighties, Woronowicz [Wo87, Wo88, Wo95] introduced the notion of compact quantum groups and generalized the classical Peter-Weyl representation theory. However, the theory of quantum groups goes back to Kac [Ka63, Ka65] in the early sixties, and his notion of ring groups in modern terms are known as finite dimensional Kac algebras. In the fundamental work [Ka68] on extensions of finite groups, Kac introduced the notion of *matched pair* of finite groups and developed the bicrossed product construction giving the first examples of semisimple Hopf algebras that are neither commutative nor cocommutative. It was later generalized by Baaĵ and Skandalis [BS93] in the context of Kac algebras and then by Vaes and Vainerman [VV03] in the framework of locally compact² (l.c. in the sequel) quantum groups; the latter was introduced by Kustermans and Vaes in [KV00]. In the classical case, i.e., in the ambience of groups, Baaĵ and Skandalis concentrated only on the case of *regular* matched pairs of l.c. groups. In [VV03], the authors extended the study to *semi-regular* matched pair of l.c. groups. The case of a general matched pair of locally compact groups was settled by Baaĵ, Skandalis and Vaes in [BSV03].

As a standing assumption, all throughout the paper, all Hilbert spaces and all C*-algebras are separable, all von Neumann algebras have separable preduals and all discrete groups are countable.

The theory of quantum groups is fathomless. In order to have a deeper insight, it is necessary to generate and study many explicit examples. The bicrossed product construction is a way to get abundant non-trivial examples of quantum groups which are very far from groups [Fi07]. A compact bicrossed product is one, in which the resulting quantum group is compact. Without being bogged technically, the bicrossed product construction in the classical case associates a l.c. quantum group to a matched pair of l.c. groups (G_1, G_2) . The associated l.c. quantum group (in the bicrossed product construction) has a Haar state, i.e., is a compact quantum group, if and only if G_1 is discrete and G_2 is compact [VV03]. In this paper, such a pair will be called as a *compact* matched pair. Moving to the quantum case, one can introduce the notion of matched pair of l.c. quantum groups, and perform an analogous bicrossed product construction to manufacture a l.c. quantum group that generalizes the classical bicrossed product construction. This construction is technical and we refer the interested reader to [VV03] for details. It is to be noted that, in the same vein, the crossed product of a compact quantum group G by a discrete group Γ acting on G by quantum automorphisms (see Section 2.2 for precise definition), as considered by Wang [Wa95b], is subsumed in the quantum bicrossed product construction and hence is a simple case of compact bicrossed product. Needless to say, that the aforesaid class of bicrossed products in the quantum setup, does not exhaust the entire class of compact bicrossed products. We point out though, that the ‘compactness’ of the matched pair in the classical case (for groups) alleviates technical obstacles which we highlight in this paper.

This paper investigates compact bicrossed products in both classical and quantum setting and their approximation and permanence properties, namely, (co)-amenability, K -amenability, (relative) (co)-Haagerup property, (relative) (co)-property (T) and rapid decay, which enables one to manufacture examples. This

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²All l.c. spaces are assumed to be Hausdorff.

paper has two major parts: one dealing with the classical case and one dealing with the quantum case. In the quantum case, we only concentrate on compact crossed products.

We provide a totally self contained and direct approach dedicated towards the construction of a compact bicrossed product arising from matched pair of groups (Γ, G) , where Γ is discrete and G is compact. An advantage with this construction is that it avoids technical intricacies that are indispensable when dealing with l.c. (quantum) groups. In the process, we observe that the compactness of G constrains the matched pair to automatically satisfy a regularity property, notably, G, Γ are subgroups of a l.c. group H such that $\Gamma G = H$ and the canonical action of either group on its complementary pair is continuous. Moreover, the action of Γ on G happens via measure preserving homeomorphisms. This regularity is not automatic in the l.c. setting and one has to compensate with ‘almost everywhere statements’. The aforesaid regularity galvanizes one to directly perform the bicrossed construction; the bicrossed product is of course known to be a Kac algebra. The continuous action of the group G on the countable set Γ yield magic unitaries, which along with the irreducible unitary representations of G and the action of Γ on G by measure preserving homeomorphisms assist us in gazetting the entire representation theory and fusion rules of the bicrossed product in an elegant fashion (Theorems 3.4 and 3.8). Some easy consequences on co-amenable (which is known from [DQV02]), K -amenability, Haagerup property is also presented.

We characterize the relative co-property (T) for the pair (G, \mathbb{G}) , where \mathbb{G} is the bicrossed product of the compact matched pair (Γ, G) in terms of the action of Γ on G . More precisely, the relative co-property (T) for the pair (G, \mathbb{G}) amounts to the existence of an asymptotically Γ -invariant sequence of Borel probability measures on G each of which assign zero weight to the identity e of the group but yet converge to the Dirac measure δ_e in the weak* topology (Theorem 4.2). In the event of existence of such a sequence measures on G , by a standard result in measure theory (due to Parthasarathy and Steernman [PS85]), the measures in the sequence versus their push forwards with respect to the group action implemented by Γ have large common support. Thus, along the way, we show that such a sequence of measures can be replaced by a one for which the Γ -action on G is nonsingular. This result generalizes the classical characterization of the relative property (T) for the pair $(H, \Gamma \rtimes H)$ (originally defined in [Ma82]), where Γ is a discrete group acting on the discrete abelian group H [CT11].

We show that if the dual of the bicrossed product has the property (T) , then Γ necessarily has Kazhdan property (T) and the converse holds when G is finite. This allows us to give concrete examples of non-commutative and non-cocommutative infinite dimensional compact quantum group whose dual has property (T) . We mention that, as far as we are aware, this is the first explicit non-trivial example of such compact quantum groups, since the twisting example of [Fi10] is based on [Fi10, Theorem 3] and the proof of this theorem breaks down.

Proceeding further, we characterize the relative co-Haagerup property for the pair (G, \mathbb{G}) , where \mathbb{G} is the bicrossed product of the compact matched pair (Γ, G) in terms of the action of Γ on G . Like before, we prove that this property is equivalent to the existence of an approximately Γ -invariant sequence of probability measures on G which converge in weak* topology to δ_e and whose Fourier transform (regarded as an element of the multiplier algebra of $C_r^*(G)$) fall in $C_r^*(G)$ (Theorem 5.3), and, like before, we show that measures can be chosen such that the action of Γ on G is nonsingular. Again, our results generalize the classical characterization of the relative Haagerup property for the pair $(H, \Gamma \rtimes H)$, where Γ is a discrete group acting on the discrete abelian group H [CT11].

In the quantum setting, a very simple and easy case of a matched pair of a classical discrete group with a compact quantum group is the pair arising in a crossed product in which the discrete group acts on the quantum group by quantum automorphisms [Wa95b]. We provide a self contained and very short approach to this construction and study all the properties mentioned above for the associated crossed product quantum group. Let (Γ, G) be a matched pair of a compact quantum group G with a discrete group Γ appearing as a crossed product pair by quantum automorphisms. Let \mathbb{G} denote the crossed product. Our characterization of the relative co-property (T) for the pair (G, \mathbb{G}) is analogous to the classical case: the approximating measures and δ_e in the characterization of the classical case are replaced in the quantum setting respectively by states on $C_m(G)$ and the counit of G . This proof is technically more involved than the classical case (Theorem 6.11). We also prove that property (T) of $\widehat{\mathbb{G}}$ forces Γ to be a property (T) group and if both G and Γ has property (T) , then so does $\widehat{\mathbb{G}}$. Analogous statements (in the quantum setting) hold for the relative co-Haagerup property of the pair (G, \mathbb{G}) as well, but in this case the approximating states in $C_m(G)$ are required

to be such that their Fourier transform is in $c_0(\widehat{G})$ (Theorem 6.15). Using dilated Chebychev polynomials of second kind, we provide examples of matched pairs of compact quantum groups and discrete groups with relative co-Haagerup property. Moreover, we generalize a result of Jolissaint regarding Haagerup property to the setup of non tracial von Neumann algebras [Jo07], which by itself is a result of independent interest. For a compact, state preserving action of a discrete group with Haagerup property on a von Neumann algebra with the same property, the crossed product has Haagerup property (Theorem 6.17). In the same vein, we show that if \widehat{G} and Γ both have Haagerup property and the action of Γ on G is compact, then \mathbb{G} has Haagerup property.

In the quantum, setting we study weak amenability of $\widehat{\mathbb{G}}$. In [KR99], it was proved that when G is Kac, then $\Lambda_{cb}(\widehat{G}) = \Lambda_{cb}(C(G)) = \Lambda_{cb}(L^\infty(G))$. Likewise, in our setup we estimate (in Theorem 6.5) the Cowling-Haagerup constants as: $\max(\Lambda_{cb}(\Gamma), \Lambda_{cb}(C(G))) \leq \Lambda_{cb}(C(\mathbb{G}))$. Moreover, if the action is compact, then $\Lambda_{cb}(C(\mathbb{G})) \leq \Lambda_{cb}(\Gamma)\Lambda_{cb}(\widehat{G})$. We also present some explicit non-trivial examples of \mathbb{G} such that $\Lambda_{cb}(\mathbb{G}) = 1$.

Rapidly decreasing functions on group C*-algebras were first studied by Jolissaint in [Jo90]. Generalizing this notion, rapid decay (RD in the sequel) for quantum groups was studied in [Ve07]. Following [Ve07], we show the permanence of RD in the setup of crossed products. To be precise, we show that in the event G is a Kac algebra, Γ acts on G via quantum automorphisms, there is a central length function on \widehat{G} which is invariant with respect to the action so that the pair (\widehat{G}, l) has RD, and Γ has RD, then $\widehat{\mathbb{G}}$ has RD (Theorem 6.9).

The lay out of the paper is as follows. In Section 2, we jot down all the notations, recall preliminary facts and basics of compact quantum groups that is used all throughout this paper. Section 3 concentrates on the bicrossed product construction from compact matched pairs of groups and studies the representation theory of classical compact bicrossed products. In Section 4 and Section 5, we respectively study (relative) Kazhdan property and (relative) Haagerup property for classical matched pairs. Section 6 is divided into many subsections, and in this section, we study the properties of crossed products of a compact quantum groups by discrete groups.

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2 Preliminaries

Notations. Throughout the paper, $\mathcal{B}(H)$ denotes the von Neumann algebra of all bounded operators on the Hilbert space H . The inner products of Hilbert spaces are assumed to be linear in the first variable. The same symbol \otimes will denote the tensor product of Hilbert spaces, the minimal tensor product of C*-algebras and as well as the tensor product of von Neumann algebras.

2.1 Compact group action on countable set

We first record some facts regarding actions of compact groups on countable sets. This will be necessary in studying the bicrossed product construction for compact matched pairs of groups.

Let X be an infinite countable set and let $S(X)$ be the group of bijections of X . It is a Polish group for the topology of pointwise convergence and the topology is generated by the sets $S_{x,y} = \{\alpha \in S(X) : \alpha(x) = y\}$ for $x, y \in X$. Since $S_{x,y}^c = \cup_{z \in X \setminus \{y\}} S_{x,z}$, these sets are clopen in $S(X)$. Moreover, for any compact subset $K \subset S(X)$ and for any $x \in X$, the orbit $K \cdot x \subset X$ is finite. Indeed, from the open cover $K \subset \cup_{y \in X} S_{x,y}$ we find $y_1, \dots, y_n \in X$ such that $K \subset \cup_{i=1}^n S_{x,y_i}$, which implies that $K \cdot x \subset \{y_1, \dots, y_n\}$.

Let $\beta : G \rightarrow S(X)$ be a continuous right action of G on X . To simplify the notations, we write $x \cdot g = \beta_g(x)$ for $g \in G$ and $x \in X$.

Observe that, since β is continuous and G is compact, every β -orbit in X is finite. Fix $r, s \in X$ and denote by $A_{r,s}$ the set

$$A_{r,s} = \{g \in G : r \cdot g = s\} = \beta^{-1}(S_{r,s}).$$

Note that, since β is continuous, $A_{r,s}$ is open and closed in G for all $r, s \in X$. Hence, $1_{A_{r,s}} \in C(G)$. Moreover, $1_{A_{r,s}} \neq 0$ if and only if r and s are in the same orbit and we have the following relations:

1. $1_{A_{s,r}}1_{A_{t,r}} = \delta_{t,s}1_{A_{s,r}}$ for all $r, s, t \in X$.
2. $1_{A_{s,r}}1_{A_{s,t}} = \delta_{r,t}1_{A_{s,r}}$ for all $r, s, t \in X$.
3. $\sum_{s \in \Gamma} 1_{A_{r,s}} = \sum_{s \in r \cdot G} 1_{A_{r,s}} = 1$ for all $r \in X$.
4. $\sum_{r \in \Gamma} 1_{A_{r,s}} = \sum_{r \in s \cdot G} 1_{A_{r,s}} = 1$ for all $r \in X$.
5. If $r \cdot G = s \cdot G$, then $\Delta_G(1_{A_{s,r}}) = \sum_{t \in s \cdot G} 1_{A_{s,t}} \otimes 1_{A_{t,r}}$.

Where Δ_G is the usual comultiplication on $C(G)$. In other words, for every orbit $x \cdot G$, the matrix $(1_{A_{r,s}})_{r,s \in x \cdot G} \in M_{|x \cdot G|}(\mathbb{C}) \otimes C(G)$ is a magic unitary and a unitary representation of G .

2.2 Compact and discrete quantum groups

In this section, we recall well known and basic facts about compact quantum groups that will be indispensable. For the general theory of compact quantum groups, we refer to [Wo87, Wo95].

For a compact quantum group G with comultiplication Δ (or Δ_G), we denote by h (or h_G) the Haar state on G and by $C(G)$ (resp. $L^\infty(G)$) the C^* -algebra (resp. von Neumann algebra) generated by the GNS construction of h . We denote by $\text{Irr}(G)$ the equivalence classes of irreducible representations of G . For $x \in \text{Irr}(G)$, we choose a representative $u^x \in \mathcal{B}(H_x) \otimes C(G)$ and we denote by $\text{Pol}(G)$ the linear span of the coefficients of $\{u^x : x \in \text{Irr}(G)\}$, which is a unital dense $*$ -subalgebra of $C(G)$. We also denote by $C_m(G)$ the enveloping C^* -algebra of $\text{Pol}(G)$, by λ (or λ_G) the canonical surjection from $C_m(G)$ to $C(G)$ and by ε (or ε_G) the counit on $C_m(G)$.

We will denote by $\text{Aut}(G)$ the set of quantum automorphisms of G . More precisely,

$$\text{Aut}(G) = \{\alpha \in \text{Aut}(C_m(G)) : \Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta\}.$$

Hence, $\text{Aut}(G)$ as a closed subgroup of the Polish³ group $\text{Aut}(C_m(G))$, is itself a Polish group.

Observe that each $\alpha \in \text{Aut}(G)$ induces a bijection $\alpha \in S(\text{Irr}(G))$. Indeed, for $x \in \text{Irr}(G)$, $\alpha(x)$ is the equivalence class of the irreducible unitary representation $(\text{id} \otimes \alpha)(u^x)$. By construction, the map $\text{Aut}(G) \rightarrow S(\text{Irr}(G))$ is a group homomorphism.

We will need the following result which is certainly well known to specialists. We include the proof since we could not find any reference in the literature.

Proposition 2.1. *The map $\text{Aut}(G) \rightarrow S(\text{Irr}(G))$ is continuous.*

Proof. We shall need the following well known lemma which is of independent interest. We include a proof for the convenience of the reader.

Lemma 2.2. *Let $u, v \in \mathcal{B}(H) \otimes C_m(G)$ be two unitary representations of G on the same finite dimensional Hilbert space H . If $\|u - v\| < 1$, then u and v are equivalent.*

Proof. Define $x = (\text{id} \otimes h)(v^*u) \in \mathcal{B}(H)$. Since u and v are unitary representations and by the property of the Haar state we get $(x \otimes 1)u = v(x \otimes 1)$. We have $u^*(x^*x \otimes 1)u = x^*x \otimes 1$. Hence, $u^*|x| \otimes 1u = |x| \otimes 1$. Now observe that $\|1 - x\| = \|(\text{id} \otimes h)(1 - v^*u)\| \leq \|1 - v^*u\| = \|v - u\| < 1$. Hence, x is invertible and, in the polar decomposition $x = w|x|$, the element w is unitary. Consequently, $v^*(w|x| \otimes 1)u = v^*(w \otimes 1)u(|x| \otimes 1) = (w \otimes 1)(|x| \otimes 1)$. By uniqueness of the polar decomposition of $x \otimes 1$, we deduce that $v^*(w \otimes 1)u = w \otimes 1$. Hence, u and v are equivalent. \square

³for the topology of pointwise norm convergence

We can now prove the proposition. Let $(\alpha_n)_n$ be a sequence in $\text{Aut}(G)$ which converges to $\alpha \in \text{Aut}(G)$. Let $F \subset \text{Irr}(G)$ be a finite subset and let $N \in \mathbb{N}$ be such that, for all $n \geq N$ we have

$$\|(\text{id} \otimes \alpha_n)(u^x) - (\text{id} \otimes \alpha)(u^x)\| < \frac{1}{2} \quad \text{for all } x \in F.$$

It follows from Lemma 2.2 that $(\text{id} \otimes \alpha_n)(u^x)$ and $(\text{id} \otimes \alpha)(u^x)$ are equivalent for all $n \geq N$ and all $x \in F$. This means that $\alpha_n(x) = \alpha(x)$ for all $x \in F$ whenever $n \geq N$. This establishes the continuity. \square

Remark 2.3. We can also define $\text{Aut}_r(G) = \{\alpha \in \text{Aut}(C(G)) : \Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta\}$ which is again a Polish group as a closed subgroup of the Polish group $\text{Aut}(C(G))$. Since any $\alpha \in \text{Aut}(G)$ preserves the Haar measure, it defines a unique element in $\text{Aut}_r(G)$. Hence we have a map $\text{Aut}(G) \rightarrow \text{Aut}_r(G)$ which is actually bijective. The inverse bijection is constructed in the following way. Since any $\alpha \in \text{Aut}_r(G)$ restrict to an automorphism of $\text{Pol}(G)$ it extends uniquely, by the universal property, to an automorphism in $\text{Aut}(G)$. It is easy to check that the map $\text{Aut}(G) \rightarrow \text{Aut}_r(G)$ is continuous.

Also, since any Δ -intertwining automorphism of $C(G)$ has a unique normal extension to $L^\infty(G)$, this gives a map $\text{Aut}_r(G) \rightarrow \text{Aut}_\infty(G)$, where $\text{Aut}_\infty(G) = \{\alpha \in \text{Aut}(L^\infty(G)) : \Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta\}$. As before, this map is a bijection and is continuous (the topology on $\text{Aut}(L^\infty(G))$ is the pointwise $\|\cdot\|_{2,h}$ convergence).

For a discrete group Γ and a compact quantum group G , a group homomorphism $\alpha : \Gamma \rightarrow \text{Aut}(G)$ is called *an action by quantum automorphisms* and is denoted by $\alpha : \Gamma \curvearrowright G$, see [Pa13, Section 6]. We call such an action *compact* if the closure of the image of Γ in $\text{Aut}(G)$ is compact. By remark 2.3, it implies that the actions of Γ on the C^* -algebra $C(G)$ (and $C_m(G)$) and the von Neumann algebra $L^\infty(G)$ are compact. By Proposition 2.1, it implies that the action of Γ on $\text{Irr}(G)$ has all orbits finite. It is shown in [MP15] that the converse is actually true: $\Gamma \curvearrowright G$ is compact if and only if the induced action of Γ on $\text{Irr}(G)$ has all orbits finite.

The associated operator algebras of the discrete dual \widehat{G} of G are denoted by

$$\ell^\infty(\widehat{G}) = \bigoplus_{x \in \text{Irr}(G)}^{\ell^\infty} \mathcal{B}(H_x) \quad \text{and} \quad c_0(\widehat{G}) = \bigoplus_{x \in \text{Irr}(G)}^{c_0} \mathcal{B}(H_x).$$

We denote by $V = \bigoplus_{x \in \text{Irr}(G)} u^x \in M(c_0(\widehat{G}) \otimes C_m(G))$ to be the maximal multiplicative unitary. Let p_x be the minimal central projection of $\ell^\infty(\widehat{G})$ corresponding to the block $\mathcal{B}(H_x)$. We say that $a \in \ell^\infty(\widehat{G})$ has *finite support* if $ap_x = 0$ for all but finitely many $x \in \text{Irr}(G)$. The set of finitely supported elements of $\ell^\infty(\widehat{G})$ is dense in $c_c(\widehat{G})$ and the latter is equal to the algebraic direct sum $c_c(\widehat{G}) = \bigoplus_{x \in \text{Irr}(G)}^{\text{alg}} \mathcal{B}(H_x)$.

When G is Kac, the Haar weight on \widehat{G} is the n.s.f. weight on $\ell^\infty(\widehat{G})$ defined by $h_{\widehat{G}}(a) = \sum_{x \in \text{Irr}(G)} \dim(x) \text{Tr}_x(ap_x)$, whenever this formula makes sense, where Tr_x is the unique trace on $\mathcal{B}(H_x)$ such that $\text{Tr}_x(1) = \dim(x)$.

3 Representation theory of bicrossed product

This section has two parts. In the first part, we discuss on matched pair of groups of which one is compact. We describe regularity of such matched pairs in terms of continuity of actions. In the second part, we study bicrossed products of compact matched pair of groups, and study the representation theory of classical bicrossed products.

3.1 Matched pairs

Definition 3.1 ([BSV03]). We say that a pair of l.c. groups (G_1, G_2) is *matched* if both G_1, G_2 are closed subgroups of a l.c. group H satisfying $G_1 \cap G_2 = \{e\}$ and such that the complement of $G_1 G_2$ in H has Haar measure zero.

From a matched pair (G_1, G_2) , one can construct a l.c. quantum group called the bicrossed product and it follows from [VV03] that the bicrossed product is compact if and only if G_1 is discrete and G_2 is compact. In the next proposition, we show some regularity properties of matched pairs (G_1, G_2) with G_2 compact.

Proposition 3.2. *Let (G_1, G_2) be a matched pair and suppose that G_2 is compact. Then $G_1G_2 = H$, and, for all $\gamma \in G_1$ and all $g \in G_2$, there exist unique $\alpha_\gamma(g) \in G_2$ and $\beta_g(\gamma) \in G_1$ such that $\gamma g = \alpha_\gamma(g)\beta_g(\gamma)$. Moreover,*

1. For $g, h \in G_2$ and $r, s \in G_1$, we have

$$\alpha_r(gh) = \alpha_r(g)\alpha_{\beta_g(r)}(h), \quad \beta_g(rs) = \beta_{\alpha_s(g)}(r)\beta_g(s) \quad \text{and} \quad \alpha_r(e) = e, \quad \beta_g(e) = e.$$

2. α is a continuous left action of G_1 on the topological space G_2 . Moreover, the Haar measure on G_2 is α -invariant whenever G_1 is discrete.
3. β is a continuous right action of G_2 on the topological space G_1 .

Proof. First observe that, since G_1 is compact, H is Hausdorff and G_2 is closed, the set G_1G_2 is closed. Hence, the complement of G_1G_2 is open and has Haar measure zero. It follows that $G_1G_2 = H = H^{-1} = G_2^{-1}G_1^{-1} = G_2G_1$. Since $G_1 \cap G_2 = \{e\}$, the existence and uniqueness of $\alpha_\gamma(g)$ and $\beta_g(\gamma)$ for all $\gamma \in G_1$ and $g \in G_2$ is obvious. Then, the relations in (1) and the facts that α (resp. β) is a left (resp. right) action as in the statement easily follow from the aforementioned uniqueness.

Let us now check the continuity of these actions. Since the subgroup G_1 is closed in the l.c. group H , so H/G_1 is a l.c. Hausdorff space with the quotient topology and the projection map $\pi : H \rightarrow H/G_1$ is continuous. Hence, $\pi|_{G_2} : G_2 \rightarrow H/G_1$ is continuous and bijective since $G_1 \cap G_2 = \{e\}$ and $G_1G_2 = H$. Since G_2 is compact, $\pi|_{G_2}$ is a homeomorphism. Let $\rho : H/G_1 \rightarrow G_2$ be the inverse of $\pi|_{G_2}$ and observe that the map $\alpha : G_1 \times G_2 \rightarrow G_2$ prescribed by $(\gamma, g) \mapsto \alpha_\gamma(g)$ is given by $\alpha = \rho \circ \pi \circ \psi$, where $\psi : G_1 \times G_2 \rightarrow H$ is the continuous map given by $\psi(\gamma, g) = \gamma g$. Consequently, the action α is continuous. Now, since for all $\gamma \in G_1$ and $g \in G_2$, we have $\beta_g(\gamma) = \alpha_\gamma(g)^{-1}\gamma g$, we deduce the continuity of $\beta : G_1 \times G_2 \rightarrow G_1$ prescribed by $(\gamma, g) \mapsto \beta_g(\gamma)$ from the continuity of α and the continuity of product and inverse operations in H .

Now suppose that G_1 is moreover discrete. Then G_1 is a co-compact lattice in H and it follows from the general theory that H is unimodular and there exists a unique H -invariant Borel probability measure ν on H/G_1 . Consider the homeomorphism $\pi|_{G_2} : G_2 \rightarrow H/G_1$ and the Borel probability measure $\mu = (\pi|_{G_2})_*(\nu)$ on G_2 . Since, for all $\gamma \in G_1$, the map $\pi|_{G_2}$ intertwines the homeomorphism α_γ of G_2 with the left translation by γ on H/G_1 and since ν is invariant under the left translation by γ , it follows that μ is invariant under α_γ . Also, $\pi|_{G_2}$ intertwines, for all $h \in G_2$, the left translation by h on G_2 with the left translation by h on H/G_1 . Hence, μ is invariant under the left translation by $h \in G_2$ for all $h \in G_2$. It follows that μ is the Haar measure. \square

Example 3.3. Let α be an action of a discrete group Γ on the compact group G by group homeomorphisms. Consider the l.c. group $H = \Gamma \rtimes G$. Then, (Γ, G) form a matched pair in the obvious way. For this matched pair the action β is trivial.

3.2 Representation theory

We first construct the bicrossed product and then study its representation theory. Along the way we prove some straight forward consequences e.g., co-amenability, K -amenability and Haagerup property of the bicrossed product.

Let (Γ, G) be a matched pair of a discrete countable group Γ and a compact group G . Associated to the continuous action β of the compact group G on the infinite countable set Γ , we have for every $\gamma \cdot G \in \Gamma/G$, a magic unitary $v^{\gamma \cdot G} = (v_{rs})_{r,s \in \gamma \cdot G} \in M_{|\gamma \cdot G|}(\mathbb{C}) \otimes C(G)$, where $v_{rs} = 1_{A_{r,s}}$ and $A_{r,s} = \{g \in G : \beta_g(r) = s\}$.

We define the C^* -algebra $A_m = \Gamma_{\alpha, \beta} \rtimes C(G)$ to be the full crossed product and the C^* -algebra $A = \Gamma_\alpha \rtimes C(G)$ to be the reduced crossed product. We still denote by α the canonical injective maps from $C(G)$ to A_m and

from $C(G)$ to A . We also denote by u_γ , $\gamma \in \Gamma$, the canonical unitaries viewed in either A_m or A . Observe that A_m is the enveloping C^* -algebra of the unital $*$ -algebra

$$\mathcal{A} = \text{Span}\{u_\gamma \alpha(u_{ij}^x) : \gamma \in \Gamma, x \in \text{Irr}(G), 1 \leq i, j \leq \dim(x)\}.$$

Let $\lambda : A_m \rightarrow A$ be the canonical surjection. Since the action α on (G, μ) is probability measure preserving, so there exists a unique faithful trace τ on A such that

$$\tau(u_\gamma \alpha(F)) = \delta_{\gamma, e} \int F d\mu, \quad \gamma \in \Gamma.$$

Theorem 3.4. *There exists a unique unital $*$ -homomorphism $\Delta_m : A_m \rightarrow A_m \otimes A_m$ such that*

$$\Delta_m \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_G \quad \text{and} \quad \Delta_m(u_\gamma) = \sum_{r \in \gamma \cdot G} u_\gamma \alpha(v_{\gamma, r}) \otimes u_r, \quad \forall \gamma \in \Gamma.$$

Moreover, $\mathbb{G} = (A_m, \Delta_m)$ is a compact quantum group and we have:

1. The Haar state of \mathbb{G} is $h = \tau \circ \lambda$, hence \mathbb{G} is Kac.
2. The set of unitary representations of \mathbb{G} of the form $V^{\gamma \cdot G} \otimes v^x$ for some $\gamma \cdot G \in \Gamma/G$ and $x \in \text{Irr}(G)$, where $V^{\gamma \cdot G} = \sum_{r, s \in \gamma \cdot G} e_{r, s} \otimes u_r \alpha(v_{r, s}) \in M_{|\gamma \cdot G|}(\mathbb{C}) \otimes \mathcal{A}$ and $v^x = (\text{id} \otimes \alpha)(u^x)$ is a complete set of irreducible unitary representations of \mathbb{G} .
3. We have $C_m(\mathbb{G}) = A_m$, $C(\mathbb{G}) = A$, $\text{Pol}(G) = \mathcal{A}$, λ is the canonical surjection and $L^\infty(\mathbb{G})$ is the von Neumann algebraic crossed product.

Proof. The uniqueness of Δ_m is obvious. To show the existence, it suffices to check that Δ_m satisfies the universal property of A_m .

Let us check that $\gamma \mapsto \Delta_m(u_\gamma)$ is a unitary representation of Γ .

Let $\gamma \in \Gamma$. We first check that $\Delta_m(u_\gamma)$ is unitary. Observe that, for all $g \in G$, we have

$$1 = \beta_g(\gamma^{-1}\gamma) = \beta_{\alpha_\gamma(g)}(\gamma^{-1})\beta_g(\gamma).$$

Hence, $(\beta_g(\gamma))^{-1} = \beta_{\alpha_\gamma(g)}(\gamma^{-1})$. From this relation it is easy to check that $\Gamma^{-1} \cdot G = \{r^{-1} : r \in \gamma \cdot G\}$ and $\alpha_\gamma(v_{\gamma, r^{-1}}) = v_{\gamma^{-1}, r}$ for all $r \in \Gamma$. It follows that

$$\Delta_m(u_\gamma)^* = \sum_{r \in \gamma \cdot G} \alpha(v_{\gamma, r}) u_{\gamma^{-1}} \otimes u_{r^{-1}} = \sum_{r \in \gamma \cdot G} u_{\gamma^{-1}} \alpha(\alpha_\gamma(v_{\gamma, r})) \otimes u_{r^{-1}} = \sum_{r \in \gamma^{-1} \cdot G} u_{\gamma^{-1}} \alpha(v_{\gamma^{-1}, r}) \otimes u_r = \Delta_m(u_{\gamma^{-1}}).$$

Let $\gamma_1, \gamma_2 \in \Gamma$. We have

$$\Delta_m(u_{\gamma_1}) \Delta_m(u_{\gamma_2}) = \sum_{r \in \gamma_1 \cdot G, s \in \gamma_2 \cdot G} u_{\gamma_1} \alpha(v_{\gamma_1, r}) u_{\gamma_2} \alpha(v_{\gamma_2, s}) \otimes u_{rs} = \sum_{r, s} u_{\gamma_1 \gamma_2} \alpha \left(\alpha_{\gamma_2^{-1}}(v_{\gamma_1, r}) v_{\gamma_2, s} \right) \otimes u_{rs}$$

Observe that $\alpha_{\gamma_2^{-1}}(v_{\gamma_1, r}) v_{\gamma_2, s} = 1_{B_{\gamma_1, \gamma_2, r, s}}$, where

$$B_{\gamma_1, \gamma_2, r, s} = \{g \in G : \beta_{\alpha_{\gamma_2}(g)}(\gamma_1) = r \text{ and } \beta_g(\gamma_2) = s\} \subset A_{\gamma_1 \gamma_2, rs} = \{g \in G : \beta_g(\gamma_1 \gamma_2) = rs\},$$

since $\beta_{\alpha_{\gamma_2}(g)}(\gamma_1) \beta_g(\gamma_2) = \beta_g(\gamma_1 \gamma_2)$. In particular, $B_{\gamma_1, \gamma_2, r, s} = \emptyset$ whenever $rs \notin \gamma_1 \gamma_2 \cdot G$; hence

$$\Delta_m(u_{\gamma_1}) \Delta_m(u_{\gamma_2}) = \sum_{t \in \gamma_1 \gamma_2 \cdot G, r \in \gamma_1 \cdot G} u_{\gamma_1 \gamma_2} \alpha \left(1_{B_{\gamma_1, \gamma_2, r, r^{-1}t}} \right) \otimes u_t = \sum_{t \in \gamma_1 \gamma_2 \cdot G} u_{\gamma_1 \gamma_2} \alpha(F_t) \otimes u_t,$$

where $F_t = \sum_r 1_{B_{\gamma_1, \gamma_2, r, r^{-1}t}} = 1_{\sqcup_r B_{\gamma_1, \gamma_2, r, r^{-1}t}} = 1_{A_{\gamma_1 \gamma_2, t}}$, where $A_{\gamma_1 \gamma_2, t} = \{g \in G : \gamma_1 \gamma_2 \cdot g = t\}$. Consequently, $1_{A_{\gamma_1 \gamma_2, t}} = v_{\gamma_1 \gamma_2, t}$ and $\Delta_m(u_{\gamma_1}) \Delta_m(u_{\gamma_2}) = \Delta_m(u_{\gamma_1 \gamma_2})$.

Since $\Delta_m(u_e) = 1$ it follows that $\gamma \mapsto \Delta_m(u_\gamma)$ is a unitary representation of Γ .

Let us now check that the relations of the crossed product are satisfied. For $\gamma \in \Gamma$ and $F \in \text{Pol}(G)$ we have:

$$\begin{aligned}
\Delta_m(u_\gamma)\Delta_m(\alpha(F))\Delta_m(u_\gamma^*) &= \sum_{r,s} (u_\gamma \otimes u_r)(\alpha \otimes \alpha) ((v_{\gamma,r} \otimes 1)\Delta_G(F)) (u_{\gamma^{-1}}\alpha(v_{\gamma^{-1},s}) \otimes u_s) \\
&= \sum_{r,s} (u_\gamma \otimes u_r)(\alpha \otimes \alpha) ((v_{\gamma,r}\alpha_{\gamma^{-1}}(v_{\gamma^{-1},s}) \otimes 1)\Delta_G(F)) (u_{\gamma^{-1}} \otimes u_s) \\
&= \sum_{r,s} (\alpha \otimes \alpha) ((\alpha_\gamma(v_{\gamma,r})v_{\gamma^{-1},s} \otimes 1)(\alpha_\gamma \otimes \alpha_r)(\Delta_G(F))) (1 \otimes u_{rs}) \\
&= \sum_{r,t} (\alpha \otimes \alpha) ((\alpha_\gamma(v_{\gamma,r})v_{\gamma^{-1},r^{-1}t} \otimes 1)(\alpha_\gamma \otimes \alpha_r)(\Delta_G(F))) (1 \otimes u_t) \\
&= \sum_t (\alpha \otimes \alpha)(H_t)(1 \otimes u_t),
\end{aligned}$$

where $H_t = \sum_r (\alpha_\gamma(v_{\gamma,r})v_{\gamma^{-1},r^{-1}t} \otimes 1)(\alpha_\gamma \otimes \alpha_r)(\Delta_G(F))$. Observe that $\alpha_\gamma(v_{\gamma,r})v_{\gamma^{-1},r^{-1}t} = 1_{B_{\gamma,r,t}}$, where

$$B_{\gamma,r,t} = \{g \in G : \beta_{\alpha_{\gamma^{-1}}(g)}(\gamma) = r \text{ and } \beta_g(\gamma^{-1}) = r^{-1}t\}.$$

Since $\beta_{\alpha_{\gamma^{-1}}(g)}(\gamma)\beta_g(\gamma^{-1}) = \beta_g(\gamma\gamma^{-1}) = \beta_g(e) = e$, we deduce that $B_{\gamma,r,t} = \emptyset$ whenever $t \neq e$, and $\bigsqcup_{r \in \gamma \cdot G} B_{\gamma,r,e} = G$. Hence, $H_t = 0$ for $t \neq e$, and for $g \in B_{\gamma,r,e}$ and $h \in G$,

$$H_e(g, h) = F(\alpha_{\gamma^{-1}}(g)\alpha_{r^{-1}}(h)) = F(\alpha_{\gamma^{-1}}(g)\alpha_{\beta_g(\gamma^{-1})}(h)) = F(\alpha_{\gamma^{-1}}(gh)).$$

It follows that $H_e = \Delta_G(\alpha_\gamma(F))$. Since $\Delta_m(u_\gamma)\Delta_m(\alpha(F))\Delta_m(u_\gamma^*) = (\alpha \otimes \alpha)(H_e)$. This completes the proof of the existence of Δ_m .

It is clear that v^x is unitary and since $(\alpha \otimes \alpha)\Delta_G = \Delta_m \circ \alpha$, we have $\Delta_m(v_{ij}^x) = \sum_k v_{ik}^x \otimes v_{kj}^x$. Observe that $V^{\gamma \cdot G} = D_\gamma(\text{id} \otimes \alpha)(v^{\gamma \cdot G}) \in M_{|\gamma \cdot G|}(\mathbb{C}) \otimes \mathcal{A}$, where D_γ is the diagonal matrix with entries u_r , $r \in \gamma \cdot G$. Hence, $V^{\gamma \cdot G}$ is unitary. Moreover,

$$\begin{aligned}
\Delta_m(V_{rs}^{\gamma \cdot G}) &= \Delta_m(u_r \alpha(v_{rs})) = \sum_{t \in r \cdot G = \gamma \cdot G} (u_r \alpha(v_{rt}) \otimes u_t)(\alpha \otimes \alpha)(\Delta_G(v_{rs})) = \sum_{t,z \in \gamma \cdot G} u_r \alpha(v_{rt}v_{rz}) \otimes u_t \alpha(v_{zs}) \\
&= \sum_{t \in \gamma \cdot G} u_r \alpha(v_{rt}) \otimes u_t \alpha(v_{ts}) = \sum_{t \in \gamma \cdot G} V_{rt}^{\gamma \cdot G} \otimes V_{ts}^{\gamma \cdot G}.
\end{aligned}$$

It follows from [Wa95a, Definition 2.1] that \mathbb{G} is a compact quantum group and $V^{\gamma \cdot G}$, v^x are unitary representations of \mathbb{G} for all $\gamma \in \Gamma/G$ and $x \in \text{Irr}(G)$.

(1) Since $\sum_s V_{rs}^{\gamma \cdot G} = u_r$, the linear span of the coefficients of the representations $V^{\gamma \cdot G} \otimes v^x$ is equal to \mathcal{A} . Hence, it suffices to check the invariance of h on the coefficients of $V^{\gamma \cdot G} \otimes v^x$. We have

$$h(V_{rs}^{\gamma \cdot G} v_{ij}^x) = h(u_r \alpha(v_{rs} v_{ij}^x)) = \delta_{r,e} \int_G v_{es} v_{ij}^x d\mu = \delta_{r,e} \delta_{s,e} \int_G v_{ij}^x d\mu = \delta_{r,e} \delta_{s,e} \delta_{x,1},$$

since $v_{es} = \delta_{s,e}1$ and v^x is irreducible. Hence, if $x \neq 1$, we have

$$(\text{id} \otimes h)\Delta_m(V_{rs}^{\gamma \cdot G} v_{ij}^x) = \sum_{t,k} V_{rt}^{\gamma \cdot G} v_{ik}^x h(V_{ts}^{\gamma \cdot G} v_{kj}^x) = 0 = \sum_{t,k} h(V_{rt}^{\gamma \cdot G} v_{ik}^x) V_{ts}^{\gamma \cdot G} v_{kj}^x = (h \otimes \text{id})\Delta_m(V_{rs}^{\gamma \cdot G} v_{ij}^x).$$

And, if $x = 1$, we have

$$(\text{id} \otimes h)\Delta_m(V_{rs}^{\gamma \cdot G}) = \sum_t V_{rt}^{\gamma \cdot G} h(V_{ts}^{\gamma \cdot G}) = \delta_{\gamma,e}1 = (h \otimes \text{id})\Delta_m(V_{rs}^{\gamma \cdot G}).$$

It follows that h is the Haar state.

(2). Since we showed that, whenever $\gamma \neq e$ and $x \neq 1$, $(\text{id} \otimes h)(V^{\gamma \cdot G} \otimes v^x) = 0$, it follows that such representations are irreducible. To simplify the notations, we write $\gamma \cdot G \otimes x = V^{\gamma \cdot G} \otimes v^x$ during this proof.

Since the linear span of the coefficients of $\gamma \cdot G \otimes x$ is equal to \mathcal{A} and hence dense, it follows that any irreducible representation of \mathbb{G} is equivalent to some $\gamma \cdot G \otimes x$. For a unitary representation u (of \mathbb{G} or G), we denote by $\chi(u) = \sum_i u_{ii}$ its character. Observe that $\chi(\gamma \cdot G \otimes x) = \chi(V^{\gamma \cdot G})\alpha(\chi(x)) = \sum_{r \in \gamma \cdot G} u_r \alpha(v_{rr})\alpha(\chi(x))$. Hence, we have, for all $\gamma, \gamma' \in \Gamma$ and all $x, y \in \text{Irr}(G)$,

$$\begin{aligned} \chi(\overline{\gamma' \cdot G \otimes y} \otimes \gamma \cdot G \otimes x) &= \chi(\gamma' \cdot G \otimes y)^* \chi(\gamma \cdot G \otimes x) = \sum_{s \in \gamma' \cdot G, r \in \gamma \cdot G} \alpha(\chi(\overline{y})v_{ss})u_{s-1r}\alpha(v_{rr}\chi(x)) \\ &= \sum_{s \in \gamma' \cdot G, r \in \gamma \cdot G} u_{s-1r}\alpha(\alpha_{r-1s}(\chi(\overline{y})v_{ss})v_{rr}\chi(x)). \end{aligned}$$

Hence,

$$h(\chi(\overline{\gamma' \cdot G \otimes y} \otimes \gamma \cdot G \otimes x)) = \delta_{\gamma' \cdot G, \gamma \cdot G} \sum_{s \in \Gamma \cdot G} \int_G v_{ss} \chi(\overline{y}) \chi(x) d\mu = \delta_{\gamma' \cdot G, \gamma \cdot G} \int_G \chi(\overline{y} \otimes x) d\mu = \delta_{\gamma' \cdot G, \gamma \cdot G} \delta x, y.$$

It follows that $\gamma \cdot x \simeq \gamma' \cdot y$ if and only if $\gamma \cdot G = \gamma' \cdot G$ and $x = y$.

(3) We have already shown that $\text{Pol}(\mathbb{G}) = \mathcal{A}$. It follows that $C_m(\mathbb{G}) = A_m$. Since λ is surjective and τ is faithful on A it follows that $C(\mathbb{G}) = A$ and $L^\infty(\mathbb{G})$ is the bicommutant of A in $\mathcal{B}(\ell^2(\Gamma) \otimes \ell^2(G))$ i.e., it is the von Neumann algebraic crossed product. Finally, since λ is the identity on $\mathcal{A} = \text{Pol}(\mathbb{G})$, it follows that it is the canonical surjection. \square

Remark 3.5. Let \mathbb{G} be the bicrossed product coming from a compact matched pair (Γ, G) . From the definition, it is easy to check that $C_m(\mathbb{G})$ is commutative if and only if the action α is trivial and Γ is abelian. Moreover, \mathbb{G} is co-commutative if and only if the action β is trivial and G is abelian.

Hence, if \mathbb{G} comes from the matched pair in Example 3.3, then $C_m(\mathbb{G})$ is not commutative if and only if α is non-trivial or Γ is non-abelian, and, since β is trivial, \mathbb{G} is non-cocommutative if and only if G is non-abelian. So we get many non-trivial compact quantum groups in this way.

Remark 3.6. The following observation is well known. Let $\alpha : \Gamma \curvearrowright A$ be an action of the countable group Γ on the unital C^* -algebra A and let C be the full crossed product which is generated by the unitaries u_γ , $\gamma \in \Gamma$, and by the copy $\alpha(A)$ of the C^* -algebra A . If A has a character $\varepsilon \in A^*$ such that $\varepsilon(\alpha_\gamma(a)) = \varepsilon(a)$ for all $a \in A$ and $\gamma \in \Gamma$, then the C^* -subalgebra $B \subset C$ generated by $\{u_\gamma : \gamma \in \Gamma\}$ is canonically isomorphic to $C^*(\Gamma)$. Indeed, it suffices to check that B satisfies the universal property of $C^*(\Gamma)$. Let $v : \Gamma \rightarrow \mathcal{U}(H)$ be a unitary representation of Γ on H . Consider the unital $*$ -homomorphism $\pi : A \rightarrow \mathcal{B}(H)$ given by $\rho(a) = \varepsilon(a)\text{id}_H$, $a \in A$. We have $v_\gamma \pi(a) v_{\gamma^{-1}} = \varepsilon(a)\text{id} = \varepsilon(\alpha_\gamma(a))\text{id} = \pi(\alpha_\gamma(a))$. Hence, we obtain a representation of C that we can restrict to B to get the universal property.

Let (Γ, G) be a matched pair. Since the map $\varepsilon : C(G) \rightarrow \mathbb{C}$ by $F \mapsto F(e)$ is an α -invariant character, it follows from the preceding observation that the C^* -subalgebra of A_m generated by the u_γ , $\gamma \in \Gamma$, is canonically isomorphic to $C^*(\Gamma)$.

The first assertion of the following corollary is already known [DQV02] but we give here a different proof. We refer to [DFSW13] for the Haagerup property of quantum groups and we refer to [Ve04] for the K-amenability of quantum groups.

Corollary 3.7. *The following holds:*

1. \mathbb{G} is co-amenable if and only if Γ is amenable.
2. If Γ is K-amenable, then $\widehat{\mathbb{G}}$ is K-amenable.
3. If $\widehat{\mathbb{G}}$ has the Haagerup property, then Γ has the Haagerup property.
4. If the action of Γ on $L^\infty(G)$ is compact and Γ has the Haagerup property, then $\widehat{\mathbb{G}}$ has the Haagerup property.

Proof. (1). If Γ is amenable, then we trivially have that λ is an isomorphism; hence, \mathbb{G} is co-amenable. If \mathbb{G} is co-amenable, then the Haar state $h = \tau \circ \lambda$ is faithful on A_m . Since $h(u_\gamma) = \delta_{\gamma,e}$, we conclude from Remark 3.6, that the canonical trace on $C^*(\Gamma)$ has to be faithful. Hence, Γ is amenable.

(2). It is an immediate consequence of [Cu83, Theorem 2.1 (c)].

(3). It follows from [DFSW13, Theorem 6.7], since $L(\Gamma)$ is a von Neumann subalgebra of $L^\infty(\mathbb{G})$.

(4). This is a direct consequence of [Jo07, Corollary 3.4] and [DFSW13, Theorem 6.7]. \square

Let us now describe the fusion rules. For $r, s \in \Gamma$, we denote by $B_{r,s} \subset G$ the clopen set

$$B_{r,s} = \{g \in G : \beta_{\alpha_s(g)}(r) = r \text{ and } \beta_g(s) = s\}.$$

We also simply write $\gamma \cdot G \in \text{Irr}(\mathbb{G})$ for the equivalence class of $V^{\gamma \cdot G}$ for $\gamma \cdot G \in \Gamma/G$, and we view $\text{Irr}(G) \subset \text{Irr}(\mathbb{G})$.

The following result provides the formulas to decompose any tensor product of irreducible representations as a direct sum of irreducible representations.

Theorem 3.8. *The following holds:*

1. *The set of unitary representation of \mathbb{G} of the form $v^x \otimes V^{\gamma \cdot G}$ for some $\gamma \cdot G \in \Gamma/G$ and $x \in \text{Irr}(G)$ is a complete set of irreducible unitary representations of \mathbb{G} . In particular, for all $\gamma \cdot G \in \Gamma/G$ and all $x \in \text{Irr}(G)$, there exists a unique $\alpha_{\gamma \cdot G}(x) \in \text{Irr}(G)$ and a unique $\beta_x(\gamma \cdot G) \in \Gamma/G$ such that*

$$\gamma \cdot G \otimes x \simeq \alpha_{\gamma \cdot G}(x) \otimes \beta_x(\gamma \cdot G).$$

Moreover, for all $\gamma \cdot G \in \Gamma/G$ and all $x \in \text{Irr}(G)$, the maps

$$\alpha_{\gamma \cdot G} : \text{Irr}(G) \rightarrow \text{Irr}(G) \quad \text{and} \quad \beta_x : \Gamma/G \rightarrow \Gamma/G$$

are bijections.

2. *For all $r, s, \gamma \in \Gamma$, $x \in \text{Irr}(G)$ we have*

$$\dim(\text{Mor}(\gamma \cdot G \otimes x, r \cdot G \otimes s \cdot G)) = \sum_{s' \in s \cdot G, r' \in r \cdot G} |\{t \in \gamma \cdot G : t = r' s'\}| \int_{B_{r',s'}} \chi(\bar{x}) d\mu.$$

Proof. (1). The proof of (1) is exactly as the proof of assertion 2 in Theorem 3.4. The second assertion is trivial, since the representations $V^{\gamma \cdot G} \otimes v^x$ are irreducible. Finally, the fact that the maps are bijective follows from uniqueness.

(2). For all $\gamma, r, s \in \Gamma$, we have

$$\begin{aligned} \chi(\overline{\gamma \cdot G \otimes x} \otimes r \cdot G \otimes s \cdot G) &= \sum_{\gamma' \in \gamma \cdot G, r' \in r \cdot G, s' \in s \cdot G} \alpha(\chi(\bar{x}) v_{\gamma' \gamma'}) u_{(\gamma')^{-1} r'} \alpha(v_{r' r'}) u_{s'} \alpha(v_{s' s'}) \\ &= \sum_{\gamma' \in \gamma \cdot G, r' \in r \cdot G, s' \in s \cdot G} u_{(\gamma')^{-1} r'} \alpha(\alpha_{(r' s')}^{-1} \gamma' (\chi(\bar{x}) v_{\gamma' \gamma'})) \alpha_{(s')^{-1}}(v_{r' r'}) v_{s' s'}. \end{aligned}$$

It follows that

$$\begin{aligned} \dim(\text{Mor}(\gamma \cdot G, r \cdot G \otimes s \cdot G)) &= h(\chi(\overline{\gamma \cdot G} \otimes r \cdot G \otimes s \cdot G)) \\ &= \sum_{\gamma' \in \gamma \cdot G, r' \in r \cdot G, s' \in s \cdot G} \delta_{\gamma', r' s'} \int_G \chi(\bar{x}) v_{r' s', r' s'} \alpha_{(s')^{-1}}(v_{r' r'}) v_{s' s'} d\mu \\ &= \sum_{s' \in s \cdot G, r' \in r \cdot G} |\{t \in \gamma \cdot G : t = r' s'\}| \int_G \chi(\bar{x}) v_{r' s', r' s'} \alpha_{(s')^{-1}}(v_{r' r'}) v_{s' s'} d\mu. \end{aligned}$$

Observe that $v_{r' s', r' s'} \alpha_{(s')^{-1}}(v_{r' r'}) v_{s' s'} = 1_{D_{r', s'}}$, where $D_{r', s'} = \{g \in G : \beta_g(r' s') = r' s'\} \cap B_{r', s'}$. Since $\beta_g(r' s') = \beta_{\alpha_{s'}(g)}(r') \beta_g(s')$, it follows that $B_{r', s'} \subset \{g \in G : \beta_g(r' s')\}$. Hence, $D_{r', s'} = B_{r', s'}$. \square

4 Property (T) and bicrossed product

This section is dedicated to relative property (T) of the pair (G, \mathbb{G}) and Kazhdan property of the bicrossed product \mathbb{G} constructed in Section 3. The results in this section generalize classical results on relative property (T) for inclusion of groups of the form $(H, \Gamma \ltimes H)$, where H is discrete abelian [CT11]. In this section, we provide the first explicit example of a non-commutative and non-cocommutative infinite dimensional compact quantum group with property (T) .

4.1 Relative property (T) for compact bicrossed product

Definition 4.1. Let G and \mathbb{G} be two compact quantum groups with an injective unital $*$ -homomorphism $\alpha : C_m(G) \rightarrow C_m(\mathbb{G})$ such that $\Delta_{\mathbb{G}} \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_G$. We say that the pair (G, \mathbb{G}) has the relative co-property (T) , if every for every representation $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$ we have $\varepsilon_{\mathbb{G}} \prec \pi \implies \varepsilon_G \subset \pi \circ \alpha$.

Observe that, by [Ky11, Proposition 2.3], $\widehat{\mathbb{G}}$ has the property (T) in the sense of [Fi10] if and only if the pair (\mathbb{G}, \mathbb{G}) has the relative co-property (T) (with $\alpha = \text{id}$). Also, if Λ, Γ are countable discrete groups and $\Lambda < \Gamma$, then the pair $(\widehat{\Lambda}, \widehat{\Gamma})$ has the relative co-property (T) if and only if the pair (Λ, Γ) has the relative property (T) in the classical sense.

Let (Γ, G) be a matched pair of a countable discrete group Γ and a compact group G . Let \mathbb{G} be the bicrossed-product. In the following result, we characterize the relative co-property (T) of the pair (G, \mathbb{G}) in terms of the action α of Γ on $C(G)$. This is a non-commutative version of [CT11, Theorem 1] and the proof is similar. We will use freely the notations and results of Section 3.

Theorem 4.2. *The following are equivalent:*

1. *The pair (G, \mathbb{G}) does not have the relative co-property (T) .*
2. *There exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of Borel probability measures on G such that*
 - (a) $\mu_n(\{e\}) = 0$ for all $n \in \mathbb{N}$;
 - (b) $\mu_n \rightarrow \delta_e$ weak*;
 - (c) $\|\alpha_\gamma(\mu_n) - \mu_n\| \rightarrow 0$ for all $\gamma \in \Gamma$.

Proof. For a representation $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$, we have $\varepsilon_G \subset \pi \circ \alpha$ if and only if $K_\pi \neq \{0\}$, where

$$K_\pi = \{\xi \in H : \pi \circ \alpha(F)\xi = F(e)\xi \text{ for all } F \in C(G)\}.$$

Let $\rho = \pi \circ \alpha : C(G) \rightarrow \mathcal{B}(H)$, and for all $\xi, \eta \in H$, let $\mu_{\xi, \eta}$ be the unique complex Borel measure on G such that $\int_G F d\mu_{\xi, \eta} = \langle \rho(F)\xi, \eta \rangle$ for all $F \in C(G)$. Let $\mathcal{B}(G)$ be the collection of Borel subsets of G and $E : \mathcal{B}(G) \rightarrow \mathcal{B}(H)$ be the projection-valued measure associated to ρ i.e., for all $B \in \mathcal{B}(G)$, the projection $E(B) \in \mathcal{B}(H)$ is the unique operator such that $\langle E(B)\xi, \eta \rangle = \mu_{\xi, \eta}(B)$ for all $\xi, \eta \in H$.

Observe that a vector $\xi \in H$ satisfies $\rho(F)\xi = F(e)\xi$ for all $F \in C(G)$, if and only if $\mu_{\xi, \eta} = \langle \xi, \eta \rangle \delta_e$ for all $\eta \in H$, which in turn is true if and only if $\langle E(\{e\})\xi, \eta \rangle = \langle \xi, \eta \rangle$ for all $\eta \in H$. Hence, $E(\{e\})$ is the orthogonal projection onto K_π .

(1) \implies (2). Suppose that the pair (G, \mathbb{G}) does not have the relative co-property (T) . Let $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$ be a representation such that $\varepsilon_{\mathbb{G}} \prec \pi$ and $K_\pi = \{0\}$. Hence, $\mu_{\xi, \eta}(\{e\}) = \langle E(\{e\})\xi, \eta \rangle = 0$ for all $\xi, \eta \in H$.

Since $\varepsilon_{\mathbb{G}} \prec \pi$, let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of unit vectors in H such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \rightarrow 0$ for all $x \in C_m(\mathbb{G})$. Define $\mu_n = \mu_{\xi_n, \xi_n}$. Then, we have $\mu_n(\{e\}) = 0$ for all $n \in \mathbb{N}$. Since μ_n is a probability measure, so $|\mu_n(F) - \delta_e(F)| = |\int_G (F - F(e)) d\mu_n| \leq \|F - F(e)\|_{L^1(\mu_n)} \leq \|F - F(e)\|_{L^2(\mu_n)}$, for all $F \in C(G)$. Moreover,

$$\|F - F(e)\|_{L^2(\mu_n)}^2 = \|\rho(F - F(e)1)\xi_n\|^2 = \|\pi(\alpha(F))\xi_n - \varepsilon_{\mathbb{G}}(\alpha(F))\xi_n\|^2 \rightarrow 0.$$

Hence, $\mu_n \rightarrow \delta_e$ weak*. Finally, for all $\gamma \in \Gamma$ and all $F \in C(G)$, we have:

$$\int_G F d\alpha_\gamma(\mu_n) = \int_G \alpha_{\gamma^{-1}}(F) d\mu_n = \langle \rho(\alpha_{\gamma^{-1}}(F))\xi_n, \xi_n \rangle = \langle \pi(u_\gamma)^* \rho(F) \pi(u_\gamma)\xi_n, \xi_n \rangle = \langle \rho(F) \pi(u_\gamma)\xi_n, \pi(u_\gamma)\xi_n \rangle.$$

It follows that

$$\begin{aligned} \left| \int_G F d\alpha_\gamma(\mu_n) - \int_G F d\mu_n \right| &= |\langle \rho(F) \pi(u_\gamma)\xi_n, \pi(u_\gamma)\xi_n \rangle - \langle \rho(F)\xi_n, \xi_n \rangle| \\ &\leq |\langle \rho(F)(\pi(u_\gamma)\xi_n - \xi_n), \pi(u_\gamma)\xi_n \rangle| + |\langle \rho(F)\xi_n, \pi(u_\gamma)\xi_n - \xi_n \rangle| \\ &\leq 2\|F\| \|\pi(u_\gamma)\xi_n - \xi_n\|, \text{ for all } F \in C(G) \text{ and } \gamma \in \Gamma. \end{aligned}$$

Hence, $\|\alpha_\gamma(\mu_n) - \mu_n\| \leq 2\|\pi(u_\gamma)\xi_n - \xi_n\| = 2\|\pi(u_\gamma)\xi_n - \varepsilon_G(u_\gamma)\xi_n\| \rightarrow 0$.

(2) \implies (1). We first prove the following claim.

Claim. *If (2) holds, then there exists a sequence $(\nu_n)_{n \in \mathbb{N}}$ of Borel probability measures on G satisfying (a), (b) and (c) and such that $\alpha_\gamma(\nu_n) \sim \nu_n$ for all $\gamma \in \Gamma$, $n \in \mathbb{N}$.*

Proof of the claim. Denote by $\ell^1(\Gamma)_{1,+}$ the set of positive ℓ^1 functions on Γ with $\|f\|_1 = 1$. For μ a Borel probability measure on G and $f \in \ell^1(\Gamma)_{1,+}$, define the Borel probability measure $f * \mu$ on G by the convex combination

$$f * \mu = \sum_{\gamma \in \Gamma} f(\gamma) \alpha_\gamma(\mu).$$

Observe that for all $\gamma \in \Gamma$, we have $\delta_\gamma * \mu = \alpha_\gamma(\mu)$ and $\alpha_\gamma(f * \mu) = f_\gamma * \mu$, where $f_\gamma \in \ell^1(\Gamma)_{1,+}$ is defined by $f_\gamma(r) = f(\gamma^{-1}r)$, $r \in \Gamma$.

Moreover, if $f \in \ell^1(\Gamma)_{1,+}$ is such that $f(\gamma) > 0$ for all $\gamma \in \Gamma$, then since $(f * \mu)(A) = \sum_{\gamma} f(\gamma) \mu(\alpha_{\gamma^{-1}}(A))$, so we have that $(f * \mu)(A) = 0$ if and only if $\mu(\alpha_\gamma(A)) = 0$ for all $\gamma \in \Gamma$. This last condition does not depend of f . Hence, if $f \in \ell^1(\Gamma)_{1,+}$ is such that $f > 0$, then since $f_\gamma(r) > 0$ for all $\gamma, r \in \Gamma$, it follows that $f * \mu \sim \alpha_\gamma(f * \mu) = f_\gamma * \mu$ for all $\gamma \in \Gamma$ as they have the same null sets: the Borel subsets A of G such that $\mu(\alpha_s(A)) = 0$ for all $s \in \Gamma$.

Finally, since $\alpha_\gamma(e) = e$ for all $\gamma \in \Gamma$, so

$$(f * \mu)(\{e\}) = \sum_{\gamma} f(\gamma) \mu(\alpha_{\gamma^{-1}}(\{e\})) = \sum_{\gamma} f(\gamma) \mu(\{e\}) = \mu(\{e\}), \text{ for all } f \in \ell^1(\Gamma)_{1,+}.$$

Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of Borel probability on G satisfying (a), (b) and (c). For all $f \in \ell^1(\Gamma)_{1,+}$ with finite support we have,

$$\|f * \mu_n - \mu_n\| \leq \sum_{\gamma} f(\gamma) \|\delta_\gamma * \mu_n - \mu_n\| = \sum_{\gamma} f(\gamma) \|\alpha_\gamma(\mu_n) - \mu_n\| \rightarrow 0. \quad (4.1)$$

Since such functions are dense in $\ell^1(\Gamma)_{1,+}$ (in the ℓ^1 -norm), it follows that $\|f * \mu_n - \mu_n\| \rightarrow 0$ for all $f \in \ell^1(\Gamma)_{1,+}$.

Let $\xi \in \ell^1(\Gamma)_{1,+}$ be any function such that $\xi > 0$ and define $\nu_n = \xi * \mu_n$. By the preceding discussion, we know that $\alpha_\gamma(\nu_n) \sim \nu_n$ for all $\gamma \in \Gamma$ and $\nu_n(\{e\}) = \mu_n(\{e\}) = 0$ for all $n \in \mathbb{N}$. Moreover, by Eq. (4.1),

$$\|\alpha_\gamma(\nu_n) - \nu_n\| = \|\xi_\gamma * \mu_n - \xi * \mu_n\| \leq \|\xi_\gamma * \mu_n - \mu_n\| + \|\mu_n - \xi * \mu_n\| \rightarrow 0, \text{ for all } \gamma \in \Gamma.$$

Finally, since $\mu_n \rightarrow \delta_e$ weak* and $\alpha_\gamma(e) = e$, one has $|\mu_n(F \circ \alpha_\gamma) - F(e)| \rightarrow 0$ for all $\gamma \in \Gamma$ and for all $F \in C(G)$. Hence, for all $F \in C(G)$, the dominated convergence Theorem implies that

$$|\nu_n(F) - \delta_e(F)| = \left| \sum_{\gamma} f(\gamma) (\mu_n(F \circ \alpha_\gamma) - F(e)) \right| \leq \sum_{\gamma} f(\gamma) |\mu_n(F \circ \alpha_\gamma) - F(e)| \rightarrow 0.$$

It follows that $\nu_n \rightarrow \delta_e$ weak* and this finishes the proof of the claim. \square

We now finish the proof of the Theorem. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of Borel probability measures on G as prescribed in the Claim. For $n \in \mathbb{N}$ and $\gamma \in \Gamma$, let $h_n(\gamma) = \frac{d\alpha_\gamma(\mu_n)}{d\mu_n}$; then $0 \leq h_n(\gamma) \leq 1$ μ_n a.e., and by uniqueness of the Radon-Nikodym derivative and since α is an action, we have for all $n \in \mathbb{N}$,

$$h_n(\gamma, g)h_n(\gamma^{-1}, \alpha_{\gamma^{-1}}(g)) = 1, \mu_n(g) \text{ a.e.}, \text{ and for all } \gamma \in \Gamma, g \in G.$$

Define $H_n = L^2(G, \mu_n)$ and let $\pi_n : \Gamma \rightarrow \mathcal{U}(H_n)$ be the unitary representation defined by

$$(u_n(\gamma)\xi)(g) = \xi(\alpha_{\gamma^{-1}}(g))h_n(\gamma, g)^{\frac{1}{2}} \quad \gamma \in \Gamma, g \in G, \xi \in H_n.$$

Consider also the representations $\rho_n : C(G) \rightarrow \mathcal{B}(H_n)$, $\rho_n(F)\xi(g) = F(g)\xi(g)$, for $\xi \in H_n$, $g \in G$ and $F \in C(G)$. Observe that the projection valued measure associated to ρ_n is given by $(E_n(B)\xi)(g) = 1_B(g)\xi(g)$ for all $B \in \mathcal{B}(G)$, $\xi \in H_n$ and $g \in G$.

Using the identity $h_n(\gamma, \cdot)h_n(\gamma^{-1}, \alpha_{\gamma^{-1}}(\cdot)) = 1$, we find $u_n(\gamma)\rho_n(F)u_n(\gamma^{-1}) = \rho_n(\alpha_\gamma(F))$ for all $\gamma \in \Gamma$, $F \in C(G)$, $g \in G$. By the universal property of A_m , there exist unique unital $*$ -homomorphisms

$$\pi_n : A_m \rightarrow \mathcal{B}(H_n) \quad \text{such that} \quad \pi_n(\gamma) = u_n(\gamma) \quad \text{and} \quad \pi_n \circ \alpha = \rho_n \text{ for all } n \in \mathbb{N}.$$

Since $\mu_n(\{e\}) = 0$, we have $E_n(\{e\}) = 0$ for all $n \in \mathbb{N}$. Hence, $K_{\pi_n} = \{0\}$ for all $n \in \mathbb{N}$. Consequently, on defining $H = \oplus_n H_n$ and $\pi = \oplus_n \pi_n : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$, it follows that $K_\pi = \{0\}$ as well. Hence, it suffices to show that $\varepsilon_G \prec \pi$.

Define the unit vectors $\xi_n = 1 \in L^2(G, \mu_n) \subset H$, $n \in \mathbb{N}$. Observe that $(\mu_n - \alpha_\gamma(\mu_n))(F) = \int_G F(1 - h_n(\gamma))d\mu_n$ for all $F \in C(G)$. Hence, $\|\mu_n - \alpha_\gamma(\mu_n)\| = \|1 - h_n(\gamma)\|_{L^1(G, \mu_n)} \rightarrow 0$ for all $\gamma \in \Gamma$. Moreover, as $0 \leq 1 - \sqrt{t} \leq \sqrt{1-t}$ for all $0 \leq t \leq 1$, it follows that

$$\|\pi(u_\gamma)\xi_n - \xi_n\|_H^2 = \|u_n(\gamma)1 - 1\|_{H_n}^2 = \int_G (1 - h_n(\gamma)^{\frac{1}{2}})^2 d\mu_n \leq \int_G (1 - h_n(\gamma))d\mu_n = \|1 - h_n(\gamma)\|_{L^1(G, \mu_n)} \rightarrow 0$$

for all $\gamma \in \Gamma$. Since $\mu_n \rightarrow \delta_e$ weak*, for all $F \in C(G)$, we also have that,

$$\|\pi(\alpha(F))\xi_n - F(e)\xi_n\|_H^2 = \|\rho_n(F)1 - F(e)1\|_{H_n}^2 = \int_G |F(g) - F(e)|^2 d\mu_n \rightarrow 0.$$

Consequently, for all $x = u_\gamma \alpha(F) \in C_m(\mathbb{G})$, we have

$$\begin{aligned} \|\pi(x)\xi_n - \varepsilon_G(x)\xi_n\| &= \|\pi(u_\gamma)\pi(\alpha(F))\xi_n - F(e)\xi_n\| \\ &\leq \|\pi(u_\gamma)(\pi(\alpha(F))\xi_n - F(e)\xi_n)\| + |F(e)| \|\pi(u_\gamma)\xi_n - \xi_n\| \\ &\leq \|\pi(\alpha(F))\xi_n - F(e)\xi_n\| + |F(e)| \|\pi(u_\gamma)\xi_n - \xi_n\| \rightarrow 0. \end{aligned}$$

By linearity and the triangle inequality, we have $\|\pi(x)\xi_n - \varepsilon_G(x)\xi_n\| \rightarrow 0$ for all $x \in \mathcal{A}$. We conclude the proof using the density of \mathcal{A} in $C_m(\mathbb{G})$. \square

4.2 Property (T)

Theorem 4.3. *The following holds:*

1. If $\widehat{\mathbb{G}}$ has property (T), then Γ has property (T).
2. If Γ has the property (T) and G is finite, then $\widehat{\mathbb{G}}$ has property (T).

Proof. (1). Let $\rho : C(G) \rightarrow C^*(\Gamma)$ be the unital $*$ -homomorphism defined by $\rho(F) = F(e)1$ and consider the canonical unitary representation of Γ given by $\Gamma \ni \gamma \mapsto \lambda_\gamma \in C^*(\Gamma)$. For all $\gamma \in \Gamma$ and $F \in C(G)$, we have

$$\rho(\alpha_\gamma(F)) = \alpha_\gamma(F)(e)1 = F(\alpha_{\gamma^{-1}}(e))1 = F(e)1 = \lambda_\gamma \rho(F) \lambda_\gamma^*.$$

Hence, there exists a unique unital $*$ -homomorphism $\pi : C_m(\mathbb{G}) \rightarrow C^*(\Gamma)$ such that

$$\pi \circ \alpha = \rho \quad \text{and} \quad \pi(u_\gamma) = \lambda_\gamma \quad \text{for all } \gamma \in \Gamma.$$

Observe that π is surjective and, for all $F \in C(G)$,

$$(\pi \otimes \pi)\Delta_{\mathbb{G}}(\alpha(F)) = (\rho \otimes \rho)(\Delta_G(F)) = \Delta_G(F)(e, e)1 \otimes 1 = F(e)1 \otimes 1 = \Delta_{\widehat{\Gamma}}(\pi(\alpha(F))).$$

Moreover, since for all $\gamma, r \in \Gamma$ one has $1_{A_{\gamma,r}}(e) = \delta_{\gamma,r}$, we find, for all $\gamma \in \Gamma$,

$$(\pi \otimes \pi)\Delta_{\mathbb{G}}(u_\gamma) = \sum_{r \in \Gamma \cdot G} \pi(u_\gamma \alpha(v_{\gamma,r}^\gamma)) \otimes \pi(u_r) = \sum_{r \in \Gamma \cdot G} \lambda_\gamma 1_{A_{\gamma,r}}(e) \otimes \lambda_r = \lambda_\gamma \otimes \lambda_\gamma = \Delta_{\widehat{\Gamma}}(\pi(u_\gamma)).$$

Hence, π intertwines the comultiplications. Now the proof of (1) follows from [Fi10, Proposition 6].

(2). Let $\pi : C_m(\mathbb{G}) = \Gamma_{\alpha,f} \times C(G) \rightarrow \mathcal{B}(H)$ be a unital $*$ -homomorphism. Consider the closed subspace given by H of $C(G)$ -invariant vectors $K = \{\xi \in H : \pi \circ \alpha(F)\xi = F(e)\xi \text{ for all } F \in C(G)\}$. Then $P = \pi(\alpha(\delta_e))$ is the orthogonal projection on K . Since

$$\pi(u_\gamma)P\pi(u_\gamma)^* = \pi(\alpha(\delta_{\alpha_g(e)})) = \pi(\alpha(\delta_e)) = P \quad \text{for all } \gamma \in \Gamma,$$

K is an invariant subspace of the unitary representation $\gamma \mapsto \pi(u_\gamma)$. Let $\gamma \mapsto v_\gamma$ be the unitary representation of Γ on K obtained by restriction. Suppose that $\varepsilon_{\mathbb{G}} \prec \pi$ and let $\xi_n \in H$ be a sequence of unit vectors such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \rightarrow 0$ for all $x \in C_m(\mathbb{G})$. Since G is finite (hence \widehat{G} has property (T)), so $K \neq \{0\}$. Moreover, since $\|P\xi_n\| - 1 \leq \|P\xi_n - \xi_n\|$, we have $\|P\xi_n\| \rightarrow 1$ and hence we may and will assume that $P\xi_n \neq 0$ for all n . Let $\eta_n = \frac{P\xi_n}{\|P\xi_n\|} \in K$. We have

$$\|v_\gamma \eta_n - \eta_n\| = \frac{1}{\|P\xi_n\|} \|P(v_\gamma \xi_n - \xi_n)\| \leq \frac{\|\pi(u_\gamma)\xi_n - \xi_n\|}{\|P\xi_n\|} \rightarrow 0.$$

Hence, $\gamma \mapsto v_\gamma$ has almost invariant vectors. Since Γ has property (T), let $\xi \in K$ be a non-zero invariant vector. Then, for all $x \in C_m(\mathbb{G})$ of the form $x = u_\gamma \alpha(F)$, we have $\pi(x)\xi = F(e)\pi(u_\gamma)\xi = F(e)\xi = \varepsilon(x)\xi$. By linearity, continuity and density of \mathcal{A} in $C_m(\mathbb{G})$ we have $\pi(x)\xi = \varepsilon(x)\xi$ for all $x \in C_m(\mathbb{G})$. \square

Example 4.4. Let \mathbb{G} be the bicrossed product coming from the matched pair associated to a group action $\alpha : \Gamma \curvearrowright G$ by group homeomorphisms as in Example 3.3. Assume that G is finite. By Theorem 4.3, \mathbb{G} has property (T) if and only if Γ has property (T). Considering any group action $\text{SL}_n(\mathbb{Z}) \curvearrowright G$ on a finite non-abelian group G for $n \geq 3$, we obtain by Remark 3.5 examples of non-commutative and non-cocommutative discrete compact quantum group \mathbb{G} with property (T).

For an explicit example, consider, for p prime, the canonical quotient $\text{SL}_n(\mathbb{Z}) \rightarrow \text{SL}_n(\mathbb{Z}/p\mathbb{Z}) = G$. Observe that G is non abelian, hence the action by conjugation of $\text{SL}_n(\mathbb{Z})$ on G is non-trivial.

5 Relative Haagerup property and bicrossed product

In this section, we study the relative Haagerup property of the pair (G, \mathbb{G}) . The main result in this section also generalizes the characterization of relative Haagerup property of the pair $(H, \Gamma \rtimes H)$, where H is discrete abelian [CT11].

Let G be a compact quantum group. For $\omega \in C_m(G)^*$, define its Fourier transform $\widehat{\omega} = (\text{id} \otimes \omega)(V) \in M(c_0(\widehat{G}))$, where $V = \bigoplus_{x \in \text{Irr}(G)} u^x \in M(c_0(\widehat{G}) \otimes C_m(G))$ is the maximal multiplicative unitary. Observe that $\omega \mapsto \widehat{\omega}$ is linear and $\|\widehat{\omega}\|_{\mathcal{B}(L^2(G))} \leq \|\omega\|_{C_m(G)^*}$ for all $\omega \in C_m(G)^*$.

When G is a classical compact group with Haar measure μ and ν is a Borel complex measure on G , then the Fourier transform $\widehat{\nu} \in M(C_r^*(G))$ is the operator $\widehat{\nu} = \int_G \lambda_g d\nu(g) \in M(C_r^*(G)) \subset \mathcal{B}(L^2(G))$.

Definition 5.1. Let G and \mathbb{G} be two CQG with an injective unital $*$ -homomorphism $\alpha : C_m(G) \rightarrow C_m(\mathbb{G})$ such that $\Delta_{\mathbb{G}} \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_G$. We say that the pair (G, \mathbb{G}) has the *relative co-Haagerup property*, if there exists a sequence of states $\omega_n \in C_m(\mathbb{G})^*$ such that $\omega_n \rightarrow \varepsilon_{\mathbb{G}}$ weak* and $\widehat{\omega_n \circ \alpha} \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$.

Observe that, for any compact quantum group G , the dual \widehat{G} has the Haagerup property in the sense of [DFSW13] if and only if the pair (G, G) has the co-Haagerup property. Moreover, it is clear that if Λ, Γ are discrete groups with $\Lambda < \Gamma$, then the pair $(C^*(\Lambda), C^*(\Gamma))$ has the relative co-Haagerup property if and only if the pair (Λ, Γ) has the relative Haagerup property in the classical sense.

Let (Γ, G) be a matched pair of a discrete countable group Γ and a compact group G . Let \mathbb{G} be the bicrossed-product. In the following theorem, we characterize the relative co-Haagerup property of the pair (G, \mathbb{G}) in terms of the action α of Γ on $C(G)$. This is a non commutative version of [CT11, Theorem 4] and the proof is similar in spirit.

However, one of the argument of the classical case does not work in our context since α_γ is not a group homomorphism and new ideas are required. Actually it is not true for a general automorphism $\pi \in \text{Aut}(C(G))$ that $\widehat{\nu} \in C_r^*(G) \Rightarrow \widehat{\pi(\nu)} \in C_r^*(G)$. However, in the specific case of automorphisms coming from the action α given by a matched pair the previous result is true.

We provide details of this idea in the next lemma. We will use freely the notations and results of Section 3.

Lemma 5.2. *Let ν be a complex Borel measure on G . If $\widehat{\nu} \in C_r^*(G)$, then $\widehat{\alpha_\gamma(\nu)} \in C_r^*(G)$ for all $\gamma \in \Gamma$.*

Proof. Observe that for all $\gamma \in \Gamma$ and all $x \in \text{Irr}(G)$ by choosing $u^{\overline{y}}$ such that $u^{\overline{y}}_{ij} = (u^y_{ji})^*$ for all i, j , we have

$$\begin{aligned} \int_G (u^y_{kl})^* u^x_{ij} \circ \alpha_\gamma d\mu &= h(\alpha(u^{\overline{y}}_{lk}) u_\gamma^{-1} \alpha(u^x_{ij}) u_\gamma) = h(u_\gamma \alpha(u^{\overline{y}}_{lk}) u_{\gamma^{-1}} \alpha(u^x_{ij})) \\ &= \sum_{r \in \gamma \cdot G, s \in \gamma^{-1} \cdot G} h(u_\gamma \alpha(v_{\gamma r} u^{\overline{y}}_{lk}) u_{\gamma^{-1}} \alpha(v_{\gamma^{-1} s} u^x_{ij})) \\ &= 0, \quad \text{whenever } y \neq \alpha_{\gamma^{-1} \cdot G}(x). \end{aligned}$$

Indeed, since each term of the above sum is the Haar state of a coefficient of $(\gamma \cdot G \otimes \overline{y}) \otimes (\gamma^{-1} \cdot G \otimes x)$ the aforesaid coefficient being 0 whenever $\overline{\gamma \cdot G \otimes \overline{y}} \neq \gamma^{-1} \cdot G \otimes x$. Again, since $\overline{\gamma \cdot G \otimes \overline{y}} \simeq y \otimes \overline{\gamma \cdot G}$ and $\gamma^{-1} \cdot G \otimes x \simeq \alpha_{\gamma^{-1} \cdot G}(x) \otimes \beta_x(\gamma^{-1} \cdot G)$, we have

$$\overline{\gamma \cdot G \otimes \overline{y}} \neq \gamma^{-1} \cdot G \otimes x \Leftrightarrow y \neq \alpha_{\gamma^{-1} \cdot G}(x) \text{ or } \overline{\gamma \cdot G} \neq \beta_x(\gamma^{-1} \cdot G).$$

Since the set $\left\{ \frac{u^x_{ij}}{\sqrt{\dim(x)}} : x \in \text{Irr}(G), 1 \leq i, j \leq \dim(x) \right\}$ is an Hilbertian basis of $L^2(G)$, we can write $u^x_{ij} \circ \alpha_\gamma$ as a finite sum by $u^x_{ij} \circ \alpha_\gamma = \sum_{k,l} \lambda_{k,l}^\gamma u^y_{kl}$, where $y = \alpha_{\gamma^{-1} \cdot G}(x)$ and $\lambda_{k,l}^\gamma \in \mathbb{C}$. Now observe that for any complex Borel measure on G , we have

$$\widehat{\nu} \in C_r^*(G) \text{ if and only if the function } x \mapsto \int_G u^x_{ij} d\nu \in c_0(\text{Irr}(G)) \text{ for all } 1 \leq i, j \leq \dim(x).$$

Let ν be a Borel complex measure on G and $\gamma \in \Gamma$. We have

$$\int_G u^x_{ij} d\alpha_\gamma(\nu) = \int_G u^x_{ij} \circ \alpha_\gamma d\nu = \sum_{k,l} \lambda_{k,l}^\gamma \int_G u^y_{kl} d\nu, \text{ where } y = \alpha_{\gamma^{-1} \cdot G}(x).$$

Suppose that $\widehat{\nu} \in C_r^*(G)$. We will show that $\alpha_\gamma(\nu) \in C_r^*(G)$. It suffices to check that

$$\left(x \mapsto \int_G u^{\alpha_{\gamma^{-1} \cdot G}(x)}_{ij} d\nu \right) \in c_0(\text{Irr}(G)).$$

Let $\epsilon > 0$. We have

$$\left\{ x \in \text{Irr}(G) : \left| \int_G u^{\alpha_{\gamma^{-1} \cdot G}(x)}_{ij} d\nu \right| \geq \epsilon \right\} = (\alpha_{\gamma^{-1} \cdot G})^{-1} \left(\left\{ x \in \text{Irr}(G) : \left| \int_G u^x_{ij} d\nu \right| \geq \epsilon \right\} \right).$$

The right hand side is a finite set since $\alpha_{\gamma^{-1} \cdot G}$ is injective and by the assumptions $\{x \in \text{Irr}(G) : \left| \int_G u^x_{ij} d\nu \right| \geq \epsilon\}$ is finite. \square

Theorem 5.3. *The following are equivalent:*

1. *The pair (G, \mathbb{G}) has the relative co-Haagerup (T) .*
2. *There exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of Borel probability measures on G such that*
 - (a) $\widehat{\mu}_n \in C_r^*(G)$ for all $n \in \mathbb{N}$;
 - (b) $\mu_n \rightarrow \delta_e$ weak*;
 - (c) $\|\alpha_\gamma(\mu_n) - \mu_n\| \rightarrow 0$ for all $\gamma \in \Gamma$.

Proof. (1) \Rightarrow (2). Let $\omega_n \in C_m(\mathbb{G})$ be a sequence of states such that $\omega_n \rightarrow \varepsilon_{\mathbb{G}}$ weak* and $\widehat{\omega_n \circ \alpha} \in C_r^*(G)$. View $\omega_n \circ \alpha \in C(G)^*$ as a Borel probability measure μ_n on G . By hypothesis, $\widehat{\mu}_n \in C_r^*(G)$ for all $n \in \mathbb{N}$ and $\mu_n \rightarrow \delta_e$ weak*. Writing (H_n, π_n, ξ_n) the GNS construction of ω_n and doing the same computation as in the proof of (1) \Rightarrow (2) of Theorem 4.2, we find

$$\left| \int_G F d\alpha_\gamma(\mu_n) - \int_G F d\mu_n \right| \leq \|F\| \|\pi_n(u_\gamma)\xi_n - \xi_n\| = \|F\| \sqrt{2(1 - \operatorname{Re}(\omega_n(u_\gamma)))}.$$

Hence, $\|\alpha_\gamma(\mu_n) - \mu_n\| \leq \sqrt{2(1 - \operatorname{Re}(\omega_n(u_\gamma)))} \rightarrow \sqrt{2(1 - \operatorname{Re}(\varepsilon_{\mathbb{G}}(u_\gamma))} = 0$.

(2) \Rightarrow (1). We first prove the following claim.

Claim. *If (2) holds, then there exists a sequence $(\nu_n)_{n \in \mathbb{N}}$ of Borel probability measures on G satisfying (a), (b) and (c) and such that $\alpha_\gamma(\nu_n) \sim \nu_n$ for all $\gamma \in \Gamma$, $n \in \mathbb{N}$.*

Proof of the claim. By the proof of the claim in Theorem 4.2, it suffices to check that whenever ν is a complex Borel measure on G and $f \in \ell^1(G)$, we have $\widehat{\nu} \in C_r^*(G) \Rightarrow \widehat{f * \nu} \in C_r^*(G)$.

Suppose that $\widehat{\nu} \in C_r^*(G)$ and $f \in c_c(\Gamma)$, then $f * \nu = \sum f(\gamma)\alpha_\gamma(\mu)$ is a finite sum and by Lemma 5.2 we find that $\widehat{f * \nu} = \sum f(\gamma)\widehat{\alpha_\gamma(\mu)} \in C_r^*(G)$.

Suppose that $\widehat{\nu} \in C_r^*(G)$ and $f \in \ell^1(\Gamma)$. Let $f_n \in c_c(\Gamma)$ be such that $\|f - f_n\|_1 \rightarrow 0$. Since for all $g \in \ell^1(\Gamma)$ and all $\nu \in C(G)^*$ the estimate $\|g * \nu\| \leq \|g\|_1 \|\nu\|$ hold, we find

$$\|\widehat{f * \nu} - \widehat{f_n * \nu}\|_{C_r^*(G)} = \|(f - f_n) * \nu\|_{C_r^*(G)} \leq \|(f - f_n) * \nu\|_{C(G)^*} \leq \|\nu\|_{C(G)^*} \|f - f_n\|_1 \rightarrow 0.$$

Consequently, as $\widehat{f_n * \nu} \in C_r^*(G)$ for all n , it follows that $\widehat{f * \nu} \in C_r^*(G)$. \square

We can now finish the proof of the Theorem. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of Borel probability measures on G as in the Claim. As in the proof of Theorem 4.2, we construct a representation $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$ with a sequence of unit vector $\xi_n \in H$ such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \rightarrow 0$ for all $x \in C_m(\mathbb{G})$ and $\int F d\mu_n = \omega_{\xi_n} \circ \pi \circ \alpha(F)$, for all $F \in C(G)$. It follows that the sequence of states $\omega_n = \omega_{\xi_n} \circ \pi \in C_m(\mathbb{G})^*$ satisfies $\omega_n \rightarrow \varepsilon_{\mathbb{G}}$ weak* and $\widehat{\omega_n \circ \alpha} = \widehat{\mu}_n \in C_r^*(G)$ for all $n \in \mathbb{N}$. \square

6 Crossed product quantum group

This section deals with a matched pair of a discrete group and a compact quantum group that arises in a crossed product, where the discrete group acts via quantum automorphisms. This section is longer and has four subsections. In this section, we analyze the quantum group structure, representation theory of such crossed products which was initially studied by Wang in [Wa95b], but unlike Wang we do not rely on free products. We also prove some consequences of the crossed products on co-amenability and K -amenability. The subsections deal with weak amenability, rapid decay, relative property (T) and relative Haagerup property for the aforesaid crossed products. The setup of this section will remain valid all throughout the subsections.

Let G be a compact quantum group, Γ a discrete group acting and $\alpha : \Gamma \curvearrowright G$ be an action by quantum automorphisms. Then, as was shown by Wang in [Wa95a], a compact quantum group structure can be given on the associated crossed product. In this section, we recall this construction and study some of its properties.

One should note that this construction is a special case of compact bicrossed product of a matched pair of a discrete group and a compact quantum group.

We will denote by the same α the action of Γ on $C_m(G)$ or $C(G)$. Let $A_m = \Gamma_{\alpha,m} \rtimes C_m(G)$ be the full crossed product and $A = \Gamma_{\alpha} \rtimes C(G)$ be the reduced crossed product. By abuse of notation, we still denote by α the canonical injective map from $C_m(G)$ to A_m and from $C(G)$ to A . We also denote by u_{γ} , for $\gamma \in \Gamma$ the canonical unitaries viewed in either A_m or A . This will be clear from the context and cause no confusion.

By the universal property of the full crossed product, we have a unique surjective unital $*$ -homomorphism $\lambda : A_m \rightarrow A$ such that $\lambda(u_{\gamma}) = u_{\gamma}$ and $\lambda(\alpha(a)) = \alpha(\lambda_G(a))$ for all $\gamma \in \Gamma$ and for all $a \in C_m(G)$.

Finally, we denote by $\omega \in A^*$ the dual state of h_G i.e., ω is the unique (faithful) state such that

$$\omega(u_{\gamma}\alpha(a)) = \delta_{e,\gamma}h_G(a) \quad \text{for all } a \in C(G), \gamma \in \Gamma.$$

Again by the universal property of the full crossed product, there exists a unique unital $*$ -homomorphism $\Delta_m : A_m \rightarrow A_m \otimes A_m$ such that $\Delta_m(u_{\gamma}) = u_{\gamma} \otimes u_{\gamma}$ and $\Delta_m \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_G$.

The following theorem is due to Wang [Wa95b]. We include a short proof for the convenience of the reader.

Theorem 6.1. $\mathbb{G} = (A_m, \Delta_m)$ is a compact quantum group and the following holds.

1. The Haar state of \mathbb{G} is $h = \omega \circ \lambda$, hence, \mathbb{G} is Kac if and only if G is Kac.
2. For all $\gamma \in \Gamma$ and all $x \in \text{Irr}(G)$, $u_{\gamma}^x = (1 \otimes u_{\gamma})(\text{id} \otimes \alpha)(u^x) \in \mathcal{B}(H_x) \otimes A_m$ is an irreducible representation of \mathbb{G} and the set $\{u_{\gamma}^x : \gamma \in \Gamma, x \in \text{Irr}(G)\}$ is a complete set of irreducible representations of \mathbb{G} .
3. One has $C_m(\mathbb{G}) = A_m$, $C(\mathbb{G}) = A$, $\text{Pol}(\mathbb{G}) = \text{Span}\{u_{\gamma}\alpha(a) : \gamma \in \Gamma, a \in \text{Pol}(G)\}$, λ is the canonical surjection from $C_m(\mathbb{G})$ to $C(\mathbb{G})$ and $L^{\infty}(\mathbb{G})$ is the von Neumann algebraic crossed product.

Proof. (1). Write $\mathcal{A} = \text{Span}\{u_{\gamma}\alpha(a) : \gamma \in \Gamma, a \in \text{Pol}(G)\}$. Since, by definition of A_m , \mathcal{A} is dense in A_m it suffices to show the invariance of h on \mathcal{A} and one has

$$\begin{aligned} (\text{id} \otimes h)(\Delta_m(u_{\gamma}\alpha(u_{ij}^x))) &= \sum_k u_{\gamma}\alpha(u_{ik}^x)h(u_{\gamma}\alpha(u_{kj}^x)) = \delta_{\gamma,e}\delta_{x,1} \\ &= h(u_{\gamma}\alpha(u_{ij}^x)) = (h \otimes \text{id})(\Delta_m(u_{\gamma}\alpha(u_{ij}^x))), \quad \gamma \in \Gamma, x \in \text{Irr}(G). \end{aligned}$$

(2). By the definition of Δ_m , it is obvious that u_{γ}^x (as above) are unitary representations of \mathbb{G} . They are moreover irreducible since we observed in (1) that $(\text{id} \otimes h)(u_{\gamma}^x) = 0$ whenever $\gamma \neq e$, $x \neq 1$. They are pairwise non-equivalent since

$$h(\chi(u_r^x)^* \chi(u_s^y)) = h(\alpha(\chi(\bar{x}))u_{r-1s}\alpha(\chi(y))) = h(u_{r-1s}\alpha(\alpha_{r-1s}(\chi(\bar{x}))\chi(y))) = \delta_{r,s}h_G(\chi(\bar{x})\chi(y)) = \delta_{r,s}\delta_{x,y}.$$

Finally, we get a complete set of irreducibles since the linear span of the coefficients of the u_{γ}^x is \mathcal{A} , which is dense in $C_m(G)$.

(3). We established in (2) that $\mathcal{A} = \text{Pol}(\mathbb{G})$. Since, by definition, A_m is the enveloping C^* -algebra of \mathcal{A} , we have $C_m(\mathbb{G}) = A_m$. Since $\lambda : A_m \rightarrow A$ is surjective and ω is faithful on A , we have $C(\mathbb{G}) = A$. Moreover, since λ is identity on $\mathcal{A} = \text{Pol}(G)$, it follows that λ is the canonical surjection. Finally, $L^{\infty}(\mathbb{G})$ is, by definition, the bicommutant of $C(\mathbb{G}) = A$ which is also the von Neumann algebraic crossed product. \square

Remark 6.2. From section 2.2, we have a group homomorphism $\Gamma \rightarrow S(\text{Irr}(G))$, $\gamma \mapsto \alpha_{\gamma}$, where $\alpha_{\gamma}(x)$, for $x \in \text{Irr}(G)$, is the class of the irreducible representation $(\text{id} \otimes \alpha_{\gamma})(u^x)$. Let $\gamma \cdot x \in \text{Irr}(\mathbb{G})$ be the class of u_{γ}^x . Observe that, we have $\gamma \otimes x \otimes \gamma^{-1} = \alpha_{\gamma}(x)$ and $\gamma \cdot x = \gamma \otimes x$, by viewing $\Gamma \subset \text{Irr}(\mathbb{G})$ and $\text{Irr}(G) \subset \text{Irr}(\mathbb{G})$. Hence, the fusion rules of \mathbb{G} are described as follows:

$$r \cdot x \otimes s \cdot y = rs \cdot \alpha_{s^{-1}}(x) \otimes y = \bigoplus_{\substack{t \in \text{Irr}(G) \\ t \subset \alpha_{s^{-1}}(x) \otimes y}} rs \cdot t, \quad \text{for all } r, s \in \Gamma, x, y \in \text{Irr}(G).$$

Moreover we have $\overline{\gamma \cdot x} = \gamma^{-1} \cdot \alpha_{\gamma}(\bar{x})$ for all $\gamma \in \Gamma$ and $x \in \text{Irr}(G)$.

The following corollary is then immediate.

Corollary 6.3. *The following hold.*

1. \mathbb{G} is co-amenable if and only if G is co-amenable and Γ is amenable.
2. If G is co-amenable and Γ is K -amenable, then $\widehat{\mathbb{G}}$ is K -amenable.

Proof. (1). Let G be co-amenable and Γ be amenable. Then as $C_m(G) = C(G)$ and since the full and the reduced crossed products are the same for actions of amenable groups, it follows from the previous theorem that \mathbb{G} is co-amenable. Now, if \mathbb{G}_m is co-amenable, its Haar state is faithful on A_m . In particular $h \circ \lambda \circ \alpha = h_G \circ \lambda_G$ must be faithful on $C_m(G)$ which implies that G is co-amenable. Since $h(u_\gamma) = \delta_{\gamma,e}$, $\gamma \in \Gamma$, we conclude, from Remark 3.6 (since the counit ε_G is an α invariant character on $C_m(G)$), that the canonical trace on $C^*(\Gamma)$ has to be faithful. Hence, Γ is amenable.

(2). Follows from [Cu83, Theorem 2.1 (c)] since $C_m(G) = C(G)$. □

6.1 Weak amenability

This subsection deals with weak amenability of \mathbb{G} constructed in Section 6. We first prove an intermediate technical result to construct finite rank u.c.p. maps from $C(G)$ to itself using compactness of the action and elements of $\ell^\infty(\widehat{G})$ of finite support. Using this construction, we provide estimates on the Cowling-Haagerup constant of \mathbb{G} and show that $C(\mathbb{G})$ is weakly amenable when both Γ and \widehat{G} are weakly amenable and when the action is compact. This enables us to compute Cowling-Haagerup constants in some explicit examples.

For $a \in \ell^\infty(\widehat{G})$ with finite support we define a linear map with finite dimensional rank $m_a : C(G) \rightarrow C(G)$ by $(\text{id} \otimes m_a)(u^x) = u^x(ap_x \otimes 1)$. We say that a net $a_i \in \ell^\infty(\widehat{G})$ converges pointwise to 1, if $\|a_i p_x - p_x\|_{\mathcal{B}(H_x)} \rightarrow 0$ for all $x \in \text{Irr}(G)$. For finitely supported a_i this is equivalent to saying that the net of linear maps (m_{a_i}) converges pointwise in norm to identity.

Recall that \widehat{G} is said to be *weakly amenable* if there exists a net of finitely supported $a_i \in \ell^\infty(\widehat{G})$ converging pointwise to 1 and such that $C = \sup_i \|m_{a_i}\|_{cb} < \infty$. The infimum of those C is denoted by $\Lambda_{cb}(\widehat{G})$ (and is, by definition, infinite if \widehat{G} is not weakly amenable). It was proved in [KR99] that, when G is Kac, we have $\Lambda_{cb}(\widehat{G}) = \Lambda_{cb}(C(G)) = \Lambda_{cb}(L^\infty(G))$.

Lemma 6.4. *Suppose that the action $\alpha : \Gamma \curvearrowright G$ is compact. Denote by $H < \text{Aut}(G)$ the compact group obtained by taking the closure of the image of Γ in $\text{Aut}(G)$. If $a \in \ell^\infty(\widehat{G})$ has finite support, then the linear map $\Psi : C(G) \rightarrow C(G)$, by $\Psi(x) = \int_H (h^{-1} \circ m_a \circ h)(x) dh$ has finite dimensional rank and $\|\Psi\|_{cb} \leq \|m_a\|_{cb}$, where dh denotes integration with respect to the normalized Haar measure on H .*

Proof. First observe that Ψ is well defined since, for all $x \in C(G)$, the map $H \ni h \mapsto (h^{-1} \circ m_a \circ h)(x) \in C(G)$ is continuous. Moreover, the linearity of Ψ is obvious. Since a has finite support, the map m_a is of the form $m_a(\cdot) = \omega_1(\cdot)y_1 + \dots + \omega_n(\cdot)y_n$, where $\omega_i \in C(G)^*$ and $y_i \in \text{Pol}(G)$. Hence, to show the finite dimensional rank assertion, it suffices to show that the map $\Psi(x) = \int_H (h^{-1} \circ \varphi \circ h)(x) dh$ has finite dimensional rank when $\varphi(\cdot) = \omega(\cdot)y$, with $\omega \in C(G)^*$ and $y \in \text{Pol}(G)$. In this case, we have $\Psi(x) = \int_H \omega(h(x))h^{-1}(y)dh$. Write y as a finite sum $y = \sum_{i=1}^N \sum_{k,l} \lambda_{i,k,l} u_{kl}^{x_i}$. Let $F = \{x_1, \dots, x_N\} \subset \text{Irr}(G)$. Since H is compact, the action of H on $\text{Irr}(G)$ has finite orbits. Writing $h \cdot x$ for the action of $h \in H$ on $x \in \text{Irr}(G)$, the set $H \cdot F = \{h \cdot x : h \in H, x \in F\} \subset \text{Irr}(G)$ is finite and, for all $h \in H$, $h^{-1}(y) \in \mathcal{F}$, where \mathcal{F} is the finite dimensional subspace of $C(G)$ generated by the coefficients of the irreducible representations $x \in H \cdot F$. Hence, the map $h \mapsto \omega(h(x))h^{-1}(y)$ takes values in \mathcal{F} , for all $x \in C(G)$. It follows that $\Psi(x) = \int_H \omega(h(x))h^{-1}(y)dh \in \mathcal{F}$ for all $x \in C(G)$. Hence, Ψ has finite dimensional rank.

Now we proceed to show that $\|\Psi\|_{cb} \leq \|m_a\|_{cb}$. For $n \in \mathbb{N}$, denote by Ψ_n the map

$$\Psi_n = \text{id} \otimes \Psi : M_n(\mathbb{C}) \otimes C(G) \rightarrow M_n(\mathbb{C}) \otimes C(G).$$

Observe that $\Psi_n(X) = \int_H (\text{id} \otimes (h^{-1} \circ m_a \circ h))(X) dh$ for all $X \in M_n(\mathbb{C}) \otimes C(G)$. Hence, for $n \in \mathbb{N}$, one has

$$\|\Psi_n(X)\| \leq \int_H \|(\text{id} \otimes (h^{-1} \circ m_a \circ h))(X)\| dh \leq \|X\| \int_H \|(h^{-1} \circ m_a \circ h)\|_{cb} dh \leq \|X\| \|m_a\|_{cb}.$$

It follows that $\|\Psi\|_{cb} \leq \|m_a\|_{cb}$. \square

Theorem 6.5. *We have $\max(\Lambda_{cb}(\Gamma), \Lambda_{cb}(C(G))) \leq \Lambda_{cb}(C(\mathbb{G}))$. Moreover, if the action $\Gamma \curvearrowright G$ is compact, then $\Lambda_{cb}(C(\mathbb{G})) \leq \Lambda_{cb}(\Gamma)\Lambda_{cb}(\widehat{G})$.*

Proof. The first inequality is obvious by the existence of conditional expectations from $C(\mathbb{G})$ to $C_r^*(\Gamma)$ and from $C(\mathbb{G})$ to $C(G)$. Let us prove the second inequality. We may assume that Γ and \widehat{G} are weakly amenable (in contrary to which the inequality is trivial). Fix $\epsilon > 0$.

Let $a_i \in \ell^\infty(\widehat{G})$ be a net of finitely supported elements such that $\sup_i \|m_{a_i}\|_{cb} \leq \Lambda_{cb}(\widehat{G}) + \epsilon$ and m_{a_i} converges pointwise in norm to identity. Consider the maps Ψ_i associated to a_i as in Lemma 6.4. Observe that the net Ψ_i converge pointwise in norm to identity. Indeed, for $x \in C(G)$,

$$\|\Psi_i(x) - x\| = \left\| \int_H ((h^{-1} \circ m_{a_i} \circ h)(x) - x) dh \right\| = \left\| \int_H (h^{-1}(m_{a_i}(h(x)) - h(x)) dh \right\| \leq \int_H \|m_{a_i}(h(x)) - h(x)\| dh.$$

Now the right hand side of the above expression is converging to 0 for all $x \in C(G)$ by the dominated convergence theorem, since $\|m_{a_i}(h(x)) - h(x)\| \rightarrow_i 0$ for all $x \in C(G)$ and all $h \in H$ and

$$\|m_{a_i}(h(x)) - h(x)\| \leq (\|m_{a_i}\|_{cb} + 1)\|x\| \leq (\Lambda_{cb}(\widehat{G}) + \epsilon + 1)\|x\| \quad \text{for all } i \text{ and all } x \in C(G).$$

By definition, the maps Ψ_i are Γ -equivariant i.e., $\Psi_i \circ \alpha_\gamma = \alpha_\gamma \circ \Psi_i$. Hence, for all i , there is a unique linear extension $\widetilde{\Psi}_i : C(\mathbb{G}) \rightarrow C(\mathbb{G})$ such that $\widetilde{\Psi}_i(u_\gamma \alpha(x)) = u_\gamma \alpha(\Psi_i(x))$ for all $x \in C(G)$ and all $\gamma \in \Gamma$. Moreover, $\|\widetilde{\Psi}_i\|_{cb} \leq \|\Psi_i\|_{cb} \leq \|m_{a_i}\|_{cb} \leq \Lambda_{cb}(\widehat{G}) + \epsilon$.

Consider a sequence of finitely supported maps $\psi_j : \Gamma \rightarrow \mathbb{C}$ going pointwise to 1 and such that $\sup \|m_{\psi_j}\|_{cb} \leq (\Lambda_{cb}(\Gamma) + \epsilon)$ and denote by $\widetilde{\psi}_j : C(\mathbb{G}) \rightarrow C(\mathbb{G})$ the unique linear extension such that $\widetilde{\psi}_j(u_\gamma \alpha(x)) = \psi_j(\gamma) u_\gamma \alpha(x)$. Then, we have $\|\widetilde{\psi}_j\|_{cb} \leq \|m_{\psi_j}\|_{cb} \leq \Lambda_{cb}(\Gamma) + \epsilon$.

Define the maps $\varphi_{i,j} = \widetilde{\psi}_j \circ \widetilde{\Psi}_i : C(\mathbb{G}) \rightarrow C(\mathbb{G})$. Then for all i, j we have $\|\varphi_{i,j}\|_{cb} \leq (\Lambda_{cb}(\Gamma) + \epsilon)(\Lambda_{cb}(\widehat{G}) + \epsilon)$. Since $\varphi_{i,j}(u_\gamma \alpha(x)) = \psi_j(\gamma) u_\gamma \alpha(\Psi_i(x))$, it is clear that $\varphi_{i,j}$ has finite dimensional rank, and $(\varphi_{i,j})_{i,j}$ is going pointwise in norm to identity. Since ϵ was arbitrary, the proof is complete. \square

Example 6.6. We explain here the example described in [Wa95b]. Let G be a compact quantum group and denote by $\chi(G)$ the set of unital $*$ -homomorphisms from $C_m(G)$ to \mathbb{C} . It is a group with the product defined by $gh = (g \otimes h) \circ \Delta$, for $g, h \in \chi(G)$. The unit of $\chi(G)$ is the counit $\varepsilon_G \in C_m(G)^*$ and the inverse of $g \in \chi(G)$ is given by $g \circ S$, where S is the antipode on $C_m(G)$. Viewing $\chi(G)$ as a closed subset of the unit ball of $C_m(G)^*$, one can consider the weak* topology on $\chi(G)$ which make $\chi(G)$ a compact group. Define, for all $g \in \chi(G)$, the map $\alpha_g = (g^{-1} \otimes \text{id} \otimes g) \circ \Delta^{(2)}$. It defines a continuous group homomorphism $\chi(G) \ni g \mapsto \alpha_g \in \text{Aut}(G) = \text{Aut}(G)$. Since $\chi(G)$ is compact, it follows that the action of $\Gamma \curvearrowright G$ is always compact. Actually, it is shown in [Pa13, Section 4] that the action of $\chi(G)$ on $\text{Irr}(G)$ is trivial. Indeed, for $g \in \chi(G)$ and $x \in \text{Irr}(G)$ a straightforward computation gives $(\text{id} \otimes \alpha_g)(u^x) = (V^* \otimes 1)u^x(V \otimes 1)$, where $V = (\text{id} \otimes g)(u^x)$.

We denote by G_Γ the crossed product.

Let us now give some explicit examples. Since the Cowling-Haagerup constant for O_N^+ and U_N^+ are both 1 [Fr13], for any subgroup $\Gamma < U(N)$ (resp. $\Gamma < O(N)$), one has $\Lambda_{cb}(\widehat{(U_N^+)_\Gamma}) = \Lambda_{cb}(\Gamma)$ (resp. $\Lambda_{cb}(\widehat{(O_N^+)_\Gamma}) = \Lambda_{cb}(\Gamma)$). Thus if, Γ is any subgroup of $\text{SL}_2(\mathbb{Z}) < O(N) < U(N)$ for $N \geq 2$ (for example $\Gamma = \mathbb{F}_n$ or $\Gamma = \text{SL}_2(\mathbb{Z})$), then $\Lambda_{cb}(\widehat{(U_N^+)_\Gamma}) = \Lambda_{cb}(\widehat{(O_N^+)_\Gamma}) = 1$.

6.2 Rapid Decay

To any length function L on a discrete group Γ , Jolissaint in [Jo90] associated a space $H_L^\infty(\Gamma)$ of rapidly decreasing functions on Γ with respect to L . He defined RD for discrete groups as: Γ has RD, when there exists a length function L on Γ such that $H_L^\infty(\Gamma) \subset C_r^*(\Gamma)$. Verginoux in [Ve07] pursued the notion of RD and extended it to the context of discrete quantum groups. Following [Ve07], in this section, we limit ourselves to Kac algebras and prove permanence of RD for \mathbb{G} (in crossed products).

We follow the definitions of [Ve07] and we assume in this subsection that G is a compact Kac algebra.

For $a \in \ell^\infty(\widehat{G})$ with finite support, define its Fourier transform as:

$$\mathcal{F}_G(a) = (h_{\widehat{G}} \otimes 1)(V(a \otimes 1)) = \sum_x \dim(x)(\text{Tr}_x \otimes \text{id})(u^x(ap_x \otimes 1)) \in \text{Pol}(G),$$

where Tr_x is the unique trace on $\mathcal{B}(H_x)$ such that $\text{Tr}_x(1) = \dim(x)$.

Let $\alpha : \Gamma \curvearrowright G$ be an action by quantum automorphisms and denote by \mathbb{G} the crossed product. Recall that $\text{Irr}(\mathbb{G}) = \{\gamma \cdot x : \gamma \in \Gamma \text{ and } x \in \text{Irr}(G)\}$, where $\gamma \cdot x$ is the equivalence class of

$$u_\gamma^x = (1 \otimes u_\gamma)(\text{id} \otimes \alpha)(u^x) \in \mathcal{B}(H_x) \otimes C(\mathbb{G}).$$

Let $V_{\gamma \cdot x} : H_{\gamma \cdot x} \rightarrow H_x$ be the unique unitary such that $u^{\gamma \cdot x} = (V_{\gamma \cdot x}^* \otimes 1)u_\gamma^x(V_{\gamma \cdot x} \otimes 1)$.

Lemma 6.7. *Let $a \in \ell^\infty(\widehat{\mathbb{G}}) = \bigoplus_{\gamma \in \Gamma, x \in \text{Irr}(G)} H_{\gamma \cdot x}$ and write $a = \sum_{\gamma \in S, x \in T} ap_{\gamma \cdot x}$ where $S \subset \Gamma$ and $T \subset \text{Irr}(G)$ are finite subsets. For $\gamma \in S$, consider the finitely supported $a_\gamma \in \ell^\infty(\widehat{G})$ by $a_\gamma = \sum_{x \in T} V_{\gamma \cdot x} ap_{\gamma \cdot x} V_{\gamma \cdot x}^* p_x$. Then:*

$$\mathcal{F}_{\mathbb{G}}(a) = \sum_{\gamma \in S} u_\gamma \alpha(\mathcal{F}_G(a_\gamma)).$$

Proof. Observe that, since $V_{\gamma \cdot x}$ is unitary, $\text{Tr}_x(V_{\gamma \cdot x}^* A V_{\gamma \cdot x} B) = \text{Tr}_{\gamma \cdot x}(A V_{\gamma \cdot x} B V_{\gamma \cdot x}^*)$ for all $\gamma \in \Gamma$, all $x \in \text{Irr}(G)$ and all $A \in \mathcal{B}(H_{\gamma \cdot x})$, $B \in \mathcal{B}(H_x)$. Hence,

$$\begin{aligned} \mathcal{F}_{\mathbb{G}}(a) &= \sum_{\gamma \in S, x \in T} \dim(\gamma \cdot x)(\text{Tr}_{\gamma \cdot x} \otimes \text{id})(u^{\gamma \cdot x}(ap_{\gamma \cdot x} \otimes 1)) \\ &= \sum_{\gamma \in S, x \in T} \dim(x)(\text{Tr}_{\gamma \cdot x} \otimes \text{id})((V_{\gamma \cdot x}^* \otimes 1)u_\gamma^x(V_{\gamma \cdot x} \otimes 1)(ap_{\gamma \cdot x} \otimes 1)) \\ &= \sum_{\gamma \in S, x \in T} \dim(x)(\text{Tr}_x \otimes \text{id})(u_\gamma^x(V_{\gamma \cdot x} ap_{\gamma \cdot x} V_{\gamma \cdot x}^* \otimes 1)) \\ &= \sum_{\gamma \in S} u_\gamma \alpha \left(\sum_{x \in T} \dim(x)(\text{Tr}_x \otimes \text{id})(u_\gamma^x(V_{\gamma \cdot x} ap_{\gamma \cdot x} V_{\gamma \cdot x}^* \otimes 1)) \right) = \sum_{\gamma \in S} u_\gamma \alpha(\mathcal{F}_G(a_\gamma)). \end{aligned}$$

□

A function $l : \text{Irr}(G) \rightarrow [0, \infty)$ is called a *length function on $\text{Irr}(G)$* if $l(\varepsilon) = 0$, $l(\overline{x}) = l(x)$ and that $l(x) \leq l(y) + l(z)$ whenever $x \subset y \otimes z$.

Lemma 6.8. *Let $\alpha : \Gamma \curvearrowright G$ is an action of Γ on G by quantum automorphisms and let l be a length function on $\text{Irr}(G)$ which is α -invariant, i.e., $l(x) = l(\alpha_\gamma(x))$ for all $\gamma \in \Gamma$ and $x \in \text{Irr}(G)$. Let \mathbb{G} be the crossed product. The function $l_0 : \text{Irr}(\mathbb{G}) \rightarrow [0, \infty)$, defined by $l_0(\gamma \cdot x) = l_\Gamma(\gamma) + l(x)$ is a length function on $\text{Irr}(\mathbb{G})$.*

Proof. We have $l_0(\varepsilon_{\mathbb{G}}) = l_\Gamma(e) + l(\varepsilon_G) = 0$ and, by remark 6.2,

$$l_0(\overline{\gamma \cdot x}) = l_0(\gamma^{-1} \cdot \alpha_\gamma(x)) = l_\Gamma(\gamma^{-1}) + l(\alpha_\gamma(x)) = l_\Gamma(\gamma) + l(x) = l_0(\gamma \cdot x).$$

From remark 6.2, $\gamma \cdot x \subset r \cdot y \otimes s \cdot z$ if and only if $\gamma = rs$ and $x \subset \alpha_{r^{-1}}(y) \otimes z$. Hence,

$$l_0(\gamma \cdot x) = l_\Gamma(\gamma) + l(x) \leq l_\Gamma(r) + l_\Gamma(s) + l(\alpha_{r^{-1}}(y)) + l(z) = l_\Gamma(r) + l(y) + l_\Gamma(s) + l(z) = l_0(r \cdot y) + l_0(s \cdot z).$$

□

Given a length function $l : \text{Irr}(G) \rightarrow [0, \infty)$, consider the element $L = \sum_{x \in \text{Irr}(G)} l(x)p_x$ which is affiliated to $c_0(\widehat{G})$. Let q_n denote the spectral projections of L associated to the interval $[n, n+1)$. We say that (\widehat{G}, l) has property (RD), if there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ and $a \in q_k c_c(\widehat{G})$ we have $\|\mathcal{F}(a)\|_{C(G)} \leq P(k)\|a\|_{2, h_{\widehat{G}}}$. Finally, \widehat{G} is said to have property (RD) if there exists a length function on $\text{Irr}(G)$ such that (\widehat{G}, l) has property (RD).

In the following theorem, we obtain permanence of property (RD) for crossed product. In case the action of the group is trivial, i.e., when the crossed product reduces to a tensor product, this result is demonstrated in [CF14, Lemma 4.5].

Theorem 6.9. *Let $\alpha : \Gamma \curvearrowright G$ be an action by quantum automorphisms and suppose G is Kac. Let l be an α -invariant length function on $\text{Irr}(G)$. If (\widehat{G}, l) has property (RD) and Γ has property (RD), then $(\widehat{\mathbb{G}}, l_0)$ has property (RD), where \mathbb{G} is the crossed product and l_0 is as in Lemma 6.8.*

Proof. Let l_Γ be any length function on Γ for which (Γ, l_Γ) has property (RD) and let l_0 be the length function on $\text{Irr}(\mathbb{G})$ defined by $l_0(\gamma \cdot x) = l_\Gamma(\gamma) + l(x)$. Let $L_0 = \sum_{\gamma \in \Gamma, x \in \text{Irr}(G)} l_0(\gamma \cdot x) = \sum_{\gamma \in \Gamma, x \in \text{Irr}(G)} (l_\Gamma(\gamma) + l(x))p_{\gamma \cdot x}$ and $L = \sum_{x \in \text{Irr}(G)} l(x)p_x$. Finally, let p_n and q_n be the spectral projections of respectively L_0 and L associated to the interval $[n, n+1)$. Let $a \in c_c(\widehat{\mathbb{G}})$ and write $a = \sum_{\gamma \in S, x \in T} a p_{\gamma \cdot x}$, where $S \subset \Gamma$ and $T \subset \text{Irr}(G)$ are finite subsets. Now suppose that $a \in p_k c_c(\widehat{\mathbb{G}})$. Since $p_k = \sum_{\gamma \in \Gamma, x \in \text{Irr}(G), k \leq l_\Gamma(\gamma) + l(x) < k+1} p_{\gamma \cdot x}$, we must have $S \subset \{\gamma \in \Gamma : l_\Gamma(\gamma) < k+1\}$ and $T \subset \{x \in \text{Irr}(G) : l(x) < k+1\}$. It follows that, for all $\gamma \in S$, the element a_γ defined in Lemma 6.7 is in $q_k c_c(\widehat{G})$, where $q_k = \sum_{j=0}^k q_j$.

Let P_1 and P_2 be polynomials witnessing (RD) respectively for (\widehat{G}, l) and (Γ, l_Γ) . Let, for $i = 1, 2$, $C_i \in \mathbb{R}_+$ and $N_i \in \mathbb{N}$ such that $P_i(k) \leq C_i(k+1)^{N_i}$ for all $k \in \mathbb{N}$. Then, for all $a \in q_k c_c(\widehat{\mathbb{G}})$ we have

$$\begin{aligned} \|\mathcal{F}_G(a)\| &\leq \sum_{j \leq k} \|\mathcal{F}_G(a q_j)\| \leq \sum_{j \leq k} P_1(j) \|a q_j\|_{2, h_{\widehat{G}}} \leq \sum_{j \leq k} C_1(j+1)^{N_1} \|a q_j\|_{2, h_{\widehat{G}}} \\ &\leq C_1(k+1)^{N_1} \sum_{j \leq k} \|a q_j\|_{2, h_{\widehat{G}}} = C_1(k+1)^{N_1} \|a\|_{2, h_{\widehat{G}}}. \end{aligned}$$

Similarly, $\|\psi * \phi\|_{\ell^2(\Gamma)}^2 \leq C_2(k+1)^{N_2} \|\psi\|_{\ell^2(\Gamma)}^2 \|\phi\|_{\ell^2(\Gamma)}^2$ for all ϕ in $\ell^2(\Gamma)$ and all function ψ on Γ (finitely) supported on words on l_Γ -length less than k .

Let y be a finite sum $y = \sum_s u_s \alpha(b_s) \in \text{Pol}(\mathbb{G})$. We have $\|y\|_{2, h_{\mathbb{G}}}^2 = \sum_s \|b_s\|_{2, h_G}^2$ and, by Lemma 6.7 and the preceding discussion,

$$\begin{aligned} \|\mathcal{F}_{\mathbb{G}}(a)y\|_{2, h_{\mathbb{G}}}^2 &= \left\| \sum_{\gamma \in S, s} u_{\gamma s} \alpha(\alpha_{s-1}(\mathcal{F}_G(a_\gamma))b_s) \right\|_2^2 = \left\| \sum_{\gamma \in S, t} u_t \alpha(\alpha_{t-1}(\mathcal{F}_G(a_\gamma))b_{\gamma^{-1}t}) \right\|_2^2 \\ &= \sum_t \left\| \sum_{\gamma \in S} \alpha_{t-1}(\mathcal{F}_G(a_\gamma))b_{\gamma^{-1}t} \right\|_2^2 \leq \sum_t \left(\sum_{\gamma \in S} \|\alpha_{t-1}(\mathcal{F}_G(a_\gamma))b_{\gamma^{-1}t}\|_2 \right)^2 \\ &\leq C_1^2(k+1)^{2N_1} \sum_t \left(\sum_{\gamma \in S} \|a_\gamma\|_2 \|b_{\gamma^{-1}t}\|_2 \right)^2 = C_1^2(k+1)^{2N_1} \|\psi * \phi\|_{\ell^2(\Gamma)}^2, \end{aligned}$$

where $\psi, \phi \in \ell^2(\Gamma)$ are defined by $\psi(\gamma) = \|a_\gamma\|_2$ and $\phi(s) = \|b_s\|_2$. We note that $\|\psi\|_{\ell^2(\Gamma)}^2 = \sum_{\gamma \in S} \|a_\gamma\|_2^2 = \|a\|_{2, h_{\widehat{G}}}^2$ and $\|\phi\|_{\ell^2(\Gamma)}^2 = \sum_s \|b_s\|_2^2 = \|y\|_2^2$. But since ψ is supported on S i.e., on elements of Γ of length less than k , we have

$$\|\mathcal{F}_{\mathbb{G}}(a)y\|_{2, h_{\mathbb{G}}}^2 \leq (C_1 C_2)^2 (k+1)^{2(N_1+N_2)} \|\psi\|_{\ell^2(\Gamma)}^2 \|\phi\|_{\ell^2(\Gamma)}^2 = P(k)^2 \|a\|_{2, h_{\widehat{G}}}^2 \|y\|_2^2,$$

where $P(x) = C_1 C_2 (x+1)^{N_1+N_2}$. As y is arbitrary, the proof is complete. \square

Example 6.10. There may not exist an α -invariant length function on $\text{Irr}(G)$. However, if $\Gamma \curvearrowright G$ is compact, then the action $\alpha : \Gamma \curvearrowright \text{Irr}(G)$ has finite orbits. Hence, for any length function l on $\text{Irr}(G)$, the length function l_α defined by $l_\alpha(x) = \sup_{\gamma \in \Gamma} l(\alpha_\gamma(x))$, for $x \in \text{Irr}(G)$, is α -invariant. Hence, \widehat{G} has (RD) whenever Γ and \widehat{G} have (RD). In particular, this happens when $\Gamma < \chi(G)$ (see Example 6.6). It was shown in [Ve07] that $\widehat{O_N^+}$ and $\widehat{O_N^+}$ have (RD). It follows that for any subgroup $\Gamma < U(N)$ (resp. $\Gamma < O(N)$) the semi-direct products $(U_N^+)_\Gamma$ (resp. $(O_N^+)_\Gamma$) has (RD) whenever Γ has (RD) (for example when $\Gamma = \mathbb{F}_n < \mathbb{F}_2 < \text{SL}_2(\mathbb{Z}) < O(N) < U(N)$ for all $N \geq 2$).

6.3 Property (T)

In the quantum setup (as in Section 6), here we characterize relative property (T) of the pair (G, \mathbb{G}) . Our results in this section generalize similar result in [CT11] concerned with inclusion of groups of the form $H \subset \Gamma \ltimes H$. We also study the scenario in which \mathbb{G} has property (T).

Let $\varepsilon_{\mathbb{G}}$ (resp. ε_G) be the counit on $C_m(\mathbb{G})$ (resp. on $C_m(G)$). Since the map $\alpha : C_m(G) \rightarrow C_m(\mathbb{G})$ intertwines the comultiplication, we have $\varepsilon_{\mathbb{G}} \circ \alpha = \varepsilon_G$.

When $\pi : A \rightarrow \mathcal{B}(H)$ is a unital $*$ -homomorphism from a unital C^* -algebra A , we denote by $\tilde{\pi} : A^{**} \rightarrow \mathcal{B}(H)$ its unique normal extension. Also, we view any state $\omega \in A^*$ as a normal state on A^{**} . Observe that if (H, π, ξ) is the GNS construction for the state ω on A , then $(H, \tilde{\pi}, \xi)$ is the GNS construction for the normal state ω on A^{**} .

Let $M = C_m(G)^{**}$ and $p_0 \in M$ be the unique central projection such that $p_0 x p_0 = \tilde{\varepsilon}_G(x) p_0$ for all $x \in M$.

In the following theorem, we characterize the relative co-property (T) of the pair (G, \mathbb{G}) in terms of the action α of Γ on G . The proof is similar to the proof of Theorem 4.2 but technically more is involved.

Theorem 6.11. *The following are equivalent:*

1. *The pair (G, \mathbb{G}) does not have the relative co-property (T).*
2. *There exists a sequence $(\omega_n)_{n \in \mathbb{N}}$ of states on $C_m(G)$ such that*
 - (a) $\omega_n(p_0) = 0$ for all $n \in \mathbb{N}$;
 - (b) $\omega_n \rightarrow \varepsilon_G$ weak*;
 - (c) $\|\alpha_\gamma(\omega_n) - \omega_n\| \rightarrow 0$ for all $\gamma \in \Gamma$.

Proof. For a representation $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$, we have $\varepsilon \subset \pi \circ \alpha$ if and only if $K_\pi \neq \{0\}$, where

$$K_\pi = \{\xi \in H : \pi \circ \alpha(a)\xi = \varepsilon_G(a)\xi \text{ for all } a \in C_m(G)\}.$$

Let $\rho = \pi \circ \alpha : C_m(G) \rightarrow \mathcal{B}(H)$ and observe that the orthogonal projection onto K_π is the projection $\tilde{\rho}(p_0)$. Indeed, for all $\xi \in H$, $a \in C_m(G)$, we have $\pi \circ \alpha(a)\tilde{\rho}(p_0)\xi = \tilde{\rho}(ap_0)\xi = \varepsilon_G(a)\tilde{\rho}(p_0)\xi$, which implies that $\text{Im}(\tilde{\rho}(p_0)) \subset K_\pi$. Moreover, if $\xi \in K_\pi$, we have $\tilde{\rho}(a)\xi = \tilde{\varepsilon}_G(a)\xi$ for all $a \in C_m(G)$. Since $C_m(G)$ is σ -weakly dense in M and the representations $\tilde{\rho}$ and $\tilde{\varepsilon}_G$ are normal, it follows that the equation $\tilde{\rho}(a)\xi = \tilde{\varepsilon}_G(a)\xi$ is valid for all $a \in M$. Hence, for $a = p_0$ we get $\tilde{\rho}(p_0)\xi = \tilde{\varepsilon}_G(p_0)\xi = \xi$, which in turn implies that $K_\pi \subset \text{Im}(\tilde{\rho}(p_0))$.

(1) \implies (2). Suppose that the pair (G, \mathbb{G}) does not have the relative co-property (T). Let $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$ be a representation such that $\varepsilon_{\mathbb{G}} \prec \pi$ and $K_\pi = \{0\}$. Denote by $\omega_{\xi, \eta} \in C_m(G)^*$ the functional given by $\omega_{\xi, \eta}(a) = \langle \pi \circ \alpha(a)\xi, \eta \rangle$. Hence, $\omega_{\xi, \eta}(p_0) = \langle \tilde{\rho}(p_0)\xi, \eta \rangle = 0$ for all $\xi, \eta \in H$.

Since $\varepsilon_{\mathbb{G}} \prec \pi$, let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of unit vectors in H such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \rightarrow 0$ for all $x \in C_m(\mathbb{G})$. Define $\omega_n = \omega_{\xi_n, \xi_n}$. Then, we have $\omega_n(p_0) = 0$ for all $n \in \mathbb{N}$. For all $a \in C_m(G)$ we have, $|\omega_n(a) - \varepsilon_G(a)| = |\langle \pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n, \xi_n \rangle| \leq \|\pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n\| \rightarrow 0$. Moreover, exactly as in the proof of Theorem 4.2, we find $\|\alpha_\gamma(\omega_n) - \omega_n\| \leq 2\|\pi(u_\gamma)\xi_n - \xi_n\| = \|\pi(u_\gamma)\xi_n - \varepsilon_{\mathbb{G}}(u_\gamma)\xi_n\| \rightarrow 0$.

(2) \implies (1). We first prove the following claim. Let $s(\omega) \in M$ be the support of a state $\omega \in C_m(G)^* = M_*$. Recall that $s(\omega) \in M$ is the unique projection in M such that $N_\omega = M(1 - s(\omega))$, where N_ω is the left ideal

defined by $N_\omega = \{x \in M : \omega(x^*x) = 0\}$. Moreover, ω is faithful on $s(\omega)Ms(\omega)$. In the sequel, we still denote by α_γ the unique $*$ -isomorphism of M which extends $\alpha_\gamma \in \text{Aut}(C_m(G))$.

Claim. *If (2) holds, then there exists a sequence $(\omega_n)_{n \in \mathbb{N}}$ of states on $C_m(G)$ satisfying (a), (b) and (c) and such that $\alpha_\gamma(s(\omega_n)) = s(\omega_n)$ for all $\gamma \in \Gamma$, $n \in \mathbb{N}$.*

Proof of the claim. Denote by $\ell^1(\Gamma)_{1,+}$ the set of positive ℓ^1 functions on Γ with $\|f\|_1 = 1$. For a state $\omega \in C_m(G)^* = M_*$ and $f \in \ell^1(\Gamma)_{1,+}$ define the state $f * \omega \in C_m(G)^*$ by the convex combination

$$f * \mu = \sum_{\gamma \in \Gamma} f(\gamma) \alpha_\gamma(\omega).$$

Observe that, for all $\gamma \in \Gamma$ we have $\delta_\gamma * \mu = \alpha_\gamma(\mu)$ and $\alpha_\gamma(f * \mu) = f_\gamma * \mu$, where $f_\gamma \in \ell^1(\Gamma)_{1,+}$ is defined by $f_\gamma(r) = f(\gamma^{-1}r)$, $r \in \Gamma$.

Moreover, if $f \in \ell^1(\Gamma)_{1,+}$ is such that $f(\gamma) > 0$ for all $\gamma \in \Gamma$, then since $(f * \omega)(x^*x) = \sum_\gamma f(\gamma) \omega(\alpha_{\gamma^{-1}}(x^*x))$, we have that $(f * \omega)(x^*x) = 0$ if and only if $\omega(\alpha_{\gamma^{-1}}(x^*x)) = 0$ for all $\gamma \in \Gamma$. It follows that

$$N_{f*\omega} = \bigcap_{\gamma \in \Gamma} \alpha_\gamma(N_\omega) = M (\wedge_{\gamma \in \Gamma} (1 - \alpha_\gamma(s(\omega)))).$$

Hence, $s(f * \omega) = 1 - \wedge_{\gamma \in \Gamma} (1 - \alpha_\gamma(s(\omega))) = \vee_{\gamma \in \Gamma} \alpha_\gamma(s(\omega))$. Hence, we have $\alpha_\gamma(s(f * \omega)) = s(f * \omega)$ for all $\gamma \in \Gamma$. Finally, since α_γ is a quantum automorphism for all $\gamma \in \Gamma$, we have $\varepsilon_G \circ \alpha_\gamma = \varepsilon_G$. Hence, $\alpha(p_0)$ is a central projection in M satisfying $a\alpha_\gamma(p_0) = \alpha_\gamma(\alpha_{\gamma^{-1}}(a)p_0) = \varepsilon_G(\alpha_{\gamma^{-1}}(a))\alpha_\gamma(p_0) = \varepsilon_G(a)\alpha_\gamma(p_0)$, $\gamma \in \Gamma$. By uniqueness of such a projection, we find $\alpha_\gamma(p_0) = p_0$ for all $\gamma \in \Gamma$. Hence, for all $f \in \ell^1(\Gamma)_{1,+}$,

$$(f * \omega)(p_0) = \sum_\gamma f(\gamma) \omega(\alpha_{\gamma^{-1}}(p_0)) = \sum_\gamma f(\gamma) \omega(p_0) = \omega(p_0).$$

Let $(\omega_n)_{n \in \mathbb{N}}$ be a sequence of states on $C_m(G)$ satisfying (a), (b) and (c). We have, for all $f \in \ell^1(\Gamma)_{1,+}$ with finite support

$$\|f * \omega_n - \omega_n\| \leq \sum_\gamma f(\gamma) \|\delta_\gamma * \omega_n - \omega_n\| = \sum_\gamma f(\gamma) \|\alpha_\gamma(\omega_n) - \omega_n\| \rightarrow 0. \quad (6.1)$$

Since such functions f are dense in $\ell^1(\Gamma)_{1,+}$ (in the ℓ^1 -norm), it follows that $\|f * \omega_n - \omega_n\| \rightarrow 0$ for all $f \in \ell^1(\Gamma)_{1,+}$.

Let $\xi \in \ell^1(\Gamma)_{1,+}$ be any function such that $\xi > 0$ and define $\nu_n = \xi * \omega_n$. By the preceding discussion, we know that $\alpha_\gamma(s(\nu_n)) = s(\nu_n)$ for all $\gamma \in \Gamma$ and $\nu_n(p_0) = \omega_n(p_0) = 0$ for all $n \in \mathbb{N}$. Moreover, by Eq. (6.1) we have $\|\alpha_\gamma(\nu_n) - \nu_n\| = \|\xi_\gamma * \omega_n - \xi * \omega_n\| \leq \|\xi_\gamma * \omega_n - \omega_n\| + \|\omega_n - \xi * \omega_n\| \rightarrow 0$ for all $\gamma \in \Gamma$.

Since $\omega_n \rightarrow \varepsilon_G$ weak* and $\varepsilon_G \circ \alpha_\gamma = \varepsilon_G$, we have, $|\omega_n(\alpha_\gamma(a)) - \varepsilon_G(a)| \rightarrow 0$ for all $a \in C_m(G)$ and all $\gamma \in \Gamma$. Hence, the Lebesgue dominated convergence theorem implies that, for all $a \in C_m(G)$,

$$|\nu_n(a) - \varepsilon_G(a)| = \left| \sum_\gamma f(\gamma) (\omega_n(\alpha_{\gamma^{-1}}(a)) - \varepsilon_G(a)) \right| \leq \sum_\gamma f(\gamma) |\omega_n(\alpha_{\gamma^{-1}}(a)) - \varepsilon_G(a)| \rightarrow 0.$$

It follows that $\nu_n \rightarrow \varepsilon_G$ weak* and this completes the proof of the claim. \square

We can now finish the proof of the Theorem. Let $(\omega_n)_{n \in \mathbb{N}}$ be a sequence of states on $C_m(G)$ as in the Claim. Let $M_n = s(\omega_n)Ms(\omega_n)$ and, since ω_n is faithful on M_n , view $M_n \subset \mathcal{B}(H_n)$ where (H_n, ξ_n) is the GNS construction of the f.n.s. ω_n on M_n . Define $\rho_n : C_m(G) \subset M \rightarrow M_n \subset \mathcal{B}(H_n)$ by $a \mapsto s(\omega_n)as(\omega_n)$. By definition, the unique normal extension of ρ_n is $\tilde{\rho}_n : M \rightarrow M_n$, $x \mapsto s(\omega_n)x s(\omega_n)$. Since $\alpha_\gamma(s(\omega_n)) = s(\omega_n)$, the action α restricts to an action, still denoted by α of Γ on M_n . Since $M_n \subset \mathcal{B}(H_n)$ is in standard form, we may consider the standard implementation (see [TakII Def 1.6]) of the action Γ on M_n to get a unitary representation $u_n : \Gamma \rightarrow \mathcal{U}(H_n)$ such that $\alpha_\gamma(x) = u_n(\gamma)xu_n(\gamma^{-1})$ for all $x \in M_n$.

By the universal property of A_m , for $n \in \mathbb{N}$ there exists a unique unital $*$ -homomorphism

$$\pi_n : A_m \rightarrow \mathcal{B}(H_n) \quad \text{such that} \quad \pi_n(\gamma) = u_n(\gamma) \quad \text{and} \quad \pi_n \circ \alpha = \rho_n.$$

Since $\omega_n(p_0) = 0$, we have $s(\omega_n)p_0s(\omega_n) = 0$. Hence, $\tilde{\rho}_n(p_0) = 0$ and $K_{\pi_n} = \{0\} \forall n \in \mathbb{N}$. It follows that, if we define $H = \bigoplus_n H_n$ and $\pi = \bigoplus_n \pi_n : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$, then $K_\pi = \{0\}$ as well. Hence it suffices to show that $\varepsilon_{\mathbb{G}} \prec \pi$. Since ξ_n is in the self-dual cone of ω_n and $u_n(\gamma)$ is the standard implementation of α_γ , it follows from [Ta00, Theorem 1.14] that $u_n(\gamma)\xi_n$ is also in the self-dual cone of ω_n for all $n \in \mathbb{N}$. Hence, we may apply [Ta00, Theorem 1.2] to get $\|u_n(\gamma)\xi_n - \xi_n\|^2 \leq \|\omega_{u_n(\gamma)\xi_n} - \omega_{\xi_n}\|$ for all $n \in \mathbb{N}$, $\gamma \in \Gamma$. Observe that $\omega_{u_n(\gamma)\xi_n}(x) = \alpha_\gamma(\omega_n)(x)$ and $\omega_{\xi_n}(x) = \omega_n(x)$ for all $x \in M$. Hence,

$$\|u_n(\gamma)\xi_n - \varepsilon_{\mathbb{G}}(u_\gamma)\xi_n\| = \|u_n(\gamma)\xi_n - \xi_n\| \leq \|\alpha_\gamma(\omega_n) - \omega_n\|^{\frac{1}{2}} \rightarrow 0.$$

Since $\omega_n \rightarrow \varepsilon_G$ weak*, it follows that for all $x = u_\gamma\alpha(a) \in C_m(\mathbb{G})$, we have

$$\begin{aligned} \|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| &= \|\pi(u_\gamma)\pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n\| \\ &\leq \|\pi(u_\gamma)(\pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n)\| + |\varepsilon_G(a)| \|\pi(u_\gamma)\xi_n - \xi_n\| \\ &\leq \|\pi(\alpha(a))\xi_n - \varepsilon_G(a)\xi_n\| + |\varepsilon_G(a)| \|u_n(\gamma)\xi_n - \xi_n\| \rightarrow 0. \end{aligned}$$

By linearity and the triangle inequality, we have $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \rightarrow 0$ for all $x \in \mathcal{A}$. We conclude the proof using the density of \mathcal{A} in $C_m(\mathbb{G})$. \square

We now turn to Property (T).

Theorem 6.12. *The following holds:*

1. If $\widehat{\mathbb{G}}$ has Property (T), then Γ has property T.
2. If \widehat{G} has Property (T) and Γ has Property (T), then $\widehat{\mathbb{G}}$ has property (T).

Proof. (1). This is the same proof of assertion 1 of Theorem 4.3: we use the counit on $C_m(G)$ and the universal property of $C_m(\mathbb{G})$ to construct a surjective *-homomorphism $C_m(\mathbb{G}) \rightarrow C^*(\Gamma)$ which intertwines the comultiplications. We conclude by [Fi10, Proposition 6].

(2). We use the notations introduced at the beginning of this section and in the proof of Theorem 6.11. Let $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$ and consider the representation $\rho = \pi \circ \alpha : C_m(G) \rightarrow \mathcal{B}(H)$ and the unitary representation $v_\gamma = \pi(u_\gamma)$ of Γ on H . Let $K_\pi = \{\xi \in H : \rho(a)\xi = \varepsilon_G(a)\xi \text{ for all } a \in C_m(G)\}$ and recall that the orthogonal projection onto K_π is $P = \tilde{\rho}(p_0)$ and that $\alpha_\gamma(p_0) = p_0$ for all $\gamma \in \Gamma$. Hence, $v_\gamma P v_{\gamma^{-1}} = \tilde{\rho}(\alpha_\gamma(p_0)) = P$ for all $\gamma \in \Gamma$, and it follows that K_π is invariant subspace of v .

Suppose that $\varepsilon_{\mathbb{G}} \prec \pi$. By property (T) of \widehat{G} , the space K_π is non-zero and we can argue exactly as in the proof of Theorem 4.3 to conclude. \square

Remark 6.13. It follows from the proof of the first assertion of the previous theorem that $C^*(\Gamma)$ is a quantum subgroup of the compact quantum group \mathbb{G} . Now, an irreducible representation of \mathbb{G} of the form u_γ^x (with dimension say m), when restricted to the subgroup $C^*(\Gamma)$, decomposes as a direct sum of m copies of γ . It now follows from [Pa13, Theorem 6.3] that $C^*(\Gamma)$ is a central subgroup (see [Pa13, Definition 6.1]). Furthermore, Γ induces an action on the chain group $c(G)$ (see definition 7.4 [Pa13, Definition 7.4]) of G and it follows from Remark 6.2 that the chain group (and hence the center, see [Pa13, section 7]) of \mathbb{G} is the semidirect product group $c(G) \rtimes \Gamma$.

Remark 6.14. (Kazhdan Pair for \mathbb{G}) Let (E_1, δ_1) be a Kazhdan pair for G and (E_2, δ_2) be a Kazhdan Pair for Γ . Then it is not hard to show that $E = (E_1 \cup E_2) \subset \text{Irr}(\mathbb{G})$ and $\delta = \min(\delta_1, \delta_2)$ is a Kazhdan pair for \mathbb{G} . Indeed, let $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$ be a *-representation having a (E, δ) -invariant (unit) vector ξ . Then restricting to the subalgebra $C_m(G)$ (and denoting the corresponding representation by π_G), we get an (E_1, δ_1) invariant vector and hence, there is an invariant vector $\eta \in H$. We may assume $\|\xi - \eta\| < 1$ (this follows from a quantum group version of Proposition 1.1.9 of [BDV08], which can be proved in an exactly similar fashion). Now, restricting π to Γ , denoting the corresponding representation by u , we have that the closed linear u -invariant subspace generated by $u_g\eta$, $g \in \Gamma$ (which we denote by H_η), is a subspace of the space

of π_G -invariant vectors (as $u_g \pi_G(a) u_g^{-1} = \pi_G(\alpha_g(a))$). Let P_{H_η} denote the orthogonal projection onto H_η . Now, the vector $P_{H_\eta} \xi$, which is non-zero, as $\|\xi - \eta\| < 1$, is an (E_2, δ_2) -invariant vector for the representation u , restricted to H_η . So, there exists an u -invariant vector $\eta_0 \in H_\eta$. This vector is, of course then, π -invariant and hence, we are done.

6.4 Haagerup property

In this section, we study the (relative) Haagerup property. We provide natural conditions that characterize the relative co-Haagerup property of the pair (G, \mathbb{G}) . We exhibit that the pair $(O_N^+, (O_N^+)_\Gamma)$ has the relative co-Haagerup property for any discrete subgroup Γ of $\chi(O_N^+)$. We extend a result of Jolissaint on Haagerup property of crossed product of a finite von Neumann algebra by a discrete group to the context of arbitrary von Neumann algebras. Thus, we can decide whether $L^\infty(\mathbb{G})$ has the Haagerup property. Finally, we also provide sufficient conditions for $\widehat{\mathbb{G}}$ to possess the Haagerup property.

For the relative Haagerup property for crossed-product we obtain the following result, similar to theorem 5.3. The proof is even simpler in the crossed-product case, since α is an action by quantum automorphisms.

Theorem 6.15. *The following are equivalent:*

1. *The pair (G, \mathbb{G}) has the relative co-Haagerup property.*
2. *There exists a sequence $(\omega_n)_{n \in \mathbb{N}}$ of states on $C_m(G)$ such that*
 - (a) $\widehat{\omega}_n \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$;
 - (b) $\omega_n \rightarrow \varepsilon_G$ weak*;
 - (c) $\|\alpha_\gamma(\omega_n) - \omega_n\| \rightarrow 0$ for all $\gamma \in \Gamma$.

Proof. (1) \Rightarrow (2). The argument is exactly the same as the proof of (1) \implies (2) of Theorem 5.3.

(2) \implies (1). We first prove the following claim

Claim. *If (2) holds, then there exists a sequence $(\nu_n)_{n \in \mathbb{N}}$ of states on $C_m(G)$ satisfying (a), (b) and (c) and such that $\alpha_\gamma(s(\nu_n)) = s(\nu_n)$ for all $\gamma \in \Gamma$, $n \in \mathbb{N}$.*

Proof of the claim. By the proof of the claim in Theorem 6.11, it suffices to check that, whenever ν is a state on $C_m(G)$ and $f \in \ell^1(G)$, we have $\widehat{\nu} \in c_0(\widehat{G}) \implies \widehat{f * \nu} \in c_0(\widehat{G})$.

We first show that $\widehat{\nu} \in c_0(\widehat{G}) \implies \widehat{\alpha_\gamma(\nu)} \in c_0(\widehat{G})$. Note that we still denote by α the action of Γ on $\text{Irr}(G)$ (see Remark 6.2). Now let ν be a state on $C_m(G)$ such that $\widehat{\nu} \in c_0(\widehat{G})$ and let $\epsilon > 0$. By assumptions, the set $F = \{x \in \text{Irr}(G) : \|(\text{id} \otimes \omega)(u^x)\|_{\mathcal{B}(H_x)} \geq \epsilon\}$ is finite. Hence, the set

$$\{x \in \text{Irr}(G) : \|(\text{id} \otimes \omega)(u^{\alpha_{\gamma^{-1}}(x)})\|_{\mathcal{B}(H_x)} \geq \epsilon\} = \{x \in \text{Irr}(G) : \alpha_{\gamma^{-1}}(x) \in F\} = \alpha_\gamma(F)$$

is also finite. Since $\widehat{\alpha_\gamma(\omega)} = \left((\text{id} \otimes \omega)(u^{\alpha_{\gamma^{-1}}(x)}) \right)_{x \in \text{Irr}(G)}$, it follows that $\widehat{\alpha_\gamma(\omega)} \in c_0(\widehat{G})$.

From this we can now conclude that for all $f \in \ell^1(G)$, we have $\widehat{\nu} \in c_0(\widehat{G}) \implies \widehat{f * \nu} \in c_0(\widehat{G})$ as in the proof of the Claim in Theorem 5.3. \square

We can now finish the proof of the Theorem. Let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence of states on $C_m(G)^*$ as in the Claim. As in the proof of Theorem 6.11, we construct a representation $\pi : C_m(\mathbb{G}) \rightarrow \mathcal{B}(H)$ with a sequence of unit vector $\xi_n \in H$ such that $\|\pi(x)\xi_n - \varepsilon_{\mathbb{G}}(x)\xi_n\| \rightarrow 0$ for all $x \in C_m(\mathbb{G})$ and $\nu_n = \omega_{\xi_n} \circ \pi \circ \alpha$. It follows that the sequence of states $\omega_n = \omega_{\xi_n} \circ \pi \in C_m(\mathbb{G})^*$ satisfies $\omega_n \rightarrow \varepsilon_{\mathbb{G}}$ weak* and $\widehat{\omega_n \circ \alpha} = \widehat{\nu}_n \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$. \square

Example 6.16. Recall from Example 6.6 that the action of $\chi(G)$ on $C_m(G)$ is given by $(\text{id} \otimes \alpha_g)(u^x) = (V^* \otimes 1)u^x(V \otimes 1)$, where $V = (\text{id} \otimes g)(u^x)$. Consequently,

$$\alpha_g(\omega)(u_{ij}^x) = \sum_{r,s} g(u_{ir}^x) \omega(u_{rs}^x) g((u_{js}^x)^*), \text{ for all } \omega \in C_m(G)^*. \quad (6.2)$$

Define the sequence of dilated Chebyshev polynomials of the second kind by the initial conditions $P_0(X) = 1$, $P_1(X) = X$ and the recursion relation $XP_k(X) = P_{k+1}(X) + P_{k-1}(X)$, $k \geq 1$. It is proved in [Br12] (see also [FV14]) that the net of states $\omega_t \in C_m(O_N^+)^*$ defined by $\omega_t(u_{ij}^k) = \frac{P_k(t)}{P_k(N)}\delta_{i,j}$, for $k \in \text{Irr}(O_N^+) = \mathbb{N}$ and $t \in (0, 1)$ realize the co-Haagerup property for O_N^+ i.e., $\widehat{\omega}_t \in c_0(\widehat{O_N^+})$ for t close to 1 and $\omega_t \rightarrow \varepsilon_{O_N^+}$ weak* when $t \rightarrow 1$. Now let $g \in \chi(O_N^+)$. By Eq. (6.2), we have

$$\alpha_g(\omega_t)(u_{ij}^k) = \frac{P_k(t)}{P_k(N)} \sum_r g(u_{ir}^k)g((u_{jr}^k)^*) = \frac{P_k(t)}{P_k(N)}\delta_{i,j} = \omega_t(u_{ij}^k).$$

Hence, $\alpha_g(\omega_t) = \omega_t$ for all $g \in \chi(G)$ and all $t \in (0, 1)$. It follows that for any subgroup $\Gamma < \chi(O_N^+) = O(N)$ the pair $(O_N^+, (O_N^+)_\Gamma)$ has the relative co-Haagerup property.

We now turn to the Haagerup property. We will need the following result which is of independent interest. This is the non-tracial version of [Jo07, Corollary 3.4] and the proof is similar. We include a slightly different proof for the convenience of the reader. We refer to [CS13, OT13] for the Haagerup property for arbitrary von Neumann algebra.

Proposition 6.17. *Let (M, ν) be a von Neumann algebra with a f.n.s. ν and let $\alpha : \Gamma \curvearrowright M$ be an action which leaves ν invariant. If α is compact, Γ and M has the Haagerup property, then $\Gamma \rtimes M$ has the Haagerup property.*

Proof. Let $H < \text{Aut}(M)$ be the closure of the image of Γ in $\text{Aut}(M)$. By assumption H is compact. Let $L^2(M)$ denote the GNS space of ν .

We first make an easy observation. Whenever $\psi : M \rightarrow M$ is a ucp, normal and ω -preserving map, then for all $x \in M$, the map $H \ni h \mapsto h^{-1} \circ \psi \circ h(x) \in M$ is σ -weakly continuous. Hence, we can define $\Psi(x) = \int_H h^{-1} \circ \psi \circ h(x) dh$, where dh is the normalized Haar measure on H . By construction, the map $\Psi : M \rightarrow M$ is ucp, ν -preserving, Γ -equivariant and normal. Moreover, for all $\xi \in L^2(M)$, the map $H \ni h \mapsto T_{h^{-1}} \circ T_\psi \circ T_h \xi \in L^2(M)$, where T_h and T_ψ are respectively the L^2 -extensions of h and ψ , is norm continuous. Consequently, the element $\int_H T_{h^{-1}} \circ T_\psi \circ T_h dh \in \mathcal{B}(L^2(M))$ and by definition of Ψ we have that the L^2 -extension of Ψ is given by $T_\Psi = \int_H T_{h^{-1}} \circ T_\psi \circ T_h dh \in \mathcal{B}(L^2(M))$. Let \mathcal{B} be the unit ball of $L^2(M)$. Consider the set $A = \{h \mapsto T_{h^{-1}} \circ T_\psi \circ T_h \xi : \xi \in \mathcal{B}\} \subset C(H, \mathcal{B})$. It is easy to check that A is equicontinuous and, since T_ψ is compact, the set $A(h) = \{f(h) : f \in A\}$ is precompact for all $h \in H$. By Ascoli's Theorem, A is precompact in $C(H, \mathcal{B})$. Since the map $H \times C(H, \mathcal{B}) \rightarrow \mathcal{B}$, by $(h, f) \mapsto f(h)$ is continuous, the image of $H \times \overline{A}$ is compact and contains $\mathcal{B}_\psi = \{T_{h^{-1}} \circ T_\psi \circ T_h(\mathcal{B}), h \in H\}$. Since the image of \mathcal{B} under T_Ψ is contained in the closed convex hull of $\overline{\mathcal{B}_\psi}$, it follows that T_Ψ is compact.

We use the standard notations $N = \Gamma \rtimes M = \{u_\gamma x : \gamma \in \Gamma, x \in M\}'' \subset \mathcal{B}(\ell^2(\Gamma) \otimes L^2(M))$. We write $\tilde{\nu}$ for the dual state of ν on N . Let ψ_i be a net of normal, ucp, ν -preserving and L^2 -compact maps on M which converge pointwise in $\|\cdot\|_{2,\nu}$ to identity. Consider the net of ν -preserving, ucp, normal, L^2 -compact and Γ -equivariant maps Ψ_i given by $\Psi_i(x) = \int_H h^{-1} \circ \psi_i \circ h(x) dh$ for all $x \in M$. Note that the $(\Psi_i)_i$ is still converging pointwise in $\|\cdot\|_{2,\nu}$ to identity since, by the dominated convergence Theorem we have,

$$\|\Psi_i(x) - x\|_{2,\nu} = \left\| \int_H h^{-1}(\psi_i(h(x)) - h(x)) dh \right\|_{2,\nu} \leq \int_H \|\psi_i(h(x)) - h(x)\|_{2,\nu} dh \rightarrow 0.$$

By the Γ -equivariance, we can consider the ucp normal $\tilde{\nu}$ -preserving maps on N given by $\widetilde{\Psi}_i(u_\gamma x) = u_\gamma \Psi_i(x)$. Observe that the net $(\widetilde{\Psi}_i)$ is still converging pointwise in $\|\cdot\|_{2,\tilde{\nu}}$ to identity and the L^2 -extension of $\widetilde{\Psi}_i$ is given by $T_{\widetilde{\Psi}_i} = 1 \otimes T_{\Psi_i} \in \mathcal{B}(\ell^2(\Gamma) \otimes L^2(M))$.

Let ϕ_i be a net of positive definite and c_0 functions on Γ converging to 1 pointwise and consider the ucp normal $\tilde{\nu}$ -preserving maps on N given by $\widetilde{\phi}_i(u_\gamma x) = \phi_i(\gamma)u_\gamma x$. Observe that the net $(\widetilde{\phi}_i)$ is converging pointwise in $\|\cdot\|_{2,\tilde{\nu}}$ to identity and the L^2 -extension of $\widetilde{\phi}_i$ is given by $T_{\widetilde{\phi}_i} = T_{\phi_i} \otimes 1 \in \mathcal{B}(\ell^2(\Gamma) \otimes L^2(M))$, where $T_{\phi_i}(\delta_\gamma) = \phi_i(\gamma)\delta_\gamma$ is a compact operator on $\ell^2(\Gamma)$.

Hence, if we define the net of ucp, $\tilde{\nu}$ -preserving map on N by $\varphi_{i,j} = \widetilde{\phi}_j \circ \widetilde{\Psi}_i$, we have $\varphi_{i,j}(u_\gamma x) = \phi_j(\gamma)u_\gamma \Psi_i(x)$; the net $(\varphi_{i,j})$ is converging pointwise in $\|\cdot\|_{2,\tilde{\nu}}$ to identity and the L^2 -extension of $\varphi_{i,j}$ is given by $T_{\varphi_{i,j}} = T_{\phi_j} \otimes T_{\Psi_i} \in \mathcal{B}(\ell^2(\Gamma) \otimes L^2(M))$ is compact. \square

Corollary 6.18. *The following holds.*

1. If $L^\infty(\mathbb{G})$ has the Haagerup property, then $L^\infty(G)$ and Γ have the Haagerup property.
2. If $L^\infty(G)$ has the Haagerup property, $\alpha : \Gamma \curvearrowright L^\infty(G)$ is compact and Γ has the Haagerup property, then $L^\infty(\mathbb{G})$ has the Haagerup property. .

Proof. (1). Follows from the fact that there exists normal (faithful), Haar-state preserving conditional expectations from $L(\mathbb{G})$ to $L(\Gamma)$ and to $L^\infty(G)$. The former is given by $u_\gamma a \mapsto h_G(a)u_\gamma$ and the latter is given by $u_\gamma a \mapsto \delta_{\gamma, \epsilon} a$.

(2). It is an immediate consequence of Proposition 6.17. \square

Remark 6.19. It is shown in [DFSW13] that for any compact quantum group G , if \widehat{G} has the Haagerup property then $L^\infty(G)$ has the Haagerup property. The converse is true when G is Kac. In general, as shown in [OOT15], one needs to assume that \widehat{G} is strongly inner amenable to get the converse.

Example 6.20. If $\Gamma = \mathrm{SL}_N(\mathbb{Z}) < O(N) < U(N)$, then the duals of $(U_N^+)_\Gamma$ and $(O_N^+)_\Gamma$ do not have Haagerup property whenever $N \geq 3$.

Theorem 6.21. *Suppose \widehat{G} has the Haagerup property and Γ has the Haagerup property, and further, suppose the action of Γ on G is compact. Then $\widehat{\mathbb{G}}$ has the Haagerup property.*

Proof. Since \widehat{G} has the Haagerup property, this assures the existence of states $(\mu_n)_{n \in \mathbb{N}}$ on $C_m(G)$ such that

1. $\widehat{\mu}_n \in c_0(\widehat{G})$ for all $n \in \mathbb{N}$;
2. $\mu_n \rightarrow \varepsilon_G$ weak*.

Our first task is to construct a sequence of α -invariant states on $C_m(G)$ satisfying (1) and (2) above. This is similar to our arguments before (while dealing with property (T) and Haagerup property). Since the action of Γ is compact, the closure of Γ in $\mathrm{Aut}(G)$ is compact, and we denote this subgroup by H . Letting dh denote the normalized Haar measure on H , we define the state on $C_m(G)$:

$$\nu_n(a) = \int_H \mu_n(h^{-1}(a)) dh, \quad a \in C_m(G).$$

It is easily seen that the ν_n are invariant under the action of Γ . Now, since the action is compact, all orbits of the induced action on $\mathrm{Irr}(G)$ are finite. We need this to show that μ_n satisfy (1) above. So, let $\epsilon > 0$. As μ_n satisfied (1), the set $L = \{x \in \mathrm{Irr}(G) : \|(\mathrm{id} \otimes \mu_n)(u^x)\| \geq \frac{\epsilon}{2}\}$ is finite and the set $K = H \cdot L \subset \mathrm{Irr}(G)$ is also finite, as all orbits are finite. For $h \in H \subset \mathrm{Aut}(G)$ and $x \in \mathrm{Irr}(G)$ write $V_{h,x} \in \mathcal{B}(H_x)$ the unique unitary such that $(\mathrm{id} \otimes h^{-1})(u^x) = (V_{h,x}^* \otimes 1)(\mathrm{id} \otimes u^{h^{-1}(x)})(V_{h,x} \otimes 1)$.

If $x \notin K$ then, for all $h \in H$, $h^{-1}(x) \notin L$. Hence, $\|(\mathrm{id} \otimes \mu_n)(u^{h^{-1}(x)})\| < \frac{\epsilon}{2}$ for all $h \in H$ and it follows that

$$\begin{aligned} \|(\mathrm{id} \otimes \nu_n)(u^x)\| &= \left\| \int_H (\mathrm{id} \otimes \mu_n)((\mathrm{id} \otimes h^{-1})(u^x)) dh \right\| = \int_H \|V_{h,x}^* (\mathrm{id} \otimes \mu_n)(u^{h^{-1}(x)}) V_{h,x}\| dh \\ &\leq \int_H \|(\mathrm{id} \otimes \mu_n)(u^{h^{-1}(x)})\| dh \leq \frac{\epsilon}{2} < \epsilon \quad \text{for all } x \notin K. \end{aligned}$$

Hence, $\{x \in \mathrm{Irr}(G) : \|(\mathrm{id} \otimes \nu_n)(u^x)\| \geq \epsilon\} \subset K$ is a finite set and (1) holds for ν_n .

To show that (2) holds, we first note that given any $a \in C_m(G)$, one has $\mu_n(h^{-1}(a)) \rightarrow \varepsilon_G(h^{-1}(a)) = \varepsilon_G(a)$ for all $h \in H$ (since the $h \in H$ are quantum automorphisms). By the dominated convergence Theorem, we see that (2) holds for ν_n .

Now, since Γ has the Haagerup property, we can construct states τ_n on $C^*(\Gamma)$ satisfying (1) and (2) above. And since the states μ_n on $C_m(G)$ are α -invariant, we can construct the crossed product states $\phi_n = \tau_n \times \mu_n$ on $C_m(\mathbb{G})$ (see [Wa95b, Proposition and Definition 3.4] and also [BO08, Exercise 4.1.4] for the case of c.c.p. maps). The straightforward computations that need to be done to see that the sequence of states $(\phi_n)_{n \in \mathbb{N}}$ satisfy (1) and (2) above, are left to the reader. This then shows that $\widehat{\mathbb{G}}$ has the Haagerup property. \square

Example 6.22. If Γ is any subgroup of $SL_2(\mathbb{Z}) < O(N) < U(N)$ for $N \geq 2$ (for example $\Gamma = \mathbb{F}_n$ or $\Gamma = SL_2(\mathbb{Z})$), then the duals of $(U_N^+)_{\Gamma}$ and $(O_N^+)_{\Gamma}$ have Haagerup property since, as shown in [Br12], the duals of U_N^+ and O_N^+ have the Haagerup property.

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