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# Violent relaxation in two-dimensional flows with varying interaction range

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Understanding the relaxation of a system towards equilibrium is a longstanding problem in statistical mechanics. Here we address the role of long-range interactions in this process by considering a class of two-dimensional or geophysical flows where the interaction between fluid particles varies with the distance as  $\sim r^{\alpha-2}$  with  $\alpha > 0$ . Previous studies in the Euler case  $\alpha = 2$  had shown convergence towards a variety of quasi-stationary states by changing the initial state. Unexpectedly, all those regimes are recovered by changing  $\alpha$  with a prescribed initial state. For small  $\alpha$ , a coarsening process leads to the formation of a sharp interface between two regions of homogenized  $\alpha$ -vorticity; for large  $\alpha$ , the flow is attracted to a stable dipolar structure through a filamentation process.

PACS numbers:

Self-gravitating systems, non-neutral plasma and two-dimensional (2D) flows are examples of long-range interacting systems, for which the two-body potential decays with the interparticle distance with an exponent smaller than the dimensionality of the embedding space [1]. Those systems share the property to self-organize spontaneously into large scale coherent structures such as globular clusters and elliptical galaxies in astrophysics or vortices and jets in geophysics [2]. Equilibrium statistical mechanics provides an explanation and a prediction for this phenomenon as the most probable result of mixing in phase space. It allows to reduce the study of the large scale organization to a few parameters, without describing the full complexity of the dynamics involving a huge number of degrees of freedom. The original idea to use statistical mechanics arguments to describe self-organization of 2D flows comes from L. Onsager himself in the framework of point vortex models [3, 4]. A statistical mechanics theory for the continuous Euler dynamics has been proposed by Miller, Robert, Sommeria [5–8] (MRS hereafter), which has led to several successful applications to geophysical flows [9–12]. This theory is the equivalent of Lynden Bell's statistical mechanics of the Vlasov dynamics [13], which has been proven useful to address self-organisation in self-gravitating systems [14], or in toy models of long-range interacting particles [15].

Equilibrium theory of long-range interacting systems is now fairly well understood [1], but determining how and when an initially unstable condition far from equilibrium actually relax towards an equilibrium state after a "violent relaxation" process [13] remains a challenging problem [16]. One key difficulty for those systems is that the set of stable states is usually much larger than the set of equilibrium states, and there exists other invariant measures than those predicted by the equilibrium theory [17]. Consequently, a class of unstable initial conditions may be attracted towards stable states different than the most probable one. Indeed, various example

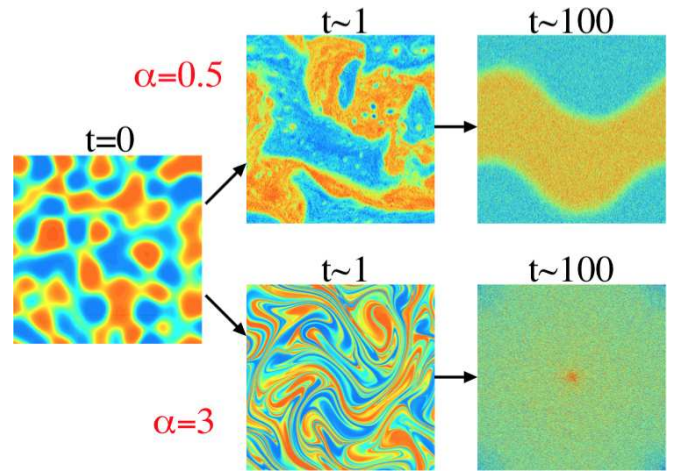


Figure 1: Temporal evolution of the  $\alpha$ -vorticity field during the violent relaxation of an unstable initial condition, for  $\alpha = 0.5$  and  $\alpha = 3$ . The initial condition is the same whatever the value of  $\alpha$ , and is characterized by a global distribution of  $\alpha$ -vorticity levels close to a doubled peaked function. Time unit is such that the maximum initial velocity is equal to one.

of ergodicity breaking in non-neutral plasma and self-gravitating system have been reported [16]. It was also found that a set of initial conditions are attracted towards quasi-stationary states better described by "core-halo" distributions than by Lynden-Bell equilibria [16]. Quasi-stationary states are stable or oscillatory states which evolve on a slow time scale related to discretization (granularity, truncation). Similarly, applications of MRS theory to freely evolving flows have led to mitigated results [18–22].

In all those previous studies, the initial conditions was varied but the range of the interaction was fixed. Here we address the role of long-range interactions during the violent relaxation of a given unstable initial condition towards quasi-stationary states by considering a class of 2D flow models introduced in Ref. [23], in which

interactions between fluid particles (infinitesimal fluid volumes) are indexed by a parameter  $\alpha > 0$ . It includes 2D Euler dynamics ( $\alpha = 2$ ), surface quasi-geostrophic dynamics ( $\alpha = 1$ ), which is relevant to describe some aspects of atmospheric and oceanic turbulence [24], and a model for mantle convection ( $\alpha = 3$ ) [25, 26]. The initial goal for studying this model was to address the locality hypothesis for turbulent cascade [23, 27, 28]. It has then been proven useful to investigate the possible emergence of finite time singularities [29] and conformal invariance [30]. We will see that it also shed new light on the dynamical effects underpinning self-organisation of long-range interacting systems. Depending on the value of  $\alpha$ , different dynamical regimes were identified in Ref. [23]. It is then natural to ask whether these different dynamical regimes lead to different quasi-stationary states following the initial violent relaxation. In particular, is surface quasi-geostrophic dynamics more prone to relax towards an equilibrium state than 2D Euler flows?

The *continuous* dynamics is expressed as the advection of the  $\alpha$ -vorticity  $q(\mathbf{r}, t)$  by a 2D incompressible velocity field  $\mathbf{v} = (-\partial_y \psi, \partial_x \psi)$ , with  $\mathbf{r} = (x, y)$ :

$$\partial_t q + \mathbf{v} \cdot \nabla q = 0, \quad q = -(-\Delta)^{\alpha/2} \psi, \quad (1)$$

where  $-(-\Delta)^{\alpha/2}$  is the fractional Laplacian defined in term of the Fourier components  $q_{\mathbf{k}} = -|\mathbf{k}|^\alpha \psi_{\mathbf{k}}$  in the case a doubly periodic domain  $\mathcal{D} = [0, 2\pi] \times [0, 2\pi]$ , with  $\mathbf{k} = (k, \ell)$  the wave vector. The dynamics conserves the  $\alpha$ -energy  $\mathcal{E}[q] \equiv -\int_{\mathcal{D}} \mathbf{dr} q \psi$  and the Casimir functionals  $\mathcal{C}_f[q] \equiv \int_{\mathcal{D}} \mathbf{dr} f(q)$ , where  $f$  is any sufficiently smooth function on  $\mathcal{D}$ , which includes the  $\alpha$ -enstrophy  $\mathcal{Z}[q] \equiv \int_{\mathcal{D}} \mathbf{dr} q^2$ .

The  $\alpha$ -energy can formally be written as a potential energy  $\mathcal{E} = \int_{\mathcal{D}} \mathbf{dr} \int_{\mathcal{D}} \mathbf{dr}' q(\mathbf{r}) V(\mathbf{r}, \mathbf{r}') q(\mathbf{r}')$ , where  $V$  is the Green function of the fractional Laplacian in two-dimensions. In the case of an infinite domain  $\mathcal{D}$ , this Green function is a Riesz potential  $V \sim r^{\alpha-2}$  with  $r = |\mathbf{r} - \mathbf{r}'|$  excepted when  $\alpha$  is even, in which case  $V \sim r^{\alpha-2} \log r$  [31, 32]. Whatever  $\alpha > 0$ , interactions between fluid particles are always long-range.

We present in this letter numerical simulations of the freely evolving *Galerkin-truncated* dynamics of these 2D flow models, which is obtained by projecting Eq. (1) on the wave-numbers  $|k| \leq k_{max}$  and  $|\ell| \leq k_{max}$ , where  $k_{max}$  is the wave-number cut-off [48]. The initial  $\alpha$ -vorticity field is the same for all numerical experiments (see Fig. 1). Two striking features of the late time self-organization are summarized in Fig.1. First, there is a scale separation in space between erratic small scale fluctuations and a well-defined large scale flow structure organized at the domain scale. Second, the large scale flow structure is drastically different depending on the value of  $\alpha$ . We show in the following that statistical mechanics of the truncated dynamics accounts for the small scale fluctuations but that statistical mechanics of the continuous

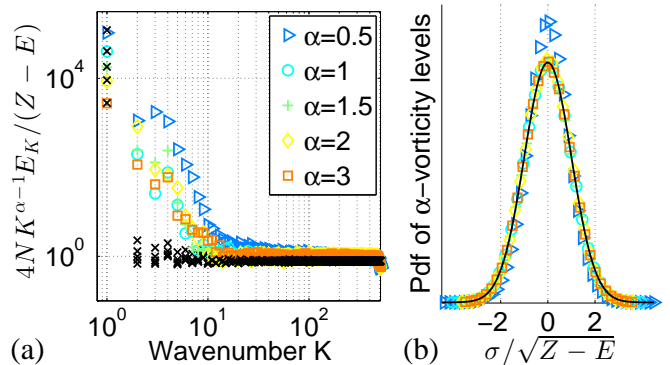


Figure 2: a) Isotropic energy spectrum  $E_K(K)$  normalized by  $(Z - E)/(4NK^{\alpha-1})$  at large time for numerical simulations at resolution  $N^2 = 1024^2$  with  $N = 2(k_{max} + 1)$ . b) Local distributions of  $\alpha$ -vorticity levels at large time. The black  $\times$  symbols on panel (a) and the black line on panel (b) are statistical mechanics predictions for the truncated system, without fitting parameters.

dynamics is more suited to describe the large scale quasi-stationary states. In any case, dynamical effects (rather than the equilibrium theory) are essential to explain the convergence towards the observed final state.

The truncated dynamics is a Hamiltonian system with  $N = (2k_{max} + 1)^2$  degrees of freedom given by the Fourier components of  $q$ , for which a Liouville theorem holds. This allows for a direct application of equilibrium statistical mechanics machinery. Among the infinite number of conserved quantities by the continuous dynamics, only the  $\alpha$ -energy  $E = \sum_{\mathbf{k}} E_{\mathbf{k}}$  and the  $\alpha$ -enstrophy  $Z = \sum_{\mathbf{k}} Z_{\mathbf{k}}$  are conserved by Galerkin-truncated models, where  $E_{\mathbf{k}} = -q_{\mathbf{k}} \psi_{\mathbf{k}}^*$  is the energy of mode  $\mathbf{k}$ , and  $Z_{\mathbf{k}} = |\mathbf{k}|^\alpha E_{\mathbf{k}}$ . Computation of equilibrium states of the truncated system in the thermodynamic limit ( $N \rightarrow +\infty$ ) is a classical result predicting condensation of the  $\alpha$ -energy in the gravest Laplacian eigenmode ( $\sum_{|\mathbf{k}|=1} E_{\mathbf{k}} = E$ ) and a concomitant loss of  $\alpha$ -enstrophy towards small scales [33, 34]. More precisely, for large  $N$ , Fourier modes other than the gravest one have a contribution to the equilibrium state given by  $\langle E_{\mathbf{k}} \rangle = \frac{1}{4N} \left( \frac{Z-E}{|\mathbf{k}|^{\alpha-1}} \right)$ , where  $\langle \cdot \rangle$  stands for a temporal average, which shows equipartition of the enstrophy  $Z - E$  among the Fourier modes for sufficiently large  $|\mathbf{k}|$  [35].

The “isotropic energy spectrum” defined as  $E(K) = \left( \sum_{|\mathbf{k}|=K} E_{\mathbf{k}} \right) / (2\pi K)$  is shown in Fig. 2-a for various values of  $\alpha$ . The spectra are computed for only one snapshot at large time, and we checked that averaging over many snapshots did not make any difference. As predicted by the theory, the energy is mostly condensed at the domain scale ( $K = 1$ ), and Fourier modes are thermalized with equipartition of  $\alpha$ -enstrophy into Fourier modes at large  $K$ , confirming previous numerical studies performed in the context of 2D Euler dynamics ( $\alpha = 2$ )

[34–36], or in the context of surface quasi-geostrophic dynamics ( $\alpha = 1$ ) [37].

Both Figs. 1 and 2-a reveal the presence of a large scale flow containing most of all the energy coexisting with wild small scale fluctuations of  $\alpha$ -vorticity. This gives a strong incentive for a mean-field theory that would predict the density probability field  $\rho(\mathbf{r}, \sigma)$  to measure the  $\alpha$ -vorticity level  $q = \sigma$  in the vicinity of point  $\mathbf{r}$ , and this is what does the MRS equilibrium statistical mechanics [7, 8]. In this framework, all conserved quantities of the continuous dynamics can be expressed in term of  $\rho(\mathbf{r}, \sigma)$ . One can then count the number of microscopic configurations associated with each macroscopic state  $\rho(\mathbf{r}, \sigma)$  and compute the most probable macrostate  $\rho(\mathbf{r}, \sigma)$  satisfying the constraints of the problem. The theory predicts a concentration of all microscopic configurations close to the most probable macrostate  $\rho(\mathbf{r}, \sigma)$  [38]. The large scale flow is then given by  $\bar{q}(\mathbf{r}) = \int d\sigma \sigma \rho$ , and the theory predicts a monotonous relation between  $\bar{q}$  and  $\psi$  [7, 8].

Using large deviation tools, it is shown in Ref. [17] that equilibrium states of the truncated system in the thermodynamic limit ( $k_{max} \rightarrow +\infty$ ) are a subclass of MRS equilibrium states, characterized by a linear  $\bar{q} - \psi$  relation and by local gaussian fluctuations of  $\alpha$ -vorticity  $\rho(\mathbf{r}, \sigma)$  with variance ( $Z - E$ ) (see also Ref. [39] for more discussions on these equilibria). By contrast, when additional invariants than the energy and the enstrophy are taken into account,  $\rho(\mathbf{r}, \sigma)$  is in general non-gaussian. For instance, the initial condition shown on Fig. 1 has been constructed in such a way that the global distribution is close to a double peaked distribution, and MRS theory predicts in that case that the local probability distributions of the equilibrium states of the continuous dynamics should also be doubled peaked distributions [11].

To the best of our knowledge, we show in Fig. 2-b the first observation of such local gaussian fluctuations for the vorticity field in numerical simulations of the Galerkin-truncated dynamics[49], confirming theoretical predictions [17, 39]. However, the success of the statistical mechanics theory of the truncated system is restricted to small scales: the theory underestimates the contribution of intermediate wavenumbers  $1 < K < 30$  in the spectra of Fig. 2-a. In addition, Fig. 3 clearly shows that the  $\bar{q} - \psi$  relation of the large scale flow is non-linear: in the case  $\alpha = 0.5$ , it has a tanh-like shape, while it has a sinh-like shape in the case  $\alpha = 3$ . In any case, such relations are not predicted by the statistical mechanics of the truncated system, which fails to account for the different large scale flow structures observed in Fig. 1.

For any monotonous functional  $\bar{q} - \psi$  relation, there exists a least one set of constraints such that the MRS equilibrium state associated with these constraints is characterized by this functional relation [40]. In that respect, the observed large scale flow is close to an equilibrium state of the continuous dynamics. However, we see in Fig. 3 that there remain fluctuations around the observed

$\bar{q} - \psi$  functional relation, and we checked that these fluctuations were independent from the wavenumber cut-off. This means that the large scale flow is not exactly stationary. In addition, given our choice of an initial condition characterized by a global distribution of  $\alpha$ -vorticity levels with a double-peaked shape, MRS theory predicts that the equilibrium state should be characterized by a tanh-like shape, whatever the value of  $\alpha$ , see e.g. [11]. This means that the large flows obtained in the case  $\alpha = 0.5$  is close to the actual equilibrium state, but not in the case  $\alpha = 3$ .

A transition from an unidirectional flow (bar state) to a dipolar flow is expected when the  $\bar{q} - \psi$  relation changes from a tanh-like shape to a sinh-like shape [41], and the large time flow structure shown in Fig. 3 are consistent with these predictions. In order to study more quantitatively the transition from one state to the other when  $\alpha$  is varied, it is useful to introduce an empirical order parameter  $\mathcal{O} = \frac{\min\{E_{(1,0)}; E_{(0,1)}\}}{\max\{E_{(1,0)}; E_{(0,1)}\}}$  comparing the energy of the gravest Laplacian eigenmodes in each direction. This parameter is zero in the case of an unidirectional state (predicted by the equilibrium theory given our initial condition), and one in the case of a purely dipolar state. Fig. 4 shows that whenever  $\alpha \geq 2$  and the resolution is sufficiently large, the flow is trapped in the dipolar state. For  $\alpha < 2$ , there is a continuous transition from the dipolar state to an unidirectional state at low  $\alpha$ . For  $\alpha \sim 1$ , the order parameter is characterized by small periodic or quasi-periodic oscillations, which are related to the presence of unmixed vortices. Such oscillations have previously been reported in the context of 2D Euler dynamics ( $\alpha = 2$ ), for particular initial conditions [20, 22, 42]. When  $\alpha \rightarrow 0$ , the unidirectional state becomes characterized by periodic oscillations corresponding to large scale oscillations of the interface between two homogeneous regions of potential vorticity, which were also reported for some range of initial condition in 2D Euler equations [42]. Strikingly, changing the interaction range between fluid particles with a prescribed initial condition allows to span all different regimes reported previously in the Euler case by varying the initial condition [18]. We show in the following that phenomenological arguments in limiting cases for the range of interactions  $\alpha$  allows to rationalize these observations.

In the limit  $\alpha \rightarrow 0$ , the  $\alpha$ -vorticity can be written at lowest order as  $q = \alpha \mathcal{L}[\psi] - \psi$ , with  $\mathcal{L}$  a definite negative operator whose eigenmodes are Laplacian eigenmodes and whose eigenvalues are increasing functions of  $|\mathbf{k}|$ . This is reminiscent of the  $1^{1/2}$  layer quasi-geostrophic model, another 2D flow model for which the advected tracer is  $q = \Delta\psi - \psi/R^2$ . In the limit  $R \rightarrow 0$ , this flow model is known to spontaneously form regions where  $q$  is homogenized, separated by sharp interfaces [43], which is expected either from cascade phenomenology [44] or from statistical mechanics arguments [45].

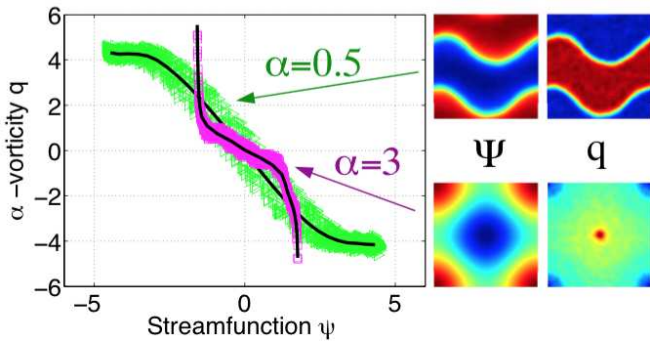


Figure 3:  $\bar{q} - \psi$  relation for the fields shown in Fig. 1 at  $t \sim 100$ . The  $\alpha$ -vorticity field has been locally coarse-grained (see inset). The black line is obtained by considering the averaged  $\alpha$ -vorticity value along a given streamline.

The formal analogy between  $1^{1/2}$  layer quasi-geostrophic model and 2D  $\alpha$ -turbulence in the limit  $\alpha \rightarrow 0$  explains therefore the phenomenon of phase separation observed in Fig. 1 for  $\alpha = 0.5$ . Once the two regions of homogenized  $\alpha$ -vorticity are formed, their interface support the existence of a neutral (or Kelvin) mode oscillating periodically [46, 47]. This prevents a complete relaxation towards the actual equilibrium state, which is characterized by a minimal interface length [45].

When  $\alpha \rightarrow +\infty$ , the streamfunction field is dominated by the gravest modes  $|\mathbf{k}| = 1$  as soon as  $q_{1,0} \neq 0$  or  $q_{0,1} \neq 0$ . The  $\alpha$ -vorticity field  $q$  is sheared by this large scale flow, excepted at the two points where straining vanishes (provided that both  $q_{1,0}$  and  $q_{0,1}$  are non-zero). Since the large scale flow is initially not stationary, the early evolution of the  $\alpha$ -vorticity field looks like chaotic mixing of a passive tracer due to a large scale flow (see Fig. 1 for  $\alpha = 3$ ). This process leads to a background of homogenized  $\alpha$ -vorticity field, with two isolated blobs of  $\alpha$ -vorticity in the vicinity of the two points where straining vanishes. Following this phenomenology, irreversible mixing in physical space through filamentation due to stretching prevents efficient mixing in phase space.

For intermediate values of  $\alpha$  there is an optimal range of interactions that favours the relaxation towards the most probable state (see the yellow region in Fig. 4-b, corresponding to a small value of the order parameter). We identify in Fig. 4 a change of regime at  $\alpha = 2$ , above which the dynamics favours the emergence of dipolar flow. This is consistent with the two dynamical regimes described in Ref. [23], who noticed that direct cascade of  $\alpha$ -enstrophy occurs through a filamentation process for  $\alpha > 2$ , but that these filaments become unstable to secondary instabilities when  $\alpha < 2$ . In that case, the filaments roll up into small blobs, which prevents irreversible mixing in physical space.

*Conclusion.* We have shown that the statistical mechanics of the *truncated* 2D flow models allows to describe quantitatively the *small scale fluctuations* of the quasi-

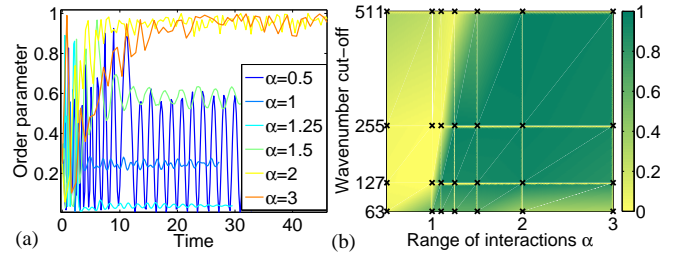


Figure 4: a) Temporal evolution of the order parameter  $\mathcal{O}$  (see text) for various values of  $\alpha$ , in the run with resolution  $1024^2$  ( $k_{max} = 511$ ) b) Temporal average of the order parameter  $\mathcal{O}$  for different values of  $\alpha$  and wavenumber cut-off  $k_{max}$ .

stationary states. By contrast, the corresponding *large scale flow* is close to a stable state of the *continuous* system, characterized by a nonlinear  $\bar{q} - \psi$  relation, which differs from statistical mechanics predictions for the *truncated* system [50].

Strong long-range interactions ( $\alpha$  large) lead to irreversible mixing in physical space through a filamentation process of  $\alpha$ -vorticity, which favours the relaxation towards a stable state different than the one predicted by equilibrium statistical mechanics of the *continuous* system. Weak long-range interactions ( $\alpha$  small) lead to a local coarsening process, which favours relaxation towards a state close to the one predicted by equilibrium statistical mechanics of the *continuous* system, but the relaxation towards equilibrium is not complete, with persistent large scale periodic oscillation around equilibrium. We see that contrary to statistical mechanics expectations, the outcome of the violent relaxation process following the instability of an initial condition far from equilibrium strongly depends on the range of interactions.

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- [48] We employ an energy-enstrophy conserving pseudo-spectral code with dealiasing using the Orzag’s 2/3 rule. Time integration were performed using Runge-Kutta scheme at order 4. We checked that energy and enstrophy decreased by less than 0.01% for all runs presented in this letter.
- [49] The pdf shown on Fig. 2-b are obtained by computing histograms of vorticity fluctuations along a given contour of stream function, and summing over many contours. We checked that they were no significant variation of the distribution from one contour to another (expected for  $\alpha < 1$ ), and that distributions obtained in the vicinity of a given point lead to similar results.
- [50] On much longer time scale increasing with the truncation cut-off  $k_{max}$ , those quasi-stationary states are expected to converge towards the equilibrium states of the *truncated* system characterized by a linear  $\bar{q} - \psi$  relation.