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2010 Mathematics Subject Classification. 35K58; 35K61.

Key words. semilinear heat equations; absorption; maximal and minimal solutions; non-uniqueness; maximal growth.

Abstract We study the admissible growth of initial data of positive solutions of $\partial_t u - \Delta u + f(u) = 0$ in $\mathbb{R}_+ \times \mathbb{R}^N$ when $f(u)$ is a continuous *mildly* superlinear function at infinity, the model being $f(u) = u \ln^\alpha(u)$ with $1 < \alpha < 2$. We prove that if the growth of the initial data is too strong, there is no more diffusion and the corresponding solution satisfies the ODE problem $\partial_t \phi + f(\phi) = 0$ on $\mathbb{R}_+$ with $\phi(0) = \infty$.

Contents

1 Introduction

Let h be a continuous nondecreasing function defined on $\mathbb{R}$ and vanishing only at 0. It is well known that for any continuous and bounded function g belonging to $C_b^+(\mathbb{R}^N)$, the

cone of bounded nonnegative continuous functions on $\mathbb{R}^N$, there exists a unique weak solution $u := u_g \in C_b^+(\mathbb{R}_+ \times \mathbb{R}^N)$ of

$$\begin{array}{rl} \partial_t u - \Delta u + uh(u) = 0 & \text{in } Q^\infty_{\mathbb{R}^N} := \mathbb{R}_+ \times \mathbb{R}^N \\ \lim\limits_{t\to 0} u(t,.) = g & \text{locally uniformly in } \mathbb{R}^N. \end{array} \tag{1.1}$$

Furthermore, the solution u satisfies

$$0 \le u(x,t) \le \Phi_{\|g\|_{L^\infty}}(t) \qquad \forall (t,x) \in Q^\infty_{\mathbb{R}^N}, \tag{1.2}$$

where Φ_a is the restriction to $\mathbb{R}_+$ of the maximal solution of

$$\begin{array}{rl} \Phi_t + \Phi h(\Phi) = 0 & \text{on } \mathbb{R} \\ \Phi(0) = a. & \end{array} \tag{1.3}$$

When g ceases to be bounded, the existence of a solution holds provided

$$\int_a^\infty \frac{ds}{sh(s)} < \infty, \tag{1.4}$$

for some $a > 0$. Furthermore there exists a minimal solution $\underline{u}_g$ which is the limit, when $n \to \infty$, of the solutions u_{g_n} of

$$\begin{array}{rl} \partial_t u - \Delta u + uh(u) = 0 & \text{in } Q^\infty_{\mathbb{R}^N} \\ \lim\limits_{t\to 0} u(t,.) = g\chi_{B_n} & \text{locally uniformly in } \mathbb{R}^N, \end{array} \tag{1.5}$$

where B_n denotes the open ball of radius n and χ_A is characteristic function of the set A. Furthermore there holds

$$0 \le \underline{u}_g \le \Phi_\infty. \tag{1.6}$$

We first prove a uniqueness result

Theorem A *Assume $r \mapsto rh(r)$ is convex and satisfies, for some $a > 0$,*

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \ \text{ where } \ H(t) = \int_0^s th(t)dt, \tag{1.7}$$

then for any $g \in C^+(\mathbb{R}^N)$, $\underline{u}_g$ is the unique nonnegative solution of (1.1).

When h is a power the class of existence and uniqueness is much larger and is associated to the notion of initial trace which has been thoroughly studied by Marcus and Véron [3], [4] and Gkikas and Véron [2].

When

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} = \infty \qquad \forall a > 0, \tag{1.8}$$

uniqueness may not hold in the class of unbounded solution. If, for any $b > 0$, V_b denotes the maximal solution of the following ODE

$$\begin{aligned} V_{rr} + \frac{N-1}{r} V_r - Vh(V) = 0 \qquad & \text{on } (0, R_a) \\ V(0) = b \qquad & \\ V_r(0) = 0 \qquad & \end{aligned} \tag{1.9}$$

then $R_b = \infty$ (see e.g. [7]). Furthermore, Nguyen Phuoc and Véron proved in [5] that if g satisfies

$$V_c(|x|) \le g(x) \le V_b(|x|) \qquad \forall x \in \mathbb{R}^N \tag{1.10}$$

for some $b > c > 0$, then there exists at least two different solutions of (1.1) defined in $Q^\infty_{\mathbb{R}^N}$: the minimal one $\underline{u}_g$ which satisfies

$$\underline{u}_g(x,t) \le \Phi_\infty(t) \qquad \forall (x,t) \in Q^\infty_{\mathbb{R}^N}, \tag{1.11}$$

and another one u_g such that

$$V_c(|x|) \le u_g(x,t) \le V_b(|x|) \qquad \forall (x,t) \in Q^\infty_{\mathbb{R}^N}. \tag{1.12}$$

Actually it is not clear wether there exists a maximal solution or not. However, if g satisfies (1.10), then there exists a minimal solution $\underline{u}_{g,c,b}$ and a maximal one $\overline{u}_{g,c,b}$ in the class $\mathcal{E}_{c,b}(g)$ of solutions of problem (1.1), satisfying inequalities (1.12). These two solutions can be constructed by the following approximate scheme. Let us define sequence $\{\underline{u}_n\}$ of solutions of the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n \\ u(t,x) = V_c(n) \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n \\ u(0,.) = g \qquad & \text{in } B_n. \end{aligned} \tag{1.13}$$

Then it is easy to check using comparison principle that the sequence $\{\underline{u}_n\}$ is increasing and converges to $\underline{u}_{g,c,b}$. Similarly, the sequence $\{\overline{u}_n\}$ of solutions of the same equation in $Q^\infty_{B_n}$ with the same initial data and boundary value $V_b(n)$ is decreasing and converges to $\overline{u}_{g,c,b}$.

When the initial data g growth at infinity faster than any function V_b with arbitrary $b < \infty$ we prove that such a solution cannot exist: For any $a > 0$ we denote by $u := u_{a,n}$ the solution of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n \\ u(t,x) = V_a(n) \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n \\ u(0,.) = \min\{V_a, g\} \qquad & \text{in } B_n, \end{aligned} \tag{1.14}$$

Theorem B *Assume $r \mapsto rh(r)$ is convex and satisfies* (1.4) *and* (1.8). *If $g \in C^+(\mathbb{R}^N)$, satisfies*

$$\lim_{|x| \to \infty} \frac{g(x)}{V_a(|x|)} = \infty \qquad \forall a > 0, \tag{1.15}$$

then the sequence $\{u_{a,n}\}_{n\in\mathbb{N}_*}$ *decreases and converges to a solution* u_a *of* (1.1) *with initial data* $\min\{V_a, g\}$. *Furthermore* $u_a(t,x) \to \infty$ *for any* $(t,x) \in Q^\infty_{\mathbb{R}^N}$ *as* $a \to \infty$.

A fundamental example of equations with nonlinearities satisfying (1.4) and (1.8) is provided by

$$\partial_t u - \Delta u + u \ln^\alpha(1+u) = 0 \qquad \text{in } Q^\infty_{\mathbb{R}^N} \tag{1.16}$$

with $1 < \alpha \leq 2$. With this specific type of nonlinearity we prove:

Theorem C *Assume* $1 < \alpha < 2$ *and* $g \in C^+(\mathbb{R}^N)$, *satisfies* (1.15). *Then the minimal solution* $\underline{u}_g$ *of* (1.16) *with initial data* g *is* Φ_∞.

2 The maximal solution

2.1 Proof of Theorem A

Let $\underline{u}_g$ be the minimal solution of (1.1) and u another solution with the same initial data g. We set $w = u - \underline{u}_g$. Since $r \mapsto rh(r)$ is convex and $u - \underline{u}_g$ is positive,

$$uh(u) \geq \underline{u}_g h(\underline{u}_g) + (u - \underline{u}_g)h(u - \underline{u}_g).$$

Therefore w is a subsolution of problem (1.1), and $w(t,x) \to 0$ as $t \to 0$, locally uniformly in $\mathbb{R}^N$. Let v_n be the minimal solution of

$$\begin{aligned} -\Delta v + vh(v) &= 0 \qquad \text{in } B_n \\ \lim_{|x|\to n} v(x) &= \infty. \end{aligned} \tag{2.17}$$

Such a solution exists by [1] or [6] because (1.7) holds, and it is radial as limit of the radial functions $v_{n,k}$, $k \in \mathbb{N}^*$, which are solutions of (2.17) with finite boundary data $v_{n,k} = k$ on ∂B_n. Moreover $v_{n,k}$, and thus v_n, is an increasing function of $|x|$. Then $v \geq 0$ and it is a stationary solution of (1.1) in $Q^\infty_{B_n}$. By the comparison principle

$$w(t,x) \leq v_n(x) \qquad \text{in } Q^\infty_{B_n}.$$

Furthermore $n \mapsto v_n$ is decreasing with limit v_∞ as $n \to \infty$. The function v_∞ verifies

$$-\Delta v + vh(v) = 0 \qquad \text{in } \mathbb{R}^N.$$

Furthermore it is nonnegative, radial and nondecreasing with respect to $|x|$. In order to prove that $v = 0$, we return to v_n which satisfies

$$v_{n\,r} = r^{1-N}\int_0^r s^{N-1} v_n(s)h(v_n(s))ds \leq v_n(r)h(v_n(r))r^{1-N}\int_0^r s^{N-1}ds = \frac{r}{N}v_n(r)h(v_n(r)).$$

Thus

$$-v_{n\,rr} + v_n(r)h(v_n(r)) = \frac{N-1}{r}v_{n\,r} \leq \left(1 - \frac{1}{N}\right) v_n(r)h(v_n(r))$$

which implies

$$-v_{n\,rr} + \frac{1}{N}v_n(r)h(v_n(r)) \leq 0.$$

Integrating twice yields

$$\int_{v_n(r)}^{\infty} \frac{ds}{\sqrt{H(t)}} \geq \sqrt{\frac{2}{N}}(n-r). \tag{2.18}$$

where H has been defined in (1.7). If we had $v_\infty(r) > 0$ for any $r > 0$, it would imply

$$\infty > \int_{v_\infty(r)}^{\infty} \frac{ds}{\sqrt{2H(t)}} \geq \infty,$$

a contradiction. Thus $v_\infty(r) = 0$ and $w(t,x) = 0$. □

2.2 Proof of Theorem B

We recall that (1.15) holds and that $u_{a,n}$ denotes the solution of (1.14). Since $V_a\lfloor_{Q^\infty_{B_n}}$ is the solution of the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n \\ u(t,x) &= V_a(n) && \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n \\ u(0,.) &= V_a && \text{in } B_n, \end{aligned} \tag{2.19}$$

it is larger than $u_{a,n}$. Thus $u_{a,n+1}\lfloor_{\partial_\ell Q^\infty_{B_n}} \leq u_{a,n}\lfloor_{\partial_\ell Q^\infty_{B_n}} = V_a$. Since $u_{a,n}(0,.) = u_{a,n+1}\lfloor_{B_n}(0,.)$ it follows that $u_{a,n+1}\lfloor_{Q^\infty_{B_n}} \leq u_{a,n}$. Then $\{u_{a,n}\}$ is a decreasing sequence, and its limit u_a is a solution of (1.1), which the first claim. By the same argument, $u_{a,n} \leq u_{b,n+1}\lfloor_{Q^\infty_{B_n}}$ in $Q^\infty_{B_n}$ for $b > a$. Hence $u_a \leq u_b$. We introduce the sequence $\{r_a\} : r_a \to \infty$ as $a \to \infty$ defined by:

$$r_a = \inf\{r > 0 : g(x) \geqslant V_a(x) \quad \forall\, |x| \geqslant r\}. \tag{2.20}$$

and, for $n \geq r_a$, we set $w_{a,n} = V_a - u_{a,n}$ By convexity it satisfies

$$\begin{aligned} \partial_t w_{a,n} - \Delta w_{a,n} + w_{a,n}h(w_{a,n}) &\leq 0 && \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n, \\ w_{a,n}(t,x) &= 0 && \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n, \\ w_{a,n}(0,x) &= (V_a - g)_+ && \text{in } B_n. \end{aligned} \tag{2.21}$$

Therefore

$$w_{a,n}(t,x) < \Phi_\infty(t) \qquad \text{in } Q^\infty_{B_n}, \tag{2.22}$$

where Φ_∞ is defined in (1.3) with $a = \infty$. Actually,

$$\int_{\Phi_\infty(t)}^{\infty} \frac{ds}{sh(s)} = t. \tag{2.23}$$

Notice also that the sequence $\{w_{a,n}\}$ is increasing and it converges, as $n \to \infty$, to $w_a = V_a - u_a$, which is dominated by Φ_∞ Thus

$$u_a(t,x) \geq V_a(x) - \Phi_\infty(t) \geq a - \Phi_\infty(t) \qquad \text{in } Q^\infty_{\mathbb{R}^N}. \tag{2.24}$$

Letting $a \to \infty$ implies the claim. □

3 The minimal solution

In this section we consider equation (1.16) with $1 < \alpha < 2$.

3.1 The stationary problem

Proposition 3.1 *Assume $1 < \alpha < 2$, $a > 0$ and V_a is the solution of*

$$\begin{aligned} V_{rr} + \frac{N-1}{r} V_r - V \ln^\alpha (V+1) = 0 \qquad & \text{in } \mathbb{R}_+ \\ V_r(0) = 0 \qquad & \\ V(a) = a. \qquad & \end{aligned} \tag{3.25}$$

Then

$$V(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}} + O(1)} \qquad \text{as } r \to \infty, \tag{3.26}$$

where $c_\alpha = \left(\frac{2-\alpha}{2}\right)^{\frac{2}{2-\alpha}}$.

Proof. We write $W = \ln(V+1)$. Since V is increasing $W > 0, W_r \geq 0$ and

$$W_{rr} + W_r^2 + \frac{N-1}{r} W_r - (1 - e^{-W}) W^\alpha = 0 \qquad \text{in } \mathbb{R}_+. \tag{3.27}$$

Thus

$$W_{rr} + W_r^2 - (1 - e^{-W}) W^\alpha \leq 0.$$

If we set $\rho = W$ and $p(\rho) = W_r(r)$, then $\rho \in [a, \infty)$ and

$$pp' + p^2 - (1 - e^{-\rho}) \rho^\alpha \leq 0.$$

This is a linear differential inequality in the unknown p^2. Integrating yields

$$p^2(\rho) \leq 2e^{-2\rho} \int_a^\rho (e^{2s} - e^s) s^\alpha ds = \rho^\alpha + O(1). \tag{3.28}$$

Thus $W_r(r) \leq W^{\frac{\alpha}{2}}(r) + O(1)$ as $r \to \infty$ which implies

$$W(r) \leq c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \qquad \text{as } r \to \infty. \tag{3.29}$$

Due to (3.29), relationship (3.28) yields also the following inequality

$$0 < W_r \leq c_\alpha^{\frac{\alpha}{2}} r^{\frac{\alpha}{2-\alpha}} (1 + o(1)).$$

Since $W(r) \to \infty$ as $r \to \infty$ it follows from equality (3.27) by using of relation (3.28) that for any $\epsilon > 0$ there exists $r_\epsilon > 0$ such that

$$W_{rr} + W_r^2 \geq (1 - \epsilon) W^\alpha \qquad \text{on } [r_\epsilon, \infty).$$

Integrating this ordinary differential inequality by standard way we get

$$W(r) \geq (1-\epsilon))c_\alpha r^{\frac{2}{2-\alpha}}(1+o(1)) \qquad \text{as } r \to \infty. \tag{3.30}$$

Since ϵ is arbitrary, we derive

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}}(1+o(1)) \qquad \text{as } r \to \infty. \tag{3.31}$$

From the above estimates, we can improve (3.30). Using (3.28) and (3.31) we deduce from (3.27):

$$pp' + p^2 = (1-e^{-\rho})\rho^\alpha - \frac{N-1}{r}W_r \geq (1-e^{-\rho})\rho^\alpha - c\rho^{\alpha-1},$$

from which it follows easily

$$p^2(\rho) \geq 2e^{-2\rho}\int_a^\rho e^{2s}(s^\alpha - c's^{\alpha-1})ds = \rho^\alpha + O(1), \tag{3.32}$$

by l'Hospital rule. Combined with (3.31), (3.29) it implies

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \qquad \text{as } r \to \infty. \tag{3.33}$$

Returning to V_a, we derive

$$V_a(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}}+O(1)} \qquad \text{as } r \to \infty. \tag{3.34}$$

□

Remark. If $\alpha = 2$, the same method yields

$$V_a(r) = e^{e^r+O(1)} \qquad \text{as } r \to \infty. \tag{3.35}$$

3.2 Proof of Theorem C

We recall that the minimal solution $\underline{u}_g$ is the limit, when $n \to \infty$ of the (increasing) sequence of solutions $\{u_{\ell_n}\}$ of

$$\begin{aligned} \partial_t u - \Delta u + u\ln^\alpha(u+1) = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} \\ u(0,.) = g\chi_{B_{\ell_n}} \qquad & \text{in } \mathbb{R}^N, \end{aligned} \tag{3.36}$$

where $\{\ell_n\}$ is any increasing sequence converging to ∞. Furthermore if we replace g by it maximal radial minorant defined by $\tilde{g}(r) := \min_{|x|=r} g(x)$, it satisfies also (1.15). Because of (1.15) there exists a sequence $\{r_n\}$ tending to infinity such that

$$r_n = \inf\{r > 0 : \tilde{g}(s) \geq V_n(s) \ \ \forall s \geq r\},$$

then $\tilde{g}(r_n) = V_n(r_n)$.

Step 1: Estimate from below. Put

$$g_n(|x|) = \begin{cases} \min\{\tilde{g}(r_n), \tilde{g}(|x|)\} & \text{if } |x| < r_n \\ \tilde{g}(r_n) & \text{if } |x| \geq r_n. \end{cases}$$

Let $\underline{u}_{g_n}$ be the minimal solution of

$$\begin{aligned} \partial_t u - \Delta u + u \ln^\alpha(u+1) = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} \\ u(0,.) = g_n \qquad & \text{in } \mathbb{R}^N. \end{aligned} \tag{3.37}$$

Then $\underline{u}_{g_n} \leq \Phi_\infty$. For any sequence $\{\ell_k\}$ converging to infinity and any fixed k, there exists n_k such that for $n \geq n_k$, there holds $g\chi_{B_{\ell_k}} \leq g_n$. Since the sequence $\{\underline{u}_{g_n}\}$ is increasing, its limit u_∞ is a solution of (1.3) in $Q^{\mathbb{R}^N}_\infty$ which is larger than u_{ℓ_k} for any ℓ_k, and therefore larger also than $\underline{u}_{\tilde{g}}$. However, since $g_n \leq \tilde{g}$, $u_\infty \leq \underline{u}_{\tilde{g}}$. This implies

$$u_\infty = \underline{u}_{\tilde{g}} \leqslant \Phi_\infty. \tag{3.38}$$

Next, since $\underline{u}_{g_n}(0,x) \leq g(r_n)$ it follows that $\underline{u}_{g_n}(t,x) \leq g(r_n)$. Let $\omega_n = \Phi_{g(r_n)}$, i.e. the solution of be the solution of (1.3) with $a = g(r_n)$, then ω_n satisfies

$$\int_{\omega_n(t)}^{g(r_n)} \frac{ds}{sh(s)} = t,$$

and $\underline{u}_{g_n} \geq w_n$ where w_n is the minimal solution of

$$\begin{aligned} \partial_t w - \Delta w + w \ln^\alpha(\omega_n + 1) = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} \\ w(0,.) = g_n \qquad & \text{in } \mathbb{R}^N. \end{aligned} \tag{3.39}$$

If we set $w_n(t,x) = e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} z_n(t,x)$, then

$$\begin{aligned} \partial_t z_n - \Delta z_n = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} \\ z_n(0,.) = g_n \qquad & \text{in } \mathbb{R}^N. \end{aligned} \tag{3.40}$$

Since

$$z_n(t,x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} g_n(y) dy,$$

we can write $w_n(t,x) = I_n(t,x) + J_n(t,x)$ where

$$I_n(t,x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}}{(4\pi t)^{\frac{N}{2}}} \int_{|y|\leq r_n} e^{-\frac{|x-y|^2}{4t}} g_n(y) dy, \tag{3.41}$$

and

$$J_n(t,x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \int_{|y|>r_n} e^{-\frac{|x-y|^2}{4t}} dy. \tag{3.42}$$

Clearly

$$\begin{aligned} J_n(t,x) &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}}\int_{|y|>r_n+|x|} e^{-\frac{|y|^2}{4t}}dy \\ &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}}\left(\int_{|z|>r_n+|x|} e^{-\frac{z^2}{4t}}dz\right)^N. \end{aligned} \tag{3.43}$$

This integral term can be estimated by introducing Gauss error function

$$\mathrm{ercf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz. \tag{3.44}$$

In dimension N, it implies easily

$$J_n(t,x) \geq e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)\left(\mathrm{ercf}\left(\frac{r_n+|x|}{2\sqrt{t}}\right)\right)^N. \tag{3.45}$$

Since

$$\mathrm{ercf}(x) = \frac{e^{-x^2}}{x\sqrt{t}}(1+O(x^{-2})) \quad \text{as } x\to\infty,$$

we derive

$$J_n(t,x) \geq \frac{\tilde{g}(r_n)}{((r_n+|x|)^2t)^{\frac{N}{2}}}e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds-\frac{N(r_n+|x|)^2}{4t}}\left(1+O\left(\frac{t}{r_n^2}\right)\right). \tag{3.46}$$

We write $\tilde{g}(r) = \exp(\gamma(r)) - 1$ and set

$$A_n(t,x) = \gamma(r_n) - \int_0^t \ln^\alpha(\omega_n(s)+1)ds - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t.$$

In order to have an estimate on $\omega_n(s)$, we fix $t\leq 1$ and $\tilde{g}(r_n)\geq 1$. There exists $a_0\geq 1$ such that

$$\min\left\{\frac{\omega_a(t)}{\omega_a(t)+1} : 0\leq t\leq 1,\, a\geq a_0\right\} \geq \frac{1}{2}.$$

In such a range of a and t,

$$\begin{aligned} \omega' + \omega\ln^\alpha(\omega+1) &= \omega' + \frac{\omega}{\omega+1}(\omega+1)\ln^\alpha(\omega+1) \\ &\geq \omega' + \frac{1}{2}(\omega+1)\ln^\alpha(\omega+1), \end{aligned}$$

which yields

$$\ln^\alpha(\omega_n(s)+1) \leq \left(\frac{2\gamma^{\alpha-1}(r_n)}{2+(\alpha-1)s\gamma^{\alpha-1}(r_n)}\right)^{\frac{\alpha}{\alpha-1}}.$$

From this inequality, we derive

$$\begin{aligned} \int_0^t \ln^\alpha(\omega_n(s)+1)ds &\leq \int_0^t \left(\frac{2\gamma^{\alpha-1}(r_n)}{2+(\alpha-1)s\gamma^{\alpha-1}(r_n)}\right)^{\frac{\alpha}{\alpha-1}} ds \\ &\leq 2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}\,d\tau. \end{aligned}$$

Therefore

$$A_n(t,x) \geq \gamma(r_n) - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t \\ - 2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}\, d\tau. \tag{3.47}$$

Step 2: The maximal admissible growth. We claim that

$$\liminf_{|x|\to\infty} |x|^{-\frac{2}{2-\alpha}} \ln \tilde{g}(|x|) > N^{\frac{1}{2-\alpha}} \Longrightarrow \lim_{n\to\infty} \underline{u}_{g_n}(t,x) = \Phi_\infty(t) \qquad \forall (t,x) \in Q^\infty_{\mathbb{R}^N}. \tag{3.48}$$

By replacing $\tau \mapsto (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}$ by its maximal value on $(0, t\gamma^{\alpha-1}(r_n))$,

$$2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}\, d\tau \leq \gamma^\alpha(r_n)t.$$

Then

$$A_n(t,x) \geq \gamma(r_n) - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t - \gamma^\alpha(r_n)t := B_n(t,x), \tag{3.49}$$

and

$$\partial_t B_n(t,x) = \frac{N(r_n+|x|)^2}{4t^2} - \frac{N}{2t} - \gamma^\alpha(r_n).$$

Thus

$$\partial_t B_n(t,x) = 0 \text{ and } t>0 \Longleftrightarrow t := t_n = \frac{N(r_n+|x|)^2}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^\alpha(r_n)}}. \tag{3.50}$$

Therefore $A_n(t_n,x)$ is bounded from below by the maximum of $B_n(t,x)$ which is achieved for $t=t_n$ and

$$B_n(t_n,x) = \gamma(r_n) - N\ln(r_n+|x|) - \frac{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^\alpha(r_n)}}{4} \\ - \frac{N(r_n+|x|)^2\gamma^\alpha(r_n)}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^\alpha(r_n)}} - \frac{N}{2}\ln\left(\frac{N(r_n+|x|)^2}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^\alpha(r_n)}}\right).$$

Since $r_n \to \infty$ as $n\to\infty$ it follows from last representation that

$$B_n(t_n,x) = r_n\gamma^{\frac{\alpha}{2}}(r_n)\left(\frac{\gamma^{1-\frac{\alpha}{2}}(r_n)}{r_n} - N^{\frac{1}{2}}(1+\nu_n(x))\right), \tag{3.51}$$

where $\nu_n(x)\to 0$ as $n\to\infty$ uniformly on any compact set in $\mathbb{R}^N$. Therefore if g satisfies

$$\liminf_{|x|\to\infty} |x|^{-\frac{2}{2-\alpha}} \ln \tilde{g}(|x|) > N^{\frac{1}{2-\alpha}}, \tag{3.52}$$

then there holds

$$J_n(t_n,x)\underset{n\to\infty}{\to}\infty\Longrightarrow\lim_{t_n\to 0}\underline{u}_{g_n}(t_n,x)=\infty, \tag{3.53}$$

uniformly on compact subsets of $\mathbb{R}^N$. We fix $m>0$, denote by λ_m the first eigenvalue of $-\Delta$ in $H_0^1(B_m)$, with corresponding eigenfunction ϕ_m normalized by $\sup_{B_m}\phi_m=1$ and set, for $\epsilon>0$,

$$W_{m,\epsilon}(t,x)=e^{-(t+\epsilon)\lambda_m}\Phi_\infty(t+\epsilon)\phi_m(x)\qquad\forall(t,x)\in Q_\infty^{B_m}.$$

Then

$$\begin{aligned}\partial_t W_{m,\epsilon}-\Delta W_{m,\epsilon}+W_{m,\epsilon}\ln^\alpha(W_{m,\epsilon}+1)&=W_{m,\epsilon}\left(\ln^\alpha(W_{m,\epsilon}+1)-\ln^\alpha(\Phi_\infty(t+\epsilon)+1)\right)\\&\le 0.\end{aligned}$$

Since $\underline{u}_{g_n}$ increases to the minimal solution $\underline{u}_{\tilde{g}}$, it follows due to (3.53) that there exists n_ϵ such that

$$\underline{u}_{\tilde{g}}(t_{n_\epsilon},x)\ge\underline{u}_{g_{n_\epsilon}}(t_{n_\epsilon},x)\ge W_{m,\epsilon}(t_{n_\epsilon}+\epsilon,x)\qquad\forall x\in B_m.$$

Last inequality in virtue of comparison principle implies

$$\underline{u}_g(t,x)\ge W_{m,\epsilon}(t+\epsilon,x)\qquad\forall(t,x)\in Q_\infty^{B_m},t\ge t_{n_\epsilon}.$$

Letting $\epsilon\to 0$ yields $\underline{u}_g\ge W_{m,0}$ in $Q_\infty^{B_m}$. Since $\lim_{m\to\infty}\phi_m(x)=1$, uniformly on any compact subset of $\mathbb{R}^N$ and $\lim_{m\to\infty}\lambda_m=0$ we derive $\underline{u}_{\tilde{g}}\geqslant\Phi_\infty$ and finally $\underline{u}_g\geqslant\Phi_\infty$. This inequality together with (3.38) leads to $\underline{u}=\Phi_\infty$. □

Remark. In the case $\alpha=2$, there holds

$$\int_0^t\ln^2(\omega_n(s)+1)ds\le 4\gamma(r_n)\int_0^{t\gamma(r_n)}(2+\tau)^{-2}d\tau\le t\gamma(r_n). \tag{3.54}$$

Therefore (3.49) is replaced by

$$A_n(t,x)\ge\gamma(r_n)-t\gamma^2(r_n)-\frac{N(r_n+|x|)^2}{4t}-N\ln(r_n+|x|)-\frac{N}{2}\ln t:=B_n(t,x). \tag{3.55}$$

A similarly, there exists $t_n>0$ where $t\mapsto B_n(t,x)$ is maximum and in that case

$$\begin{aligned}B_n(t_n,x)=\gamma(r_n)-N\ln(r_n+|x|)-\frac{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}}{4}\\-\frac{N(r_n+|x|)^2\gamma^2(r_n)}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}}-\frac{N}{2}\ln\left(\frac{N(r_n+|x|)^2}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}}\right),\end{aligned}$$

which yields

$$B_n(t_n,x)=\gamma(r_n)-r_n\gamma(r_n)(N^{\frac{1}{2}}-\nu_n(x)), \tag{3.56}$$

where $\nu_n(x)\to 0$ as $n\to\infty$ uniformly on any compact set in $\mathbb{R}^N$. Thus $B_n(t_n,x)\to-\infty$ as $n\to\infty$. A similar type of computation shows that the expression $I_n(t,x)$ defined in (3.41) converges to 0, whatever is the sequence $\{r_n\}$ considered which converges to ∞.

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