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Admissible initial growth for diffusion equations with weakly superlinear absorption

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Abstract We study the admissible growth of initial data of positive solutions of $\partial_t u - \Delta u + f(u) = 0$ in $\mathbb{R}_+ \times \mathbb{R}^N$ when $f(u)$ is a continuous weakly superlinear function *mildly* at infinity, the model being $f(u) = u \ln^\alpha(u)$ with $1 < \alpha < 2$. We prove that if the growth of the initial data is too strong, there is no more diffusion and the corresponding solution satisfies the ODE problem $\partial_t \phi + f(\phi) = 0$ on \mathbb{R}_+ with $\phi(0) = \infty$.

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1 Introduction

Let h be a continuous nondecreasing function defined on \mathbb{R} and vanishing only at 0. It is well known that for any continuous and bounded function g belonging to $C_b^+(\mathbb{R}^N)$, the

cone of bounded nonnegative continuous functions on \mathbb{R}^N , there exists a unique weak solution $u := u_g \in C_b^+(\mathbb{R}_+ \times \mathbb{R}^N)$ of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty := \mathbb{R}_+ \times \mathbb{R}^N \\ \lim_{t \rightarrow 0} u(t, \cdot) &= g && \text{locally uniformly in } \mathbb{R}^N. \end{aligned} \quad (1.1)$$

Furthermore, the solution u satisfies

$$0 \leq u(x, t) \leq \Phi_{\|g\|_{L^\infty}}(t) \quad \forall (t, x) \in Q_{\mathbb{R}^N}^\infty, \quad (1.2)$$

where Φ_a is the restriction to \mathbb{R}_+ of the maximal solution of

$$\begin{aligned} \Phi_t + \Phi h(\Phi) &= 0 && \text{on } \mathbb{R}_+ \\ \Phi(0) &= a. \end{aligned} \quad (1.3)$$

When g ceases to be bounded, the existence of a solution holds provided

$$\int_a^\infty \frac{ds}{sh(s)} < \infty, \quad (1.4)$$

for some $a > 0$. Furthermore there exists a minimal solution \underline{u}_g which is the limit when $n \rightarrow \infty$ of the solutions u_{g_n} of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty \\ \lim_{t \rightarrow 0} u(t, \cdot) &= g\chi_{B_n} && \text{locally uniformly in } \mathbb{R}^N, \end{aligned} \quad (1.5)$$

where B_n denotes the open ball of radius n . Furthermore there holds

$$0 \leq \underline{u}_g \leq \Phi_\infty. \quad (1.6)$$

We first prove a uniqueness result

Theorem A *Assume $r \mapsto rh(r)$ is convex and satisfies, for some $a > 0$,*

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \quad \text{where } H(t) = \int_0^t th(t)dt, \quad (1.7)$$

then for any $g \in C^+(\mathbb{R}^N)$, \underline{u}_g is the unique nonnegative solution of (1.1).

When h is a power the class of existence and uniqueness is much larger and is associated to the notion of initial trace which has been thoroughly studied by Marcus and Véron [3], [4] and Gkikas and Véron [2].

When

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} = \infty \quad \forall a > 0, \quad (1.8)$$

uniqueness may not hold in the class of unbounded solution. If, for any $b > 0$, V_b denotes the maximal solution of the following ode

$$\begin{aligned} V_{rr} + \frac{N-1}{r}V_r - Vh(V) &= 0 & \text{on } (0, R_a) \\ V(0) &= b \\ \partial_r V(0) &= 0 \end{aligned} \tag{1.9}$$

then $R_b = \infty$ (see e.g. [7]). Furthermore, Nguyen Phuoc and Véron proved in [5] that if g satisfies

$$V_c(|x|) \leq g(x) \leq V_b(|x|) \quad \forall x \in \mathbb{R}^N \tag{1.10}$$

for some $b > c > 0$, then there exists at least two different solutions of (1.1) defined in $Q_{\mathbb{R}^N}^\infty$: the minimal one \underline{u}_g which satisfies

$$\underline{u}_g(x, t) \leq \Phi_\infty(t) \quad \forall (x, t) \in Q_{\mathbb{R}^N}^\infty \tag{1.11}$$

and another one u_g such that

$$V_c(|x|) \leq u(x, t) \leq V_b(|x|) \quad \forall (x, t) \in Q_{\mathbb{R}^N}^\infty. \tag{1.12}$$

Actually it is not clear whether there exists a maximal solution or not. However, if g satisfies (1.10), there exists a minimal solution $\underline{u}_{g,c,b}$ and a maximal one $\bar{u}_{g,c,b}$ in the class $\mathcal{E}_{c,b}(g)$ of solutions with initial data g which satisfy (1.12). These two solutions are constructed using an approximate scheme of solutions $u := \underline{u}_n$ of Cauchy-Dirichlet problems, for example with

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 & \text{in } Q_{B_n}^\infty := \mathbb{R}_+ \times B_n \\ u(t, x) &= V_c(n) & \text{in } \partial_t Q_{B_n}^\infty := \mathbb{R}_+ \times \partial B_n \\ u(0, \cdot) &= g & \text{in } B_n, \end{aligned} \tag{1.13}$$

then $\{\underline{u}_n\}$ is increasing and converges to $\underline{u}_{g,c,b}$. Similarly, the sequence $\{\bar{u}_n\}$ of solutions of the same equation in $Q_{B_n}^\infty$ with the same initial data and boundary value $V_b(n)$ is decreasing and converges to $\bar{u}_{g,c,b}$.

When the initial data g growth at infinity faster than any function V_b we prove that such a solution cannot exist: For any $a > 0$ we denote by $u := u_{a,n}$ the solution of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 & \text{in } Q_{B_n}^\infty := \mathbb{R}_+ \times B_n \\ u(t, x) &= V_a(n) & \text{in } \partial_t Q_{B_n}^\infty := \mathbb{R}_+ \times \partial B_n \\ u(0, \cdot) &= \min\{V_a, g\} & \text{in } B_n, \end{aligned} \tag{1.14}$$

Theorem B *Assume $r \mapsto rh(r)$ is convex and satisfies (1.4) and (1.8). If $g \in C^+(\mathbb{R}^N)$, satisfies*

$$\lim_{|x| \rightarrow \infty} \frac{g(x)}{V_a(|x|)} = \infty \quad \forall a > 0, \tag{1.15}$$

then the sequence $\{u_{a,n}\}_{n \in \mathbb{N}_*}$ decreases and converges to a solution u_a of (1.1) with initial data $\min\{V_a, g\}$. Furthermore $u_a(t, x) \rightarrow \infty$ for any $(t, x) \in Q_{\mathbb{R}^N}^\infty$ as $a \rightarrow \infty$.

A fundamental example of equations with nonlinearities satisfying (1.4) and (1.8) is provided by

$$\partial_t u - \Delta u + u \ln^\alpha(1 + u) = 0 \quad \text{in } Q_{\mathbb{R}^N}^\infty \quad (1.16)$$

with $1 < \alpha \leq 2$. With this specific type of nonlinearity we prove:

Theorem C *Assume $1 < \alpha < 2$ and $g \in C^+(\mathbb{R}^N)$, satisfies (1.15). Then the minimal solution \underline{u}_g of (1.16) with initial data g is Φ_∞ .*

2 The maximal solution

2.1 Proof of Theorem A

Let \underline{u}_g be the minimal solution of (1.1) and u another solution with the same initial data g . We set $w = u - \underline{u}_g$. Since $r \mapsto rh(r)$ is convex and $u - \underline{u}_g$ is positive,

$$uh(u) \geq \underline{u}_g h(\underline{u}_g) + (u - \underline{u}_g)h(u - \underline{u}_g).$$

Therefore w is a subsolution of problem (1.1), and $w(t, x) \rightarrow 0$ as $t \rightarrow 0$, locally uniformly in \mathbb{R}^N . Let v_n be the minimal solution of

$$\begin{aligned} -\Delta v + vh(v) &= 0 & \text{in } B_n \\ \lim_{|x| \rightarrow n} v(x) &= \infty. \end{aligned} \quad (2.17)$$

Such a solution exists by [1], [6] because (1.7) holds and it is radial as limit of the radial functions $v_{n,k}$, $k \in \mathbb{N}^*$, which are solutions of (2.17) with finite boundary data $v_{n,k} = k$ on ∂B_n . Moreover $v_{n,k}$, and thus v_n , is an increasing function of $|x|$. Then $v \geq 0$ and it is a stationary solution of (1.1) in $Q_\infty^{B_n}$. By the comparison principle

$$w(t, x) \leq v_n(x) \quad \text{in } B_\infty^{B_n}.$$

Furthermore $n \mapsto v_n$ is decreasing with limit v_∞ as $n \rightarrow \infty$. The function v_∞ is a nonnegative, radial and nondecreasing with respect to $|x|$ solution of

$$-\Delta v + vh(v) = 0 \quad \text{in } \mathbb{R}^N.$$

In order to prove that $v = 0$, we return to v_n which satisfies

$$v_{nr} = r^{1-N} \int_0^r s^{N-1} v_n(s) h(v_n(s)) ds \leq v_n(r) h(v_n(r)) r^{1-N} \int_0^r s^{N-1} ds = \frac{r}{N} v_n(r) h(v_n(r)),$$

thus

$$-v_{nrr} + v_n(r) h(v_n(r)) = \frac{N-1}{r} v_{nr} \leq \left(1 - \frac{1}{N}\right) v_n(r) h(v_n(r))$$

which implies

$$-v_{nrr} + \frac{1}{N} v_n(r) h(v_n(r)) \leq 0.$$

Integrating twice yields

$$\int_{v_n(r)}^{\infty} \frac{ds}{\sqrt{\int_0^s th(t)dt}} \geq \sqrt{\frac{2}{N}}(n-r). \quad (2.18)$$

If we had $v_\infty(r) > 0$ for any $r > 0$, it would imply

$$\infty > \int_{v_\infty(r)}^{\infty} \frac{ds}{\sqrt{2\int_0^s th(t)dt}} \geq \infty,$$

a contradiction. Thus $v_\infty(r) = 0$ and $w(t, x) = 0$. \square

2.2 Proof of Theorem B

We recall that (1.15) holds and that $u_{a,n}$ denotes the solution of (1.14). Since $V_a|_{Q_{B_n}^\infty}$ is the solution of the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q_{B_n}^\infty := \mathbb{R}_+ \times B_n \\ u(t, x) &= V_a(n) && \text{in } \partial_\ell Q_{B_n}^\infty := \mathbb{R}_+ \times \partial B_n \\ u(0, \cdot) &= V_a && \text{in } B_n, \end{aligned} \quad (2.19)$$

it is larger than $u_{a,n}$. Thus $u_{a,n+1}|_{\partial_\ell Q_{B_n}^\infty} \leq u_{a,n}|_{\partial_\ell Q_{B_n}^\infty} = V_a$. Since $u_{a,n}(0, \cdot) = u_{a,n+1}|_{B_n}(0, \cdot)$ it follows that $u_{a,n+1}|_{Q_{B_n}^\infty} \leq u_{a,n}$. Then $\{u_{a,n}\}$ is a decreasing sequence, and its limit u_a is a solution of (1.1), which the first claim. By the same argument, $u_{a,n} \leq u_{b,n+1}|_{Q_{B_n}^\infty}$ in $Q_{B_n}^\infty$ for $b > a$. Hence $u_a \leq u_b$. We introduce the sequence $\{r_a\} : r_a \rightarrow \infty$ as $a \rightarrow \infty$ defined by:

$$r_a = \inf\{r > 0 : g(x) \geq V_a(x) \quad \forall |x| \geq r\}. \quad (2.20)$$

and, for $n \geq r_a$, we set $w_{a,n} = V_a - u_{a,n}$. By convexity it satisfies

$$\begin{aligned} \partial_t w_{a,n} - \Delta w_{a,n} + w_{a,n}h(w_{a,n}) &\leq 0 && \text{in } Q_{B_n}^\infty := \mathbb{R}_+ \times B_n \\ w_{a,n}(t, x) &= 0 && \text{in } \partial_\ell Q_{B_n}^\infty := \mathbb{R}_+ \times \partial B_n \\ w_{a,n} &= (V_a - g)_+ && \text{in } B_n, \end{aligned} \quad (2.21)$$

Therefore

$$w_{a,n}(t, x) < \Phi_\infty(t) \quad \text{in } Q_{B_n}^\infty. \quad (2.22)$$

where Φ_∞ is defined in (1.3) with $a = \infty$. Actually,

$$\int_{\Phi_\infty(t)}^{\infty} \frac{ds}{sh(s)} = t \quad (2.23)$$

Notice also that the sequence $\{w_{a,n}\}$ is increasing and it converges, as $n \rightarrow \infty$, to $w_a = V_a - u_a$, which is dominated by Φ_∞ . Thus

$$u_a(t, x) \geq V_a(x) - \Phi_\infty(t) \geq a - \Phi_\infty(t) \quad \text{in } Q_{\mathbb{R}^N}^\infty. \quad (2.24)$$

Letting $a \rightarrow \infty$ implies the claim. \square

3 The minimal solution

In this section we consider equation (1.16) with $1 < \alpha < 2$

3.1 The stationary problem

Proposition 3.1 *Assume $1 < \alpha \leq 2$, $a > 0$ and V_a is the solution of*

$$\begin{aligned} V_{rr} + \frac{N-1}{r}V_r - V \ln^\alpha(V+1) &= 0 && \text{in } \mathbb{R}_+ \\ V_r(0) &= 0 \\ V(a) &= a. \end{aligned} \tag{3.25}$$

Then

$$V(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}} + O(1)} \quad \text{as } r \rightarrow \infty, \tag{3.26}$$

where $c_\alpha = \left(\frac{2-\alpha}{2}\right)^{\frac{2}{2-\alpha}}$.

Proof. We write $W = \ln(V+1)$. Since V is increasing $W > 0$, $W_r \geq 0$ and

$$W_{rr} + W_r^2 + \frac{N-1}{r}W_r - (1 - e^{-W})W^\alpha = 0 \quad \text{in } \mathbb{R}_+. \tag{3.27}$$

Thus

$$W_{rr} + W_r^2 - (1 - e^{-W})W^\alpha \leq 0.$$

If we set $\rho = W$ and $p(\rho) = W_r(r)$, then $\rho \in [a, \infty)$ and

$$pp' + p^2 - (1 - e^{-\rho})\rho^\alpha \leq 0.$$

This is a linear differential inequality in the unknown p^2 . Integrating yields

$$p^2(\rho) \leq 2e^{-2\rho} \int_a^\rho (e^{2s} - e^s)s^\alpha ds = \rho^\alpha + O(1). \tag{3.28}$$

Thus $W_r(r) \leq W^{\frac{\alpha}{2}}(r) + O(1)$ as $r \rightarrow \infty$ which implies

$$W(r) \leq c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \quad \text{as } r \rightarrow \infty. \tag{3.29}$$

By (3.27),

$$0 < W_r \leq c_\alpha^{\frac{\alpha}{2}} r^{\frac{\alpha}{2-\alpha}} (1 + o(1)).$$

We derive that for any $\epsilon > 0$ there exists $r_\epsilon > 0$ such that

$$W_{rr} + W_r^2 \geq (1 - \epsilon)W^\alpha \quad \text{on } [r_\epsilon, \infty),$$

which in turn implies

$$W(r) \geq (1 - \epsilon)c_\alpha r^{\frac{2}{2-\alpha}} (1 + o(1)) \quad \text{as } r \rightarrow \infty. \tag{3.30}$$

Since ϵ is arbitrary, we derive

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}}(1 + o(1)) \quad \text{as } r \rightarrow \infty. \quad (3.31)$$

From the above estimates, we can improve (3.30) and get

$$pp' + p^2 = (1 - e^{-\rho})\rho^\alpha - \frac{N-1}{r}W_r \geq (1 - e^{-\rho})\rho^\alpha - c\rho^{\alpha-1},$$

from which it follows easily

$$p^2(\rho) \geq 2e^{-2\rho} \int_a^\rho e^{2s}(s^\alpha - c's^{\alpha-1})ds = \rho^\alpha + O(1), \quad (3.32)$$

by l'Hospital rule. Combined with (3.31) it implies

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \quad \text{as } r \rightarrow \infty. \quad (3.33)$$

Returning to V_a , we derive

$$V_a(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}} + O(1)} \quad \text{as } r \rightarrow \infty. \quad (3.34)$$

□

Remark. If $\alpha = 2$, the same method yields

$$V_a(r) = e^{e^r + O(1)} \quad \text{as } r \rightarrow \infty. \quad (3.35)$$

3.2 Proof of Theorem C

We recall that the minimal solution \underline{u}_g is the limit, when $n \rightarrow \infty$ of the (increasing) sequence of solutions $\{u_{\ell_n}\}$ of

$$\begin{aligned} \partial_t u - \Delta u + u \ln^\alpha(u+1) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty \\ u(0, \cdot) &= g\chi_{B_{\ell_n}} && \text{in } \mathbb{R}^N, \end{aligned} \quad (3.36)$$

where $\{\ell_n\}$ is any increasing sequence converging to ∞ . Furthermore if we replace g by it maximal radial minorant defined by $\tilde{g}(r) := \min_{|x|=r} g(x)$, it satisfies also (1.8). Because of (1.8) there exists a sequence $\{r_n\}$ tending to infinity such that

$$r_n = \tilde{g}(r_n) \leq \tilde{g}(r) \quad \forall r \geq r_n.$$

Step 1: Estimate from below. Put

$$g_n(|x|) = \begin{cases} \inf\{\tilde{g}(r_n), \tilde{g}(|x|)\} & \text{if } |x| < r_n \\ \tilde{g}(r_n) & \text{if } |x| \geq r_n. \end{cases}$$

Let \underline{u}_{g_n} be the minimal solution of

$$\begin{aligned} \partial_t u - \Delta u + u \ln^\alpha(u+1) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty \\ u(0, \cdot) &= g_n && \text{in } \mathbb{R}^N. \end{aligned} \quad (3.37)$$

Then $\underline{u}_{g_n} \leq \Phi_\infty$. For any sequence $\{\ell_k\}$ converging to infinity and any fixed k , there exists n_k such that for $n \geq n_k$, there holds $g\chi_{B_{\ell_k}} \leq g_n$. Since the sequence $\{\underline{u}_{g_n}\}$ is increasing, its limit u_∞ is a solution of (1.3) in $Q_\infty^{\mathbb{R}^N}$ which is larger than u_{ℓ_k} for any ℓ_k , and therefore larger also than $\underline{u}_{\tilde{g}}$. However, since $g_n \leq \tilde{g}$, $u_\infty \leq \underline{u}_{\tilde{g}}$. This implies

$$u_\infty = \underline{u}_{\tilde{g}}. \quad (3.38)$$

Next, since $\underline{u}_{g_n}(0, x) \leq g(r_n)$ it follows that $\underline{u}_{g_n}(t, x) \leq g(r_n)$. Let $\omega_n = \Phi_{g(r_n)}$, i.e. the solution of be the solution of (1.3) with $a = g(r_n)$, then ω_n satisfies

$$\int_{\omega_n(t)}^{g(r_n)} \frac{ds}{\sqrt{\int_0^s th(t)dt}} = t,$$

and $\underline{u}_{g_n} \geq w_n$ where w_n is the minimal solution of

$$\begin{aligned} \partial_t w - \Delta w + w \ln^\alpha(\omega_n + 1) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty \\ w(0, \cdot) &= g_n && \text{in } \mathbb{R}^N. \end{aligned} \quad (3.39)$$

If we set $w_n(t, x) = e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} z_n(t, x)$, then

$$\begin{aligned} \partial_t z_n - \Delta z_n &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty \\ z_n(0, \cdot) &= g_n && \text{in } \mathbb{R}^N. \end{aligned} \quad (3.40)$$

Since

$$z_n(t, x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} g_n(y) dy,$$

we can write $w_n(t, x) = I_n(t, x) + J_n(t, x)$ where

$$I_n(t, x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}}{(4\pi t)^{\frac{N}{2}}} \int_{|y| \leq r_n} e^{-\frac{|x-y|^2}{4t}} g_n(y) dy, \quad (3.41)$$

and

$$J_n(t, x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \int_{|y| > r_n} e^{-\frac{|x-y|^2}{4t}} dy. \quad (3.42)$$

Clearly

$$\begin{aligned} J_n(t, x) &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \int_{|y| > r_n + |x|} e^{-\frac{|y|^2}{4t}} dy \\ &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \left(\int_{|y| > r_n + |x|} e^{-\frac{z^2}{4t}} dz \right)^N. \end{aligned} \quad (3.43)$$

This integral term can be estimated by introducing Gauss error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} dz. \quad (3.44)$$

In dimension N , it implies easily

$$J_n(t, x) \geq e^{-\int_0^t \ln^\alpha(\omega_n(s)+1) ds} \tilde{g}(r_n) \left(\operatorname{erfc} \left(\frac{r_n + |x|}{2\sqrt{t}} \right) \right)^N. \quad (3.45)$$

Since

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{t}}(1 + O(x^{-2})) \quad \text{as } x \rightarrow \infty,$$

we derive

$$J_n(t, x) \geq \frac{\tilde{g}(r_n)}{((r_n + |x|)^2 t)^{\frac{N}{2}}} e^{-\int_0^t \ln^\alpha(\omega_n(s)+1) ds - \frac{N(r_n + |x|)^2}{4t}} \left(1 + O \left(\frac{t}{r_n^2} \right) \right). \quad (3.46)$$

We write $\tilde{g}(r) = \exp(\gamma(r)) - 1$ and set

$$A_n(t, x) = \gamma(r_n) - \int_0^t \ln^\alpha(\omega_n(s) + 1) ds - \frac{N(r_n + |x|)^2}{4t} - N \ln(r_n + |x|) - \frac{N}{2} \ln t.$$

In order to have an estimate on $\omega_n(s)$, we fix $t \leq 1$ and $\tilde{g}(r_n) \geq 1$. Since

$$\begin{aligned} \omega' + \omega \ln^\alpha(\omega + 1) &= \omega' + \frac{\omega}{\omega + 1} (\omega + 1) \ln^\alpha(\omega + 1) \\ &\geq \omega' + \frac{1}{2} (\omega + 1) \ln^\alpha(\omega + 1), \end{aligned}$$

because

$$\min \left\{ \frac{\omega_a(t)}{\omega_a(t) + 1} : 0 \leq t \leq 1, a \geq 1 \right\} = \frac{1}{2},$$

we obtain

$$\ln^\alpha(\omega_n(s) + 1) \leq \left(\frac{2\gamma^{\alpha-1}(r_n)}{2 + (\alpha - 1)s\gamma^{\alpha-1}(r_n)} \right)^{\frac{\alpha}{\alpha-1}}.$$

From this inequality, we derive

$$\begin{aligned} \int_0^t \ln^\alpha(\omega_n(s) + 1) ds &\leq \int_0^t \left(\frac{2\gamma^{\alpha-1}(r_n)}{2 + (\alpha - 1)s\gamma^{\alpha-1}(r_n)} \right)^{\frac{\alpha}{\alpha-1}} ds \\ &\leq 2^{\frac{\alpha}{\alpha-1}} \gamma(r_n) \int_0^{t\gamma^{\alpha-1}(r_n)} (2 + (\alpha - 1)\tau)^{-\frac{\alpha}{\alpha-1}} d\tau. \end{aligned}$$

Therefore

$$\begin{aligned} A_n(t, x) &\geq \gamma(r_n) - \frac{N(r_n + |x|)^2}{4t} - N \ln(r_n + |x|) - \frac{N}{2} \ln t \\ &\quad - 2^{\frac{\alpha}{\alpha-1}} \gamma(r_n) \int_0^{t\gamma^{\alpha-1}(r_n)} (2 + (\alpha - 1)\tau)^{-\frac{\alpha}{\alpha-1}} d\tau. \end{aligned} \quad (3.47)$$

Step 2: The maximal admissible growth. We claim that

$$\liminf_{|x| \rightarrow \infty} |x|^{-\frac{2}{2-\alpha}} \ln \tilde{g}(|x|) > N^{\frac{2}{2-\alpha}} \implies \lim_{n \rightarrow \infty} \underline{u}_{g_n}(t, x) = \Phi_\infty(t) \quad \forall (t, x) \in Q_{\mathbb{R}^N}^\infty. \quad (3.48)$$

By replacing $\tau \mapsto (2 + (\alpha - 1)\tau)^{-\frac{\alpha}{\alpha-1}}$ by its maximal value on $(0, t\gamma^{\alpha-1}(r_n))$,

$$2^{\frac{\alpha}{\alpha-1}}\gamma(r_n) \int_0^{t\gamma^{\alpha-1}(r_n)} (2 + (\alpha - 1)\tau)^{-\frac{\alpha}{\alpha-1}} d\tau \leq \gamma^\alpha(r_n)t.$$

Then

$$A_n(t, x) \geq \gamma(r_n) - \frac{N(r_n + |x|)^2}{4t} - N \ln(r_n + |x|) - \frac{N}{2} \ln t - \gamma^\alpha(r_n)t := B_n(t, x), \quad (3.49)$$

and

$$\partial_t B_n(t, x) = \frac{N(r_n + |x|)^2}{4t^2} - \frac{N}{2t} - \gamma^\alpha(r_n).$$

Thus

$$\partial_t B_n(t, x) = 0 \text{ and } t > 0 \iff t := t_n = \frac{N(r_n + |x|)^2}{N + \sqrt{N^2 + 4N(r_n + |x|)^2\gamma^\alpha(r_n)}}. \quad (3.50)$$

Therefore $A_n(t_n, x)$ is bounded from below by the maximum of $B_n(t, x)$ which is achieved for $t = t_n$ and

$$B_n(t_n, x) = \gamma(r_n) - N \ln(r_n + |x|) - \frac{N + \sqrt{N^2 + 4N(r_n + |x|)^2\gamma^\alpha(r_n)}}{4} - \frac{N(r_n + |x|)^2\gamma^\alpha(r_n)}{N + \sqrt{N^2 + 4N(r_n + |x|)^2\gamma^\alpha(r_n)}} - \frac{N}{2} \ln \left(\frac{N(r_n + |x|)^2}{N + \sqrt{N^2 + 4N(r_n + |x|)^2\gamma^\alpha(r_n)}} \right).$$

Therefore

$$B_n(t_n, x) = r_n \gamma^{\frac{\alpha}{2}}(r_n) \left(\frac{\gamma^{1-\frac{\alpha}{2}}(r_n)}{r_n} - N(1 + o(1)) \right). \quad (3.51)$$

If g satisfies

$$\liminf_{|x| \rightarrow \infty} |x|^{-\frac{2}{2-\alpha}} \ln \tilde{g}(|x|) > N^{\frac{2}{2-\alpha}}, \quad (3.52)$$

there holds

$$\lim_{t_n \rightarrow 0} \underline{u}_{g_n}(t_n, x) = \infty, \quad (3.53)$$

uniformly on compact subsets of \mathbb{R}^N . We fix $m > 0$, denote by λ_m the first eigenvalue of $-\Delta$ in $H_0^1(B_m)$, with corresponding eigenfunction ϕ_m normalized by $\sup_{B_m} \phi_m = 1$ and set, for $\epsilon > 0$,

$$W_{m,\epsilon}(t, x) = e^{-(t+\epsilon)\lambda_m} \Phi_\infty(t + \epsilon) \phi_m(x) \quad \forall (t, x) \in Q_\infty^{B_m}.$$

Then

$$\begin{aligned} \partial_t W_{m,\epsilon} - \Delta W_{m,\epsilon} + W_{m,\epsilon} \ln^\alpha(W_{m,\epsilon} + 1) &= W_{m,\epsilon} \left(\ln^\alpha(W_{m,\epsilon} + 1) - \ln^\alpha(\phi_\infty(t + \epsilon) + 1) \right) \\ &\leq 0. \end{aligned}$$

Since \underline{u}_{g_n} increases to the minimal solution $\underline{u}_{\tilde{g}}$, there exists n_ϵ such that

$$\underline{u}_{\tilde{g}}(t_{n_\epsilon}, x) \geq \underline{u}_{g_{n_\epsilon}}(t_{n_\epsilon}, x) \geq W_{m,\epsilon}(t_{n_\epsilon} + \epsilon, x) \quad \forall x \in B_m$$

which implies

$$\underline{u}_g(t, x) \geq W_{m,\epsilon}(t + \epsilon, x) \quad \forall (t, x) \in Q_\infty^{B_m}, t \geq t_{n_\epsilon}.$$

Letting $\epsilon \rightarrow 0$ yields $\underline{u}_g \geq W_{m,0}$ in $Q_\infty^{B_m}$. Since $\lim_{m \rightarrow \infty} \phi_m(x) = 1$, uniformly on any compact subset of \mathbb{R}^N and $\lim_{m \rightarrow \infty} \lambda_m = 1$ we derive $\underline{u}_{\tilde{g}} = \Phi_\infty$ and finally $\underline{u}_g = \Phi_\infty$. \square

Remark. In the case $\alpha = 2$, there holds

$$\int_0^t \ln^2(\omega_n(s) + 1) ds \leq 4\gamma(r_n) \int_0^{t\gamma(r_n)} (2 + \tau)^{-2} d\tau \leq t\gamma(r_n). \quad (3.54)$$

Therefore (3.49) is replaced by

$$A_n(t, x) \geq \gamma(r_n) - t\gamma^2(r_n) - \frac{N(r_n + |x|)^2}{4t} - N \ln(r_n + |x|) - \frac{N}{2} \ln t := B_n(t, x). \quad (3.55)$$

A similarly, there exists $t_n > 0$ where $t \mapsto B_n(t, x)$ is maximum and in that case

$$B_n(t_n, x) = \gamma(r_n) - N \ln(r_n + |x|) - \frac{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^2(r_n)}}{4} - \frac{N(r_n + |x|)^2 \gamma^2(r_n)}{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^2(r_n)}} - \frac{N}{2} \ln \left(\frac{N(r_n + |x|)^2}{N + \sqrt{N^2 + 4N(r_n + |x|)^2 \gamma^2(r_n)}} \right),$$

which yields

$$B_n(t_n, x) = \gamma(r_n) - r_n \gamma(r_n) (1 = o(1)) \quad \text{as } r_n \rightarrow \infty. \quad (3.56)$$

Thus $B_n(t_n, x) \rightarrow 0$ as $n \rightarrow \infty$. A similar type of computation shows that the expression $I_n(t, x)$ defined in (3.41) converges to 0, whatever is the sequence $\{r_n\}$ considered which converges to ∞ .

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