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Andrey Shishkov, Laurent Véron. Admissible initial growth for diffusion equations with weakly superlinear absorption. 2015. hal-01136836v2

**HAL Id: hal-01136836**

**<https://hal.science/hal-01136836v2>**

Preprint submitted on 8 May 2015 (v2), last revised 9 Sep 2015 (v3)

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# Admissible initial growth for diffusion equations with weakly superlinear absorption

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*2010 Mathematics Subject Classification.* 35K58; 35K61.

*Key words.* semilinear heat equations; absorption; maximal and minimal solutions; non-uniqueness; maximal growth.

**Abstract** We study the admissible growth of initial data of positive solutions of  $\partial_t u - \Delta u + f(u) = 0$  in  $\mathbb{R}_+ \times \mathbb{R}^N$  when  $f(u)$  is a continuous *mildly* superlinear function at infinity, the model being  $f(u) = u \ln^\alpha(u)$  with  $1 < \alpha < 2$ . We prove that if the growth of the initial data is too strong, there is no more diffusion and the corresponding solution satisfies the ODE problem  $\partial_t \phi + f(\phi) = 0$  on  $\mathbb{R}_+$  with  $\phi(0) = \infty$ .

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## 1 Introduction

Let  $h$  be a continuous nondecreasing function defined on  $\mathbb{R}$  and vanishing only at 0. It is well known that for any continuous and bounded function  $g$  belonging to  $C_b^+(\mathbb{R}^N)$ , the

cone of bounded nonnegative continuous functions on  $\mathbb{R}^N$ , there exists a unique weak solution  $u := u_g \in C_b^+(\mathbb{R}_+ \times \mathbb{R}^N)$  of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty := \mathbb{R}_+ \times \mathbb{R}^N \\ \lim_{t \rightarrow 0} u(t, \cdot) &= g && \text{locally uniformly in } \mathbb{R}^N. \end{aligned} \quad (1.1)$$

Furthermore, the solution  $u$  satisfies

$$0 \leq u(x, t) \leq \Phi_{\|g\|_{L^\infty}}(t) \quad \forall (t, x) \in Q_{\mathbb{R}^N}^\infty, \quad (1.2)$$

where  $\Phi_a$  is the restriction to  $\mathbb{R}_+$  of the maximal solution of

$$\begin{aligned} \Phi_t + \Phi h(\Phi) &= 0 && \text{on } \mathbb{R} \\ \Phi(0) &= a. \end{aligned} \quad (1.3)$$

When  $g$  ceases to be bounded, the existence of a solution holds provided

$$\int_a^\infty \frac{ds}{sh(s)} < \infty, \quad (1.4)$$

for some  $a > 0$ . Furthermore there exists a minimal solution  $\underline{u}_g$  which is the limit, when  $n \rightarrow \infty$ , of the solutions  $u_{g_n}$  of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty \\ \lim_{t \rightarrow 0} u(t, \cdot) &= g\chi_{B_n} && \text{locally uniformly in } \mathbb{R}^N, \end{aligned} \quad (1.5)$$

where  $B_n$  denotes the open ball of radius  $n$  and  $\chi_A$  is characteristic function of the set  $A$ . Furthermore there holds

$$0 \leq \underline{u}_g \leq \Phi_\infty. \quad (1.6)$$

We first prove a uniqueness result

**Theorem A** *Assume  $r \mapsto rh(r)$  is convex and satisfies, for some  $a > 0$ ,*

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \text{ where } H(t) = \int_0^t th(t)dt, \quad (1.7)$$

*then for any  $g \in C^+(\mathbb{R}^N)$ ,  $\underline{u}_g$  is the unique nonnegative solution of (1.1).*

When  $h$  is a power the class of existence and uniqueness is much larger and is associated to the notion of initial trace which has been thoroughly studied by Marcus and Véron [3], [4] and Gkikas and Véron [2].

When

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} = \infty \quad \forall a > 0, \quad (1.8)$$

uniqueness may not hold in the class of unbounded solution. If, for any  $b > 0$ ,  $V_b$  denotes the maximal solution of the following ODE

$$\begin{aligned} V_{rr} + \frac{N-1}{r}V_r - Vh(V) &= 0 & \text{on } (0, R_a) \\ V(0) &= b \\ V_r(0) &= 0 \end{aligned} \tag{1.9}$$

then  $R_b = \infty$  (see e.g. [7]). Furthermore, Nguyen Phuoc and Véron proved in [5] that if  $g$  satisfies

$$V_c(|x|) \leq g(x) \leq V_b(|x|) \quad \forall x \in \mathbb{R}^N \tag{1.10}$$

for some  $b > c > 0$ , then there exists at least two different solutions of (1.1) defined in  $Q_{\mathbb{R}^N}^\infty$ : the minimal one  $\underline{u}_g$  which satisfies

$$\underline{u}_g(x, t) \leq \Phi_\infty(t) \quad \forall (x, t) \in Q_{\mathbb{R}^N}^\infty, \tag{1.11}$$

and another one  $u_g$  such that

$$V_c(|x|) \leq u_g(x, t) \leq V_b(|x|) \quad \forall (x, t) \in Q_{\mathbb{R}^N}^\infty. \tag{1.12}$$

Actually it is not clear whether there exists a maximal solution or not. However, if  $g$  satisfies (1.10), then there exists a minimal solution  $\underline{u}_{g,c,b}$  and a maximal one  $\overline{u}_{g,c,b}$  in the class  $\mathcal{E}_{c,b}(g)$  of solutions of problem (1.1), satisfying inequalities (1.12). These two solutions can be constructed by the following approximate scheme. Let us define sequence  $\{\underline{u}_n\}$  of solutions of the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 & \text{in } Q_{B_n}^\infty := \mathbb{R}_+ \times B_n \\ u(t, x) &= V_c(n) & \text{in } \partial_t Q_{B_n}^\infty := \mathbb{R}_+ \times \partial B_n \\ u(0, \cdot) &= g & \text{in } B_n. \end{aligned} \tag{1.13}$$

Then it is easy to check using comparison principle that the sequence  $\{\underline{u}_n\}$  is increasing and converges to  $\underline{u}_{g,c,b}$ . Similarly, the sequence  $\{\overline{u}_n\}$  of solutions of the same equation in  $Q_{B_n}^\infty$  with the same initial data and boundary value  $V_b(n)$  is decreasing and converges to  $\overline{u}_{g,c,b}$ .

When the initial data  $g$  growth at infinity faster than any function  $V_b$  with arbitrary  $b < \infty$  we prove that such a solution cannot exist: For any  $a > 0$  we denote by  $u := u_{a,n}$  the solution of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 & \text{in } Q_{B_n}^\infty := \mathbb{R}_+ \times B_n \\ u(t, x) &= V_a(n) & \text{in } \partial_t Q_{B_n}^\infty := \mathbb{R}_+ \times \partial B_n \\ u(0, \cdot) &= \min\{V_a, g\} & \text{in } B_n, \end{aligned} \tag{1.14}$$

**Theorem B** *Assume  $r \mapsto rh(r)$  is convex and satisfies (1.4) and (1.8). If  $g \in C^+(\mathbb{R}^N)$ , satisfies*

$$\lim_{|x| \rightarrow \infty} \frac{g(x)}{V_a(|x|)} = \infty \quad \forall a > 0, \tag{1.15}$$

then the sequence  $\{u_{a,n}\}_{n \in \mathbb{N}_*}$  decreases and converges to a solution  $u_a$  of (1.1) with initial data  $\min\{V_a, g\}$ . Furthermore  $u_a(t, x) \rightarrow \infty$  for any  $(t, x) \in Q_{\mathbb{R}^N}^\infty$  as  $a \rightarrow \infty$ .

A fundamental example of equations with nonlinearities satisfying (1.4) and (1.8) is provided by

$$\partial_t u - \Delta u + u \ln^\alpha(1 + u) = 0 \quad \text{in } Q_{\mathbb{R}^N}^\infty \quad (1.16)$$

with  $1 < \alpha \leq 2$ . With this specific type of nonlinearity we prove:

**Theorem C** *Assume  $1 < \alpha < 2$  and  $g \in C^+(\mathbb{R}^N)$ , satisfies (1.15). Then the minimal solution  $\underline{u}_g$  of (1.16) with initial data  $g$  is  $\Phi_\infty$ .*

## 2 The maximal solution

### 2.1 Proof of Theorem A

Let  $\underline{u}_g$  be the minimal solution of (1.1) and  $u$  another solution with the same initial data  $g$ . We set  $w = u - \underline{u}_g$ . Since  $r \mapsto rh(r)$  is convex and  $u - \underline{u}_g$  is positive,

$$uh(u) \geq \underline{u}_g h(\underline{u}_g) + (u - \underline{u}_g)h(u - \underline{u}_g).$$

Therefore  $w$  is a subsolution of problem (1.1), and  $w(t, x) \rightarrow 0$  as  $t \rightarrow 0$ , locally uniformly in  $\mathbb{R}^N$ . Let  $v_n$  be the minimal solution of

$$\begin{aligned} -\Delta v + vh(v) &= 0 & \text{in } B_n \\ \lim_{|x| \rightarrow n} v(x) &= \infty. \end{aligned} \quad (2.17)$$

Such a solution exists by [1] or [6] because (1.7) holds, and it is radial as limit of the radial functions  $v_{n,k}$ ,  $k \in \mathbb{N}^*$ , which are solutions of (2.17) with finite boundary data  $v_{n,k} = k$  on  $\partial B_n$ . Moreover  $v_{n,k}$ , and thus  $v_n$ , is an increasing function of  $|x|$ . Then  $v \geq 0$  and it is a stationary solution of (1.1) in  $Q_{B_n}^\infty$ . By the comparison principle

$$w(t, x) \leq v_n(x) \quad \text{in } Q_{B_n}^\infty.$$

Furthermore  $n \mapsto v_n$  is decreasing with limit  $v_\infty$  as  $n \rightarrow \infty$ . The function  $v_\infty$  verifies

$$-\Delta v + vh(v) = 0 \quad \text{in } \mathbb{R}^N.$$

Furthermore it is nonnegative, radial and nondecreasing with respect to  $|x|$ . In order to prove that  $v = 0$ , we return to  $v_n$  which satisfies

$$v_{nr} = r^{1-N} \int_0^r s^{N-1} v_n(s) h(v_n(s)) ds \leq v_n(r) h(v_n(r)) r^{1-N} \int_0^r s^{N-1} ds = \frac{r}{N} v_n(r) h(v_n(r)).$$

Thus

$$-v_{nrr} + v_n(r) h(v_n(r)) = \frac{N-1}{r} v_{nr} \leq \left(1 - \frac{1}{N}\right) v_n(r) h(v_n(r))$$

which implies

$$-v_{nrr} + \frac{1}{N} v_n(r) h(v_n(r)) \leq 0.$$

Integrating twice yields

$$\int_{v_n(r)}^{\infty} \frac{ds}{\sqrt{H(t)}} \geq \sqrt{\frac{2}{N}}(n-r). \quad (2.18)$$

where  $H$  has been defined in (1.7). If we had  $v_{\infty}(r) > 0$  for any  $r > 0$ , it would imply

$$\infty > \int_{v_{\infty}(r)}^{\infty} \frac{ds}{\sqrt{2H(t)}} \geq \infty,$$

a contradiction. Thus  $v_{\infty}(r) = 0$  and  $w(t, x) = 0$ .  $\square$

## 2.2 Proof of Theorem B

We recall that (1.15) holds and that  $u_{a,n}$  denotes the solution of (1.14). Since  $V_a|_{Q_{B_n}^{\infty}}$  is the solution of the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q_{B_n}^{\infty} := \mathbb{R}_+ \times B_n \\ u(t, x) &= V_a(n) && \text{in } \partial_{\ell} Q_{B_n}^{\infty} := \mathbb{R}_+ \times \partial B_n \\ u(0, \cdot) &= V_a && \text{in } B_n, \end{aligned} \quad (2.19)$$

it is larger than  $u_{a,n}$ . Thus  $u_{a,n+1}|_{\partial_{\ell} Q_{B_n}^{\infty}} \leq u_{a,n}|_{\partial_{\ell} Q_{B_n}^{\infty}} = V_a$ . Since  $u_{a,n}(0, \cdot) = u_{a,n+1}|_{B_n}(0, \cdot)$  it follows that  $u_{a,n+1}|_{Q_{B_n}^{\infty}} \leq u_{a,n}$ . Then  $\{u_{a,n}\}$  is a decreasing sequence, and its limit  $u_a$  is a solution of (1.1), which the first claim. By the same argument,  $u_{a,n} \leq u_{b,n+1}|_{Q_{B_n}^{\infty}}$  in  $Q_{B_n}^{\infty}$  for  $b > a$ . Hence  $u_a \leq u_b$ . We introduce the sequence  $\{r_a\} : r_a \rightarrow \infty$  as  $a \rightarrow \infty$  defined by:

$$r_a = \inf\{r > 0 : g(x) \geq V_a(x) \quad \forall |x| \geq r\}. \quad (2.20)$$

and, for  $n \geq r_a$ , we set  $w_{a,n} = V_a - u_{a,n}$ . By convexity it satisfies

$$\begin{aligned} \partial_t w_{a,n} - \Delta w_{a,n} + w_{a,n}h(w_{a,n}) &\leq 0 && \text{in } Q_{B_n}^{\infty} := \mathbb{R}_+ \times B_n, \\ w_{a,n}(t, x) &= 0 && \text{in } \partial_{\ell} Q_{B_n}^{\infty} := \mathbb{R}_+ \times \partial B_n, \\ w_{a,n}(0, x) &= (V_a - g)_+ && \text{in } B_n. \end{aligned} \quad (2.21)$$

Therefore

$$w_{a,n}(t, x) < \Phi_{\infty}(t) \quad \text{in } Q_{B_n}^{\infty}, \quad (2.22)$$

where  $\Phi_{\infty}$  is defined in (1.3) with  $a = \infty$ . Actually,

$$\int_{\Phi_{\infty}(t)}^{\infty} \frac{ds}{sh(s)} = t. \quad (2.23)$$

Notice also that the sequence  $\{w_{a,n}\}$  is increasing and it converges, as  $n \rightarrow \infty$ , to  $w_a = V_a - u_a$ , which is dominated by  $\Phi_{\infty}$ . Thus

$$u_a(t, x) \geq V_a(x) - \Phi_{\infty}(t) \geq a - \Phi_{\infty}(t) \quad \text{in } Q_{\mathbb{R}^N}^{\infty}. \quad (2.24)$$

Letting  $a \rightarrow \infty$  implies the claim.  $\square$

### 3 The minimal solution

In this section we consider equation (1.16) with  $1 < \alpha < 2$ .

#### 3.1 The stationary problem

**Proposition 3.1** *Assume  $1 < \alpha < 2$ ,  $a > 0$  and  $V_a$  is the solution of*

$$\begin{aligned} V_{rr} + \frac{N-1}{r}V_r - V \ln^\alpha(V+1) &= 0 & \text{in } \mathbb{R}_+ \\ V_r(0) &= 0 \\ V(a) &= a. \end{aligned} \tag{3.25}$$

Then

$$V(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}} + O(1)} \quad \text{as } r \rightarrow \infty, \tag{3.26}$$

where  $c_\alpha = \left(\frac{2-\alpha}{2}\right)^{\frac{2}{2-\alpha}}$ .

*Proof.* We write  $W = \ln(V+1)$ . Since  $V$  is increasing  $W > 0, W_r \geq 0$  and

$$W_{rr} + W_r^2 + \frac{N-1}{r}W_r - (1 - e^{-W})W^\alpha = 0 \quad \text{in } \mathbb{R}_+. \tag{3.27}$$

Thus

$$W_{rr} + W_r^2 - (1 - e^{-W})W^\alpha \leq 0.$$

If we set  $\rho = W$  and  $p(\rho) = W_r(r)$ , then  $\rho \in [a, \infty)$  and

$$pp' + p^2 - (1 - e^{-\rho})\rho^\alpha \leq 0.$$

This is a linear differential inequality in the unknown  $p^2$ . Integrating yields

$$p^2(\rho) \leq 2e^{-2\rho} \int_a^\rho (e^{2s} - e^s)s^\alpha ds = \rho^\alpha + O(1). \tag{3.28}$$

Thus  $W_r(r) \leq W^{\frac{\alpha}{2}}(r) + O(1)$  as  $r \rightarrow \infty$  which implies

$$W(r) \leq c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \quad \text{as } r \rightarrow \infty. \tag{3.29}$$

Due to (3.29), relationship (3.28) yields also the following inequality

$$0 < W_r \leq c_\alpha^{\frac{\alpha}{2}} r^{\frac{\alpha}{2-\alpha}} (1 + o(1)).$$

Since  $W(r) \rightarrow \infty$  as  $r \rightarrow \infty$  it follows from equality (3.27) by using of relation (3.28) that for any  $\epsilon > 0$  there exists  $r_\epsilon > 0$  such that

$$W_{rr} + W_r^2 \geq (1 - \epsilon)W^\alpha \quad \text{on } [r_\epsilon, \infty).$$

Integrating this ordinary differential inequality by standard way we get

$$W(r) \geq (1 - \epsilon)c_\alpha r^{\frac{2}{2-\alpha}}(1 + o(1)) \quad \text{as } r \rightarrow \infty. \quad (3.30)$$

Since  $\epsilon$  is arbitrary, we derive

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}}(1 + o(1)) \quad \text{as } r \rightarrow \infty. \quad (3.31)$$

From the above estimates, we can improve (3.30). Using (3.28) and (3.31) we deduce from (3.27):

$$pp' + p^2 = (1 - e^{-\rho})\rho^\alpha - \frac{N-1}{r}W_r \geq (1 - e^{-\rho})\rho^\alpha - c\rho^{\alpha-1},$$

from which it follows easily

$$p^2(\rho) \geq 2e^{-2\rho} \int_a^\rho e^{2s}(s^\alpha - c's^{\alpha-1})ds = \rho^\alpha + O(1), \quad (3.32)$$

by l'Hospital rule. Combined with (3.31), (3.29) it implies

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \quad \text{as } r \rightarrow \infty. \quad (3.33)$$

Returning to  $V_a$ , we derive

$$V_a(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}} + O(1)} \quad \text{as } r \rightarrow \infty. \quad (3.34)$$

□

*Remark.* If  $\alpha = 2$ , the same method yields

$$V_a(r) = e^{e^r + O(1)} \quad \text{as } r \rightarrow \infty. \quad (3.35)$$

### 3.2 Proof of Theorem C

We recall that the minimal solution  $\underline{u}_g$  is the limit, when  $n \rightarrow \infty$  of the (increasing) sequence of solutions  $\{u_{\ell_n}\}$  of

$$\begin{aligned} \partial_t u - \Delta u + u \ln^\alpha(u+1) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty \\ u(0, \cdot) &= g\chi_{B_{\ell_n}} && \text{in } \mathbb{R}^N, \end{aligned} \quad (3.36)$$

where  $\{\ell_n\}$  is any increasing sequence converging to  $\infty$ . Furthermore if we replace  $g$  by its maximal radial minorant defined by  $\tilde{g}(r) := \min_{|x|=r} g(x)$ , it satisfies also (1.15). Because of (1.15) there exists a sequence  $\{r_n\}$  tending to infinity such that

$$r_n = \inf\{r > 0 : \tilde{g}(s) \geq V_n(s) \quad \forall s \geq r\},$$

then  $\tilde{g}(r_n) = V_n(r_n)$ .



Step 1: Estimate from below. Put

$$g_n(|x|) = \begin{cases} \min\{\tilde{g}(r_n), \tilde{g}(|x|)\} & \text{if } |x| < r_n \\ \tilde{g}(r_n) & \text{if } |x| \geq r_n. \end{cases}$$

Let  $\underline{u}_{g_n}$  be the minimal solution of

$$\begin{aligned} \partial_t u - \Delta u + u \ln^\alpha(u+1) &= 0 & \text{in } Q_{\mathbb{R}^N}^\infty \\ u(0, \cdot) &= g_n & \text{in } \mathbb{R}^N. \end{aligned} \quad (3.37)$$

Then  $\underline{u}_{g_n} \leq \Phi_\infty$ . For any sequence  $\{\ell_k\}$  converging to infinity and any fixed  $k$ , there exists  $n_k$  such that for  $n \geq n_k$ , there holds  $g \chi_{B_{\ell_k}} \leq g_n$ . Since the sequence  $\{\underline{u}_{g_n}\}$  is increasing, its limit  $u_\infty$  is a solution of (1.3) in  $Q_\infty^{\mathbb{R}^N}$  which is larger than  $u_{\ell_k}$  for any  $\ell_k$ , and therefore larger also than  $\underline{u}_{\tilde{g}}$ . However, since  $g_n \leq \tilde{g}$ ,  $u_\infty \leq \underline{u}_{\tilde{g}}$ . This implies

$$u_\infty = \underline{u}_{\tilde{g}} \leq \Phi_\infty. \quad (3.38)$$

Next, since  $\underline{u}_{g_n}(0, x) \leq g(r_n)$  it follows that  $\underline{u}_{g_n}(t, x) \leq g(r_n)$ . Let  $\omega_n = \Phi_{g(r_n)}$ , i.e. the solution of be the solution of (1.3) with  $a = g(r_n)$ , then  $\omega_n$  satisfies

$$\int_{\omega_n(t)}^{g(r_n)} \frac{ds}{sh(s)} = t,$$

and  $\underline{u}_{g_n} \geq w_n$  where  $w_n$  is the minimal solution of

$$\begin{aligned} \partial_t w - \Delta w + w \ln^\alpha(\omega_n + 1) &= 0 & \text{in } Q_{\mathbb{R}^N}^\infty \\ w(0, \cdot) &= g_n & \text{in } \mathbb{R}^N. \end{aligned} \quad (3.39)$$

If we set  $w_n(t, x) = e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} z_n(t, x)$ , then

$$\begin{aligned} \partial_t z_n - \Delta z_n &= 0 & \text{in } Q_{\mathbb{R}^N}^\infty \\ z_n(0, \cdot) &= g_n & \text{in } \mathbb{R}^N. \end{aligned} \quad (3.40)$$

Since

$$z_n(t, x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} g_n(y) dy,$$

we can write  $w_n(t, x) = I_n(t, x) + J_n(t, x)$  where

$$I_n(t, x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}}{(4\pi t)^{\frac{N}{2}}} \int_{|y| \leq r_n} e^{-\frac{|x-y|^2}{4t}} g_n(y) dy, \quad (3.41)$$

and

$$J_n(t, x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \int_{|y| > r_n} e^{-\frac{|x-y|^2}{4t}} dy. \quad (3.42)$$

Clearly

$$\begin{aligned} J_n(t, x) &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \int_{|y|>r_n+|x|} e^{-\frac{|y|^2}{4t}} dy \\ &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \left( \int_{|z|>r_n+|x|} e^{-\frac{z^2}{4t}} dz \right)^N. \end{aligned} \quad (3.43)$$

This integral term can be estimated by introducing Gauss error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} dz. \quad (3.44)$$

In dimension  $N$ , it implies easily

$$J_n(t, x) \geq e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n) \left( \operatorname{erfc} \left( \frac{r_n + |x|}{2\sqrt{t}} \right) \right)^N. \quad (3.45)$$

Since

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{t}} (1 + O(x^{-2})) \quad \text{as } x \rightarrow \infty,$$

we derive

$$J_n(t, x) \geq \frac{\tilde{g}(r_n)}{((r_n + |x|)^2 t)^{\frac{N}{2}}} e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds - \frac{N(r_n+|x|)^2}{4t}} \left( 1 + O\left(\frac{t}{r_n^2}\right) \right). \quad (3.46)$$

We write  $\tilde{g}(r) = \exp(\gamma(r)) - 1$  and set

$$A_n(t, x) = \gamma(r_n) - \int_0^t \ln^\alpha(\omega_n(s) + 1) ds - \frac{N(r_n + |x|)^2}{4t} - N \ln(r_n + |x|) - \frac{N}{2} \ln t.$$

In order to have an estimate on  $\omega_n(s)$ , we fix  $t \leq 1$  and  $\tilde{g}(r_n) \geq 1$ . There exists  $a_0 \geq 1$  such that

$$\min \left\{ \frac{\omega_a(t)}{\omega_a(t) + 1} : 0 \leq t \leq 1, a \geq a_0 \right\} \geq \frac{1}{2}.$$

In such a range of  $a$  and  $t$ ,

$$\begin{aligned} \omega' + \omega \ln^\alpha(\omega + 1) &= \omega' + \frac{\omega}{\omega + 1} (\omega + 1) \ln^\alpha(\omega + 1) \\ &\geq \omega' + \frac{1}{2} (\omega + 1) \ln^\alpha(\omega + 1), \end{aligned}$$

which yields

$$\ln^\alpha(\omega_n(s) + 1) \leq \left( \frac{2\gamma^{\alpha-1}(r_n)}{2 + (\alpha - 1)s\gamma^{\alpha-1}(r_n)} \right)^{\frac{\alpha}{\alpha-1}}.$$

From this inequality, we derive

$$\begin{aligned} \int_0^t \ln^\alpha(\omega_n(s) + 1) ds &\leq \int_0^t \left( \frac{2\gamma^{\alpha-1}(r_n)}{2 + (\alpha - 1)s\gamma^{\alpha-1}(r_n)} \right)^{\frac{\alpha}{\alpha-1}} ds \\ &\leq 2^{\frac{\alpha}{\alpha-1}} \gamma(r_n) \int_0^{t\gamma^{\alpha-1}(r_n)} (2 + (\alpha - 1)\tau)^{-\frac{\alpha}{\alpha-1}} d\tau. \end{aligned}$$

Therefore

$$A_n(t, x) \geq \gamma(r_n) - \frac{N(r_n + |x|)^2}{4t} - N \ln(r_n + |x|) - \frac{N}{2} \ln t - 2^{\frac{\alpha}{\alpha-1}} \gamma(r_n) \int_0^{t\gamma^{\alpha-1}(r_n)} (2 + (\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}} d\tau. \quad (3.47)$$

*Step 2: The maximal admissible growth.* We claim that

$$\liminf_{|x| \rightarrow \infty} |x|^{-\frac{2}{2-\alpha}} \ln \tilde{g}(|x|) > N^{\frac{1}{2-\alpha}} \implies \lim_{n \rightarrow \infty} \underline{u}_{g_n}(t, x) = \Phi_\infty(t) \quad \forall (t, x) \in Q_{\mathbb{R}^N}^\infty. \quad (3.48)$$

By replacing  $\tau \mapsto (2 + (\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}$  by its maximal value on  $(0, t\gamma^{\alpha-1}(r_n))$ ,

$$2^{\frac{\alpha}{\alpha-1}} \gamma(r_n) \int_0^{t\gamma^{\alpha-1}(r_n)} (2 + (\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}} d\tau \leq \gamma^\alpha(r_n)t.$$

Then

$$A_n(t, x) \geq \gamma(r_n) - \frac{N(r_n + |x|)^2}{4t} - N \ln(r_n + |x|) - \frac{N}{2} \ln t - \gamma^\alpha(r_n)t := B_n(t, x), \quad (3.49)$$

and

$$\partial_t B_n(t, x) = \frac{N(r_n + |x|)^2}{4t^2} - \frac{N}{2t} - \gamma^\alpha(r_n).$$

Thus

$$\partial_t B_n(t, x) = 0 \text{ and } t > 0 \iff t := t_n = \frac{N(r_n + |x|)^2}{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^\alpha(r_n)}}. \quad (3.50)$$

Therefore  $A_n(t_n, x)$  is bounded from below by the maximum of  $B_n(t, x)$  which is achieved for  $t = t_n$  and

$$B_n(t_n, x) = \gamma(r_n) - N \ln(r_n + |x|) - \frac{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^\alpha(r_n)}}{4} - \frac{N(r_n + |x|)^2 \gamma^\alpha(r_n)}{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^\alpha(r_n)}} - \frac{N}{2} \ln \left( \frac{N(r_n + |x|)^2}{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^\alpha(r_n)}} \right).$$

Since  $r_n \rightarrow \infty$  as  $n \rightarrow \infty$  it follows from last representation that

$$B_n(t_n, x) = r_n \gamma^{\frac{\alpha}{2}}(r_n) \left( \frac{\gamma^{1-\frac{\alpha}{2}}(r_n)}{r_n} - N^{\frac{1}{2}}(1 + \nu_n(x)) \right), \quad (3.51)$$

where  $\nu_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly on any compact set in  $\mathbb{R}^N$ . Therefore if  $g$  satisfies

$$\liminf_{|x| \rightarrow \infty} |x|^{-\frac{2}{2-\alpha}} \ln \tilde{g}(|x|) > N^{\frac{1}{2-\alpha}}, \quad (3.52)$$

then there holds

$$J_n(t_n, x) \rightarrow \infty \implies \lim_{t_n \rightarrow 0} \underline{u}_{g_n}(t_n, x) = \infty, \quad (3.53)$$

uniformly on compact subsets of  $\mathbb{R}^N$ . We fix  $m > 0$ , denote by  $\lambda_m$  the first eigenvalue of  $-\Delta$  in  $H_0^1(B_m)$ , with corresponding eigenfunction  $\phi_m$  normalized by  $\sup_{B_m} \phi_m = 1$  and set, for  $\epsilon > 0$ ,

$$W_{m,\epsilon}(t, x) = e^{-(t+\epsilon)\lambda_m} \Phi_\infty(t + \epsilon) \phi_m(x) \quad \forall (t, x) \in Q_\infty^{B_m}.$$

Then

$$\begin{aligned} \partial_t W_{m,\epsilon} - \Delta W_{m,\epsilon} + W_{m,\epsilon} \ln^\alpha(W_{m,\epsilon} + 1) &= W_{m,\epsilon} \left( \ln^\alpha(W_{m,\epsilon} + 1) - \ln^\alpha(\Phi_\infty(t + \epsilon) + 1) \right) \\ &\leq 0. \end{aligned}$$

Since  $\underline{u}_{g_n}$  increases to the minimal solution  $\underline{u}_{\tilde{g}}$ , it follows due to (3.53) that there exists  $n_\epsilon$  such that

$$\underline{u}_{\tilde{g}}(t_{n_\epsilon}, x) \geq \underline{u}_{g_{n_\epsilon}}(t_{n_\epsilon}, x) \geq W_{m,\epsilon}(t_{n_\epsilon} + \epsilon, x) \quad \forall x \in B_m.$$

Last inequality in virtue of comparison principle implies

$$\underline{u}_g(t, x) \geq W_{m,\epsilon}(t + \epsilon, x) \quad \forall (t, x) \in Q_\infty^{B_m}, t \geq t_{n_\epsilon}.$$

Letting  $\epsilon \rightarrow 0$  yields  $\underline{u}_g \geq W_{m,0}$  in  $Q_\infty^{B_m}$ . Since  $\lim_{m \rightarrow \infty} \phi_m(x) = 1$ , uniformly on any compact subset of  $\mathbb{R}^N$  and  $\lim_{m \rightarrow \infty} \lambda_m = 0$  we derive  $\underline{u}_{\tilde{g}} \geq \Phi_\infty$  and finally  $\underline{u}_g \geq \Phi_\infty$ . This inequality together with (3.38) leads to  $\underline{u} = \Phi_\infty$ .  $\square$

*Remark.* In the case  $\alpha = 2$ , there holds

$$\int_0^t \ln^2(\omega_n(s) + 1) ds \leq 4\gamma(r_n) \int_0^{t\gamma(r_n)} (2 + \tau)^{-2} d\tau \leq t\gamma(r_n). \quad (3.54)$$

Therefore (3.49) is replaced by

$$A_n(t, x) \geq \gamma(r_n) - t\gamma^2(r_n) - \frac{N(r_n + |x|)^2}{4t} - N \ln(r_n + |x|) - \frac{N}{2} \ln t := B_n(t, x). \quad (3.55)$$

A similarly, there exists  $t_n > 0$  where  $t \mapsto B_n(t, x)$  is maximum and in that case

$$\begin{aligned} B_n(t_n, x) &= \gamma(r_n) - N \ln(r_n + |x|) - \frac{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^2(r_n)}}{4} \\ &\quad - \frac{N(r_n + |x|)^2 \gamma^2(r_n)}{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^2(r_n)}} - \frac{N}{2} \ln \left( \frac{N(r_n + |x|)^2}{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^2(r_n)}} \right), \end{aligned}$$

which yields

$$B_n(t_n, x) = \gamma(r_n) - r_n \gamma(r_n) (N^{\frac{1}{2}} - \nu_n(x)), \quad (3.56)$$

where  $\nu_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly on any compact set in  $\mathbb{R}^N$ . Thus  $B_n(t_n, x) \rightarrow -\infty$  as  $n \rightarrow \infty$ . A similar type of computation shows that the expression  $I_n(t, x)$  defined in (3.41) converges to 0, whatever is the sequence  $\{r_n\}$  considered which converges to  $\infty$ .

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