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L^1-minimization for mechanical systems

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Abstract

Second order systems whose drift is defined by the gradient of a given potential are considered, and minimization of the L^1-norm of the control is addressed. An analysis of the extremal flow emphasizes the role of singular trajectories of order two [25, 29]; the case of the two-body potential is treated in detail. In L^1-minimization, regular extremals are associated with controls whose norm is bang-bang; in order to assess their optimality properties, sufficient conditions are given for broken extremals and related to the no-fold conditions of [20]. An example of numerical verification of these conditions is proposed on a problem coming from space mechanics.

Keywords. L^1-minimization, second order mechanical systems, order two singular trajectories, no-fold conditions for broken extremals, two-body problem

MSC classification. 49K15, 70Q05

1 Introduction

This paper is concerned with the optimal control of mechanical systems of the following form:

$$\ddot{q}(t) + \nabla_q V(q(t)) = \frac{u(t)}{M(t)}\,, \quad \dot{M}(t) = -\beta|u(t)|,$$

where q is valued in an open subset Q of $\mathbf{R}^m$, $m \geq 2$, on which the potential V is defined. The second equation describes the variation of the mass, M, of the system when a control is used (β is some nonnegative constant). The finite dimensional norm is Euclidean,

$$|u| = \sqrt{u_1^2 + \cdots + u_m^2}$$

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and a constraint on the control is assumed,

$$|u(t)| \leq \varepsilon, \quad \varepsilon > 0. \tag{1}$$

Given boundary conditions in the n-dimensional state (phase) space $X := TQ \simeq Q \times \mathbf{R}^m$ ($n = 2m$), the problem of interest is the minimization of consumption, that is the maximization of the final mass $M(t_f)$ for a fixed final time. Clearly, this amounts to minimizing the L^1-norm of the control,

$$\int_0^{t_f} |u(t)|\,\mathrm{d}t \to \min. \tag{2}$$

Up to some rescaling, there are actually two cases, $\beta = 1$ or $\beta = 0$. In the second one, the mass is constant; though maximizing the final mass does not make sense anymore, the Lagrange cost (2) is still meaningful. Actually, as propellant is only a limited fraction of the total mass, one can expect this idealized constant mass model to capture the main features of the original problem. We shall henceforth assume $\beta = 0$, so the state reduces to $x := (q, v)$ with $v := \dot{q}$.

In finite dimensions, ℓ^1-minimization is well-known to generate sparse solutions having a lot of zero components; this fact translates here into the existence of subintervals of time where the control vanishes, as is clear when applying the maximum principle (see §2). This intuitively goes along well with the idea of minimizing consumption: There are privileged values of the state where the control is more efficient and should be switched on (*burn* arcs), while there are some others where it should be switched off (*cost* arcs). (See also [5] for a different kind of interpretation in a biological setting, again with L^1-minimization.) The resulting sparsity of the solution is then tuned by the ratio of the fixed final time over the minimum time associated with the boundary conditions: While a simple consequence of the form of the dynamics (and of the ball constraint on the control) is that the min. time control norm is constant and maximum everywhere for the constant mass model,[1] the extra amount of time available allows for some optimization that results in the existence of subarcs of the trajectory with zero control. (See Proposition 1, in this respect.) A salient peculiarity of the infinite dimensional setting is the existence of subarcs with intermediate value of the norm of the control, namely singular arcs. This was analyzed in the seminal paper of Robbins [25] in the case of the two-body potential, providing yet another example of the fruitful exchanges between space mechanics and optimal control in the early years of both disciplines. The consequence of these singular arcs being of order two was further realized by Marchal who studied chattering in [18]; this example comes probably second after the historical one of Fuller [12] and has been thoroughly investigated by Zelikin and Borisov in [29, 30].

A typical example of second order controlled system is the restricted three-body problem [8] where, in complex notation ($\mathbf{R}^2 \simeq \mathbf{C}$),

$$V_\mu(t, q) := -\frac{1-\mu}{|q + \mu e^{it}|} - \frac{\mu}{|q - (1-\mu)e^{it}|}.$$

In this case, μ is the ratio of the masses of the two primary celestial bodies, in circular motion around their common center of mass. The controlled third body

[1] See, *e.g.*, [10]; this fact remains true for time minimization if the mass is varied provided the mass at final time is left free [*ibid*].

is a spacecraft gravitating in the potential generated by the two primaries, but not influencing their motion. When $\mu = 0$, the potential is autonomous and one retrieves the standard controlled two-body problem. The study of "continuous" (as opposed to impulsive) strategies for the control began in the 60's; see, *e.g.*, the work of Lawden [16], or Beletsky's book [3] where the importance of low thrust (small ε in (1)) to spiral out from a given initial orbit was foreseen. There is currently a strong interest for low-thrust missions with, *e.g.*, the Lisa Pathfinder [17] one of ESA[2] towards the L_1 Lagrange point of the Sun-Earth system, or BepiColombo [4] mission of ESA and JAXA[3] to Mercury.

An important issue in optimal control is the ability to verify sufficient optimality conditions. In L^1-minimization, the first candidates for optimality are controls whose norm is bang-bang, switching from zero to the bound prescribed by (1) (more complicated situations including singular controls). Second order conditions in the bang-bang case have received quite an extensive treatment; references include the paper of Sarychev [26], followed by [2] and [19, 22, 23]. On a similar line, the stronger notion of state optimality was introduced in [24] for free final time. More recently, a regularization procedure has been developed in [27] for single-input systems. These papers consider controls valued in polyhedra; the standing assumptions allow to define a finite dimensional accessory optimization problem in the switching times only. Then, checking a second order sufficient condition on this auxiliary problem turns to be sufficient to ensure strong local optimality of the bang-bang controls. A byproduct of the analysis is that conjugate times, where local optimality is lost, are switching times. A different approach, based on Hamilton-Jacobi-Bellman and the method of characteristics in optimal control, has been proposed by Noble and Schättler in [20]. Their results encompass the case of broken extremals with conjugate points occuring at *or* between switching times. We provide a similar analysis by requiring some generalized (with respect to the smooth case) disconjugacy condition on the Jacobi fields, and using instead a Hamiltonian point of view reminiscent of [11, 15]. Treating the case of such broken extremals is crucial for L^1-minimization: As the finite dimensional norm of the control involved in the constraint (1) and in the cost (2) is an ℓ^2-norm, the control is valued in the Euclidean ball of $\mathbf{R}^m$, not a polyhedron if $m > 1$. When $m = 1$, the situation is degenerate, and one can for instance set $u = u_+ - u_-$, with $u_+, u_- \geq 0$. (This approach also works for $m > 1$ when an ℓ^1 or ℓ^∞-norm is used for the values of the control; see, *e.g.*, [28].) When $m > 1$, it is clear using spherical coordinates that although the norm of the control might be bang-bang, the variations of the control component on $\mathbf{S}^{m-1}$ preclude the reduction to a finite dimensional optimization problem. (The same remark holds true for any ℓ^p-norm of the control values with $1 < p < \infty$.) An example of conjugacy occuring between switching times is provided in §4.

The paper is organized as follows. In section 2, the extremal lifts of L^1-minimizing trajectories are studied for an arbitrary potential in the constant mass case; the properties of the flow are encoded by the Poisson structure defined by two Hamiltonians. In section 3, sufficient conditions for strong local optimality of broken extremals with regular switching points are given in terms of jumps on the Jacobi fields; these conditions are related to the no-fold con-

[2]European Space Agency.
[3]Japan Aerospace Exploration Agency.

dition of [20]. In section 4 some numerical results illustrating the verification of these sufficient conditions for L^1-minimizing trajectories are given. The two-body mechanical potential is considered, completing the study of Gergaud and Haberkorn [13] where the first numerical computation of fuel minimizing controls with hundreds of switchings (for low thrust) was performed using a clever combination of shooting and homotopy techniques. (See also [21] in the case of a few switchings.) The classical construction of fields of extremals in the smooth case is reviewed in an appendix.

2 Singularity analysis of the extremal flow

By renormalizing the time and the potential, one can assume $\varepsilon = 1$ in (1), so we consider the L^1-minimum control of

$$\ddot{q}(t) + \nabla V(q(t)) = u(t), \quad |u(t)| \leq 1,$$

with $x(t) = (q(t), v(t)) \in X = TQ$ ($v(t) = \dot{q}(t)$), Q an open subset of $\mathbf{R}^m$ ($m \geq 2$), and make the following assumptions on the boundary conditions:

$$x(0) = x_0 \quad \text{and} \quad x(t_f) \in X_f \subset X$$

where (i) x_0 does not belong to the terminal submanifold X_f, (ii) X_f is invariant wrt. the flow of the drift,[4]

$$F_0(q, v) = v\frac{\partial}{\partial q} - \nabla V(q)\frac{\partial}{\partial v},$$

and (iii) the fixed final time t_f is supposed strictly greater than the minimum time $\bar{t}_f(x_0, X_f) < \infty$ of the problem. As the cost is not differentiable for $u = 0$, rather than using a non-smooth maximum principle (compare, *e.g.*, [5]) we make a simple desingularization: In spherical coordinates, $u = \rho w$ where $\rho \in [0, 1]$ and $w \in \mathbf{S}^{m-1}$; the change of coordinates amounts to adding an $\mathbf{S}^{m-1}$ fiber above the singularity $u = 0$ of the cost. In these coordinates, the dynamics write

$$\dot{x}(t) = F_0(x(t)) + \rho(t) \sum_{i=1}^{m} w_i(t) F_i(x(t))$$

with canonical $F_i = \partial/\partial v_i$, $i = 1, \ldots, m$, and the criterion is linearized:

$$\int_0^{t_f} \rho(t)\, \mathrm{d}t \to \min.$$

The Hamiltonian of the problem is

$$H(x, p, \rho, w) = p^0 \rho + H_0(x, p) + \rho \sum_{i=1}^{m} w_i \psi_i(x, p)$$

where $H_0(x, p) := pF_0(x)$ and the $\psi_i(x, p) := pF_i(x)$ are the Hamiltonian lifts of the F_i, $i = 1, \ldots, m$. Readily, $H \leq H_0 + \rho H_1$ with

$$H_1 := p^0 + \sqrt{\sum_{i=1}^{m} \psi_i^2},$$

[4] This assumption can be weakened; it is only used to ensure that a time minimizing control extended by zero beyond the min. time is admissible. (See Lemma 1.)

and the equality can always be achieved for some $w \in \mathbf{S}^{m-1}$: $w = \psi/|\psi|$ whenever $\psi := (\psi_1, \ldots, \psi_m)$ is not zero, any w on the sphere otherwise. By virtue of the maximum principle, if (ρ, w) is a measurable minimizing control then the associated trajectory is the projection of an integral curve $(x, p) : [0, t_f] \to T^*X$ of $H_0 + \rho H_1$ such that, a.e.,

$$H_0(x(t), p(t)) + \rho(t) H_1(x(t), p(t)) = \max_{r \in [0,1]} H_0(x(t), p(t)) + r H_1(x(t), p(t)). \quad (3)$$

Moreover, the constant p^0 is nonpositive and $(p^0, p) \neq (0, 0)$. Either $p^0 = 0$ (abnormal case), or p^0 can be set to -1 by homogeneity (normal case).

Proposition 1 (Gergaud *et al.* [13])**.** *There are no abnormal extremals.*

Lemma 1. *The function ψ evaluated along an extremal has only isolated zeros.*

Proof. As a function of time when evaluated along an extremal, ψ is absolutely continous and, a.e. on $[0, t_f]$,

$$\dot{\psi}_i(t) = p(t)[F_0, F_i](x(t)), \quad i = 1, \ldots, m.$$

As a result, ψ is a $\mathscr{C}^1$ function of time and, if $\psi(t) = 0$, then $\dot{\psi}(t) \neq 0$; indeed, the rank of $\{F_1, \ldots, F_m, [F_0, F_1], \ldots, [F_0, F_m]\}$ is maximum everywhere, so $p(t)$ would otherwise be zero. Since p is solution of a linear ode, this would imply that it vanishes identically; necessarily $p^0 < 0$, so ρ would also be zero a.e. because of the maximization condition (3). This is impossible because $x_0 \notin X_f$. □

Proof of the Proposition. By contradiction: Assume $p^0 = 0$; as ψ has only isolated zeros according to the previous lemma, $\rho = 1$ a.e. by maximization. The resulting cost is equal to t_f. Now, the target submanifold X_f is invariant by the drift, so any minimizing control extended by $u = 0$ on $[\bar{t}_f, t_f]$ (where $\bar{t}_f$ denotes the min. time) remains admissible. It has a cost equal to $\bar{t}_f < t_f$, hence the contradiction. □

We set $p^0 = -1$, so $H_1 = |\psi| - 1$. In contrast with the minimum time case, the singularity $\psi = 0$ does not play any role in L^1-minimization. In the neighbourhood of t such that $\psi(t) = 0$, H_1 is negative, so $\rho = 0$. Locally, the control vanishes and the extremal is smooth. The only effect of the singularity is a discontinuity in the $\mathbf{S}^{m-1}$ fiber over $u = 0$ in which $w(t+) = -w(t-)$ (see [10]).

The important remaining singularity is $H_1 = 0$. As opposed to the standard single-input case, H_1 is not the lift of a vector field on X; the properties of the extremal flow depend on H_0, H_1, and their Poisson brackets. (See also §4 for the consequences in terms of second order conditions.) We denote by H_{01} the bracket $\{H_0, H_1\}$, and so forth. The following result is standard (see [6], *e.g.*) and accounts for the intertwining of arcs along which $\rho = 0$ (labeled γ_0) with arcs such that $\rho = 1$ (labeled γ_+).

Proposition 2. *In the neighbourhood of z_0 in $\{H_1 = 0\}$ such that $H_{01}(z_0) \neq 0$, every extremal is locally bang-bang of the form $\gamma_0\gamma_+$ or $\gamma_+\gamma_0$, depending on the sign of $H_{01}(z_0)$.*

Proof. As $H_{01}(z_0) \neq 0$, H_1 must be a submersion at z_0, so $\{H_1 = 0\}$ is locally a codimension one submanifold splitting T^*X into $\{H_1 < 0\}$ and $\{H_1 > 0\}$. Evaluated along an extremal, H_1 is a $\mathscr{C}^1$ function of time since

$$\dot{H}_1(t) = \{H_0 + \rho(t)H_1, H_1\} = H_{01}(t).$$

Through z_0 passes only one extremal, and it is of the form $\gamma_0\gamma_+$ if $H_{01}(z_0) > 0$ (*resp.* $\gamma_+\gamma_0$ if $H_{01}(z_0) < 0$). The bracket condition allows to use the implicit function theorem to prove that neighbouring extremals also cross $\{H_1 = 0\}$ transversally. □

Such switching points are termed *regular* and are studied in §3 from the point of view of second order optimality conditions. Besides the occurence of γ_0 arcs resulting in the parsimony of solutions as explained in the introduction, the peculiarity of the control setting is the existence of singular arcs along which H_1 vanishes identically. On such arcs, ρ may take arbitrary values in $[0, 1]$.

Theorem 1 (Robbins [25]). *Singular extremals are at least of order two, and minimizing singulars of order two are contained in*

$$\{z = (q, v, p_q, p_v) \in T^*X \mid V''(q)p_v^2 \geq 0,\ V'''(q)p_v^3 > 0\}.$$

Proof. One has $H_0 = (p_q|v) - (p_v|\nabla V(q))$, and $H_1 = 0$ along a singular so,

$$0 = H_{01} = -\frac{1}{|p_v|}(p_q|p_v)$$

along the arc.

Lemma 2. *On T^*X, $H_{101} = H_{1001} = 0$.*

Proof. Computing,

$$H_{101} = \{H_1, H_{01}\} = \{|p_v| - 1, -\frac{1}{|p_v|}(p_q|p_v)\} = 0,$$

and it is standard that

$$\begin{aligned} H_{1001} &= \{H_1, \{H_0, H_{01}\}\} \\ &= \{-H_{01}, H_{01}\} + \{H_0, H_{101}\} \\ &= 0 \end{aligned}$$

using Leibniz rule. □

Then $0 = \dot{H}_{01} = H_{001} + \rho H_{101}$ implies $H_{001} = 0$ along a singular arc. Iterating, $0 = \dot{H}_{001} = H_{0001} + \rho H_{1001}$ so, by the previous lemma again, $0 = H_{0001}$. Eventually, $0 = \dot{H}_{0001} = H_{00001} + \rho H_{10001}$. Set $f := H_0$, $g := H_1$, $h := -(p_q|p_v)$, so that $H_{01} = \beta h$ with $\beta = 1/|p_v|$. Using Leibniz rule, the following is clear.

Lemma 3.

$$(\mathrm{ad}^k f)(\beta h)|_{(\mathrm{ad}^i f)h=0,\ 0\leq i<k} = \beta(\mathrm{ad}^k f)h$$

$$\{g, (\mathrm{ad}^k f)(\beta h)\}|_{(\mathrm{ad}^i f)h=0,\ 0\leq i\leq k} = \beta\{g, (\mathrm{ad}^k f)h\}$$

Computing, one obtains

$$(\mathrm{ad}f)h = -V''(q)p_v^2 + |p_q|^2,$$

so $0 = H_{001}$ implies $V''(q)p_v^2 \geq 0$, and

$$(\mathrm{ad}^2 f)h = -V'''(q)(v, p_v, p_v) + 4V''(q)(p_q, p_v),$$

$$\{g, (\mathrm{ad}^2 f)h\} = -\frac{1}{|p_v|}V'''(q)p_v^3.$$

Through a point z_0 such that the last quantity does not vanish, there passes a so-called order two singular extremal that is an integral curve of the Hamiltonian $H_s := H_0 + \rho_s H_1$ with the dynamic feedback

$$\rho_s := -\frac{H_{00001}}{H_{10001}}\cdot$$

Along such a minimizing singular arc, the generalized Legrendre condition must hold, $H_{10001} \leq 0$, so $V'''(q)p_v^3 > 0$. □

Corollary 1. *In the case of the two-body potential $V(q) = -1/|q|$ $(q \neq 0)$, along an order two singular arc one has either $\alpha \in (\pi/2, \alpha_0]$ or $\alpha \in [-\alpha_0, -\pi/2)$ where α is the angle of the control with the radial direction, and $\alpha_0 = \mathrm{acos}(1/\sqrt{3})$.*

Proof. One has

$$V'(q)p_v = \frac{(p_v|q)}{|q|^3}\,, \quad V''(q)p_v^2 = \frac{|p_v|^2}{|q|^3} - \frac{3(p_v|q)^2}{|q|^5}\,,$$

$$V'''(q)p_v^3 = -\frac{9(p_v|q)|p_v|^2}{|q|^5} + \frac{15(p_v|q)^3}{|q|^7}\cdot$$

On $Q = \mathbf{R}^m \backslash \{0\}$, $\mathbf{S}^{m-1} \ni w = p_v$ since $|p_v| = 1$ along a singular arc, so $\cos\alpha = (p_v|q)/|q|$. The condition $V''(q)p_v^2 \geq 0$ reads $1 - 3\cos^2\alpha \geq 0$, and $V'''(q)p_v^3 > 0$ is fulfilled if and only if

$$\cos\alpha(3 - 5\cos^2\alpha) < 0$$

that is provided $\cos\alpha < 0$ in addition to the previous condition. Hence the two cases (in exclusion since the singular control is smooth) for the angle. □

The existence of order two singular arcs in the two-body case results in the well-known Fuller or chattering phenomenon [18, 29]. The same phenomenon actually persists for the restricted three body problem as is explained in [30]. Altough these singular trajectories are contained in some submanifold of the cotangent space with codimension > 1, their existence rules out the possibility to bound globally the number of switchings of regular extremals described by Proposition 2. The next section is devoted to giving sufficient optimality conditions for such bang-bang extremals.

3 Sufficient conditions for extremals with regular switchings

Let X be an open subset of $\mathbf{R}^n$, U a nonempty subset of $\mathbf{R}^m$, f a vector field on X parameterized by $u \in U$, and $f^0 : X \times U \to \mathbf{R}$ a cost function, all smooth. Consider the following minimization problem with fixed final time t_f: Find $(x, u) : [0, t_f] \to X \times U$, x absolutely continous, u measurable and bounded, such that

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, t_f] \text{ (a.e)},$$
$$x(0) = x_0, \quad x(t_f) = x_f,$$

and such that

$$\int_0^{t_f} f^0(x(t), u(t))\, \mathrm{d}t$$

is minimized. The maximum principle asserts that, if $(\overline{x}, \overline{u})$ is such a pair, there exists an absolutely continuous lift $(\overline{x}, \overline{p}) : [0, t_f] \to T^*X$ and a nonpositive scalar $\overline{p}^0$, $(\overline{p}^0, \overline{p}) \neq (0, 0)$, such that a.e. on $[0, t_f]$

$$\dot{\overline{x}}(t) = \frac{\partial H}{\partial p}(\overline{x}(t), \overline{p}(t), \overline{u}(t)), \quad \dot{\overline{p}}(t) = -\frac{\partial H}{\partial x}(\overline{x}(t), \overline{p}(t), \overline{u}(t)),$$

and

$$H(\overline{x}(t), \overline{p}(t), \overline{u}(t)) = \max_U H(\overline{x}(t), \overline{p}(t), \cdot)$$

where $H : T^*X \times U \to \mathbf{R}$ is the Hamiltonian of the problem,

$$H(x, p, u) := p^0 f^0(x, u) + p f(x, u).$$

We first assume that

(A0) The reference extremal is normal.

Accordingly, p^0 can be set to -1. Let $H_1, H_2 : T^*X \to \mathbf{R}$ be two smooth functions, and denote $\Sigma := \{H_1 = H_2\}$, $\Omega_1 := \{H_1 > H_2\}$ ($\Omega_2 := \{H_2 > H_1\}$, *resp.*) We assume that

$$\max_U H(z, \cdot) = H_i, \quad z \in \Omega_i, \quad i = 1, 2, \tag{4}$$

and follow the point of view of [11] that these two Hamiltonians are competing Hamiltonians. Let $(\overline{x}, \overline{p}, \overline{u})$ be a reference extremal having only one contact with Σ at $\overline{z}_1 := \overline{z}(\bar{t}_1)$, $\bar{t}_1 \in (0, t_f)$ ($\overline{z} := (\overline{x}, \overline{p})$). We denote $H_{12} = \{H_1, H_2\}$ the Poisson bracket of H_1 with H_2 and make the following assumption:

(A1) $H_{12}(\overline{z}_1) > 0$.

In [15] terms, $\overline{z}_1$ is a *regular* (or *normal*) switching point. This condition is called the *strict bang-bang Legendre* condition in [2]. The analysis of this section readily extends to a finite number of such switchings.

Lemma 4. *$\overline{z}$ is the concatenation of the flows of H_1 and then H_2.*

Proof. The extremal having only one contact with Σ at $\bar{z}_1$, $\bar{z}(t)$ is either in Ω_1 or Ω_2 for $t \neq \bar{t}_1$. Because of (4), the maximization condition of the maximum principle implies that $\bar{z}$ is given by the flow of either H_1 or H_2 on $[0, \bar{t}_1]$. In both cases,

$$\frac{\mathrm{d}}{\mathrm{d}t}(H_1 - H_2)(\bar{z}(t))|_{t=\bar{t}_1} = -H_{12}(\bar{z}_1) < 0,$$

so $H_1 > H_2$ before $\bar{t}_1$ ($H_2 > H_1$ after $\bar{t}_1$, *resp.*) and the only possibility is an H_1 then H_2 concatenation of flows. □

As a result of (A1), Σ is a codimension one submanifold in the nbd of $\bar{z}_1$, and one can define locally a function $z_0 \mapsto t_1(z_0)$ such that $z_1(t_1(z_0), z_0)$ belongs to Σ for z_0 in a nbd of $\bar{z}_0 := \bar{z}(0)$. As we have just done, we will denote

$$z_i(t, z_0) = e^{t\overrightarrow{H}_i}(z_0), \quad i = 1, 2,$$

the Hamiltonian flows of H_1 and H_2. These flows will be assumed complete for the sake of simplicity. We will denote $' = \partial/\partial z$ for flows. Clearly,

Lemma 5.

$$t_1'(z_0) = \frac{(H_1 - H_2)'}{H_{12}}(z_1(t_1(z_0), z_0))z_1'(t_1(z_0), z_0).$$

One then defines locally $z_0 \mapsto z(t, z_0) = (x(t, z_0), p(t, z_0)) := z_1(t, z_0)$ if $t \leq t_1(z_0)$, and $z(t, z_0) := z_2(t - t_1(z_0), z_1(t_1(z_0), z_0))$ if $t \geq t_1(z_0)$. We recall the following standard computation:

Lemma 6. *For $t > t_1(z_0)$,*

$$\frac{\partial z}{\partial z_0}(t, z_0) = z_2'(t - t_1(z_0), z_1(t_1(z_0), z_0))(I + \sigma(z_0))z_1'(t_1(z_0), z_0)$$

with

$$\sigma(z_0) = \overrightarrow{H_1 - H_2}\frac{(H_1 - H_2)'}{H_{12}}(z_1(t_1(z_0), z_0)). \tag{5}$$

Proof. The derivative is equal to (arguments omitted)

$$-\dot{z}_2 t_1' + z_2'(\dot{z}_1 t_1' + z_1'),$$

hence the result by factoring out z_2' and using Lemma 5 plus the fact that the adjoint action of a flow is idempotent on its generator,

$$(z_2'(s, z))^{-1}\overrightarrow{H}_2(z_2(s, z)) = \overrightarrow{H}_2(z), \quad (s, z) \in \mathbf{R} \times T^*X.$$

□

The function

$$\delta(t) := \det \frac{\partial x}{\partial p_0}(t, \bar{z}_0), \quad t \neq \bar{t}_1, \tag{6}$$

is piecewise continuous along the reference extremal, and we make the additional assumption that

(A2) $\delta(t) \neq 0$, $t \in (0, \bar{t}_1) \cup (\bar{t}_1, t_f]$, and $\delta(\bar{t}_1+)\delta(\bar{t}_1-) > 0$.

This condition means that we assume disconjugacy on $(0, \bar{t}_1]$ and $[\bar{t}_1, t_f]$ along the linearized flows of H_1 and H_2, respectively, and that the jump (encoded by the matrix $\sigma(z_0)$) in the Jacobi fields is such that there is no sign change in the determinant. This is exactly the condition one is able to check numerically by computing Jacobi fields (see [6, 7], *e.g.*). As will be clear from the proof of the result below, geometrically this assumption is the *no-fold condition* of [20] (no fold outside $\bar{t}_1$, no *broken* fold at $\bar{t}_1$).

Theorem 2. *Under assumptions (A0)-(A2), the reference trajectory is a $\mathscr{C}^0$-local minimizer among all trajectories with same endpoints.*

Proof. We proceed in five steps.

Step 1. According to (A2), $\partial x_1/\partial p_0(t, \bar{z}_0)$ is invertible for $t \in (0, \bar{t}_1]$; one can then construct a Lagrangian perturbation $\mathscr{L}_0$ transverse to $T^*_{x_0}X$ containing $\bar{z}_0$ such that $\partial x_1/\partial z_0(t, \bar{z}_0)$ is invertible for $t \in [0, \bar{t}_1]$, $t = 0$ included, $\partial/\partial z_0$ denoting the n partials wrt. $z_0 \in \mathscr{L}_0$. (See appendix A.) For $\varepsilon > 0$ small enough define

$$\mathscr{L}_1 := \{(t, z) \in \mathbf{R} \times T^*X \mid (\exists z_0 \in \mathscr{L}_0) : t \in (-\varepsilon, t_1(z_0) + \varepsilon) \text{ s.t. } z = z_1(t, z_0)\}.$$

By restricting $\mathscr{L}_0$ if necessary, $\Pi : \mathbf{R} \times T^*X \to \mathbf{R} \times X$, $(t, z) \mapsto (t, x)$ induces a diffeomorphism of $\mathscr{L}_1$ onto its image. Similarly, (A2) implies that

$$\frac{\partial}{\partial p_0} \left[x_2(t - t_1(z_0), z_1(t_1(z_0), z_0))\right] |_{z_0 = \bar{z}_0}$$

is invertible for $t \in [\bar{t}_1, t_f]$; restricting again $\mathscr{L}_0$ if necessary, one can assume that Π also induces a diffeomorphism from

$$\mathscr{L}_2 := \{(t, z) \in \mathbf{R} \times T^*X \mid (\exists z_0 \in \mathscr{L}_0) : t \in (t_1(z_0) - \varepsilon, t_f + \varepsilon)$$
$$\text{s.t. } z = z_2(t - t_1(z_0), z_1(t_1(z_0), z_0))\}$$

onto its image.

Step 2. Define $\Sigma_1 := \mathscr{L}_1 \cap (\mathbf{R} \times \Sigma)$. As $(t, z_0) \mapsto (t, x_1(t, z_0))$ is a diffeomorphism from $\mathbf{R} \times \mathscr{L}_0$ onto $\Pi(\mathscr{L}_1)$, there exists an inverse function $z_0(t, x)$ such that $\Pi(\Sigma_1) = \{\psi = 0\}$ with

$$\psi(t, x) := t - t_1(z_0(t, x)).$$

Denote $\psi(t) := \psi(t, \bar{x}(t))$ the evaluation of this function along the reference trajectory. By construction, $\dot{\psi}(\bar{t}_1-) = 1 > 0$ and (compare [20])

$$\dot{\psi}(\bar{t}_1+) = 1 + \frac{\partial t_1}{\partial z_0}(\bar{z}_0) \left(\frac{\partial x_1}{\partial z_0}(\bar{t}_1, \bar{z}_0)\right)^{-1} \nabla_p (H_1 - H_2)(\bar{z}_1).$$

Lemma 7.

$$\delta(\bar{t}_1+) = \delta(\bar{t}_1-) \left(1 + \frac{\partial t_1}{\partial p_0}(\bar{z}_0) \left(\frac{\partial x_1}{\partial p_0}(\bar{t}_1, \bar{z}_0)\right)^{-1} \nabla_p (H_1 - H_2)(\bar{z}_1)\right). \quad (7)$$

Proof. By virtue of Lemma 6,

$$\frac{\partial x}{\partial p_0}(\bar t_1+,\bar z_0) = \frac{\partial x_1}{\partial p_0}(\bar t_1,\bar z_0) + \nabla_p(H_1-H_2)\underbrace{\frac{(H_1-H_2)'}{H_{12}}(\bar z_1)\frac{\partial z_1}{\partial p_0}(\bar t_1,\bar z_0)}_{=\frac{\partial t_1}{\partial p_0}(\bar z_0)}$$

(the second equality coming from Lemma 5). Assumption (A2) implies $\delta(\bar t_1-)\neq 0$ so, taking determinants,

$$\begin{aligned}\delta(\bar t_1+) &= \delta(\bar t_1-)\det\left(I+\left(\frac{\partial x_1}{\partial p_0}(\bar t_1,\bar z_0)\right)^{-1}\nabla_p(H_1-H_2)(\bar z_1)\frac{\partial t_1}{\partial p_0}(\bar z_0)\right)\\ &= \delta(\bar t_1-)\left(1+\frac{\partial t_1}{\partial p_0}(\bar z_0)\left(\frac{\partial x_1}{\partial p_0}(\bar t_1,\bar z_0)\right)^{-1}\nabla_p(H_1-H_2)(\bar z_1)\right)\end{aligned}$$

as $\det(I+x^ty)=1+(x|y)$. □

Since $\delta(\bar t_1+)$ and $\delta(\bar t_1-)$ have the same sign, the quantity in brackets in (7) must be positive. Accordingly, $\dot\psi(\bar t_1+)>0$ as $\mathscr{L}_0$ can be taken arbitrarily close to $T^*_{x_0}X$. So, locally, $\Pi(\Sigma_1)$ is a submanifold that splits $\mathbf{R}\times X$ in two and, by restricting $\mathscr{L}_0$ if necessary, every extremal of the field $t\mapsto x(t,z_0)$ for $z_0\in\mathscr{L}_0$ crosses $\Pi(\Sigma_1)$ transversally. Defining

$$\mathscr{L}_1^- := \{(t,z)\in\mathbf{R}\times T^*X \mid (\exists z_0\in\mathscr{L}_0): t\in[0,t_1(z_0)] \text{ s.t. } z=z_1(t,z_0)\}$$

and

$$\begin{aligned}\mathscr{L}_2^+ := \{(t,z)\in\mathbf{R}\times T^*X \mid (\exists z_0\in\mathscr{L}_0): t\in[t_1(z_0),t_f]\\ \text{s.t. } z=z_2(t-t_1(z_0),z_1(t_1(z_0),z_0))\},\end{aligned}$$

one can hence piece together the restrictions of Π to $\mathscr{L}_1^-$ and $\mathscr{L}_2^+$ into a continuous bijection from $\mathscr{L}_1^-\cup\mathscr{L}_2^+$ into $\Pi(\mathscr{L}_1^-\cup\mathscr{L}_2^+)$. By restricting to a compact neighbourhood of the graph of $\bar z$, one may assume that Π induces a homeomorphism on its image.

Step 3. Denote $\alpha_i := p\,\mathrm{d}x - H_i(z)\mathrm{d}t$, $i=1,2$, the Poincaré-Cartan forms associated with H_1 and H_2, respectively. To prove that α_1 is exact on $\mathscr{L}_1$, it is enough to prove that it is closed. Indeed, if $\gamma(s):=(t(s),z_1(t(s),z_0(s)))$ is a closed curve on $\mathscr{L}_1$, it retracts continuously on $\gamma_0(s):=(0,z_0(s))$ so that, provided α_1 is closed,

$$\int_\gamma\alpha_1 = \int_{\gamma_0}\alpha_1 = \int_{\gamma_0}p\,\mathrm{d}x = 0$$

because $z_0(s)$ belongs to $\mathscr{L}_0$ that can be chosen such that $p\,\mathrm{d}x$ is exact on it. (Compare [1, §17].) Similarly, to prove that α_2 is exact on $\mathscr{L}_2$, it suffices to prove that it is closed: If $\gamma(s):=(t(s),z_2(t(s)-t_1(z_0(s)),z_1(t_1(z_0(s)),z_0(s))))$ is a closed curve in $\mathscr{L}_2$, it readily retracts continuously on the curve $\gamma_1(s):=(t_1(z_0(s)),z_1(t_1(z_0(s)),z_0(s)))$ in Σ_1, which retracts continuously on $\gamma_0(s):=(0,z_0(s))$ again. Then, as $H_1=H_2$ on Σ,

$$\int_\gamma\alpha_2 = \int_{\gamma_1}\alpha_2 = \int_{\gamma_1}\alpha_1 = \int_{\gamma_0}\alpha_1$$

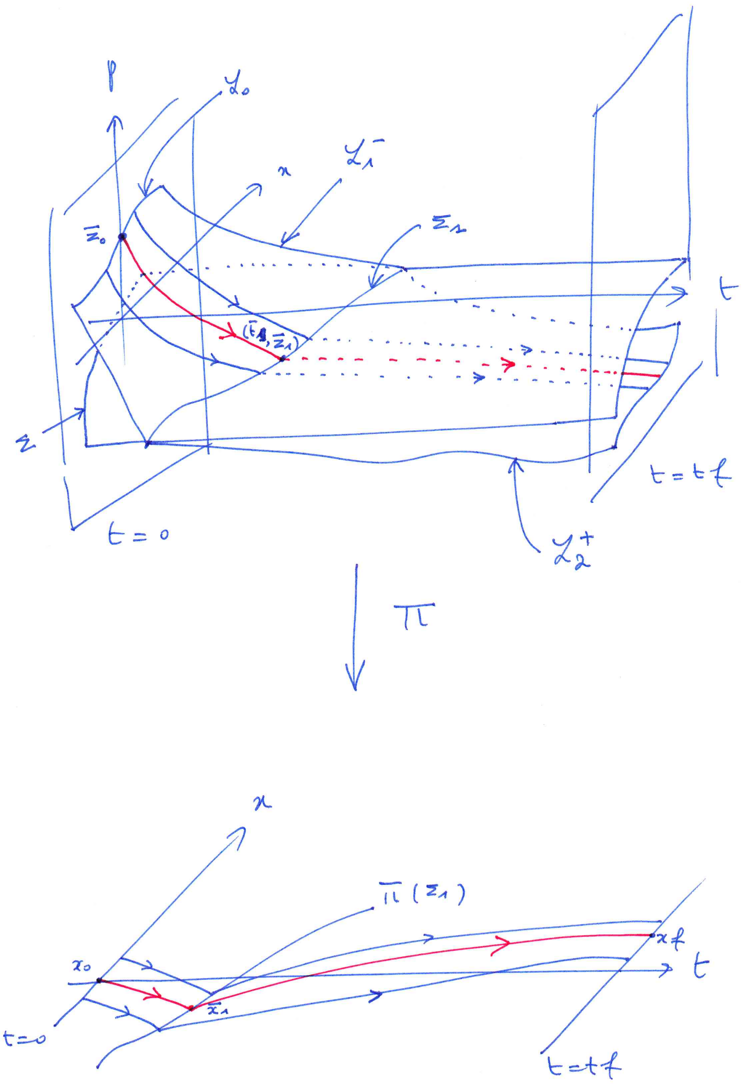

Figure 1: The field of extremals.

that vanishes as before. To prove that α_1 is closed, consider tangent vectors at $(t, z) \in \mathscr{L}_1$; a parameterization of this tangent space is

$$(\delta t, \overrightarrow{H}_1(z)\delta t + z_1'(t, z_0)\delta z_0), \quad (\delta t, \delta z_0) \in \mathbf{R} \times T_{z_0}\mathscr{L}_0$$

where $z_0 \in \mathscr{L}_0$ is such that $z = z_1(t, z_0)$. For two such vectors v_1, v_2,

$$\begin{aligned} \mathrm{d}\alpha_1(t, z)(v_1, v_2) &= (\mathrm{d}p \wedge \mathrm{d}x - \mathrm{d}H_1(z)\mathrm{d}t)(v_1, v_2) \\ &= \mathrm{d}p \wedge \mathrm{d}x(z_1'(t, z_0)\delta z_0^1, z_1'(t, z_0)\delta z_0^2) \\ &= \mathrm{d}p \wedge \mathrm{d}x(\delta z_0^1, \delta z_0^2) \\ &= 0 \end{aligned}$$

because $\exp(t\overrightarrow{H}_1)$ is symplectic and $\mathscr{L}_0$ is Lagrangian. Regarding α_2, the tangent space at $(t,z) \in \mathscr{L}_2$ is parameterized according to

$$(\delta t, \overrightarrow{H}_2(z)\delta t + z_2'(t - t_1(z_0), z_1(t_1(z_0), z_0))(I + \sigma(z_0))z_1'(t, z_0)\delta z_0)$$

with $(\delta t, \delta z_0) \in \mathbf{R} \times T_{z_0}\mathscr{L}_0$, and where $z_0 \in \mathscr{L}_0$ is such that $z = z_2(t - t_1(z_0), z_1(t_1(z_0), z_0))$. For two such vectors v_1, v_2,

$$\begin{aligned} \mathrm{d}\alpha_2(t,z)(v_1,v_2) &= (\mathrm{d}p \wedge \mathrm{d}x - \mathrm{d}H_2(z)\mathrm{d}t)(v_1,v_2) \\ &= \mathrm{d}p \wedge \mathrm{d}x((I + \sigma(z_0))z_1'(t,z_0)\delta z_0^1, (I + \sigma(z_0))z_1'(t,z_0)\delta z_0^2) \\ &= \mathrm{d}p \wedge \mathrm{d}x(z_1'(t,z_0)\delta z_0^1, z_1'(t,z_0)\delta z_0^2) \end{aligned}$$

because $\exp(t\overrightarrow{H}_2)$ is symplectic and because

Lemma 8.

$$I + \sigma(z_0) \in \mathrm{Sp}(2n, \mathbf{R}).$$

Proof. For any $z \in \mathbf{R}^{2n}$,

$${}^t(I + Jz\,{}^tz)J(I + Jz\,{}^tz) = J - z\,{}^tz + z\,{}^tz + z(\underbrace{{}^tzJz}_{0})\,{}^tz = J.$$

This proves the lemma because of the definition (5) of $\sigma(z_0)$. $\square$

One then concludes as before that α_2 is closed using the fact that $\exp(t\overrightarrow{H}_1)$ is symplectic and $\mathscr{L}_0$ is Lagrangian.

Step 4. Let $(x,u) : [0, t_f] \to X \times U$ be an admissible pair. We first assume that x is of class $\mathscr{C}^1$ and that its graph has only one isolated contact with $\Pi(\Sigma_1)$, at some point point $(t_1, x(t_1))$. For x close enough to $\overline{x}$ in the $\mathscr{C}^0$-topology, this graph has a unique lift $t \mapsto (t, x(t), p(t))$ in $\mathscr{L}_1^- \cup \mathscr{L}_2^+$. As a gluing at t_1 of two absolutely continous functions, $z := (x,p) : [0, t_f] \to T^*X$ is absolutely continous. Denote γ_1 and γ_2 the two pieces of this lift. Denote similarly $\overline{\gamma}_1$ and $\overline{\gamma}_2$ the pieces of the graph of the extremal $\overline{z}$ (see Fig. 2). One has

$$\begin{aligned} \int_0^{t_f} f^0(x(t), u(t))\,\mathrm{d}t &= \left(\int_0^{t_1} + \int_{t_1}^{t_f}\right)(p(t)\dot{x}(t) - H(x(t), p(t), u(t)))\,\mathrm{d}t \\ &\geq \int_0^{t_1} (p(t)\dot{x}(t) - H_1(x(t), p(t)))\,\mathrm{d}t \\ &+ \int_{t_1}^{t_f} (p(t)\dot{x}(t) - H_2(x(t), p(t)))\,\mathrm{d}t \\ &= \int_{\gamma_1} \alpha_1 + \int_{\gamma_2} \alpha_2 \end{aligned}$$

since $z(t)$ belongs to Ω_1 for $t \in [0, t_1)$ (*resp.* to Ω_2 for $t \in (t_1, t_f]$). By connectedness, there exists a smooth curve $\gamma_{12} \subset \Sigma_1$ connecting $(\overline{t}_1, \overline{z}(\overline{t}_1))$ to $(t_1, z(t_1))$;

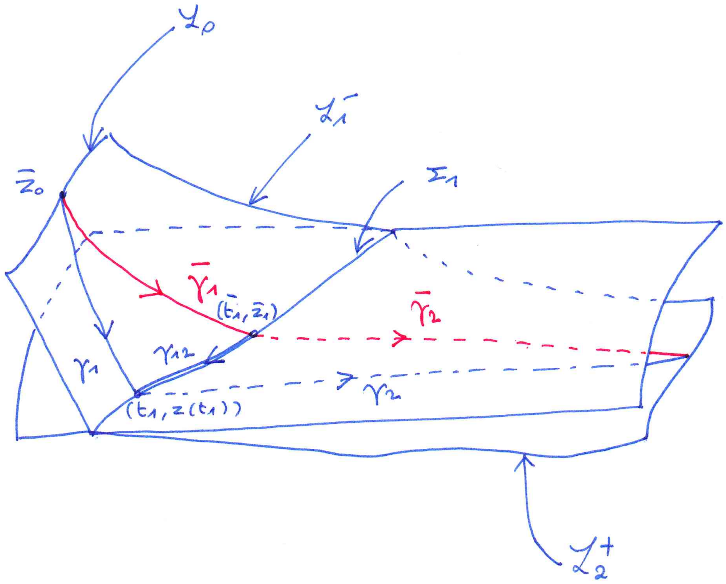

Figure 2: Integration paths.

having the same endpoints, γ_1 and $\overline{\gamma}_1 \cup \gamma_{12}$ (*resp.* γ_2 and $-\gamma_{12} \cup \overline{\gamma}_2$) are homotopic. Since α_1 and α_2 are exact one forms on $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively,

$$\begin{aligned}
\int_{\gamma_1} \alpha_1 + \int_{\gamma_2} \alpha_2 &= \int_{\overline{\gamma}_1 \cup \gamma_{12}} \alpha_1 + \int_{-\gamma_{12} \cup \overline{\gamma}_2} \alpha_2 \\
&= \int_{\overline{\gamma}_1} \alpha_1 + \int_{\overline{\gamma}_2} \alpha_2 \\
&= \int_0^{t_f} f^0(\overline{x}(t), \overline{u}(t))\, \mathrm{d}t
\end{aligned}$$

since $H_1 = H_2$ on Σ.

Step 5. Consider finally an admissible pair (x, u), x close enough to $\overline{x}$ in the $\mathscr{C}^0$-topology. One can find $\widetilde{x}$ of class $\mathscr{C}^1$ arbitrarily close to x in the $\mathrm{W}^{1,\infty}$-topology such that $\widetilde{x}(0) = x_0$ and $\widetilde{x}(t_f) = x_f$. Moreover, as $\Pi(\Sigma_1)$ is a locally a smooth manifold, up to some $\mathscr{C}^1$-small perturbation one can assume that the graph of $\widetilde{x}$ has only transverse intersections with $\Pi(\Sigma_1)$. Let $\widetilde{z} := (\widetilde{x}, \widetilde{p})$ denote the associated lift; one has

$$f^0(\widetilde{x}(t), u(t)) = (\widetilde{p}(t)\dot{\widetilde{x}}(t) - H(\widetilde{x}(t), \widetilde{p}(t), u(t))) + \widetilde{p}(t)(f(\widetilde{x}(t), u(t)) - \dot{\widetilde{x}}(t)),$$

and the second term in the right-hand side can be made arbitrarily small when $\widetilde{x}$ gets closer to x in the $\mathrm{W}^{1,\infty}$-topology since $(t, \widetilde{z}(t)) = \Pi^{-1}(t, \widetilde{x}(t))$ remains bounded by continuity of the inverse of Π. Let then $\varepsilon > 0$; as a result of the previous discussion, there exists $\widetilde{x}$ of class $\mathscr{C}^1$ with same endpoints as x and whose graph has only isolated contacts with $\Pi(\Sigma_1)$ such that

$$\int_0^{t_f} f^0(x(t), u(t))\, \mathrm{d}t \geq \int_0^{t_f} f^0(\widetilde{x}(t), u(t))\, \mathrm{d}t - \varepsilon,$$

$$\int_0^{t_f} f^0(\widetilde{x}(t), u(t))\,\mathrm{d}t \geq \int_0^{t_f} (\widetilde{p}(t)\dot{\widetilde{x}}(t) - H(\widetilde{x}(t), \widetilde{p}(t), u(t)))\,\mathrm{d}t - \varepsilon.$$

One can extend straightforwardly the analysis of the previous step to finitely many contacts with $\Pi(\Sigma_1)$, and bound below the integral in the right-hand side of the second inequality by the cost of the reference trajectory. As ε is arbitrary, this allows to conclude. □

4 Numerical example: The two-body potential

Following [13], we consider the two-body controlled problem in dimension three. Restricting to negative energy, orbits of the uncontrolled motion are ellipses, and the issue is to realize minimum fuel transfer between non-coplanar orbits around a fixed center of mass. The potential is $V(q) := -\mu/|q|$ defined on $Q := \{q \in \mathbf{R}^3 \mid q \neq 0\}$, and we actually restrict to

$$X := \{(q, v) \in TQ \mid |v|^2/2 - \mu/|q| < 0,\ q \wedge v > 0\}.$$

(The last condition on the momentum avoids collisional trajectories and orientates the elliptic orbits.) The constant μ is the gravitational constant that depends on the attracting celestial body. To keep things clear, a medium thrust case is presented below; the final time is fixed to 1.3 times the minimum time, approximately, which already ensures a satisfying gain of consumption [13]. In order to have fixed endpoints to perform a conjugate point test according to §3 result, initial and final positions are fixed on the orbits (fixed longitudes[5]). A more relevant treatment would leave the final longitude free (in accordance with assumption (ii) on the target in §2); this would require a focal point test that could be done much in the same way (see, *e.g.*, [9]). See Tab. 1 for a summary of the physical constants.

As explained in §2, the L^1-minimization results in a competition between two Hamiltonians: H_0 (coming from the drift, only), and $H_0 + H_1$ (assuming the control bound is normalized to 1 after some rescaling). Both Hamiltonians are smooth and fit in the framework set up in §3 to check sufficient optimality conditions. Restricting to bang-bang (in the norm of the control) extremals, regularity of the switchings is easily verified numerically, while normality is taken care of by Proposition 1. Then one has to check the no-fold condition on the Jacobi fields. The optimal solution (see Fig. 3) and these fields are computed using the `hampath` software [14]; as in [9, 13], a regularization by homotopy is used to capture the switching structure and initialize the computation of the bang-bang extremal by single shooting. We are then able to check condition (A2) directly on this extremal by a simple sign test (including the jumps on the Jacobi fields at the regular switchings) on the determinant of the fields (see Fig. 4). An alternative approach would be to establish a convergence result as in [27], and to verify the second order conditions on the sequence of regularized extremals. As underlined in §1 and §3, conjugate times may occur at or between

[5]Precisely, the longitude l is defined as the sum of three broken angles: $l = \Omega + \theta + \varpi$, where Ω is the longitude of the ascending node (first Euler angle of the orbit plane with the equatorial plane; the second Euler angle defines the inclination of the orbit), θ is the argument of perigee (angle of the semi-major axis of the ellipse, equal to the third Euler angle of the orbit plane), and ϖ is the true anomaly (polar angle with respect to the semi-major axis in the orbit plane). Here, $\Omega = \theta = 0$ on the initial and final orbits.

Table 1: Summary of physical constants used for the numerical commputation.

Gravitational constant μ of the Earth:			398600.47 $\mathrm{Km}^3\mathrm{s}^{-2}$
Mass of the spacecraft:	1500 Kg	Thrust:	10 Newtons
Initial perigee:	6643 Km	Final perigee:	42165 Km
Initial apogee:	46500 Km	Final apogee:	42165 Km
Initial inclination:	0.1222 rad	Final inclination:	0 rad
Initial longitude:	π rad	Final longitude:	56.659 rad
Minimum time:	110.41 hours	Fixed final time:	147.28 hours
L^1 cost achieved (normalized):			67.617

switching times. On the example treated, no conjugate point is detected on $[0, t_f]$, ensuring strong local optimality. The extremal is then extended up to $3.5\, t_f$, and a conjugate point is detected about $3.2\, t_f$, at a switching point (sign change occurint at the jump). A second test is provided Fig. 5; by perturbing slightly the endpoint conditions, one observes that conjugacy occurs not at a switching anymore, but along a burn arc.

Remark 1. As H_0 is the lift of a vector field, the determinant of Jacobi fields is either identically zero or non-vanishing along a cost arc ($\rho = 0$). (Compare with the case of polyhedral control set; see also Corollary 3.9 in [20].) Moreover, coming from a mechanical system, the drift F_0 is the symplectic gradient of the energy function,

$$E(q, v) := \frac{1}{2}|v|^2 + V(q).$$

Accordingly, the $\delta x = (\delta q, \delta v)$ part of the Jacobi field (see appendix) along an integral arc of $\overrightarrow{H}_0$ verifies

$$\delta\dot{x}(t) = \overrightarrow{E}'(\overline{x}(t))\delta x(t),$$

so δx has a constant determinant along such an arc since the associated flow is symplectic. In particular, the disconjugacy condition (A2) implies that the optimal solution starts with a burn arc.

Conclusion

We have reviewed some of the particularities of L^1-minimization in the control setting. Among these, the existence of singular controls valued in the interior of the Euclidean ball comes in strong contrast with the finite dimensional case. Moreover, these singular extremals are at least of order two, entailing existence of chattering [29]. By changing coordinates on the control, one can reduce the system to a single control, namely the norm of the original one. This emphasizes the role played by the Poisson structure of two Hamiltonians, the second not the lift of a vector field; this fact accounts for the possibility of conjugacy happening

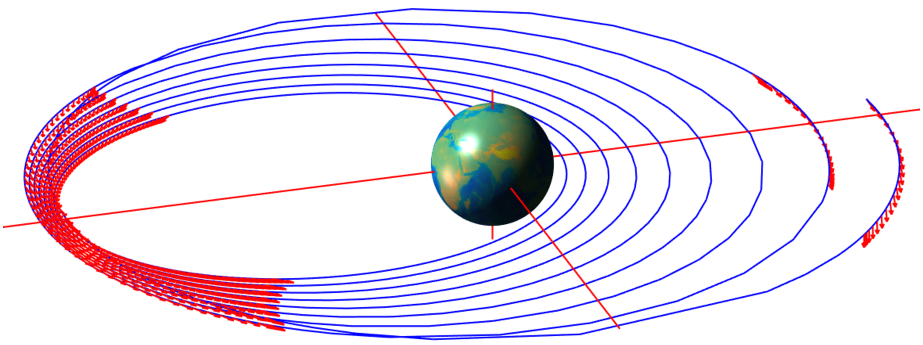

Figure 3: L^1 minimum trajectory. The graph displays the trajectory (blue line), as well as the action of the control (red arrows). The initial orbit is strongly eccentric (0.75) and slightly inclined (7 degrees). The geostationary target orbit around the Earth is reached at $t_f \simeq 147.28$ hours. The sparse structure of the control is clearly observed, with burn arcs concentrated around perigees and apogees (see [13]). The minimization leads to thrust only 46% of the time.

not necessarily at switching times, as opposed to the simpler case of bang-bang controls valued in polyhedra. Sufficient conditions for this type of extremals have been given; they rely on a simple and numerically verifiable check on the discontinuous Jacobi fields of the system. They are essentially equivalent to the no-fold conditions of [20], formulated here in a Hamiltonian setting. The example of L^1-minimization for the three-dimensional two-body potential illustrates the interest of the approach. Future work include the treatment of mass varying systems (that is of maximization of the final mass) for more general problems such as the restricted three-body one.

A Sufficient conditions in the smooth case

Consider the same minimization problem as in §3. Suppose that

(B0) The reference extremal is normal.

Having fixed p^0 to -1, we make the stronger assumption that the maximized Hamiltonian is well defined and smooth, and set

$$h(z) := \max_U H(z, \cdot), \quad z \in T^*X.$$

Scholium. *For almost all* $t \in [0, t_f]$,

$$h'(\overline{z}(t)) = \frac{\partial H}{\partial z}(\overline{z}(t), \overline{u}(t)), \quad \nabla^2 h(\overline{z}(t)) - \nabla^2_{zz} H(\overline{z}(t), \overline{u}(t)) \geq 0.$$

Proof. For a.a. $t \in [0, t_f]$, $h(\overline{z}(t)) - H(\overline{z}(t), \overline{u}(t)) = 0$, while

$$h(z) - H(z, \overline{u}(t)) \geq 0, \quad z \in T^*X,$$

by definition of h. Applying the first and second order necessary conditions for optimality on T^*X at $z = \overline{z}(t)$ gives the result. □

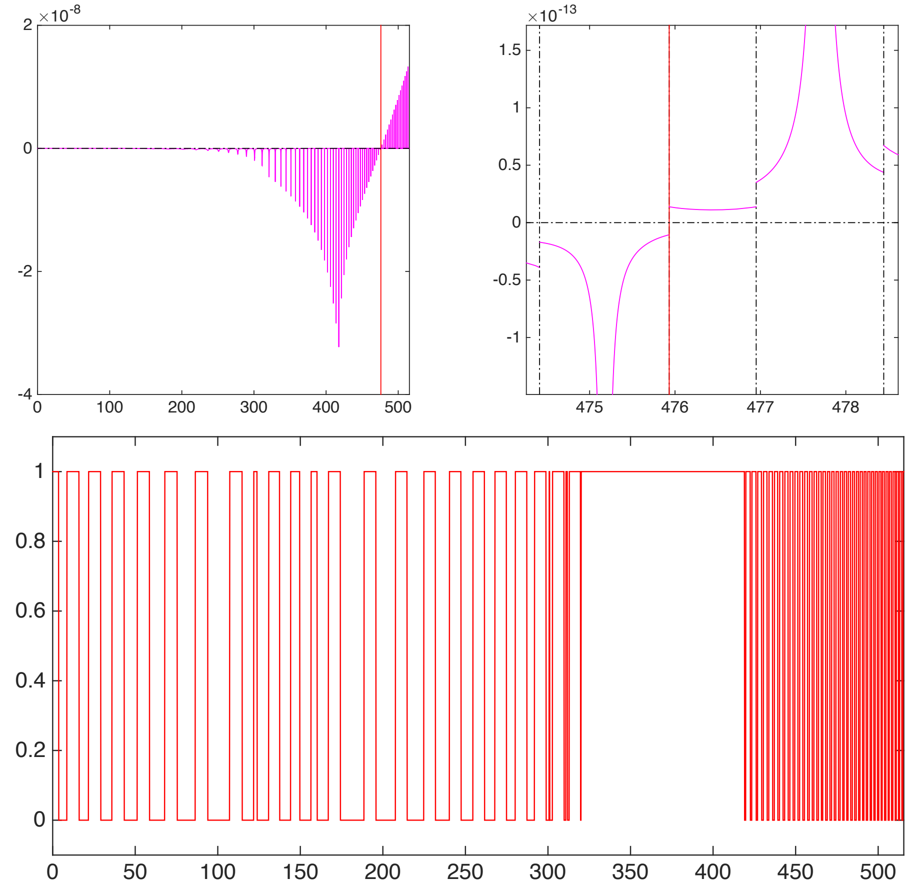

Figure 4: Conjugate point test on the bang-bang L[1]-extremal extended to $[0, 3.5\,t_f]$. The value of the determinant of Jacobi fields (6) along the extremal is plotted against time on the upper left subgraph. The first conjugate point occurs at $t_{1c} \simeq 475.93$ hours $> t_f$; optimality of the reference extremal on $[0, t_f]$ follows. On the upper right subgraph, a zoom is provided to show the jumps on the Jacobi fields (then on their determinant) around the first conjugate time; several jumps are observed, the first one leading to a sign change at the conjugate time. Note that in accordance with Remark 1, the determinant must be constant along the cost arcs ($\rho = 0$) provided the symplectic coordinates $x = (q, v)$ are used; this is not the case here as the so called equinoctial elements [10] are used for the state—hence the slight change in the determinant. The bang-bang norm of the control, rescaled to belong to $[0, 1]$ and extended to $3.5\,t_f$, is portrayed on the lower graph. On the extended time span, there are already more than 70 switchings though the thrust is just a medium one. For low thrusts, hundreds of switchings occur.

We make the following assumption on the smooth reference extremal.

(B1) $\partial x/\partial p_0(t, \bar{z}_0)$ is invertible for $t \in (0, t_f]$.

Theorem 3. *Under assumptions (B0)-(B1), the reference trajectory is a $\mathscr{C}^0$-local minimizer among all trajectories with same endpoints.*

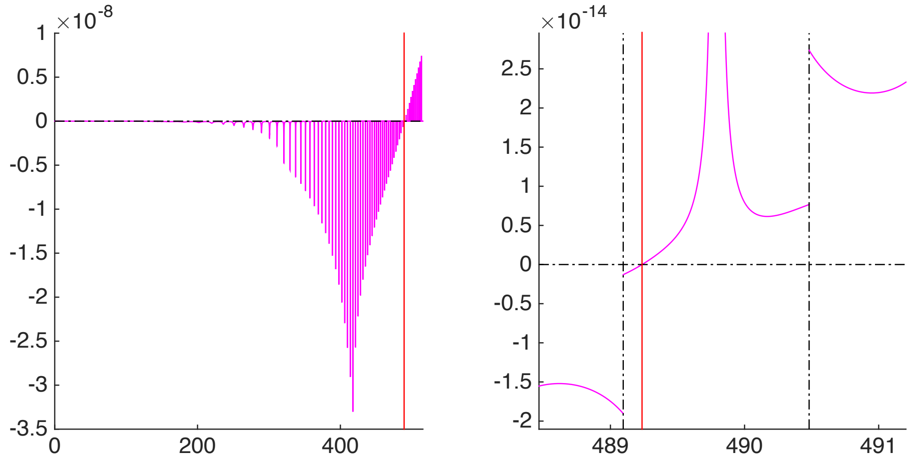

Figure 5: Conjugate point test on a perturbed bang-bang L^1-extremal extended to $[0, 3.5\, t_f]$. The value of the determinant of Jacobi fields (6) along the extremal is plotted against time (detail on the right subgraph). The endpoint conditions x_0, x_f given in Tab. 1 are perturbed according to $x \leftarrow x + \Delta x$, $|\Delta x| \simeq 1e-5$, leading to conjugacy not at but between switching points—along a burn arc ($\rho = 1$). The first conjugate point occurs at $t_{1c} \simeq 489.23$ hours $> t_f$, ensuring again optimality of the reference extremal on $[0, t_f]$.

Note that no Legendre type assumption is made, and that the disconjugacy condition (B1) can be numerically verified (*e.g.*, by a rank test while integrating the variational system along the reference extremal). For the sake of completeness, we provide a proof that essentially goes along the lines of [1, §21].

Proof. For S_0 symmetric of order n, $L_0 := \{\delta x_0 = S_0 \delta p_0\}$ is a Lagrangian subspace of $T_{\overline{z}_0}(T^*X)$. Denote by $\delta z = (\delta x, \delta p)$ the solution of the linearized system

$$\delta \dot{z}(t) = \overrightarrow{h}'(\overline{z}(t))\delta z(t), \quad \delta z(0) = (S_0, I),$$

and set $\delta\widetilde{z}(t) = (\delta\widetilde{x}(t), \delta\widetilde{p}(t)) := \Phi_t^{-1}\delta z(t)$ where Φ_t is the fundamental solution of the linearized system

$$\dot{\Phi}_t = \frac{\partial \overrightarrow{H}}{\partial z}(\overline{z}(t), \overline{u}(t))\Phi_t, \quad \Phi_0 = I.$$

As $\delta p(0) = \delta\widetilde{p}(0) = I$,

$$S(t) := \delta\widetilde{x}(t)\delta\widetilde{p}(t)^{-1}$$

is well defined for small enough $t \geq 0$. Since

$$L_t := \exp(t\overrightarrow{h})'(\overline{z}(t))(L_0) \quad \text{and} \quad \Phi_t^{-1}(L_t)$$

are Lagrangian as images of L_0 through linear symplectic mappings, $S(t)$ must be symmetric.

Lemma 9. $\dot{S}(t) \geq 0$

Proof. Let $t_1 \geq 0$ such that $S(t_1)$ is well defined, and let $\xi \in \mathbf{R}^n$. Set

$$\xi_0 := \delta\widetilde{p}(t_1)^{-1}\xi \quad \text{and} \quad \delta\widetilde{z}_1(t) := \delta\widetilde{z}(t)\xi_0.$$

Then $\delta\widetilde{z}_1(t_1) = (S(t_1)\xi, \xi)$, and $\delta\widetilde{x}_1(t) = S(t)\delta\widetilde{p}_1(t)$. Differentiating the previous relation and using $S(t)$ symmetry leads to

$$(\dot{S}(t)\delta\widetilde{p}_1(t)|\delta\widetilde{p}_1(t)) = \omega(\delta\widetilde{z}_1(t), \delta\dot{\widetilde{z}}_1(t)).$$

Differentiating now

$$\delta\widetilde{z}_1(t) = \Phi_t^{-1}\delta z(t)\xi_0,$$

one gets

$$\begin{aligned} \delta\dot{\widetilde{z}}_1(t) &= \Phi_t^{-1}(\overrightarrow{h}'(\overline{z}(t)) - \frac{\partial\overrightarrow{H}}{\partial z}(\overline{z}(t), \overline{u}(t)))\Phi_t\delta\widetilde{z}_1(t) \\ &= J^t\Phi_t(\underbrace{\nabla^2 h(\overline{z}(t)) - \nabla^2_{zz}H(\overline{z}(t), \overline{u}(t))}_{\geq 0})\Phi_t\delta\widetilde{z}_1(t). \end{aligned}$$

(J denotes the standard symplectic matrix.) Evaluating at $t = t_1$, one eventually gets $(\dot{S}(t_1)\xi|\xi) \geq 0$. □

For $S_0 = 0$, there is $\eta > 0$ such that $S(t)$ is well defined on $[0, \eta]$, which remains true for $S_0 > 0$, $|S_0|$ small enough. By the lemma before, $S_t > 0$ on $[0, \eta]$. In particular, it is an invertible matrix, which ensures that $\Phi_t^{-1}(L_t)$ is transversal to $\ker \pi'(\overline{z}_0)$ ($\pi : T^*X \to X$ being the canonical projection), that is L_t is transversal to $\ker \pi'(\overline{z}(t))$ by virtue of

Scholium. $\Phi_t(\ker \pi'(\overline{z}_0)) = \ker \pi'(\overline{z}(t))$

Proof. Note that in the linearized system defining Φ_t,

$$\begin{aligned} \delta\dot{x}(t) &= \nabla^2_{xp}H(\overline{z}(t), \overline{u}(t))\delta x(t), \\ \delta\dot{p}(t) &= -\nabla^2_{xx}H(\overline{z}(t), \overline{u}(t))\delta x(t) - \nabla^2_{px}H(\overline{z}(t), \overline{u}(t))\delta p(t), \end{aligned}$$

the equation on δx is linear. Hence $\delta x(0) = 0$ implies $\delta x \equiv 0$. □

By restricting if necessary $|S_0|$, (B1) allows to assume that $\delta x(t)$ remains invertible for $t \in [\eta, t_f]$, so transversality of L_t holds on $[0, t_f]$. As a result, one can devise a Lagrangian submanifold $\mathscr{L}_0$ of T^*X whose tangent space at $\overline{z}_0$ is L_0; then

$$\mathscr{L} := \{(t, z) \in \mathbf{R} \times T^*X \mid (\exists z_0 \in \mathscr{L}_0) : t \in (-\varepsilon, t_f + \varepsilon) \text{ s.t. } z = \exp(t\overrightarrow{h})(z_0)\}$$

is well defined for ε small enough, and such that $\Pi : \mathbf{R} \times T^*X \to \mathbf{R} \times X$ induces a diffeomorphism from $\mathscr{L}$ onto its image. One can moreover choose $\mathscr{L}_0$ such that $p\,\mathrm{d}x$ is not only closed but an exact form on it, in order that the Poincaré-Cartan form $p\,\mathrm{d}x - h(z)\mathrm{d}t$ is exact on $\mathscr{L}$. This, together with assumption (B0), allows to conclude as usual that the reference trajectory is optimal with respect to $\mathscr{C}^0$-neighbouring trajectories with same endpoints. □

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