A probabilistic approach to large time behaviour of parabolic equations with Neumann boundary conditions

Ying Hu, Pierre-Yves Madec

A probabilistic approach to large time behaviour of parabolic equations with Neumann boundary conditions

Ying Hu* Pierre-Yves Madec†

March 26, 2015

Abstract: This paper is devoted to the study of the large time behaviour of viscosity solutions of parabolic equations with Neumann boundary conditions. This work is the sequel of [12] in which a probabilistic method was developped to show that the solution of a parabolic semilinear PDE behaves like a linear term λT shifted with a function v, where (v, λ) is the solution of the ergodic PDE associated to the parabolic PDE. We adapt this method in finite dimension by a penalization method in order to be able to apply an important basic coupling estimate result and with the help of a regularization procedure in order to avoid the lack of regularity of the coefficients in finite dimension.

Keywords: Backward stochastic differential equations; Ergodic backward stochastic differential equations; HJB equations; Large time bahaviour; Viscosity solutions;.

AMS classification: 35B40, 35K10, 60H30, 93E20.

1 Introduction

We are concerned with the large time behaviour of solutions of the Cauchy problem with Neumann boundary conditions:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f(x, \nabla u(t,x)\sigma), & \forall (t,x) \in \mathbb{R}_+ \times G, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in \mathbb{R}_+ \times \partial G \\ u(0,x) = h(x), & \forall x \in \overline{G}, \end{cases} \tag{1.1}$$

where, at least formally, $\forall \psi : \overline{G} \to \mathbb{R}$,

$$(\mathscr{L}\psi)(x) = \frac{1}{2}\operatorname{Tr}(\sigma^t \sigma \nabla^2 \psi(x)) + \langle b(x), \nabla \psi(x) \rangle,$$

and $G = \{\phi > 0\}$ is a bounded convex open set of $\mathbb{R}^d$ with regular boundary. $u : \mathbb{R}_+ \times \overline{G} \to \mathbb{R}$ is the unknown function. We will assume that b is Lipschitz and σ is invertible. h is continuous and $g \in \mathscr{C}^1_{\text{lip}}(\overline{G})$. Furthermore we will assume that the non-linear term $f(x,z) : \mathbb{R}^d \times \mathbb{R}^{1\times d} \to \mathbb{R}$

*IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France (ying.hu@univ-rennes1.fr), partially supported by Lebesgue center of mathematics ("Investissements d'avenir" program - ANR-11-LABX-0020-01)

†IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France (pierre-yves.madec@univ-rennes1.fr), partially supported by Lebesgue center of mathematics ("Investissements d'avenir" program - ANR-11-LABX-0020-01)

is continuous in the first variable for all z and there exists $C > 0$ such that for all $x \in \mathbb{R}^d$, $\forall z_1, z_2 \in \mathbb{R}^{1\times d}$, $|f(x,z_1)-f(x,z_2)| \leq C|z_1-z_2|$. Finally in order to obtain uniqueness for viscosity solutions of (1.1), we assume that ∂G is $W^{3,\infty}$ and that there exists $\exists m \in \mathscr{C}((0,+\infty),\mathbb{R})$, $m(0^+) = 0$ such that $\forall x, y \in \overline{G}, \forall z \in \mathbb{R}^{1\times d}$,

$$|f(x,z) - f(y,z)| \leq m\left((1+|z|)|x-y|\right).$$

A lot of papers deal with the large time behaviour of parabolic PDEs (see for e.g. [20], [10], [14], [9] or [13]), but there are not a lot of them which deal with Neumann boundary conditions. In [3], Benachour and Dabuleanu study the large time behaviour of the Cauchy problem with zero Neumann boundary condition

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + a|\nabla u(t,x)|^p, & \forall (t,x) \in \mathbb{R}_+^* \times G, \\ \frac{\partial u(t,x)}{\partial n} = 0, & \forall (t,x) \in \mathbb{R}_+ \times \partial G \\ u(0,x) = h(x), & \forall x \in \overline{G}, \end{cases} \tag{1.2}$$

where $a \in \mathbb{R}$, $a \neq 0$, $p > 0$ and G is a bounded open set with smooth boundary of $\mathscr{C}^3$ class. The large time behaviour depends on the exponent p. Thus if $p \in (0,1)$, and if h is a periodic function, then the solution is constant from a finite time. That is, there exist $T^* > 0$ and $c \in \mathbb{R}$ such that $u(t,x) = c$, for all $t > T^*$. When $p \geq 1$, any solution of (1.2) converges uniformly to a constant, as $t \to +\infty$.

In [15], Ishii establishes a result about the large time behaviour of a parabolic PDE in a bounded set with an Hamiltonian of first order $H(x,p)$, convex and coercive in p and with Neumann boundary coniditons.

In [2], Barles and Da Lio give a result for the large time behaviour of (1.1). However, the result about the large time behaviour has been improved by Da Lio in [5] under the same hypotheses. In this last paper, the author studies the large time behaviour of non linear parabolic equation with Neumann boundary conditions on a smooth bounded domain $\overline{G}$:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + F(x, \nabla u(t,x), \nabla^2 u(t,x)) = \lambda, & \forall (t,x) \in \mathbb{R}_+ \times G \\ L(x, \nabla u(t,x)) = \mu & \forall (t,x) \in \mathbb{R}_+ \times \overline{G}, \\ u(0,x) = h(x), & \forall x \in G. \end{cases} \tag{1.3}$$

The spirit of this paper is slightly different from our work. Indeed, the result says that $\forall \lambda \in \mathbb{R}$, there exists $\mu \in \mathbb{R}$ such that (1.3) has a continuous viscosity solution. Moreover there exits a unique $\widetilde{\lambda}$ such that $\mu(\widetilde{\lambda}) = \widetilde{\lambda}$ for which the solution of (1.3) remains uniformly bounded in time $\widetilde{u}$. Then, there exits $\widetilde{u}_\infty$ solution of the ergodic PDE associated to (1.3) such that

$$\widetilde{u}(t,x) \underset{t\to+\infty}{\longrightarrow} \widetilde{u}_\infty(x), \quad \text{uniformly in } \overline{G}.$$

Our method is purely probabilistic, which can be described as follows. First, let us consider $(X_t^x, K_t^x)_{t\geq 0}$ the solution of the following reflected SDE with values in $\overline{G} \times \mathbb{R}_+$

$$\begin{cases} X_t = x + \int_0^t b(X_s^x)\mathrm{d}s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x + \sigma \mathrm{d}W_t, & t \in \geq 0, \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}} \mathrm{d}K_s^x, & \forall t \geq 0. \end{cases}$$

where W is a $\mathbb{R}^d$-valued standard brownian motion. Let (v,λ) be the solution of the following ergodic PDE.

$$\begin{cases} \mathscr{L}(x)v + f(x, \nabla v(x)\sigma) - \lambda = 0, & \forall x \in G, \\ \frac{\partial v(t,x)}{\partial n} + g(x) = 0, & \forall x \in \partial G. \end{cases}$$

Let $(Y^{T,x}, Z^{T,x})$ be the solution of the BSDE:

$$\begin{cases} \mathrm{d}Y_s^{T,x} = -f(X_s^x, Z_s^{T,x})\mathrm{d}s - g(X_s^x)\mathrm{d}K_s^x + Z_s^{T,x}\mathrm{d}W_s, \\ Y_T^{T,x} = h(X_T^x), \end{cases}$$

and (Y, Z, λ) be solution of the EBSDE:

$$\mathrm{d}Y_s = -(f(X_s^x, Z_s^x) - \lambda)\mathrm{d}s - g(X_s^x)\mathrm{d}K_s^x + Z_s^x\mathrm{d}W_s.$$

Then we have the following probabilistic representation

$$\begin{cases} Y_s^{T,x} = u(T-s, X_s^x), \\ Y_s^x = v(X_s^x). \end{cases}$$

Then, in order to apply the method exposed in [12], we penalize and regularize the reflected process in order to apply the basic coupling estimates. Then, the use of a stability argument for BSDE helps us to conclude. Finally, we deduce that there exists a constant $L \in \mathbb{R}$ such that for all $x \in \mathbb{R}^d$,

$$Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to+\infty}{\longrightarrow} L,$$

i.e.

$$u(T,x) - \lambda T - v(x) \underset{T\to+\infty}{\longrightarrow} L.$$

Our method uses not only purely probabilistic arguments, but also gives a rate of convergence:

$$|u(T,x) - \lambda T - v(x)| \leq Ce^{-\hat{\eta}T}.$$

The paper is organized as follows: In section 2, we introduce some notations. In section 3, we recall some existence and uniqueness results about a perturbed SDE, a reflected SDE, a BSDE and an EBSDE that will be useful for what follows in the paper. We recall how such BSDE and EBSDE are linked with PDE. In section 4, we study the large time behaviour of the solution of the BSDE taken at initial time when the horizon T of the BSDE increases. Then, we obtain a more precise result with an explicite rate of convergence in the Markovian case. In section 5, we apply our results to an optimal ergodic control problem.

2 Notations

We introduce some notations. Let E be an Euclidian space. We denote by $\langle \cdot, \cdot \rangle$ its scalar product and by $|\cdot|$ the associated norm. We denote by $B(x, M)$ the ball of center $x \in E$ and radius $M > 0$. Given $\phi \in B_b(E)$, the space of bounded and measurable functions $\phi : E \to \mathbb{R}$, we denote by $||\phi||_0 = \sup_{x\in E} |\phi(x)|$. If a function f is continuous and defined on a compact and convex subset $\overline{G}$ of $\mathbb{R}^d$, we define $f_{\mathbb{R}^d} := f(\Pi(x))$ where Π is the projection on $\overline{G}$. Note that $f_{\mathbb{R}^d}$ is continuous and bounded. $\mathscr{C}^k_{\mathrm{lip}}(\overline{G})$ denotes the set of the functions of class $\mathscr{C}^k$ whose partial derivatives of order k are Lipschitz functions.

Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\mathscr{F}_t$, we consider the following classes of stochastic processes.

1. $L^p_{\mathscr{P}}(\Omega, \mathscr{C}([0,T];E))$, $p \in [1,\infty)$, $T > 0$, is the space of predictable processes Y with continuous paths on $[0,T]$ such that

$$|Y|_{L^p_{\mathscr{P}}(\Omega,\mathscr{C}([0,T];E))} = \mathbb{E} \sup_{t\in[0,T]} |Y_t|^p < \infty.$$

2. $L^p_{\mathscr{P}}(\Omega, L^2([0,T];E))$, $p \in [1,\infty)$, $T>0$, is the space of predictable processes Y on $[0,T]$ such that

$$|Y|_{L^p_{\mathscr{P}}(\Omega,L^2([0,T],E))} = \mathbb{E}\left(\int_0^T |Y_t|^2 \mathrm{d}t\right)^{p/2} < \infty.$$

3. $L^2_{\mathscr{P},\mathrm{loc}}(\Omega, L^2([0,\infty);K))$ is the space of predictable processes Y on $[0,\infty)$ which belong to the space $L^2_{\mathscr{P}}(\Omega, L^2([0,T];E))$ for every $T>0$. We define in the same way $L^p_{\mathscr{P},\mathrm{loc}}(\Omega, \mathscr{C}([0,\infty);E))$.

In the following, we consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a standard Brownian motion denoted by $(W_t)_{t\geq 0}$ with values in $\mathbb{R}^k$. $(\mathscr{F}_t)_{0\leq t}$ will denote the natural filtration of W augmented with the family of $\mathbb{P}$-null sets of $\mathscr{F}$.

In this paper, C denotes a generic constants for which we specify the dependency on some parameters when it is necessary to do so. In this paper, we will consider only continuous viscosity solutions.

3 Preliminaries

3.1 The perturbed forward SDE

Let us consider the following stochastic differential equation with values in $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_t = d(X_t)\mathrm{d}t + b(t,X_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t\geq 0, \\ X_0 = x \in \mathbb{R}^d. \end{cases} \tag{3.1}$$

We will assume the following about the coefficients of the SDE:

Hypothesis 3.1. 1. $d : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz, strict dissipative (i.e. there exists $\eta > 0$ such that for every $x, y \in \mathbb{R}^d$, $< d(x) - d(y), x - y > \leq -\eta|x-y|^2$) and with polynomial growth (i.e. there exists $\mu > 0$ such that for every $x \in \mathbb{R}^d$, $|d(x)| \leq C(1+|x|^\mu)$).

2. $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and measurable.

3. $\sigma \in \mathbb{R}^{d\times d}$ is invertible.

Definition 3.1. *We say that the SDE* (3.1) *admits a weak solution if there exists a new $\mathscr{F}$-Brownian motion $(\widehat{W}^x)_{t\geq 0}$ with respect to a new probability measure $\widehat{\mathbb{P}}$ (absolutely coninuous with respect to $\mathbb{P}$), and an $\mathscr{F}$-adapted process $(\widehat{X}^x)_{t\geq 0}$ with continuous trajectories for which* (3.1) *holds with $(W)_{t\geq 0}$ replaced by $(\widehat{W}^x)_{t\geq 0}$.*

Lemma 3.2. *Assume that Hypothesis 3.1 holds true and that $b(t,\cdot)$ is Lipschitz for every $t \geq 0$. Then for every $x \in \mathbb{R}^d$, equation(3.1) admits a unique strong solution, that is, an adapted $\mathbb{R}^d$-valued process denoted by X^x with continuous paths satisfying $\mathbb{P}$-a.s.,*

$$X^x_t = x + \int_0^t d(X^x_s)\mathrm{d}s + \int_0^t b(s,X^x_s)\mathrm{d}s + \int_0^t \sigma \mathrm{d}W_s, \quad \forall t \geq 0.$$

Furthermore, we have the following estimate:

$$\mathbb{E}[|X^x_s|^p] \leq C(1+|x|^p), \tag{3.2}$$

If F is only bounded and measurable then there exists a weak solution $(\widehat{X}, \widehat{W})$ and unicity in law holds. Furthermore, (3.2) *still hold (with respect to the new probability measure).*

Proof. For the first part of the lemma see [11], Theorem 3.3 in chapter 1 or [18], Theorem 3.5. Estimates (3.2) is a simple consequence of Ito's formula. Weak existence and unicity in law are a direct consequence of a Girsanov's transformation. □

We define the Kolmogorov semigroup associated to Eq. (3.1) as follows: $\forall \phi : \mathbb{R}^d \to \mathbb{R}$ measurable with polynomial growth

$$\mathscr{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x).$$

Lemma 3.3 (Basic coupling estimate)**.** *Assume that Hypothesis 3.1 hold true and that $\forall t \geq 0$, $b(t,\cdot)$ is Lipschitz. Then there exists $\hat{c} > 0$ and $\hat{\eta} > 0$ such that for all $\phi : \mathbb{R}^d \to \mathbb{R}$ measurable and bounded (i.e. $\exists C, \mu > 0$ such that $\forall x \in \mathbb{R}^d$, $\phi(x) \leq C(1 + |x|^\mu)$),*

$$|\mathscr{P}_t[\phi](x) - \mathscr{P}[\phi](y)| \leq \hat{c}e^{-\hat{\eta}t}. \tag{3.3}$$

We stress the fact that $\hat{c}$ and $\hat{\eta}$ depend on b only through $\sup_{t\geq 0}\sup_{x\mathbb{R}^d}|b(t,x)|$.

Proof. See [17]. □

Corollary 3.4. *Relation (3.3) can be extended to the case in which b is only bounded and measurable and for all $t \geq 0$, there exits a uniformly bounded sequence of Lipschitz functions in x, $(b_n(t,\cdot))_{n\geq 1}$ (i.e. $\forall t \geq 0, \forall n \in \mathbb{N}$, $b_n(t,\cdot)$ is Lipschitz and $\sup_n \sup_t \sup_x |b_n(t,x)| < +\infty$) such that*

$$\lim_n b_n(t,x) = b(t,x), \qquad \forall t \geq 0, \forall x \in R^d.$$

Clearly in this case in the definition of $\mathscr{P}[\phi]$ the mean value is taken with respect to the new probability measure $\widehat{\mathbb{P}}$.

Proof. It is enough to adapt the proof of Corollary 2.5 in [6]. The goal is to show that, if $\mathscr{P}^n$ denotes the Kolmogorov semigroup corresponding to equation (3.1) but with b replaced by $\widetilde{b}_n$, then $\forall x \in \mathbb{R}^d$, $\forall t \geq 0$,

$$\mathscr{P}_t^n[\phi](x) \underset{n\to+\infty}{\longrightarrow} \mathscr{P}_t[\phi](x).$$

□

Remark 3.5. Similarly, if for every $t \geq 0$, there exits a uniformly bounded sequence of Lipschitz functions $(b_{m,n}(t,\cdot))_{m\in\mathbb{N},n\in\mathbb{N}}$ (i.e. $\forall t \geq 0, \forall n \in \mathbb{N}, \forall m \in \mathbb{N}$, $F_{m,n}(t,\cdot)$ is Lipschitz and $\sup_m \sup_n \sup_t \sup_x |b_{m,n}(t,x)| < +\infty$) such that

$$\lim_m \lim_n b_{m,n}(t,x) = b(t,x), \qquad \forall t \geq 0, \forall x \in \mathbb{R}^d,$$

then, if $\mathscr{P}^{m,n}$ is the Kolmogorov semigroup corresponding to equation (3.1) but with F replaced by $b_{m,n}$, we have $\forall t \geq 0$, $\forall x \in \mathbb{R}^d$,

$$\lim_m \lim_n \mathscr{P}_t^{m,n}[\phi](x) = \mathscr{P}[\phi](x),$$

which shows that relation (3.3) still hold.

We will need to apply the lemma above to some functions with particular form.

Lemma 3.6. *Let $f : \mathbb{R}^d \times \mathbb{R}^{1\times k} \to \mathbb{R}$ be continuous in the first variable and Lipschitz in the second one and ζ, ζ' be two continuous functions: $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{1\times k}$ be such that for all $s \geq 0$, $\zeta(s,\cdot)$ and $\zeta'(s,\cdot)$ are continuous. We define, for every $s \geq 0$ and $x \in \mathbb{R}^d$,*

$$\Upsilon(x) = \begin{cases} \frac{f(x,\zeta(s,x))-f(x,\zeta'(s,x))}{|\zeta(s,x)-\zeta'(x)|^2}{}^t(\zeta(s,x)-\zeta'(s,x)), & \text{if } \zeta(s,x) \neq \zeta'(s,x), \\ 0, & \text{if } \zeta(s,x) = \zeta'(s,x). \end{cases}$$

Then for all $s \geq 0$, there exists a uniformly bounded sequence of Lipschitz functions $(\Upsilon_{m,n}(s,\cdot))_{m\in\mathbb{N},n\in\mathbb{N}}$ (i.e., for every $m \in \mathbb{N}^*$ and $n \in \mathbb{N}^*$, $\Upsilon_n(s,\cdot)$ is Lipschitz and $\sup_m \sup_n \sup_s \sup_x |\Upsilon(s,x)| < +\infty$) such that for every $s \geq 0$ and for every $x \in \mathbb{R}^d$,

$$\forall x \in \mathbb{R}^d, \lim_{m\to+\infty} \lim_{n\to+\infty} \Upsilon_{m,n}(s,x) = \Upsilon(s,x).$$

Proof. See the proof of Lemma 3.5 in [6]. □

3.2 The reflected SDE

We consider a process X_t^x reflected in $\overline{G} = \{\phi > 0\}$. Let $(X_t^x, K_t^x)_{t\geq 0}$ denotes the unknown of the following SDE:

$$\begin{cases} X_t = x + \int_0^t b(X_s^x)\mathrm{d}s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x + \sigma\mathrm{d}W_t, \quad t \in \mathbb{R}_+, \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x. \end{cases} \tag{3.4}$$

Hypothesis 3.2. 1. $b : \overline{G} \to \mathbb{R}^d$ is Lipschitz,

2. $\sigma \in \mathbb{R}^{d\times d}$ is invertible.

We will make the following assumptions about G.

Hypothesis 3.3. 1. G is a bounded convex open set of $\mathbb{R}^d$,

2. $\phi \in \mathscr{C}^2_{\mathrm{lip}}(\mathbb{R}^d)$ and $G = \{\phi > 0\}$, $\partial G = \{\phi = 0\}$ and $\forall x \in \partial G$, $|\nabla\phi(x)| = 1$.

Remark 3.7. Let us denote by $\Pi(x)$ the projection of $x \in \mathbb{R}^d$ on $\overline{G}$. Let us extend the definition of b to $\mathbb{R}^d$ by setting, $\forall x \in \mathbb{R}^d$,

$$\widetilde{b}(x) := -x + (b(\Pi(x)) + \Pi(x))$$

Note that $d(x) := -x$ is dissipative and that $p(x) := b(\Pi(x)) + \Pi(x)$ is Lipschitz bounded. Therefore, $\widetilde{b}$ is weakly dissipative and satisfy Hypothesis 3.1.

Let us denote by $(X_t^{x,n})$ the solution of the following penalized SDE associated to 3.4:

$$X_t^{x,n} = x + \int_0^t \left[\widetilde{b}(X_s^{x,n}) + F_n(X_s^{x,n})\right]\mathrm{d}s + \int_0^t \sigma\mathrm{d}W_s,$$

where $\forall x \in \mathbb{R}^d$, $F_n(x) = -2n(x - \Pi(x))$.

Lemma 3.8. *Assume that Hypotheses 3.2 and 3.3 hold true. Then for every $x \in \overline{G}$ there exists a unique pair of processes $(X_t^x, K_t^x)_{t\geq 0}$ with values in $(\overline{G} \times \mathbb{R}_+)$ and which belongs to the space $L^p_{\mathscr{P}}(\Omega, \mathscr{C}([0,+\infty[;\mathbb{R}^d)) \times L^p_{\mathscr{P}}(\Omega, \mathscr{C}([0,+\infty[;\mathbb{R}_+))$, $\forall p \in [1,+\infty[$, satisfying (3.4) and such that*

$$\eta_t^x := \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \qquad \text{has bounded variation on } [0,T],\ \forall 0 \leq T < +\infty, \quad \eta_0^x = 0,$$

and for all process z continuous and progressively measurable taking values in the closure $\overline{G}$ we have

$$\int_0^T (X_s^x - z_s)\mathrm{d}K_s^x \leq 0, \qquad \forall T \geq 0.$$

Finally the following estimates holds for the convergence of the penalized process, for any $1 < q < p/2$, for any $T \geq 0$ there exists $C \geq 0$ such that

$$\mathbb{E} \sup_{0\leq t\leq T} |X_t^{x,n} - X_t^x|^p \leq \frac{C}{n^q}.$$

Proof. See Lemma 4.2 in [17]. □

3.3 The BSDE

Let us fix $T > 0$ and let us consider the following BSDE in finite horizon for an unknown process $(Y_s^{T,t,x}, Z_s^{T,t,x})_{s\in[t,T]}$ with values in $\mathbb{R} \times \mathbb{R}^{1\times d}$:

$$Y_s^{T,x} = \xi^T + \int_s^T f(X_r^{t,x}, Z_r^{T,t,x})\mathrm{d}r + \int_s^T g(X_r^{t,x})\mathrm{d}K_r^{t,x} - \int_s^T Z_r^{T,t,x}\mathrm{d}W_r, \qquad \forall s \in [t,T], \tag{3.5}$$

where $(X_s^{t,x}, K_s^{t,x})_{s\geq[t,T]}$ is the solution of the SDE (3.4) starting from x at time t. If $t = 0$, we use the following standard notations $X_s^x = X_s^{0,x}$,$K_s^x = K_s^{0,x}$, $Y_s^{T,x} := Y_s^{T,0,x}$ and $Z_s^{T,x} = Z_s^{T,0,x}$. We will assume the following assumptions.

Hypothesis 3.4 (Path dependent case)**.** There exists $C > 0$, such that the function $f : \overline{G} \times \mathbb{R}^{1,d} \to \mathbb{R}$ and ξ^T satisfy:

1. ξ^T is a $\mathbb{R}$ valued random variable $\mathscr{F}_T$ measurable and $|\xi^T| \leq C$,
2. $\forall x \in \overline{G}$, $\forall z_1, z_2 \in \mathbb{R}^{1\times d}$, $|f(x, z_1) - f(x, z_2)| \leq C|z_1 - z_2|$,
3. $f(\cdot, z)$ is continuous,
4. $g \in \mathscr{C}^1_{\mathrm{lip}}(\overline{G})$.

Lemma 3.9. *Assume that Hypothesis 3.2, 3.3 and 3.4 hold true, then there exists a unique solution $(Y_s^{T,t,x}, Z_s^{T,t,x}) \in L^2_{\mathscr{P}}(\Omega, \mathscr{C}([0,T];\mathbb{R})) \times L^2_{\mathscr{P}}(\Omega, L^2([0,T];\mathbb{R}^{1\times d}))$.*

Proof. See Theorem 1.7 in [21]. □

Hypothesis 3.5 (Markovian case)**.** There exists $C > 0$ such that

1. $\xi^T = h(X_T^x)$, where $h : \overline{G} \to \mathbb{R}$ is continuous,
2. $\forall x \in \overline{G}$, $\forall z, z' \in \mathbb{R}^{1\times d}$, $|f(x, z) - f(x, z')| \leq C|z - z'|$,
3. $\forall z \in \mathbb{R}^{1\times d}$, $f(\cdot, z)$ is continuous,
4. $g \in \mathscr{C}^1_{\mathrm{lip}}(\overline{G})$.

Let us consider the following semilinear PDE:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathscr{L}u(t,x) + f(x, \nabla u(t,x)\sigma) = 0, & \forall (t,x) \in [0,T] \times G, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in [0,T] \times \partial G \\ u(T,x) = h(x), & \forall x \in G, \end{cases} \tag{3.6}$$

where $\mathscr{L}u(t,x) = \frac{1}{2}\operatorname{Tr}(\sigma^t\sigma\nabla^2 u(t,x)) + \langle b(x), \nabla u(t,x)\rangle$.

Lemma 3.10 (Existence)**.** *Assume that Hypotheses 3.2, 3.3 and 3.5 hold true then there exists a viscosity solution to the PDE (3.6) given by*

$$u_T(t,x) = Y_t^{T,t,x}.$$

Proof. In the general case, $\left((t,x) \mapsto \int_0^t g(X_s^{t,x})\mathrm{d}K_s^{t,x}\right)$ may fail to be continuous, and therefore $\left((t,x) \mapsto Y_t^{T,t,x}\right)$ may also fail to be continuous. However in our framework, $u_T(t,x) \in \mathscr{C}([0,T]\times \overline{G};\mathbb{R})$. Indeed, first as in the proof of Theorem 3.1 in [22], we deduce the existence of a function $v^1 : \overline{G} \to \mathbb{R}$ which belongs to the space $\mathscr{C}^2_{\mathrm{lip}}(\overline{G})$ and which is solution of the Helmholtz's equation for some $\alpha \in \mathbb{R}$

$$\begin{cases} \Delta v^1(x) - \alpha v^1(x) = 0, \\ \frac{\partial v^1(x)}{\partial n} + g(x) = 0 \end{cases}$$

We set $Y_s^{1,t,x} = v^1(X_s^{t,x})$ and $Z_s^{1,t,x} = \nabla v^1(X_s^{t,x})\sigma$. These processes verify $\forall s \in [t,T]$,

$$Y_s^{1,t,x} = v^1(X_T^{t,x}) + \int_s^T [-\mathscr{L}v^1(X_r^{t,x})]\mathrm{d}r + \int_s^T g(X_r^{t,x})\mathrm{d}K_r^{t,x} - \int_s^T Z_r^{1,t,x}\mathrm{d}W_r.$$

where

$$(\mathscr{L}v^1(x)) = \frac{1}{2}\operatorname{Tr}(\sigma^t\sigma\nabla^2 v^1(x)) + \langle b(x), \nabla v^1(x)\rangle.$$

Then, if we define

$$\widetilde{Y}_s^{T,t,x} = Y_s^{T,t,x} - v^1(X_s^{t,x})$$
$$\widetilde{Z}_s^{T,t,x} = Z_s^{T,t,x} - \nabla v^1(X_s^{t,x})\sigma,$$

$(\widetilde{Y}^{T,t,x}, \widetilde{Z}^{T,t,x})$ satisfies the BSDE, $\forall s \in [t,T]$:

$$\widetilde{Y}_s^{T,t,x} = (h - v^1)(X_T^{t,x}) + \int_s^T \left[f(X_r^{t,x}, \widetilde{Z}_r^{T,t,x} + \nabla v^1(X_r^{t,x})\sigma) + \mathscr{L}(v^1(X_r^{t,x}))\right]\mathrm{d}r - \int_s^T \widetilde{Z}_r^{T,t,x}\mathrm{d}W_r,$$

which shows, since $v^1 \in \mathscr{C}^2_{\mathrm{lip}}(\overline{G})$, that $\left((t,x) \mapsto \widetilde{Y}_t^{T,t,x}\right)$ is continuous. To show that $u_T(t,x)$ is a viscosity solution of 3.6 see [21], Theorem 4.3. □

Uniqueness for solutions of (3.6) holds under additional assumptions in our framework.

Hypothesis 3.6.

1. ∂G is of class $W^{3,\infty}$,
2. $\exists m \in \mathscr{C}((0,+\infty),\mathbb{R})$, $m(0^+) = 0$ such that $\forall x, y \in \overline{G}, \forall z \in \mathbb{R}^{1\times d}$,

$$|f(x,z) - f(y,z)| \le m\left((1+|z|)|x-y|\right).$$

Lemma 3.11 (Uniqueness)**.** *Assume that Hypotheses 3.2, 3.3, 3.5 and 3.6 hold true. Then, uniqueness holds for viscosity solutions of* (3.6)*.*

Proof. See Theorem II.1 in [1]. □

Remark 3.12. By the following change of time: $\widetilde{u}_T(t,x) := u_T(T-t,x)$, we remark that $\widetilde{u}_T(t,x)$ is the unique viscosity solution of (1.1). Now remark that $\widetilde{u}_T(T,x) = u_T(0,x) = Y_0^{T,0,x} = Y_0^{T,x}$, therefore the large time behaviour of $Y_0^{T,x}$ is the same as that of the solution of equation (1.1). This justifies our approach.

3.4 The EBSDE

In this section, we consider the following ergodic BSDE for an unknown process $(Y_t^x, Z_t^x, \lambda)_{t\geq 0}$ with values in $\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{R}$:

$$Y_t^x = Y_T^x + \int_t^T (f(X_s^x, Z_s^x) - \lambda)\mathrm{d}s + \int_t^T g(X_s^x)\mathrm{d}K_s^x - \int_t^T Z_s^x \mathrm{d}W_s, \quad \forall 0\leq t\leq T<+\infty. \tag{3.7}$$

Hypothesis 3.7. There exists $C>0$ and $\mu>0$ such that,

1. $\forall x\in\overline{G}$, $\forall z,z'\in\mathbb{R}^{1\times d}$, $|f(x,z)-f(x,z')|\leq C|z-z'|$,
2. $\forall z\in\mathbb{R}^{1\times d}$, $f(\cdot,z)$ is continuous,
3. $g\in\mathscr{C}^1_{\mathrm{lip}}(\overline{G})$.

Lemma 3.13 (Existence when Neumann boundary conditions are null)**.** *Assume that $g\equiv 0$ and that Hypotheses 3.2, 3.3 and 3.7 hold true. Then there exists a solution $(Y^x, Z^x, \lambda)\in L^2_{\mathscr{P}}(\Omega,\mathscr{C}([0,+\infty[;\mathbb{R}))\times L^2_{\mathscr{P}}(\Omega, L^2([0,+\infty[;\mathbb{R}^{1\times d}))\times\mathbb{R}$ to (3.7). Moreover there exists $v:\overline{G}\to\mathbb{R}$ and $\xi:\overline{G}\to\mathbb{R}^{1\times d}$ measurable such that for every $x,y\in\overline{G}$, for all $t\geq 0$*

$$\begin{aligned}
&Y_t^x = v(X_t^x), Z_t^x = \xi(X_t^x),\\
&v(0)=0,\\
&|v(x)-v(y)|\leq C,\\
&|v(x)-v(y)|\leq C|x-y|.
\end{aligned}$$

Proof. First let us recall that by Remark 3.7, one can replace b by its extension $\widetilde{b}$ which is weakly dissipative. Therefore, replacing f by $f_{\mathbb{R}^d}$, we obtain, by Theorem 4.4 in [17] that there exists $v:\overline{G}\to\mathbb{R}$ and $\xi:\overline{G}\to\mathbb{R}^{1\times d}$ measurable such that for every $x,y\in\overline{G}$, for all $t\geq 0$,

$$\begin{aligned}
&Y_t^x = v(X_t^x), Z_t^x = \xi(X_t^x),\\
&v(0)=0,\\
&|v(x)-v(y)|\leq C(1+|x|^{1+\mu}+|y|^{1+\mu}),\\
&|v(x)-v(y)|\leq C(1+|x|^{1+\mu}+|y|^{1+\mu})|x-y|.
\end{aligned}$$

And the result follows by the boundedness of $\overline{G}$. □

Remark 3.14. Note that even if $0\notin\overline{G}$, the method used in [17] constructs a function v defined on the whole $\mathbb{R}^d$ and such that $v(x)=0$.

Lemma 3.15 (Existence)**.** *Assume that Hypotheses 3.2, 3.3 and 3.7 hold true. Then there exists a solution* $(Y^x, Z^x, \lambda) \in L^2_{\mathscr{P}}(\Omega, \mathscr{C}([0,+\infty[;\mathbb{R})) \times L^2_{\mathscr{P}}(\Omega, L^2([0,+\infty[;\mathbb{R}^{1\times d})) \times \mathbb{R}$ *to the EBSDE (3.7). Moreover there exists* $v : \overline{G} \to \mathbb{R}$ *such that for every* $x, y \in \overline{G}$*, for all* $t \geq 0$*,*

$$
\begin{aligned}
&Y^x_t = v(X^x_t),\\
&|v(x)| \leq C,\\
&|v(x) - v(y)| \leq C|x-y|.
\end{aligned}
$$

Proof. First as in the proof of Theorem 3.1 in [22], we deduce the existence of a function $v^1 : \overline{G} \to \mathbb{R}$ which belongs to the space $\mathscr{C}^2_{\text{lip}}(\overline{G})$ and solutions of the Helmholtz's equation for some $\alpha \in \mathbb{R}$

$$
\begin{cases}
\Delta v^1(x) - \alpha v^1(x) = 0,\\
\frac{\partial v^1(x)}{\partial n} + g(x) = 0
\end{cases}
$$

Then, if we define $(Y^1_t := v^1(X^x_t), Z^1_t := \nabla v^1(X^x_t)\sigma)$ satisfy, for every $0 \leq t \leq T < +\infty$:

$$
Y^1_t = Y^1_T + \int_t^T \left[-\mathscr{L}v^1(X^x_s)\right] \mathrm{d}s + \int_t^T g(X^x_s)\mathrm{d}K^x_s - \int_t^T Z^1_s \mathrm{d}W_s, \tag{3.8}
$$

where

$$
(\mathscr{L}v^1)(x) = \frac{1}{2}\,\mathrm{Tr}(\sigma^t\sigma\nabla^2 v^1) + \langle \widetilde{b}(x), \nabla v^1\rangle.
$$

Now consider the following EBSDE:

$$
Y^2_t = Y^2_T + \int_t^T [f^2(X^x_s, Z^2_s) - \lambda]\mathrm{d}s - \int_t^T Z^2_s \mathrm{d}W_s, \quad \forall 0 \leq t \leq T < +\infty, \tag{3.9}
$$

with $f^2(x,z) := \mathscr{L}v^1(x) + f(x, z + \nabla v^1(x)\sigma)$. Since $f^2(\cdot, 0)$ is continuous and since for every $x \in \overline{G}$, $f^2(x,\cdot)$ is Lipschitz, one an apply Lemma 3.13 to obtain the existence of a solution $(Y^2_t = v^2(X^x_t), Z^2_t = \xi^2(X^x_t))$ to EBSDE (3.9) such that v^2 is continuous. We set

$$
\begin{aligned}
&Y^x_t = Y^1_t + Y^2_t = v^1(X^x_t) + v^2(X^x_t),\\
&Z^x_t = Z^1_t + Z^2_t = \nabla v^1(X^x_t)\sigma + \xi^2(X^x_t).
\end{aligned}
$$

Then (Y^x, Z^x, λ) is a solution of the EBSDE (3.7). □

Theorem 3.16 (Uniqueness of λ)**.** *If* (Y^1, Z^1, λ^1) *and* (Y^2, Z^2, λ^2) *denote two solutions of the EBSDE* (3.7) *in the class of solutions* (Y, Z, λ) *such that* $\forall t \geq 0, |Y_t| \leq C$*,* $\mathbb{P}$*-a.s. and* $Z \in L^2_{\mathscr{P},loc}(\Omega, L^2([0,\infty[;\mathbb{R}^{1\times d})$*, then*

$$
\lambda^1 = \lambda^2.
$$

Proof. See Theorem 4.6 in [7]. □

Theorem 3.17 (Uniqueness of solutions (Y, Z, λ))**.** *Uniqueness hold for solutions* (Y, Z, λ) *of the EBSDE* (3.7) *in the class of solutions such that there exits* $v : \overline{G} \to \mathbb{R}$ *continuous such that* $Y_s = v(X^x_s)$ *with* $v(0) = 0$*, and* $Z \in L^2_{\mathscr{P},loc}(\Omega, L^2([0,\infty[;\mathbb{R}^{1\times d})$*.*

Proof. Let $(Y^1 = v^1(X^x), Z^1, \lambda^1)$ and $(Y^2 = v^2(X^x), Z^2, \lambda^2)$ denote two solutions. Then from Theorem 3.16, we deduce that $\lambda^1 = \lambda^2 =: \lambda$.

Now, let us denote by $v : \overline{G} \to \mathbb{R}$, $v \in \mathscr{C}^2_{\text{lip}}(\overline{G})$ and solutions of the Helmholtz's equation for some $\alpha \in \mathbb{R}$

$$\begin{cases} \Delta v(x) - \alpha v(x) = 0, \\ \frac{\partial v(x)}{\partial n} + g(x) = 0. \end{cases}$$

Then, if we define $(Y_t := v(X_t^x), Z_t := \nabla v(X_t^x)\sigma)$ satisfy, for every $0 \leq t \leq T < +\infty$:

$$Y_t = Y_T + \int_t^T \left[-\mathscr{L}v(X_s^x)\right] \mathrm{d}s + \int_t^T g(X_s^x)\mathrm{d}K_s^x - \int_t^T Z_s \mathrm{d}W_s, \tag{3.10}$$

where

$$(\mathscr{L}v)(x) = \frac{1}{2}\operatorname{Tr}(\sigma^t\sigma\nabla^2 v) + \langle \widetilde{b}(x), \nabla v\rangle.$$

Therefore, $(\widehat{Y}_t^1 = Y_t^1 - v(X_t^x), \widehat{Z}_t^1 = Z_t^1 - {}^t\nabla v(X_t^x)\sigma)$ satisfies the EBSDE, $\forall 0 \leq s \leq T < +\infty$,

$$\widehat{Y}_t^1 = \widehat{Y}_T^1 + \int_t^T \left[\widehat{f}(X_s^x, \widehat{Z}_s^x) - \lambda\right] \mathrm{d}s - \int_t^T \widehat{Z}_s^1 \mathrm{d}W_s,$$

where $\forall x, z \in \mathbb{R}^d \times \mathbb{R}^{1\times d}$,

$$\widehat{f}(x,z) = f\left(x, z + {}^t\nabla v^1(x)\sigma\right) - \lambda - f\left(x, z + {}^t\nabla v^1(x)\sigma\right).$$

Then, let $(Y^{1,T,t,x}, Z^{1,T,t,x})$ be the solution of the following BSDE, $\forall s \in [t,T]$,

$$\widehat{Y}_s^{1,T,t,x} = (v^1 - v)(X_T^{t,x}) + \int_t^T \widehat{f}(X_s^x, Z_s^{1,T,t,x})\mathrm{d}s - \int_t^T Z_s^{1,T,t,x}\mathrm{d}W_s.$$

By uniqueness of solutions to BSDE, we deduce that

$$v^1(x) - v(x) = \widehat{Y}_0^{1,T,0,x}.$$

Now, we fix infinitely differentiable functions $\rho_\varepsilon : \mathbb{R}^d \to \mathbb{R}^+$ bounded together with their derivatives of all order, such that: $\int_{\mathbb{R}^d} \rho_\varepsilon(x)\mathrm{d}x = 1$ and

$$\operatorname{supp}(\rho_\varepsilon) \subset \left\{\xi \in \mathbb{R}^d : |\xi| \leq \varepsilon\right\}$$

where supp denotes the support. Then we define $\forall n \in \mathbb{N}$,

$$(F_n)_\varepsilon(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(y) F_n(x-y)\mathrm{d}y,$$
$$\widetilde{b}_\varepsilon(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(y).\widetilde{b}(x-y)\mathrm{d}y$$

Let us denote by $X^{t,x,n,\varepsilon}$ the solution of the following SDE, $\forall s \geq t$

$$X_s^{t,x} = x + \int_t^s \left(\widetilde{b}_\varepsilon + (F_n)_\varepsilon\right)(X_r^{t,x,n,\varepsilon})\mathrm{d}r + \int_t^s \sigma \mathrm{d}W_r,$$

and let $(Y^{1,T,t,x,n,\varepsilon}, Z^{1,T,t,x,n,\varepsilon})$ be the solution of the following BSDE, $\forall s \in [t,T]$

$$Y_s^{1,T,t,x,n,\varepsilon} = (v^1 - v)(X_s^{t,x,n,\varepsilon}) + \int_t^s \widehat{f}(X_r^{t,x,n,\varepsilon}, Z_r^{1,T,t,x,n,\varepsilon})\mathrm{d}r - \int_s^T Z_r^{1,T,t,x,n,\varepsilon}\mathrm{d}W_s.$$

Then by a stability result, (see for e.g. Lemma 2.3 of [4]), we deduce that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to+\infty} Y_0^{1,T,0,x,n,\varepsilon} = \widehat{Y}_0^{1,T,0,x} = v^1(x) - v(x). \tag{3.11}$$

Similarly, defining $(Y^{2,T,t,x}, Z^{2,T,t,x})$ and $(Y^{2,T,t,x,n,\varepsilon}, Z^{2,T,t,x,n,\varepsilon})$ in the same way, we deduce that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to+\infty} Y_0^{2,T,0,x,n,\varepsilon} = \widehat{Y}_0^{2,T,0,x} = v^2(x) - v(x).$$

Furthermore, by Theorem 4.2 in [16] (note that it can be extended to the case in which the terminal condition and the generator is continuous in x and with polynomial growth in x exactly by the same arguments exposed in the Theorem 4.2 in [8], the only difference coming from the fact that the authors of this last paper work in infinite dimension for the SDE), if we define $u^{1,T,n,\varepsilon}(t,x) := Y_t^{1,T,t,x,n,\varepsilon}$, then $(x \mapsto u^{1,T,n,\varepsilon}(t,x))$ is continuously differentiable for all $t \in [0,T[$, and $\forall s \in [t,T[$,

$$Z_s^{1,T,t,x,n,\varepsilon} = {}^t\nabla u^{1,T,n,\varepsilon}(s, X_s^{t,x,n,\varepsilon})\sigma.$$

Similarly, we define $u^{2,T,n,\varepsilon}(t,x) := Y_t^{2,T,t,x,n,\varepsilon}$ and then

$$Z_s^{2,T,t,x,n,\varepsilon} = {}^t\nabla u^{2,T,n,\varepsilon}(s, X_s^{t,x,n,\varepsilon})\sigma.$$

Therefore, taking $t = 0$, $\forall T > 0$

$$\begin{aligned} u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) &= (v^1 - v^2)(X_T^{x,n\varepsilon}) - \int_0^T (Z_s^{1,T,t,x,n,\varepsilon} - Z_s^{2,T,t,x,n,\varepsilon})\mathrm{d}W_s \\ &\quad + \int_0^T \left[\widehat{f}(X_s^{x,n,\varepsilon}, Z_s^{1,T,t,x,n,\varepsilon}) - \widehat{f}(X_s^{x,n,\varepsilon}, Z_s^{2,T,t,x,n,\varepsilon})\right]\mathrm{d}s \\ &= (v^1 - v^2)(X_T^{x,n\varepsilon}) - \int_0^T (Z_s^{1,T,t,x,n,\varepsilon} - Z_s^{2,T,t,x,n,\varepsilon})(-\beta(s, X_s^{x,n,\varepsilon})\mathrm{d}s + \mathrm{d}W_s), \end{aligned}$$

where

$$\beta^T(s,x) = \begin{cases} \frac{f(x,{}^t\nabla u^{1,T,n,\varepsilon}(X_s^{x,n,\varepsilon})\sigma) - f(x,{}^t\nabla u^{2,T,n,\varepsilon}(X_s^{x,n,\varepsilon})\sigma)}{|{}^t\nabla u^{1,T,n,\varepsilon}(X_s^{x,n,\varepsilon})\sigma - {}^t\nabla u^{2,T,n,\varepsilon}(X_s^{x,n,\varepsilon})\sigma|^2}\mathbb{1}_{t<T}, & \text{if } \nabla u^{1,T,n,\varepsilon}(t,x) \neq \nabla u^{2,T,n,\varepsilon}(t,x), \\ 0, & \text{otherwise.} \end{cases}$$

The process β^T is progressively measurable and bounded, therefore, we can apply Girsanov's Theorem to obtain that there exits a new probability measure $\mathbb{Q}^T$ equivalent to $\mathbb{P}$ under which $(W_t - \int_0^t \beta(s, X_s^{x,n,\varepsilon})\mathrm{d}s)_{t\in[0,T]}$ is a Brownian motion. Therefore, denoting by $E^{\mathbb{Q}^T}$ the expectation with respect to the probability $\mathbb{Q}^T$,

$$\begin{aligned} u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) &= \mathbb{E}^{\mathbb{Q}^T}\left[(v^1 - v^2)(X_T^{x,n\varepsilon})\right] \\ &= \mathscr{P}_T[v^1 - v^2](x), \end{aligned}$$

where $\mathscr{P}_t$ is the Kolmogorov semigroup associated to the following SDE, $\forall t \geq 0$,

$$U_t^x = x + \int_0^t \left(\widetilde{b}_\varepsilon + (F_n)_\varepsilon\right)(U_s^x)\mathrm{d}s + \int_0^t \sigma\beta(s, U_s^x)\mathrm{d}s + \int_t^s \sigma \mathrm{d}W_s.$$

By Corollary 3.4 and Remark 3.6, we deduce that

$$|u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) - (u^{1,T,n,\varepsilon}(0,0) - (u^{2,T,n,\varepsilon}(0,0))| \leq Ce^{-\hat{\eta}T}.$$

Therefore, thanks to (3.11),

$$|v^1(x) - v^2(x) - (v^1(0) - v^2(0))| \leq Ce^{-\hat{\eta}T}.$$

Therefore, since $v^1(0) = v^2(0) = 0$, letting $T \to +\infty$ implies that

$$v^1(x) = v^2(x), \forall x \in \overline{G}.$$

□

We recall the link of such EBSDE with ergodic PDE. Let us consider the following ergodic semilinear PDE for which the unknown is a pair (v, λ):

$$\begin{cases} \mathscr{L}v(x) + f(x, \nabla v(x)\sigma) - \lambda = 0, & \forall x \in G, \\ \frac{\partial v(x)}{\partial n} + g(x) = 0, & \forall x \in \partial G. \end{cases} \tag{3.12}$$

Lemma 3.18 (Existence of ergodic viscosity solutions)**.** *Assume that Hypotheses 3.2, 3.3 and 3.7 hold true then the solution (v, λ) of Lemma 3.15 is a viscosity solution of* (3.12).

Proof. Note that v is continuous by Lemma 3.15. The proof of this result is very classical and can be easily adapted from [21]. □

Lemma 3.19 (Uniqueness of ergodic viscosity solutions)**.** *Assume that Hypothesis 3.2, 3.3, 3.6 and 3.7 hold true. Then uniqueness holds for viscosity solutions (v, λ) of* (3.12) *in the class of (continuous) viscosity solutions such that $\exists a \in \mathbb{R}^d$, $v^1(a) = v^2(a)$.*

Proof. Let (v^1, λ^1) and (v^2, λ^2) be two continuous viscosity solutions of (3.12). First we show that $\lambda^1 = \lambda^2$. Let us fix $0 \leq t < T < +\infty$, and let us consider $(Y^{1,T,t,x}, Z^{1,T,t,x})$ the solution of the following BSDE in finite horizon, $\forall s \in [t, T]$,

$$Y_s^{1,T,t,x} = v^1(X_T^{t,x}) + \int_s^T [f(X_r^{t,x}, Z_r^{1,T,t,x}) - \lambda^1]\mathrm{d}r + \int_s^T g(X_r^{t,x})\mathrm{d}K_r^{t,x} - \int_s^T Z_r^{1,T,t,x}\mathrm{d}W_r.$$

And we define $(Y^{2,T,t,x}, Z^{2,T,t,x})$ similarly, replacing λ^1 by λ^2. By Lemma 3.10, we deduce that $u^{1,T}(t,x) = Y_t^{1,T,t,x}$ is a viscosity solution of (3.6). Since v^1 is also a viscosity solution of (3.6), it follows by Lemma 3.11 that $\forall t \in [0,T], \forall x \in \overline{G}$,

$$u^{1,T}(t,x) = v^1(x).$$

Of course, similarly, $\forall t \in [0,T]$, $\forall x \in \overline{G}$,

$$u^{2,T}(t,x) = v^2(x).$$

Then, taking $t = 0$, $\forall T > 0$,

$$\begin{aligned} u^{1,T}(0,x) - u^{2,T}(0,x) &= v^1(X_T^x) - v^2(X_T^x) + \int_0^T [f(X_s^x, Z_s^{1,T,x}) - f(X_s^x, Z_s^{2,T,x})]\mathrm{d}s \\ &\quad + (\lambda^2 - \lambda^1)T - \int_0^T Z_s^{1,T,x} - Z_s^{2,T,x}\mathrm{d}W_s \\ &= v^1(X_T^x) - v^2(X_T^x) + \int_0^T (Z_s^{1,T,x} - Z_s^{2,T,x})(\beta_s + \mathrm{d}W_s), \end{aligned}$$

where, $\forall s \in [0,T]$

$$\beta_s = \begin{cases} \frac{(f(X_s^x, Z_s^{1,T,x}) - f(X_s^x, Z_s^{2,T,x}))^t (Z_s^{1,T,x} - Z_s^{2,T,x})}{|Z_s^{1,T,x} - Z_s^{2,T,x}|^2}, & \text{if } Z_s^{1,T,x} \neq Z_s^{2,T,x} \\ 0, & \text{otherwise.} \end{cases}$$

Since $(\beta_s)_{s\in[0,T]}$ is a measurable and bounded process, by the Girsanov's theorem, there exists a new probability $\mathbb{Q}^T$ equivalent to $\mathbb{P}$ under which $(W_t + \int_0^t \beta_s \mathrm{d}s)_{t\in[0,T]}$. Taking the expectation with respect to this new probability, we get

$$\frac{u^{1,T}(0,x) - u^{2,T}(0,x)}{T} = \frac{\mathbb{E}^{Q^T}(v^1(X_T^x) - v^2(X_T^x))}{T} + \lambda^2 - \lambda^1.$$

Since v^1 and v^2 are continuous and therefore bounded on $\overline{G}$, letting $T \to +\infty$ implies that

$$\lambda^1 = \lambda^2.$$

Then, since $\lambda^1 = \lambda^2$, it implies that v^1 and v^2 are viscosity solutions of the parabolic PDE:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathscr{L}u(t,x) + f(x, \nabla u(t,x)\sigma) - \lambda = 0, & \forall x \in G, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall x \in \partial G. \end{cases}$$

Then by the strong maximum principle for this parabolic equation (see Proposition 2.1 in [5]; note that v^1 and v^2 are assumed to be continuous on $\overline{G}$ and therefore $w = v^1 - v^2$ is bounded on $\overline{G}$), we derive that there exists $c \in \mathbb{R}$ such that $v^1 - v^2 = c$. Since $v^1(a) = v^2(a)$, it implies that $c = 0$, which concludes the proof. □

Remark 3.20. Uniqueness for viscosity solutions of elliptic PDE (3.12) implies uniqueness for Markovian solutions $(Y^x = v(X^x), Z^x, \lambda)$ of (3.7). Indeed, if $(Y^{1,x} = v^1(X^x), Z^{1,x}, \lambda^1)$ and $(Y^{2,x} = v^2(X^x), Z^{2,x}, \lambda^2)$ are Markovian solutions of (3.7) then, defining v^1 and v^2 are continuous by applying the same method as in the proof of Lemma 3.15. Then it follows that (v^1, λ) and (v^2, λ^2) are viscosity solutions of (3.12). By uniqueness of viscosity solutions, it follows that $v^1 = v^2$ and $\lambda^1 = \lambda^2$. Then remarking that $Y^{1,x} = v^1(X^x)$ and $Y^{2,x} = v^2(X^x)$ implies that $Y^{1,x} = Y^{2,x}$. Therefore $Z^{1,x} = Z^{2,x}$ by Itô's formula applied to $|Y_t^{1,x} - Y_t^{2,x}|^2$.

4 Large time behaviour

4.1 First behaviour

We recall that $(Y_s^{T,x}, Z_s^{T,x})_{s\geq 0}$ denotes the solution of the finite horizon BSDE (3.5) with $t = 0$ and that $(Y_s^x, Z_s^x, \lambda)_{s\geq 0}$ denotes the solution of the EBSDE (3.7).

Theorem 4.1. *Assume that Hypotheses 3.2, 3.3 and 3.4 hold true (path dependent case), then,* $\forall x \in \overline{G}$, $\forall T > 0$*:*

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \frac{C}{T}.$$

In particular,

$$\frac{Y_0^{T,x}}{T} \underset{T \to +\infty}{\longrightarrow} \lambda,$$

uniformly in $\overline{G}$*.*
Assume that Hypotheses 3.2, 3.3, 3.5 and 3.6 hold true (Markovian case). Then, $\forall x \in \overline{G}$, $\forall T > 0$*:*

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \frac{C}{T}.$$

i.e.

$$\left| \frac{u(T,x)}{T} - \lambda \right| \leq \frac{C}{T},$$

where u *is the viscosity solution of* (1.1)*. In particular,*

$$\frac{u(T,x)}{T} = \frac{Y_0^{T,x}}{T} \underset{T \to +\infty}{\longrightarrow} \lambda,$$

uniformly in $\overline{G}$*.*

Proof. The proof is identical to the proof of Theorem 4.1 in [12]. Note that the proof is even simpler since we work with a bounded subset G of $\mathbb{R}^d$ and then for any probability $\mathbb{Q}^T$, $\mathbb{E}^{\mathbb{Q}^T}[\sup_{0 \leq t \leq T} |X_t|^\mu] \leq C$, where C depends only on G and μ. Note that the proof make appears an important result

$$|u_T(0,x) - \lambda T - v(x)| \leq C, \tag{4.1}$$

which will be useful for what follows. Finally note that for the Markovian case, Hypothesis 3.6 is added in order to obtain uniqueness of viscosity solutions of (1.1). □

4.2 Second and third behaviour

In this section we introduce a new set of hypothesis without loss of generality. Note that it is the same as Hypothesis 3.5 but with modified assumptions for b. However we write it again for reader's convenience. The remark immediately following this new set of hypothesis justify the fact that there is no loss of generality. Let us denote by $(Y_s^{t,x}, Z_s^{t,x}, \lambda)_{s \geq 0}$ the solution of the EBSDE (3.7) when X^x is replaced by $X^{t,x}$. We recall that this solution satisfies

$$Z_s^{t,x} = \nabla v^1(X_s^{t,x})\sigma + Z_s^2. \tag{4.2}$$

Hypothesis 4.1. There exists $C > 0$ such that

1. $b : \mathbb{R}^d \to \mathbb{R}^d$ is $\mathscr{C}^1$ Lipschitz and dissipative (i.e. $\exists \eta > 0$ such that $\forall x, y \in \mathbb{R}^d$, $\langle b(x) - b(y), x - y\rangle \leq -\eta |x-y|^2$),
2. $\xi^T = h(X_T^x)$, where $h : \overline{G} \to \mathbb{R}$ is continuous,
3. $\forall x \in \overline{G}$, $\forall z, z' \in \mathbb{R}^{1\times k}$, $|f(x,z) - f(x,z')| \leq C|z - z'|$,
4. $\forall z \in \mathbb{R}^{1\times k}$, $f(\cdot, z)$ is continuous,
5. $g \in \mathscr{C}^1_{\text{lip}}(\overline{G})$.

Remark 4.2. Note that asking b to be $\mathscr{C}^1$ Lipschitz and dissipative is not restrictive. Indeed, let us consider $b : \overline{G} \to \mathbb{R}^d$ only Lipschitz. Let us recall that the purpose of this paper is to study the large time behaviour of the viscosity solution of

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L} u(t,x) + f(x, \nabla u(t,x) G), & \forall (t,x) \in \mathbb{R}_+ \times \overline{G}, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in \mathbb{R}_+ \times \partial G \\ u(0,x) = h(x), & \forall x \in G. \end{cases}$$

Now, we define, $\forall x \in \mathbb{R}^d$, $\widetilde{b}(x) := -x + (b(\Pi(x)) + \Pi(x))$. Note that $\widetilde{b}$ is equal to b on $\overline{G}$. Furthermore,

$$\langle \widetilde{b}(x), \nabla u(t,x)\rangle + f(x, \nabla u(t,x)\sigma) = \langle -x, \nabla u(t,x)\rangle + \widetilde{f}(x, \nabla u(t,x)\sigma),$$

where $\widetilde{f}(x,z) = f(x,z) + \langle b(\Pi(x)) + \Pi(x), z\sigma^{-1}\rangle$ is a continuous function in x and Lipschitz in z. Therefore, under our assumptions, we can always consider the case b being $\mathscr{C}^1$ Lipschitz and dissipative by replacing b by $(x \mapsto -x)$ and f by $\widetilde{f}$ if necessary. Note that it only changes the stochastic problem, not the deterministic one.

Theorem 4.3. *Assume that Hypotheses 3.2, 3.3, 3.6 and 4.1 hold true. Then there exists $L \in \mathbb{R}$ such that,*

$$\forall x \in \overline{G},\ Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to+\infty}{\longrightarrow} L,$$

i.e.

$$\forall x \in \overline{G},\ u(T,x) - \lambda T - v(x) \underset{T\to+\infty}{\longrightarrow} L,$$

where u is the viscosity solution of (1.1) *and v is the viscosity solution of* (3.12). *Furthermore the following rate of convergence holds*

$$|Y_0^{T,x} - \lambda T - Y_0^x - L| \leq Ce^{-\eta T},$$

i.e.

$$|u_T(0,x) - \lambda T - v(x) - L| \leq Ce^{-\eta T}.$$

Proof. Let us start by defining

$$u_T(t,x) := Y_t^{T,t,x}$$
$$w_T(t,x) := u_T(t,x) - \lambda(T-t) - v(x).$$

We recall that $Y_s^{T,t,x} = u_T(s, X_s^{t,x})$ and that $Y_s^x = v(X_s^x)$.

Note that $(x \mapsto w_T(0,x))$ is continuous and bounded uniformly in T by (4.1). Therefore one can extend the definition of $w_T(0,x)$ to the whole $\mathbb{R}^d$ into a continuous and uniformly bounded in T function by setting $w_{T,\mathbb{R}^d}(0,x) := w_T(0,\Pi(x))$ where Π is the projection on $\overline{G}$.

We recall that for all $T, S \geq 0$, u_T is the unique solution of

$$\begin{cases} \frac{\partial u_T(t,x)}{\partial t} + \mathscr{L}u_T(t,x) + f(x, \nabla u_T(t,x)G) = 0, & \forall (t,x) \in [0,T] \times G, \\ \frac{\partial u_T(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in [0,T] \times \partial G, \\ u_T(T,x) = h(x), & \forall x \in G, \end{cases}$$

and that u_{T+S} is the unique solution of

$$\begin{cases} \frac{\partial u_{T+S}(t,x)}{\partial t} + \mathscr{L}u_{T+S}(t,x) + f(x, \nabla u_{T+S}(t,x)\sigma) = 0, & \forall (t,x) \in [0,T+S] \times G, \\ \frac{\partial u_{T+S}(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in [0,T+S] \times \partial G, \\ u_{T+S}(T+S,x) = h(x), & \forall x \in G. \end{cases}$$

By uniqueness of viscosity solutions, it implies that $u_T(0,x) = u_{T+S}(S,x)$, for all $x \in \overline{G}$, and then,

$$w_T(0,x) = w_{T+S}(S,x), \forall x \in \overline{G}. \tag{4.3}$$

For every $T \geq t$, the process $(w_T(s, X_s^{t,x}))_{s \in [t,T]}$ satisfies the following BSDE in infinite horizon, $\forall t \leq s \leq T < +\infty$,

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T, X_T^{t,x}) + \int_s^T [f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x})]\mathrm{d}r - \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r \\ &= g(X_T^{t,x}) - v(X_T^{t,x}) + \int_s^T [f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x})]\mathrm{d}r - \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r. \end{aligned}$$

Since we do not have a basic coupling estimate Lemma for the reflected process $X^{t,x}$, we will use an approximation procedure. We fix infinitely differentiable functions $\rho_\varepsilon : \mathbb{R}^d \to \mathbb{R}^+$ bounded together with their derivatives of all order, such that: $\int_{\mathbb{R}^d} \rho_\varepsilon(x)\mathrm{d}x = 1$ and

$$\mathrm{supp}(\rho_\varepsilon) \subset \left\{\xi \in \mathbb{R}^d : |\xi| \leq \varepsilon\right\}$$

where supp denotes the support. Then we define $\forall n \in \mathbb{N}$,

$$(F_n)_\varepsilon(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(y) F_n(x-y)\mathrm{d}y.$$

It is well known that $(F_n)_\varepsilon$ is $\mathscr{C}^\infty$. Furthermore, $(F_n)_\varepsilon$ is still 0-dissipative. Let $(X_s^{t,x,n})_{s \geq t}$ be the solution of

$$X_s^{t,x,n,\varepsilon} = x + \int_t^s (b + (F_n)_\varepsilon)(X_r^{t,x,n,\varepsilon})\mathrm{d}r + \int_t^s \sigma \mathrm{d}W_r, \quad \forall s \geq t$$

and $(Y_s^{2,t,x,n,\varepsilon}, Z_s^{2,t,x,n,\varepsilon})_{s \geq t}$ be the solution of the following monotonic BSDE in infinite horizon, $for all t \leq s \leq T < +\infty$

$$Y_s^{2,t,x,n,\varepsilon} = Y_T^{2,t,x,n,\varepsilon} + \int_s^T \left[f(X_r^{t,x,n,\varepsilon}, Z_r^{2,t,x,n,\varepsilon}) - \alpha Y_r^{2,t,x,n,\varepsilon}\right]\mathrm{d}r - \int_s^T Z_r^{2,t,x,n,\varepsilon}\mathrm{d}W_r.$$

By Theorem 4.4 in [17], there exists sequences $\varepsilon_m \underset{m\to+\infty}{\longrightarrow} 0$, $\beta(n) \underset{p\to+\infty}{\longrightarrow} +\infty$ and $\alpha_k \underset{k\to+\infty}{\longrightarrow} 0$ such that for all $T \geq t$,

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\lim_{m\to+\infty}\mathbb{E}\int_t^T \left|Z_s^{2,t,x,\alpha_k,\beta(n),\varepsilon_m} - Z_s^2\right|^2 \mathrm{d}s = 0. \tag{4.4}$$

In what follows, we will use the following notation. If $q^{\alpha,n,\varepsilon}$ denotes a function depending of the parameters α, n and ε, then

$$\lim_{\alpha,n,\varepsilon} q^{\alpha,n,\varepsilon} := \lim_{k\to+\infty}\lim_{n\to+\infty}\lim_{m\to+\infty} q^{\alpha_k,\beta(n),\varepsilon_m}.$$

Let us come back to the equation (4.2). Then if we define, for all $s \geq t$,

$$\widetilde{Z}_s^{t,x,\alpha,n} := (\nabla v^1)_{\mathbb{R}^d}(X_s^{t,x,n,\varepsilon})\sigma + Z_s^{2,t,x,\alpha,n,\varepsilon},$$

by the dominated convergence theorem and thanks to (4.4), for all $T \geq t$

$$\lim_{\alpha,n,\varepsilon}\mathbb{E}\int_t^T |\widetilde{Z}_s^{t,x,\alpha,n,\varepsilon} - Z_s^{t,x}|^2\mathrm{d}s = 0. \tag{4.5}$$

Note that by Theorem (4.2) in [16], if we define $v^{2,\alpha,n}(x) := Y_0^{x,\alpha,n,\varepsilon}$, then v^2 is $\mathscr{C}^1$ and $\forall s \geq t$

$$Z_s^{2,t,x,\alpha,n,\varepsilon} = \nabla v^{2,\alpha,n}(X_s^{t,x,n,\varepsilon})\sigma.$$

Therefore, we have the following representation, $\forall s \geq t$,

$$\begin{aligned}\widetilde{Z}_s^{t,x,\alpha,n,\varepsilon} &= \nabla(v^1)_{\mathbb{R}^d}(X_s^{t,x,n,\varepsilon})\sigma + \nabla v^{2,\alpha,n,\varepsilon}(X_s^{t,x,n,\varepsilon})\sigma \\ &=: \nabla\widetilde{v}^{\alpha,n,\varepsilon}(X_s^{t,x,n,\varepsilon})\sigma \end{aligned}\tag{4.6}$$

Let us denote by $(\overline{Y}_s^{T,t,x,\alpha,n,\varepsilon}, \overline{Z}_s^{T,t,x,\alpha,n,\varepsilon})_{s\geq t}$ the solution of the following BSDE in finite horizon, $\forall s \in [t,T]$

$$\begin{aligned}\overline{Y}_s^{T,t,x,\alpha,n,\varepsilon} &= w_{T,\mathbb{R}^d}(T, X_{T'}^{t,x,n,\varepsilon}) + \int_s^T \left[f(X_r^{t,x,n,\varepsilon}, \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} - \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon})\right]\mathrm{d}r \\ &\qquad - \int_s^T \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon}\mathrm{d}W_r \\ &= (h+v)_{\mathbb{R}^d}(X_T^{t,x,n,\varepsilon}) + \int_s^T \widetilde{f}^{\alpha,n,\varepsilon}(s, \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon})\mathrm{d}r - \int_s^T \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon}\mathrm{d}W_r.\end{aligned}$$

where, for all $z \in \mathbb{R}^{1\times d}$

$$\widetilde{f}^{\alpha,n,\varepsilon}(r,z) := f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, z - \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}).$$

We define, for all $z \in \mathbb{R}^{1\times d}$,

$$\widetilde{f}(r,z) := f_{\mathbb{R}^d}(X_r^{t,x}, z - Z_r^{t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{t,x}).$$

The assumption (A2) of [4] is satisfied, indeed:

$$\forall z_1, z_2 \in \mathbb{R}^{1\times d}, |\widetilde{f}^{\alpha,n,\varepsilon}(s,z_1) - \widetilde{f}(s,z_2)| \leq C|z_1 - z_2|,$$
$$\mathbb{E}\left[\int_t^T |\widetilde{f}^{\alpha,n,\varepsilon}(s,0)|^2 \mathrm{d}s\right] + \sup_{t\leq s\leq T} \mathbb{E}\left[|X_s^{x,n}|^2\right] \leq C.$$

Now we show that the assumption (A3) of [4] is satisfied. We have, $\forall s \in [t,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\int_s^T |\widetilde{f}^{\alpha,n,\varepsilon}(r, Z_r^{t,x}) - \widetilde{f}(r, Z_r^{t,x})|^2 \mathrm{d}r\right] \\
&= \mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{t,x} - \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x},0) + f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{t,x})|^2 \mathrm{d}r\right] \\
&\leq 2\mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{t,x} - \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x},0)|^2 \mathrm{d}r\right] \\
&\quad + 2\mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{t,x})|^2 \mathrm{d}r\right] \\
&\leq C\mathbb{E}\left[\int_s^T (|\widetilde{Z}_r^{t,x,\alpha,n,\varepsilon} - Z_r^{t,x}|^2 + |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon},0) - f_{\mathbb{R}^d}(X_r^{t,x},0)|^2) \mathrm{d}r\right] \\
&\quad + C\mathbb{E}\left[\int_s^T (|\widetilde{Z}_r^{t,x,\alpha,n,\varepsilon} - Z_r^{t,x}|^2 + |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{t,x})|^2) \mathrm{d}r\right].
\end{aligned}$$

Then, with the help of (4.5), since $f_{\mathbb{R}^d}(\cdot, z)$ is continuous for all $z \in \mathbb{R}^{1\times d}$, since $\lim_n \mathbb{E}\sup_{t\leq s\leq T} |X_s^{t,x,n,\varepsilon} - X_s^{t,x}|^2 = 0$ and since $|f_{\mathbb{R}^d}(x,0)|$ is bounded, we obtain by the dominated convergence theorem

$$\lim_{\alpha,n,\varepsilon} \mathbb{E}\left[\int_s^T |\widetilde{f}^{\alpha,n,\varepsilon}(r, Z_r^{t,x}) - \widetilde{f}(r, Z_r^{t,x})|^2 \mathrm{d}r\right] = 0.$$

Furthermore, by continuity and boundedness of $(g+v)_{\mathbb{R}^d}$, we deduce by the dominated convergence theorem:

$$\lim_{\alpha,n,\varepsilon} \mathbb{E}\left[|(g+v)_{\mathbb{R}^d}(X_T^{t,x,n}) - (g+v)_{\mathbb{R}^d}(0, X_T^{t,x})|^2\right].$$

Thus assumption (A3) of [4] is satisfied. Therefore, by Lemma 2.3 of [4], we obtain:

$$\lim_{\alpha,n,\varepsilon} \overline{Y}_t^{T,t,x,\alpha,n,\varepsilon} = w_{T,\mathbb{R}^d}(t,x).$$

Thus, $\forall x \in \overline{G}$,

$$\lim_{\alpha,n,\varepsilon} \overline{Y}_t^{T,t,x,\alpha,n,\varepsilon} = w_T(t,x). \tag{4.7}$$

Now we are in force to apply the method exposed in [12] for the quantity $\overline{Y}_0^{T,0,x,\alpha,n,\varepsilon}$ with slight modifications. Let us write the full proof for reader convenience.

First we establish the following Lemma.

Lemma 4.4. *Under the hypotheses of Theorem 4.3,* $\exists C > 0$, $\forall x, y \in \overline{G}$, $\forall T > 0$, $\forall 0 < T' \leq T$, $\exists C_{T'}$,

$$
\begin{aligned}
&|\overline{Y}_0^{T,x,\alpha,n,\varepsilon}| \leq C,\\
&|\nabla_x \overline{Y}_0^{T,x,\alpha,n,\varepsilon}| \leq \frac{C_{T'}}{\sqrt{\overline{T'}}},\\
&|\overline{Y}_0^{T,x,\alpha,n,\varepsilon} - \overline{Y}_0^{T,y,\alpha,n,\varepsilon}| \leq Ce^{-\hat{\eta}T}.
\end{aligned}
$$

We stress the fact that C *depends only* η, σ, $\overline{G}$. *The constants* $C_{T'}$ *depends only on the same constant and* T'.

Proof. The first estimate is a direct consequence a the Girsanov's theorem. Indeed, we have,

$$
\begin{aligned}
\overline{Y}_0^{T,x,\alpha,n,\varepsilon} &= (g+v)_{\mathbb{R}^d}(X_T^{x,n,\varepsilon}) + \int_0^T \left[f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \overline{Z}_r^{T,x,\alpha,n,\varepsilon} - \widetilde{Z}_r^{x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \widetilde{Z}_r^{x,\alpha,n,\varepsilon}) \right] \mathrm{d}r\\
&\qquad - \int_0^T \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} \mathrm{d}W_r\\
&= w_T(0, X_T^{t,x,n,\varepsilon}) + \int_0^T \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon}(\beta_r \mathrm{d}r + \mathrm{d}W_r),
\end{aligned}
$$

where

$$
\beta_r := \begin{cases} \frac{(f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \overline{Z}_r^{T,x,\alpha,n,\varepsilon} - \widetilde{Z}_r^{x,\alpha,n}) - f_{\mathbb{R}^d}(X_r^{x,n}, \widetilde{Z}_r^{x,\alpha,n,\varepsilon}))^t (\overline{Z}_r^{T,x,\alpha,n,\varepsilon})}{|\overline{Z}_r^{T,x,\alpha,n,\varepsilon}|^2}, & \text{if } \overline{Z}_r^{T,x,\alpha,n,\varepsilon} \neq 0,\\ 0 & \text{otherwise.} \end{cases}
$$

Since β is a measurable and bounded process, there exists a new probability equivalent to $\mathbb{P}$, $\mathbb{Q}^{T,\alpha,n,\varepsilon}$ under which $(W_s + \int_0^s \beta_r \mathrm{d}r)_{r\in[0,T]}$ is a Brownian motion. Therefore, thanks to estimate (4.1):

$$
\begin{aligned}
|\overline{Y}_0^{T,x,\alpha,n,\varepsilon}| &\leq \mathbb{E}^{Q^{T,\alpha,n,\varepsilon}} |w_{T-T'}(0, X_{T'}^{x,n,\varepsilon})|\\
&\leq C.
\end{aligned}
$$

Let us establish the second and third inequality of the lemma. First we notice that thanks to equation (4.3), $\forall 0 \leq T' < T$, $\forall s \in [t, T']$,

$$
\begin{aligned}
\overline{Y}_s^{T,t,x,\alpha,n,\varepsilon} &= w_{T,\mathbb{R}^d}(T', X_{T'}^{t,x,n,\varepsilon}) + \int_s^{T'} \left[f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} - \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) \right] \mathrm{d}r\\
&\qquad - \int_s^{T'} \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} \mathrm{d}W_r\\
&= w_{T-T',\mathbb{R}^d}(0, X_{T'}^{t,x,n,\varepsilon}) + \int_s^{T'} \widetilde{f}^{\alpha,n,\varepsilon}(s, \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon}) \mathrm{d}r - \int_s^{T'} \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} \mathrm{d}W_r
\end{aligned}
$$

We recall that we have the following representation:

$$
\widetilde{Z}_s^{t,x,\alpha,n,\varepsilon} = \nabla \widetilde{v}(X_s^{t,x,n,\varepsilon})\sigma.
$$

Furthermore, by Theorem 4.2 in [16] (note that it can be extended to the case in which the terminal condition and the generator is continuous in x and with polynomial growth in x exactly

by the same arguments exposed in the Theorem 4.2 in [8], the only difference coming from the fact that the authors of this last paper work in infinite dimension for the SDE), if we define $\overline{u}_T^{\alpha,n,\varepsilon}(t,x) := \overline{Y}_t^{T,t,x,\alpha,n,\varepsilon}$, then $(x \mapsto \overline{u}_T^{\alpha,n,\varepsilon}(t,x))$ is continuously differentiable for all $t \in [0,T[$ and $\forall s \in [t,T[$,

$$\overline{Z}_s^{t,x,\alpha,n,\varepsilon} = \nabla \overline{u}_T^{\alpha,n,\varepsilon}(s, X_s^{t,x,n,\varepsilon})\sigma.$$

Therefore, we can apply the same method as exposed in [12] to obtain the second and third estimate. □

Let us conclude the proof. From Lemma 4.4, we derive that, thanks to equation (4.7) that, $\exists C > 0$, $\forall x, y \in \overline{G}$, $\forall T > 0$, $\forall 0 < T' \leq T$, $\exists C_{T'}$,

$$|w_T(0,x)| \leq C \tag{4.8}$$

$$|w_T(0,x) - w_T(0,y)| \leq \frac{C_{T'}}{\sqrt{T'}}|x-y| \tag{4.9}$$

$$|w_T(0,x) - w_T(0,y)| \leq Ce^{-\hat{\eta}T} \tag{4.10}$$

By estimate (4.8) one can construct, by a diagonale procedure, a sequence $(T_i)_i \nearrow +\infty$ such that for a function $w : D \to \mathbb{R}$ defined on a countable dense subset D of $\overline{G}$ the following holds

$$\forall x \in D, \lim_{i \to +\infty} w_{T_i}(0,x) = w(x).$$

By using inequality (4.9) it is possible to extend w to the whole $\overline{G}$. Indeed if $x \notin D$, then there exists $(x_p)_{p\in\mathbb{N}} \in D^{\mathbb{N}}$ such that $x_p \to x$. Thus if we set $w(x) := \lim_{p\to+\infty} w(x_p)$, one can check that $w_{T_i}(x) \underset{i\to+\infty}{\longrightarrow} w(x)$ for any $x \in \mathbb{R}^d$. Now, by the third estimate in Lemma 4.4, we deduce that $w(x)$ is a constant function, that is there exits $L \in \mathbb{R}$ such that $\forall x \in \overline{G}$, $w(x) = L$. Since $\overline{G}$ is compact, $\{w_T(0,\cdot); T > 1\}$ is a relatively compact subspace of the space of continuous function $\overline{G} \to \mathbb{R}$ for the uniform distance (denoted by $(\mathscr{C}(\overline{G},\mathbb{R}), ||\cdot||_\infty)$) thanks to estimates (4.8) and (4.9). Note now that L is an accumulation point of $\{w_T(0,\cdot); T > 1\}$ since w_{T_i} converges uniformly toward L by estimate (4.9).

Therefore, if we show that $\{w_T(0,\cdot); T > 1\}$ admits only one accumulation point, it will imply that

$$\lim_{T\to+\infty} w_T(0,x) = L.$$

Now we claim that the accumulation point is unique. Let us assume that there exists another subsequence $(T_i')_{i\in\mathbb{N}} \nearrow +\infty$ and $w_2(\cdot) \in \mathscr{C}(\overline{G},\mathbb{R})$ such that

$$||w_{T_i'}(0,x) - w_2||_\infty \underset{i\to+\infty}{\longrightarrow} 0.$$

Then, by estimate (4.10), there exists $L_2 \in \mathbb{R}$ such that $\forall x \in \overline{G}$, $w_2(x) = L_2$. In other words,

$$\lim_i w_{T_i'}(0,x)(0,x) = L_2.$$

Let us write, $\forall x \in \overline{G}$, $\forall T, S > 0$,

$$\begin{aligned}\overline{Y}_0^{T+S,x,\alpha,n,\varepsilon} &= w_{T+S,\mathbb{R}^d}(S, X_S^{x,n,\varepsilon}) - \int_0^S \overline{Z}_r^{T+S,x,\alpha,n,\varepsilon} \mathrm{d}W_r \\ &\quad + \int_0^S \left[f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \overline{Z}_r^{T+S,x,\alpha,n,\varepsilon} - \widetilde{Z}_r^{x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \widetilde{Z}_r^{x,\alpha,n,\varepsilon}) \right] \mathrm{d}r \\ &= w_{T+S,\mathbb{R}^d}(S, X_S^{x,n,\varepsilon}) + \int_0^S \overline{Z}_r^{T+S,x,\alpha,n,\varepsilon} \mathrm{d}\widetilde{W}_r^{T,S},\end{aligned}$$

where

$$\widetilde{W}_t^{T,S} = -\int_0^t \beta^{T,S,\alpha,n,\varepsilon}(s, X_s^{x,n,\varepsilon}) \mathrm{d}s + W_t, \quad \forall t \in [0, S],$$

and where

$$\beta^{T,S,\alpha,n,\varepsilon}(t,x) = \begin{cases} \frac{f_{\mathbb{R}^d}(x, \nabla \overline{u}_{T+S}^{\alpha,n,\varepsilon}(t,x)\sigma - \nabla \widetilde{v}^{\alpha,n,\varepsilon}(x)\sigma) - f_{\mathbb{R}^d}(x, \nabla \widetilde{v}^{\alpha,n,\varepsilon}(x)\sigma)^t (\nabla \overline{u}_{T+S}^{\alpha,n,\varepsilon}(t,x)\sigma)}{|\nabla \overline{u}_{T+S}^{\alpha,n,\varepsilon}(t,x)\sigma|^2} \mathbb{1}_{t \le S}, & \text{if} \nabla \overline{u}_{T+S}^{\alpha,n,\varepsilon}(t,x)\sigma \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Taking the expectation with respect to the probability $\mathbb{Q}^{T,\alpha,n}$ under which $\widetilde{W}^{T,S}$ is a Brownian motion, we get

$$\begin{aligned}\overline{Y}_0^{T+S,x,\alpha,n,\varepsilon} &= \mathbb{E}^{\mathbb{Q}^{T,\alpha,n,\varepsilon}}(w_{T+S,\mathbb{R}^d}(S, X_S^{x,n,\varepsilon})) \\ &= \mathbb{E}^{\mathbb{Q}^{T,\alpha,n,\varepsilon}}(w_{T+S,\mathbb{R}^d}(0, X_S^{x,n,\varepsilon})) \\ &= \mathscr{P}_S[w_{T,\mathbb{R}^d}(0,\cdot)](x),\end{aligned}$$

where $\mathscr{P}_t$ is the Kolmogorov semigroup of the following SDE, defined $\forall t \in \mathbb{R}_+$:

$$\mathrm{d}U_t^x = [b(U_t^x) + \sigma \beta^{T,S,\alpha,n,\varepsilon}(t, U_t^x)]\mathrm{d}t + \sigma \mathrm{d}W_t, \quad U_0^x = x.$$

This implies, substituting T by T_i' and S by $T_i - T_i'$, (up to a subsequence for $(T_i)_{i \in \mathbb{N}}$ such that $T_i > T_i'$), for all $x \in \overline{G}$,

$$\overline{Y}_0^{T_i,x,\alpha,n,\varepsilon} = \mathscr{P}_{T_i - T_i'}[w_{T_i'}(0,\cdot)](x). \tag{4.11}$$

Then,

$$w_{T_i}(0,x) = \lim_{\alpha,n,\varepsilon} \mathscr{P}_{T_i - T_i'}[w_{T_i'}(0,\cdot)](x).$$

Since $w_{T_i}(0,x) \underset{i \to +\infty}{\longrightarrow} L$, we just have to show that

$$\lim_{i \to +\infty} \lim_{\alpha,n,\varepsilon} \mathscr{P}_{T_i - T_i'}[w_{T_i'}(0,\cdot)](x) = L_2.$$

We have for all $x \in \overline{G}$,

$$|\mathscr{P}_{T_i - T_i'}[w_{T_i'}(0,\cdot)](x) - L_2| \le |\mathscr{P}_{T_i - T_i'}[w_{T_i'}(0,\cdot)](x) - \overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}| + |\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon} - L_2|.$$

We have

$$\lim_{i\to+\infty}\lim_{\alpha,n,\varepsilon}|\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}-L_2|=\lim_{i\to+\infty}w_{T_i'}(0,x)=0.$$

Furthermore, if we denote by $U^{x,p,m}$ the solution of

$$\mathrm{d}U_t^{x,p,m}=[b(U_t^{x,p,m})+\sigma\beta_{p,m}^{T,S,\alpha,n,\varepsilon}](t,U_t^{x,p,m})\mathrm{d}t+\sigma\mathrm{d}W_t,\qquad U_0^{x,p,m}=0,$$

where $(\beta_{p,m}^{T,S,\alpha,n,\varepsilon})_{p\in\mathbb{N}^*,m\in\mathbb{N}^*}$ is the sequence of functions obtained by Lemma 3.6, then we have

$$\begin{aligned}|\mathscr{P}_{T_i-T_i'}[w_{T_i'}(0,\cdot)](x)-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}|&=\lim_{p,m}|\mathbb{E}(w_{T_i'}(0,U_{T_i-T_i'}^{x,p,m}))-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}|\\&=\lim_{p,m}\left|\mathbb{E}\left[\lim_{\alpha',n',\varepsilon'}\overline{Y}_0^{T_i',U_{T_i-T_i'}^{x,p,m},\alpha',n',\varepsilon'}\right]-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}\right|\end{aligned}$$

Then, thanks to the first estimate of Lemma 4.4 and the convergence (4.7), one can apply the dominated convergence theorem to obtain

$$\begin{aligned}|\mathscr{P}_{T_i-T_i'}[w_{T_i'}(0,\cdot)](x)-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}|&=\lim_{p,m}\lim_{\alpha',n',\varepsilon'}\left|\mathbb{E}\left[\overline{Y}_0^{T_i',U_{T_i-T_i'}^{x,p,m},\alpha',n',\varepsilon'}\right]-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}\right|\\&\le\limsup_{p,m}\limsup_{\alpha',n',\varepsilon'}\left[\left|\mathbb{E}\left(\overline{Y}_0^{T_i',U_{T_i'}^{x,p,m},\alpha',n',\varepsilon'}\right)-\overline{Y}_0^{T_i',x,\alpha',n',\varepsilon'}\right|+\left|\overline{Y}_0^{T_i',x,\alpha',n',\varepsilon'}-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}\right|\right].\end{aligned}$$

Therefore, by the third estimate of Lemma 4.4, we deduce that

$$\begin{aligned}|\mathscr{P}_{T_i-T_i'}[w_{T_i'}(0,\cdot)](x)-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}|&\le Ce^{-\hat{\eta}T_i'}+\limsup_{\alpha',n',\varepsilon',\varepsilon'}|\overline{Y}_0^{T_i',x,\alpha',n',\varepsilon'}-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}|\\&\le Ce^{-\hat{\eta}T_i'}+|w_{T_i'}(0,x)-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}|.\end{aligned}$$

This implies that

$$\lim_i\lim_{\alpha,n,\varepsilon}|\mathscr{P}_{T_i-T_i'}[w_{T_i'}(0,\cdot)](x)-\overline{Y}_0^{T_i',x,\alpha,n,\varepsilon}|=0.$$

We have shown that $L=L_2$, which implies that for all $x\in\overline{G}$,

$$\lim_{T\to+\infty}w_T(0,x)=L.$$

Finally, we prove that this convergence holds with an explicit rate of convergence. Let us write $\forall x\in\overline{G}$, $\forall T>0$,

$$\begin{aligned}|w_T(0,x)-L|&=\lim_{V\to+\infty}|w_T(0,x)-w_V(0,x)|\\&=\lim_V\lim_{\alpha,n,\varepsilon}|\overline{Y}_0^{T,x,\alpha,n,\varepsilon}-\overline{Y}_0^{V,x,\alpha,n,\varepsilon}|.\end{aligned}$$

Then, by equation (4.11):

$$|w_T(0,x)-L|=\lim_V\lim_{\alpha,n,\varepsilon}\left|\overline{Y}_0^{T,x,\alpha,n,\varepsilon}-\mathscr{P}_{V-T}\left[w_T(0,\cdot)\right](x)\right|,$$

where $\mathscr{P}_t$ is the Kolmogorov semigroup of the following SDE, defined $\forall t\in\mathbb{R}_+$:

$$\mathrm{d}U_t^x=[b(U_t^x)+\sigma\beta^{T,V-T,\alpha,n,\varepsilon}(t,U_t^x)]\mathrm{d}t+\sigma\mathrm{d}W_t,\qquad U_0^x=x.$$

Now, if we denote by $U^{x,p,m}$ the solution of the following SDE, $\forall t \geq$,

$$\mathrm{d}U_t^{x,p,m} = [b(U_t^{x,p,m}) + \sigma\beta_{p,m}^{T,V-T,\alpha,n,\varepsilon}(t, U_t^{x,p,m})]\mathrm{d}t + \sigma\mathrm{d}W_t, \quad U_0^x = x,$$

where $(\beta_{p,m}^{T,V-T,\alpha,n,\varepsilon})_{p\in\mathbb{N}^*,m\in\mathbb{N}^*}$ is the sequence of functions obtained by Lemma 3.6, then we have

$$\begin{aligned}
|w_T(0,x) - L| &= \lim_V \lim_{\alpha,n,\varepsilon} \lim_{p,m} \left|\overline{Y}_0^{T,x,\alpha,n,\varepsilon} - \mathbb{E}\left[w_T(0, U_{V-T}^{x,p,m})\right]\right| \\
&= \lim_V \lim_{\alpha,n,\varepsilon} \lim_{p,m} \lim_{\alpha',n',\varepsilon'} \left|\overline{Y}_0^{T,x,\alpha,n,\varepsilon} - \mathbb{E}\left[\overline{Y}_0^{T,U_{V-T}^{x,p,m},\alpha',n',\varepsilon'}\right]\right| \\
&\leq \limsup_V \limsup_{\alpha,n,\varepsilon} \limsup_{p,m} \limsup_{\alpha',n',\varepsilon'} \left[\left|\overline{Y}_0^{T,x,\alpha,n,\varepsilon} - \overline{Y}_0^{T,x,\alpha',n',\varepsilon'}\right| + \left|\overline{Y}_0^{T,x,\alpha',n',\varepsilon'} - \mathbb{E}\left[\overline{Y}_0^{T,U_{V-T}^{x,p,m},\alpha',n',\varepsilon'}\right]\right.\right.
\end{aligned}$$

where we have applied the dominated convergence theorem for the second line thanks to (4.7) and the first estimate in Lemma 4.4. Since

$$\lim_V \lim_{\alpha,n,\varepsilon} \lim_{p,m} \lim_{\alpha',n',\varepsilon'} \left|\overline{Y}_0^{T,x,\alpha,n,\varepsilon} - \overline{Y}_0^{T,x,\alpha',n',\varepsilon'}\right| = 0,$$

we deduce, by the third estimate of Lemma 4.4,

$$|w_T(0,x) - L| \leq Ce^{-\hat{\eta}T}.$$

which concludes the proof of the Theorem. □

5 Application to an ergodic control problem

In this section, we show how we can apply our results to an ergodic control problem. We assume that Hypotheses 3.2 and 3.3 hold. Let U be a separable metric space. We define a control a as an $(\mathscr{F}_t)_{t\geq 0}$-predictable U-valued process. We will assume the following.

Hypothesis 5.1. The functions $R : U \to \overline{G}$, $L : \overline{G} \times U \to \mathbb{R}$ and $g_0 : \overline{G} \to \mathbb{R}$ are measurable and satisfy, for some $C > 0$,

1. $|R(a)| \leq C \quad \forall a \in U$;
2. $L(\cdot, a)$ is continuous in x uniformly with respect to $a \in U$; furthermore $|L(\cdot, a)| \leq C$, furthermore $|L(x,a)| \leq C \quad \forall x \in \overline{G}, \forall x \in \overline{G}, \forall a \in U$;
3. $h_0(\cdot)$ is continuous,
4. $g \in \mathscr{C}^1(\overline{G})$.

We denote by $(X_t^x)_{t\geq 0}$ the solution of (3.4). Given an arbitrary control a and $T > 0$, we introduce the Girsanov density

$$\rho_T^{x,a} = \exp\left(\int_0^T \sigma^{-1}R(a_s)\mathrm{d}W_s - \frac{1}{2}\int_0^T |\sigma^{-1}R(a_s)|^2\mathrm{d}s\right)$$

and the probability $\mathbb{P}_T^a = \rho_T^a\mathbb{P}$ on $\mathscr{F}_T$. We introduce two costs. The first one is the cost in finite horizon:

$$J^T(x,a) := \mathbb{E}^{a,T}\left[\int_0^T L(X_s^x, a_s)\mathrm{d}s + \int_0^T g(X_s^x)\mathrm{d}K_s^x\right] + \mathbb{E}^{a,T}h_0(X_T^x),$$

where $\mathbb{E}^{a,T}$ denotes the expectation with respect to $\mathbb{P}_T^a$. The associated optimal control problem is to minimize the cost $J^T(x,a)$ over all controls $a^T : \Omega \times [0,T] \to U$, progressively measurable. The second one is called the ergodic cost and is the time averaged finite horizon cost:

$$J(x,a) := \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}^{a,T}\left[\int_0^T L(X_s^x, a_s)\mathrm{d}s + \int_0^T g(X_s^x)\mathrm{d}K_s^x\right].$$

The associated optimal control problem is to minimize the cost $J(x,a)$ over all controls $a : \Omega \times [0,+\infty[\to +\infty$, progressively measurable.

We notice that $w_t^a = W_t - \int_0^t \sigma^{-1}R(a_s)\mathrm{d}s$ is a Brownian motion on $[0,T]$ under $\mathbb{P}_T^a$ and that

$$\mathrm{d}X_t^x = (b(X_t^x) + R(a_t))\mathrm{d}t + \sigma\mathrm{d}W_t^a \quad \forall t \in [0,T],$$

and this justifies our formulation of the control problem.

We want to show how our results can be applied to such an optimization problem to get an asymptotic expansion of the finite horizon cost involving the ergodic cost.

To apply our results, we first define the Hamiltonian in the usual way

$$f_0(x,z) = \inf_{a\in U}\left\{L(x,a) + z\sigma^1 R(a)\right\}, \tag{5.1}$$

and we note that , if for all x, z the infimum is attained in (5.1) then by the Filippov theorem (see [19]), there exits a measurable function $\gamma\overline{G} \times \mathbb{R}^{1\times d}$ such that

$$f_0(x,z) = L(x,\gamma(x,z)) + z\sigma^{-1}R(\gamma(x,z)).$$

Lemma 5.1. *Under the above assumptions, the Hamiltonian f_0 satisfies assumptions on f in hypotheses 3.4, 3.5, 3.7, or 4.1.*

Proof. See Lemma 5.2 in [8]. □

We recall the following results about the finite horizon cost:

Lemma 5.2. *Assume that Hypotheses 3.1, 3.3, 3.6 and 5.1 hold true. Then for arbitrary control $a^T : \Omega \times [0,T] \to U$,*

$$J^T(x,a^T) \geq u(T,x),$$

where $u(t,x)$ is the viscosity solution of

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f_0(x,\nabla u(t,x)G), & \forall (t,x) \in \mathbb{R}_+ \times G, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in \mathbb{R}_+ \times \partial G \\ u(0,x) = h_0(x), & \forall x \in G, \end{cases}$$

Furthermore, if $\forall x$, z the infimum is attained in (5.1) *then we have the equality:*

$$J^T(x,\overline{a}^T) = u(T,x),$$

where $\overline{a}_t^T = \gamma(X_t^x, \nabla u(t,X_t^x)\sigma)$.

Proof. The proof of this result is similar to the proof of Theorem 7.1 in [7], so we omit it. □

Similarly, for the ergodic cost we have the following result.

Lemma 5.3. *Assume that Hypotheses 3.2, 3.3, 3.6 and 5.1 hold true, then for arbitrary control* $a : \Omega \times [0, +\infty[\to U$*,*

$$J(x, a) \geq \lambda,$$

where (v, λ) *is the viscosity solution of*

$$\begin{cases} \mathscr{L}v + f_0(x, \nabla v(x)\sigma) - \lambda = 0, & \forall x \in G, \\ \frac{\partial v(t,x)}{\partial n} + g(x) = 0, & \forall x \in \partial G. \end{cases}$$

Furthermore, if $\forall x$*,* z *the infimum is attained in* (5.1) *then we have the equality:*

$$J^T(x, \overline{a}) = \lambda,$$

where $\overline{a}_t = \gamma(X_t^x, \nabla v(X_t^x)\sigma)$*.*

Finally, we apply our result to obtain the following theorem.

Theorem 5.4. *Assume that Hypotheses 3.2, 3.3, 3.6 and 5.1 hold true. Then, for any control* $a : \Omega \times [0, T] \to U$*, we have*

$$\liminf_{T \to +\infty} \frac{J^T(x, a^T)}{T} \geq \lambda.$$

Furthermore, if $\forall x$*,* z *the infimum is attained in* (5.1) *then*

$$|J^T(x, \overline{a}^T) - J(x, \overline{a})T - v(x) + L| \leq Ce^{-\hat{\eta}T}$$

Proof. The proof is a straightforward consequence of the two previous lemmas above and of Theorem 4.3. □

References

[1] G. Barles. Fully nonlinear Neumann type boundary conditions for second-order elliptic and parabolic equations. *J. Differential Equations*, vol. 106, no. 1, pp. 90-106, 1993.

[2] G. Barles and F. Da Lio. On the boundary ergodic problem for fully nonlinear equations in bounded domains with general nonlinear Neumann boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, vol. 22, no. 5, pp. 521-541, 2005.

[3] S. Benachour and S. Dabuleanu. Large time behavior for a viscous Hamilton-Jacobi equation with Neumann boundary condition. *J. Differential Equations*, vol. 216, no. 1, pp. 223-258, 2005.

[4] P. Briand and Y. Hu.Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs. *J. Funct. Anal.*, vol. 155, no. 2, pp. 455-494, 1998.

[5] F. Da Lio. Large time behavior of solutions to parabolic equations with Neumann boundary conditions. *J. Math. Anal. Appl.*, vol. 339, no. 1, pp. 384-398, 2008.

[6] A. Debussche, Y. Hu and G. Tessitore. Ergodic BSDEs under weak dissipative assumptions. *Stochastic Process. Appl.*, vol. 121, no. 3, pp. 407-426, 2011.

[7] M. Fuhrman, Y. Hu and G. Tessitore. Ergodic BSDES and optimal ergodic control in Banach spaces. *SIAM J. Control Optim.*, vol. 48, no. 3, pp. 1542-1566, 2009.

[8] M. Fuhrman and G. Tessitore. The Bismut-Elworthy formula for backward SDEs and applications to nonlinear Kolmogorov equations and control in infinite dimensional spaces. *Stoch. Stoch. Rep.*, vol. 74, no. 1-2, pp. 429-464, 2002.

[9] Y. Fujita, H. Ishii and P. Loreti. Asymptotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator. *Comm. Partial Differential Equations*, vol. 31, no. 4-6, pp. 827-848, 2006.

[10] Y. Fujita and P. Loreti. Long-time behavior of solutions to Hamilton-Jacobi equations with quadratic gradient term. *NoDEA Nonlinear Differential Equations Appl.*, vol. 16, no. 6, pp. 771-791, 2009.

[11] R. Z. Has'minskiĭ. Stochastic stability of differential equations. *Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis*, 1980.

[12] Y. Hu, P.-Y. Madec and A. Richou.A Probabilistic Approach to Large Time Behavior of Mild Solutions of HJB Equations in Infinite Dimension. *SIAM J. Control Optim.*, vol. 53, no. 1, pp. 378-398, 2015.

[13] N. Ichihara and S.-J. Sheu. Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradients. *SIAM J. Math. Anal.*, vol. 45, no. 1, pp. 279-306, 2013.

[14] H. Ishii. Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, vol. 25, no. 2, pp. 231-266, 2008.

[15] H. Ishii. Long-time asymptotic solutions of convex Hamilton-Jacobi equations with Neumann type boundary conditions. *Calc. Var. Partial Differential Equations*, vol. 42, no. 1-2, pp. 189-209, 2011.

[16] J. Ma and J. Zhang. Representation theorems for backward stochastic differential equations. *Ann. Appl. Probab.*, vol. 12, no. 4, pp. 1390-1418, 2002.

[17] P.-Y. Madec.Ergodic BSDEs and related PDEs with Neumann boundary conditions under weak dissipative assumptions. *Stochastic Process. Appl.*, vol. 125, no. 5, pp. 1821-1860, 2015.

[18] X. Mao.Stochastic differential equations and applications. *Horwood Publishing Limited, Chichester.*, 2008.

[19] E. J. McShane and Jr. R. B. Warfield. On Filippov's implicit functions lemma. *Proc. Amer. Math. Soc.*, vol. 18, pp. 41-47, 1967.

[20] G. Namah and J.-M. Roquejoffre. Convergence to periodic fronts in a class of semilinear parabolic equations. *NoDEA Nonlinear Differential Equations Appl.*, vol. 4, no. 4, pp. 521-536, 1997.

[21] É. Pardoux and S. Zhang. Generalized BSDEs and nonlinear Neumann boundary value problems. *Probab. Theory Related Fields*, vol. 110, no. 4, pp. 535-558, 1998.

[22] A. Richou. Ergodic BSDEs and related PDEs with Neumann boundary conditions. *Stochastic Process. Appl.*, vol. 119, no. 9, pp. 2945-2969, 2009.