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Persistent random walks

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1 Introduction

We would like to investigate the asymptotic behaviour of random walks in a particular random environment, where the latter is partially the trajectory of the walk itself (the past) which acts as a (possibly) reinforcement: we consider a walker that at each step keeps the same direction (or changes) with a probability that directly depends on the time already spent in the direction the walker is currently moving.

Classical random walks are defined by

$$S_t := \sum_{n=1}^t X_n, \tag{1.1}$$

for $t \in \mathbb{N}$ and for i.i.d. increments $(X_n)_{n \in \mathbb{N}}$. When the increments $(X_n)_{n \in \mathbb{N}}$ are defined as a one-order Markov chain, a short memory in the dynamics of the stochastic paths is introduced and the random walk S itself is no longer Markovian; such a process is called in the literature a *persistent random walk* or also a *Kac walk* (see Eckstein et al. [4], Renshaw and Henderson [9], Weiss [12; 13]).

In order to take into account possibly infinite reinforcements, we consider here a process $(X_n)_{n \in \mathbb{N}}$ with a *variable memory*, for which the dependency from the past is unbounded.

A *Variable Length Markov Chain* (VLMC) can be defined as follows (this probabilistic presentation comes from Cénac et al. [1], the model was introduced in Rissanen [10] and an overview on VLMC can be found in Galves and Löcherbach [7]). Let $\mathcal{L} = \mathcal{A}^{-\mathbb{N}}$ be the set of left-infinite words on the alphabet \mathcal{A} . Consider a complete (each node has 0 or $|\mathcal{A}|$ children) \mathcal{A} -ary tree whose finite leaves \mathcal{C} are words

on the alphabet \mathcal{A} . To each leaf c (not necessarily finite) is attached a Bernoulli distribution denoted by q_c . Each leaf is called a *context* and this probabilized tree is called a *context tree*. Different context trees

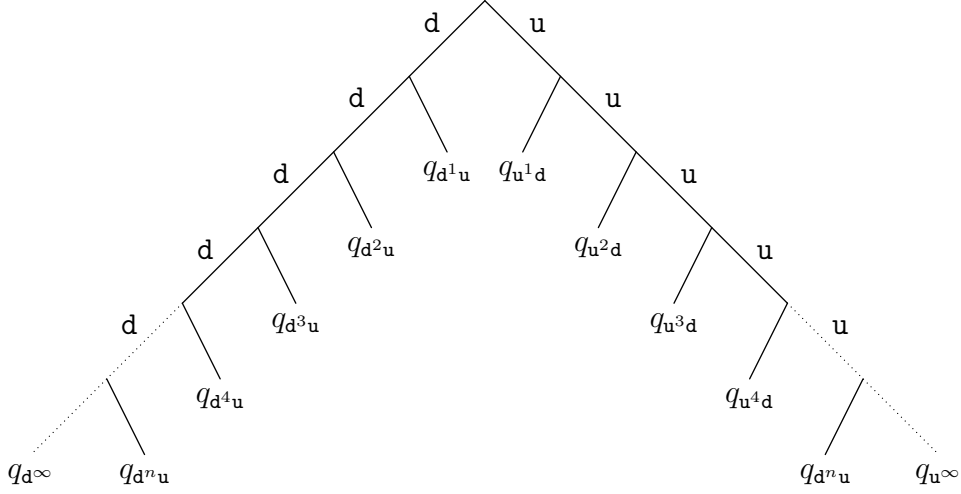


Figure 1.1: Context tree of the double infinite comb.

correspond to different probabilistic impacts of the past and different dependencies; the model of (very) persistent random walk we propose here appears to correspond to a so called *double infinite comb* (see Figure 1.1). The set of leaves \mathcal{C} is defined with a binary alphabet $\mathcal{A} := \{u, d\}$ consisting of the letters 'u' for moving *up* and 'd' for moving *down*. The increments X_n , for $n \in \mathbb{N}$, take thus place in \mathcal{A} and the set of leaves \mathcal{C} is defined by

$$\mathcal{C} := \{u^n d, n \geq 0\} \cup \{u^\infty\} \cup \{d^n u, n \geq 0\} \cup \{d^\infty\}$$

where $u^n d$ represents the sequence $uu \dots ud$ composed with n characters *up* 'u' and one character *down* 'd'. By convention $u^0 d = d$. The set of leaves contains two infinite leaves u^∞ and d^∞ and a countable set of finite leaves $u^n d, d^n u$. The *prefix function* $\overleftarrow{\text{pref}} : \mathcal{L} = \mathcal{A}^{-\mathbb{N}} \rightarrow \mathcal{C}$ indicates the length of the last run: for instance, after $n = 3$ consecutive moves up, one has to consider the prefix

$$\overleftarrow{\text{pref}}(\dots duuu) = uuud = u^3 d.$$

For a general context tree and for any left-infinite word U , we define $\overleftarrow{\text{pref}}(U)$ in a similar way as the first suffix of U reading from right to left appearing as a leaf of the context tree. The associated VLMC is the \mathcal{L} -valued Markov chain $(U_n)_{n \geq 0}$ defined by the transitions

$$\mathbb{P}(U_{n+1} = U_n \ell | U_n) = q_{\overleftarrow{\text{pref}}(U_n)}(\ell) \quad (1.2)$$

where ℓ is any letter of the alphabet \mathcal{A} . Notice that the VLMC is entirely determined by the data $q_c, c \in \mathcal{C}$. Moreover the order of dependence (the *memory*) depends on the past itself.

For a given VLMC $(U_n)_{n \geq 0}$, define X_n as the last letter of U_n for any $n \geq 0$. When the context tree associated with (U_n) is infinite, then the letter process $(X_n)_{n \geq 0}$ is non Markovian, because the transition probabilities (1.2) indicate that X_{n+1} depends on a variable and unbounded number of previous letters. The corresponding random walk (S_t) defined by (1.1) is no longer Markovian, it is somehow *very persistent*.

In this paper, we want to study some asymptotic properties of this (very) persistent random walk. On which conditions on the "transition" probabilities (to keep on the same direction or to change) the corresponding persistent random walks are recurrent or transient ?

In Cénac et al. [2], a strong law of large numbers is established for (S_t) , as well as a central limit theorem under stronger assumptions on the distributions $q_c, c \in \mathcal{C}$ attached to each leaf of the context tree, expressed in terms of conditions on the first and second moments on some (random) persistence time in the same direction. In this paper, we first provide recurrence vs transience conditions for the persistent random walk (S_t) without assuming that the latter admits any stationary probability. A summary of the different situations is given after Theorem 3.2, in Table 3.1.

In particular, examples are exhibited when this process is recurrent even if the parameters $q_c, c \in \mathcal{C}$ are chosen slightly asymmetrical. In fact, in the case when the time taken to change directions is integrable, the condition of recurrence is a condition of null drift. When these time changes of direction are not summable, we can slightly perturb the symmetric configuration while remaining recurrent. Contrary to the well defined drift case for which a small perturbation on the parameters of a recurrent persistent random walk will lead in general to a transient behaviour, in the case of an undefined drift the persistent random walk may stay recurrent as long as the perturbation remains controlled asymptotically.

The paper is organized as follows. Section 2 is devoted to the presentation of the model and the assumptions. In Section 3 we give the recurrence/transience criteria.

2 Persistent random walk in one dimension

2.1 The model

Let us consider the alphabet $\mathcal{A} = \{u, d\}$, where the letter u codes for a rise and is associated with a jump of length $+1$ on \mathbb{Z} and the letter d codes for a way down and is associated to a jump of length -1 on \mathbb{Z} . With each letter $\ell \in \mathcal{A}$ we associate a sequence $(\alpha_n^\ell)_{n \geq 1}$ of real numbers belonging to $[0, 1]$, called an admissible sequence, where α_n^ℓ stands for the probability of changing letter after a run of length n of letter ℓ . Defining

$$q_{u^nd}(d) = 1 - q_{u^nd}(u) := \alpha_n^u \quad \text{and} \quad q_{d^nu}(u) = 1 - q_{d^nu}(d) := \alpha_n^d, \quad (2.1)$$

one builds a VLMC on the double infinite comb $(U_n)_{n \in \mathbb{N}}$ defined for all $n \geq 1$ by the transitions probabilities $(q_c)_{c \in \mathcal{C}}$ above. Furthermore, denote by $(X_n)_{n \in \mathbb{N}}$ the sequence of coding numbers of the last letters of $(U_n)_{n \in \mathbb{N}}$, i.e. $X_n := +1$, if the last letter of U_n is 'u', and $X_n = -1$, if the last letter of U_n is 'd'. Then we can introduce the main object of our study which is the random walk with unbounded variable memory (S_t) defined by (1.1).

2.2 Settings and assumptions

In the sequel, we assume $q_{u^\infty}(u) \neq 1$, and $q_{d^\infty}(d) \neq 1$ so that the trivial situations where the persistent walk stays a.s. frozen in one of the two directions are thus excluded.

Let us consider the sequence of breaking times $(T_n)_{n \in \mathbb{N}}$ defined by

$$T_0 = \inf \{k \geq 0 : X_{k+1} \neq X_k\}, \quad (2.2)$$

$$\text{and } \forall n \geq 1, \quad T_n = \inf \{k > T_{n-1}, X_{k+1} \neq X_k\}. \quad (2.3)$$

Figure 2.1 illustrates one realization of (S_t) and the above notations. For the sake of simplicity, we deal throughout this paper with the conditional probability of the event:

$$T_0 = 0, \quad X_0 = +1 \quad (\text{and thus } X_1 = -1). \quad (2.4)$$

Conditioning by (2.4), the length of rises $(\tau_n^u)_{n \geq 0}$ and of descents $(\tau_n^d)_{n \geq 0}$ can be defined by

$$\forall n \geq 0, \quad \tau_n^d := T_{2n+1} - T_{2n} \quad \text{and} \quad \tau_n^u := T_{2n+2} - T_{2n+1}. \quad (2.5)$$

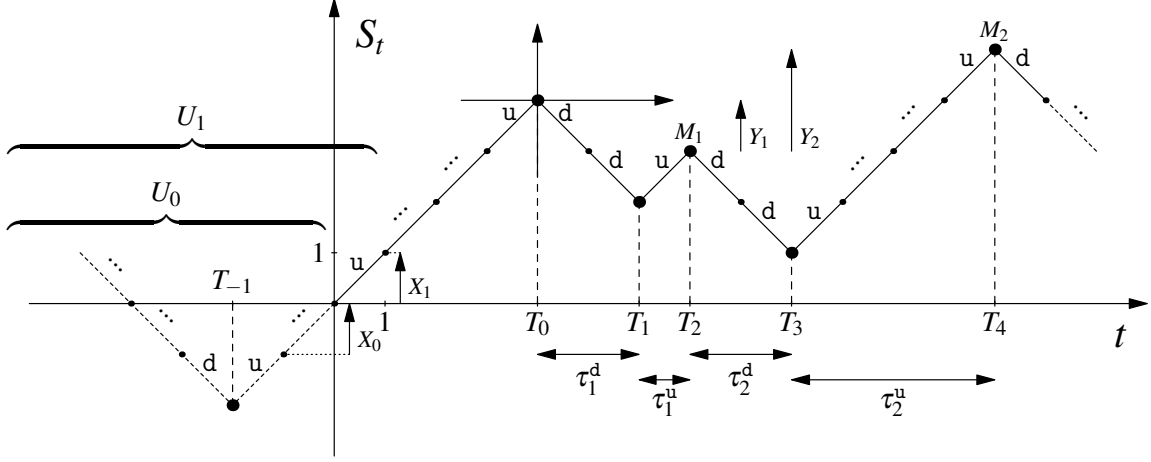


Figure 2.1: One realization of the random walk (S_t) .

Due to the renewal properties of the chosen variable length Markov chain (see Proposition 2.3 in Cénac et al. [2]), it turns out that $(\tau_n^d)_{n \in \mathbb{N}}$ and $(\tau_n^u)_{n \in \mathbb{N}}$ are independent sequences of i.i.d. random variables. Besides, an easy computation leads to their tail distributions and expectations, given for any letter $\ell \in \mathcal{A}$ and all integer $n \geq 1$ by

$$\mathcal{F}^\ell(n) := \mathbb{P}(\tau_0^\ell \geq n) = \prod_{k=1}^{n-1} (1 - \alpha_k^\ell) \quad \text{and} \quad \Theta_\ell := \mathbb{E}[\tau_0^\ell] = \sum_{n=1}^{\infty} \prod_{k=1}^{n-1} (1 - \alpha_k^\ell). \quad (2.6)$$

Remark 2.1. As a matter of facts, the persistent random walk $(S_n)_{n \geq 0}$ can be equivalently defined either via the distribution tails $(\mathcal{F}^\ell(n))_{n \geq 1}$ or probabilities $(\alpha_n^\ell)_{n \geq 1}$. Thus, depending on the context, we will choose the more suitable description of the parameters of the model.

In the sequel, we make the following assumption.

Assumption 2.1 (finiteness of the length of runs).

$$\forall \ell \in \mathcal{A}, \quad \prod_{k=1}^{\infty} (1 - \alpha_k^\ell) = 0. \quad (2.7)$$

Assumption 2.1 is equivalent to

$$\forall \ell \in \mathcal{A}, \quad \left(\exists n \geq 1, \alpha_n^\ell = 1 \right) \quad \text{or} \quad \left(\sum_{k=1}^{\infty} \alpha_k^\ell = \infty \right). \quad (2.8)$$

Thanks to Assumption 2.1 and to the renewal properties of the chosen variable length Markov chain, the condition (2.4) can be done without loss of generality and has no fundamental importance in the study of the asymptotic behaviour of S in terms of recurrence/transience criteria or scaling properties.

Denoting for any integer $p \geq 1$ the composition p times of the function \log by $\log_p := \log \circ \dots \circ \log$, the right hand side of (2.8) holds for instance if

$$\frac{1}{n \log(n) \cdots \log_p(n)} = \mathcal{O}(\alpha_n^\ell) \quad \text{as } n \rightarrow \infty, \quad (2.9)$$

whereas it is not satisfied if there exists $\varepsilon > 0$ and $\ell \in \mathcal{A}$ such that

$$\alpha_n^\ell = \mathcal{O} \left(\frac{1}{n \log(n) \cdots \log_{p-1}(n) (\log_p(n))^{1+\varepsilon}} \right) \quad \text{as } n \rightarrow \infty. \quad (2.10)$$

In order to deal with a more tractable random walk built with possibly unbounded but i.i.d. increments, we consider the underlying skeleton random walk $(M_n)_{n \geq 1}$ of even breaking times defined by

$$M_n := \sum_{k=1}^n Y_k, \quad \text{where } \forall n \geq 1, \quad Y_n := \tau_n^u - \tau_n^d. \quad (2.11)$$

In the case when Θ_u or Θ_d is finite, let us define the drift of M , resp. of S by:

$$\mathbf{d}_M := \Theta_u - \Theta_d \in \overline{\mathbb{R}} \quad \text{resp.} \quad \mathbf{d}_S := \frac{\Theta_u - \Theta_d}{\Theta_u + \Theta_d} \in [-1, 1]. \quad (2.12)$$

For instance, they are well-defined if there exist $\ell \in \mathcal{A}$, $p \geq 0$ and $\varepsilon > 0$ such that for n sufficiently large

$$\alpha_n^\ell \geq \frac{1}{n} + \frac{1}{n \log(n)} + \cdots + \frac{1}{n \log(n) \cdots \log_{p-1}(n)} + \frac{1 + \varepsilon}{n \log(n) \cdots \log_p(n)}, \quad (2.13)$$

and there are undefined if for any $\ell \in \mathcal{A}$ there exists $p \geq 0$ such that for n sufficiently large

$$\alpha_n^\ell \leq \frac{1}{n} + \frac{1}{n \log(n)} + \cdots + \frac{1}{n \log(n) \cdots \log_p(n)}. \quad (2.14)$$

3 Recurrence and transience

3.1 Equivalent criteria and comparison lemma

Let us recall that a stochastic process $(S_n)_{n \geq 0}$ on \mathbb{Z} is said to be recurrent if

$$\forall x \in \mathbb{Z}, \quad \sup\{n \geq 0 : S_n = x\} = \infty \quad \text{a.s.}, \quad (3.1)$$

and transient (resp. transient to ∞ , transient to $-\infty$) if

$$\lim_{n \rightarrow \infty} |S_n| = \infty \quad \left(\text{resp. } \lim_{n \rightarrow \infty} S_n = \infty, \lim_{t \rightarrow \infty} S_t = -\infty \right) \quad \text{a.s.} \quad (3.2)$$

Also recall the following well-known Theorem 3.1. We can refer for instance to Feller [6, Theorem 1., Chap. XII and Theorem 4., Chap. VI].

Theorem 3.1. *Any random walk M on \mathbb{Z} is either almost surely constant or of type:*

a) *oscillating when*

$$\limsup_{n \rightarrow \infty} M_n = \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} M_n = -\infty \quad \text{a.s.}; \quad (3.3)$$

b) *drifting to ∞ when*

$$\lim_{n \rightarrow \infty} M_n = \infty \quad \text{a.s.}; \quad (3.4)$$

c) *drifting to $-\infty$ when*

$$\lim_{n \rightarrow \infty} M_n = -\infty \quad \text{a.s.} \quad (3.5)$$

Furthermore, if the drift \mathbf{d}_M is well-defined, then M is oscillating if and only if $\mathbf{d}_M = 0$. Moreover, one has almost surely

$$\lim_{n \rightarrow \infty} \frac{M_n}{n} = \mathbf{d}_M. \quad (3.6)$$

Our strategy to study recurrence *versus* transience consists in reducing the determination of the type of the persistent random walk defined in Section 2 by studying some properties of the underlying skeleton random walk $(M_n)_{n \geq 1}$. This is made clear by the following lemma.

Lemma 3.1. *The persistent random walk S defined by (1.1) is either recurrent or transient according to the type of the underlying skeleton random walk M of even breaking times (see (2.11)). More precisely, one has:*

- a) S is recurrent if and only if M is oscillating;
- b) S is transient to ∞ if and only if M is drifting to ∞ ;
- c) S is transient to $-\infty$ if and only if M is drifting to $-\infty$;

Proof of Lemma 3.1. If M is oscillating then S is recurrent. Also, if M is drifting to $-\infty$, then so is S , since due to condition (2.4), the trajectory of S is always under the breaking line formed by the M_n 's.

Let us introduce the skeleton at odd breaking times as follows

$$M_n^o = M_0 - \tau_1^d + \sum_{k=1}^n (\tau_k^u - \tau_{k+1}^d). \quad (3.7)$$

Up to the random variable $-\tau_1^d$, M^o and M are equal in distribution. The second point follows using the same argument as above. ■

With respect to the considerations above, it seems natural to distinguish the following three different cases :

- Θ_u and Θ_d are both finite,
- Θ_u or Θ_d is finite,
- both are infinite.

The first and second cases correspond to the cases in which the drift \mathbf{d}_M is *well-defined*. They will be considered together in Section 3.2. The third case, when the definition of the drift in (2.12) is meaningless, will be considered apart in Section 3.3.

Finally, let us end this part with a comparison lemma which will be useful for the second case and in section 3.4 for the perturbations of the parameters.

Lemma 3.2. *Consider two persistent random walks S and \tilde{S} defined by (1.1) and built from double infinite combs. Assume that the distribution tails $(\mathcal{T}^\ell(n))$ and $(\tilde{\mathcal{T}}^\ell(n))$, defined in (2.6), related to S and \tilde{S} respectively, with $\ell \in \mathcal{A}$, satisfy for all $n \geq 1$,*

$$\mathcal{T}^u(n) \leq \tilde{\mathcal{T}}^u(n) \quad \text{and} \quad \mathcal{T}^d(n) \geq \tilde{\mathcal{T}}^d(n). \quad (3.8)$$

Then there exists a coupling (S, \tilde{S}) such that, for all $t \geq 1$,

$$S_t \leq \tilde{S}_t \quad \text{a.s.} \quad (3.9)$$

Remark 3.1. *The above lemma can equivalently be stated in terms of the transition probabilities with the same conclusions, by considering instead of (3.8) the equivalent hypothesis that (α_n^ℓ) and $(\tilde{\alpha}_n^\ell)$ with $\ell \in \mathcal{A}$ are admissible sequences such that for all $n \geq 1$,*

$$\tilde{\alpha}_n^u \leq \alpha_n^u \quad \text{and} \quad \tilde{\alpha}_n^d \geq \alpha_n^d. \quad (3.10)$$

Proof of Lemma 3.2. For any $\ell \in \mathcal{A}$ and $n \geq 1$ let (τ_n^ℓ) and $(\tilde{\tau}_n^\ell)$ be the associated lengths of runs of S and \tilde{S} (see (2.5)) and introduce G^ℓ and \tilde{G}^ℓ the corresponding left continuous inverse of the cumulative distributions. Inequalities (3.8) yield for all $x \in [0, 1]$,

$$G^u(x) \leq \tilde{G}^u(x) \quad \text{and} \quad G^d(x) \geq \tilde{G}^d(x). \quad (3.11)$$

This means that the lengths of rises of S are stochastically bounded below by those of \tilde{S} and the lengths of descents of S are stochastically bounded above by those of \tilde{S} . Then we construct a coupling (see for example Thorisson [11]) of the lengths of runs such that for all $n \geq 1$,

$$\tau_n^u \leq \tilde{\tau}_n^u \quad \text{and} \quad \tau_n^d \geq \tilde{\tau}_n^d \quad \text{a.s.} \quad (3.12)$$

To this end, consider two independent sequences (V_n^ℓ) of uniform random variables on $[0, 1]$ and set

$$\tau_n^\ell := G^\ell(V_n^\ell) \quad \text{and} \quad \tilde{\tau}_n^\ell := \tilde{G}^\ell(V_n^\ell). \quad (3.13)$$

Then one can build a coupling of the persistent random walks S and \tilde{S} satisfying inequality (3.9) since there are entirely determined by these runs. ■

3.2 Well-defined Drift case

In this part, we assume that the drift is well defined, that is Θ_u or Θ_d is finite.

Example 3.1. Consider the case in which α_n^ℓ is equivalent to λ/n , for some $\lambda > 0$. Then Θ_ℓ is finite if $\lambda > 1$ and infinite otherwise.

Proposition 3.1 (Recurrence criterium and LLN). *Under assumption 2.1 and assuming that the drift d_M defined in (2.12) is well-defined, the persistent random walk S defined by (1.1) is recurrent if and only if $\Theta_u = \Theta_d$. Moreover, one has almost surely*

$$\lim_{t \rightarrow \infty} \frac{S_t}{t} = \mathbf{d}_S. \quad (3.14)$$

Proof. First remark that in this well-defined drift case, recurrence criterium is a straightforward consequence of Theorem 3.1 and Lemma 3.1. Besides, the law of large numbers (3.14) when Θ_u and Θ_d are both finite is already proved in Cénac et al. [2, Proposition 4.5, p. 33].

Assume $\Theta_u = \infty$ and $\Theta_d < \infty$ (thus $\mathbf{d}_S = 1$). To prove (3.14), it is sufficient to prove the minoration since the majoration is obvious because $S_t \leq t$. To this end, we shall construct for any $0 < \varepsilon < 1$, a persistent random walk S^ε with finite drift, such that for all $t \in \mathbb{N}^*$,

$$S_t^\varepsilon \leq S_t \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{S_t^\varepsilon}{t} \geq 1 - \varepsilon \quad \text{a.s.} \quad (3.15)$$

More specifically, we can choose $N \geq 1$ such that

$$\frac{\Theta_u^N - \Theta_d}{\Theta_u^N + \Theta_d} \geq 1 - \varepsilon, \quad \text{where} \quad \Theta_u^N := \sum_{n=1}^N \prod_{k=1}^{n-1} (1 - \alpha_k^u). \quad (3.16)$$

Thus considering the persistent random walk S^ε associated with the admissible sequences

$$\begin{aligned} \alpha_n^{u,\varepsilon} &:= \begin{cases} \alpha_n^u & \text{if } 1 \leq n \leq N-1, \\ 1 & \text{if } n \geq N, \end{cases} \\ \alpha_n^{d,\varepsilon} &:= \alpha_n^d \quad \forall n \geq 1. \end{aligned}$$

Lemma 3.2 leads to (3.15). Thus, for all $\varepsilon > 0$, almost surely, $\liminf_{t \rightarrow \infty} \frac{S_t}{t} \geq 1 - \varepsilon$ which concludes the proof of (3.14).

The case when $\Theta_d = \infty$ and $\Theta_u < \infty$ can be treated with the same arguments by symmetry. ■

3.3 Undefined drift case

In this section, we consider the remaining case in which both Θ_u and Θ_d are infinite (for instance, consider $0 < \lambda \leq 1$ in the example 3.1). In this case, the information given by the expectation of one increment is no longer sufficient to discriminate between transience or recurrence. In fact, following Erickson [5], the oscillating or drifting behaviour of the skeleton random walk is characterized through the cumulative distribution function of its increments. However, Erickson's criteria do not fit well to our context since the distribution of an increment of $(M_n)_{n \geq 1}$ is the convolution of the distributions of τ_1^u and $-\tau_1^d$. To avoid these difficulties, we introduce a sequence $(\xi_n)_{n \geq 0}$ of i.i.d. Bernoulli random variables with parameter $p \in [0, 1]$, independent of the sequences (τ_n^u) and (τ_n^d) , and define the following random walk

$$M_n^\xi := \sum_{k=1}^n J_k^\xi, \text{ with } J_k^\xi = \xi_k \tau_k^u - (1 - \xi_k) \tau_k^d. \quad (3.17)$$

It turns out that the randomly modified random walk $(M_n^\xi)_{n \geq 1}$ and the skeleton random walk $(M_n)_{n \geq 1}$ share the same behaviour.

Lemma 3.3. *If $p = 1/2$ then the random walks M and M^ξ are of the same type in the sense of Theorem 3.1. For arbitrary $p \in (0, 1)$, if M^ξ is oscillating then M is also oscillating.*

Proof Lemma 3.3. First let $p = 1/2$. If M^ξ is drifting, then so is M , since

$$M^\xi \stackrel{\mathcal{L}}{=} M^{1-\xi} \quad \text{and} \quad M = M^\xi + M^{1-\xi}.$$

By symmetry and Theorem 3.1, it is only needed to prove that if the supremum limit of M^ξ is ∞ , then so is for M .

Let us define for $i \in \{0, 1\}$, $n \geq 1$ and $t \geq 0$,

$$A_{n,t}^i := \left\{ \tau_n^u \geq \sum_{k=1}^n (1 - \xi_k) \tau_k^d + t, \xi_n = i \right\}. \quad (3.18)$$

Introducing the random walk,

$$R_n := \sum_{k=1}^n (1 - \xi_k) (\tau_k^u - \tau_k^d) \quad (3.19)$$

one can check that the subordinated process $(M_{N_n})_{n \geq 0}$, where

$$N_n := \#\{1 \leq k \leq n : \xi_k = 0\}, \quad (3.20)$$

is equal in distribution to R_n . Under the assumption

$$\mathbb{P}(\limsup A_{n,0}^0) = 1, \quad (3.21)$$

the random walk R is infinitely often positive and so is $(M_{N_n})_{n \geq 0}$. Consequently, by Theorem 3.1, the supremum limit of the latter is infinite so that the Lemma is reduced to the statement (3.21).

As a direct consequence of Durrett [3, Theorem 5.3.2, p. 205]) it follows that, for $i \in \{0, 1\}$,

$$\limsup A_{n,t}^i = \left\{ \sum_{n=1}^{\infty} \mathbb{E} \left(1_{\{A_{n,t}^i\}} | \mathcal{F}_{n-1} \right) = \infty \right\}, \quad (3.22)$$

where $(\mathcal{F}_n)_{n \geq 1}$ is the natural filtration associated with the random processes τ^d , τ^u and ξ .

Using independence of τ_n^u with respect to $\mathcal{F}_{n-1} \vee \tau_n^d$, we obtain

$$\begin{aligned}
\mathbb{E}\left(1_{\{A_{n,0}^0\}} \mid \mathcal{F}_{n-1}\right) &= \mathbb{P}(\xi_1 = 0) \mathbb{E}\left(1_{\{\tau_n^u \geq \sum_{k=1}^{n-1} (1-\xi_k) \tau_k^d + \tau_n^d\}} \mid \mathcal{F}_{n-1}\right) \\
&= \mathbb{P}(\xi_1 = 0) \mathbb{E}\left(\mathbb{E}\left(1_{\{\tau_n^u \geq \sum_{k=1}^{n-1} (1-\xi_k) \tau_k^d + \tau_n^d\}} \mid \mathcal{F}_{n-1} \vee \tau_n^d\right) \mid \mathcal{F}_{n-1}\right) \\
&= \mathbb{P}(\xi_1 = 0) \mathbb{E}\left(\mathcal{T}^u\left(\sum_{k=1}^{n-1} (1-\xi_k) \tau_k^d + \tau_n^d\right) \mid \mathcal{F}_{n-1}\right) \\
&= \mathbb{P}(\xi_1 = 0) \sum_{\ell \geq 1} \mathbb{P}(\tau_n^d = \ell) \mathcal{T}^u\left(\sum_{k=1}^{n-1} (1-\xi_k) \tau_k^d + \ell\right).
\end{aligned} \tag{3.23}$$

Summing over $n \geq 1$ on both sides, we get

$$\sum_{n \geq 1} \mathbb{E}\left(1_{\{A_{n,0}^0\}} \mid \mathcal{F}_{n-1}\right) = \mathbb{P}(\xi_1 = 0) \sum_{n \geq 1} \sum_{\ell \geq 1} \mathbb{P}(\tau_n^d = \ell) \mathcal{T}^u\left(\sum_{k=1}^{n-1} (1-\xi_k) \tau_k^d + \ell\right). \tag{3.24}$$

By similar arguments, we obtain, by changing the order of summation

$$\sum_{n \geq 1} \mathbb{E}\left(1_{\{A_{n,0}^0\}} \mid \mathcal{F}_{n-1}\right) = \mathbb{P}(\xi_1 = 0) \sum_{\ell \geq 1} \mathbb{P}(\tau_1^d = \ell) \sum_{n \geq 1} \mathbb{E}\left(1_{\{A_{n,\ell}^1\}} \mid \mathcal{F}_{n-1}\right). \tag{3.25}$$

A straightforward application of Kesten [8, Theorem 5., p. 1190] combined with (3.22) implies the summand on the right hand side is almost surely infinite for all $t \geq 0$ which ends the proof. \blacksquare

In order to obtain the oscillating or drifting property of $(M_n)_{n \geq 0}$, it suffices now to apply the criteria of Erickson [5] to $(M_n^{\xi})_{n \geq 0}$ — with $p = 1/2$ — whose cumulative distribution function of an increment has the following simple form :

$$F_p(t) = p \mathcal{T}^u(\lceil t \rceil) 1_{\{t > 0\}} + (1-p) \mathcal{T}^d(\lfloor -t \rfloor) 1_{\{t \leq 0\}}. \tag{3.26}$$

Roughly speaking, the criteria of Erickson tell that the persistent random walk $(S_n)_{n \geq 0}$ will be recurrent if the distribution tails of the positive and negative runs are comparable, transient otherwise. It turns out that these comparisons can be made through the quantities defined for $\ell_1, \ell_2 \in \mathcal{A}$ by

$$J_{\ell_1 | \ell_2} := \sum_{n=1}^{\infty} \left(1 - \frac{n \mathcal{T}^{\ell_2}(n)}{\sum_{k=1}^n \mathcal{T}^{\ell_2}(k)}\right) \frac{\mathcal{T}^{\ell_1}(n)}{\sum_{k=1}^n \mathcal{T}^{\ell_2}(k)}. \tag{3.27}$$

Theorem 3.2. *Under assumption 2.1 and assuming that both Θ_u and Θ_d are infinite, the persistent random walk S defined by (1.1) is recurrent if and only if*

$$J_{u|d} = \infty \quad \text{and} \quad J_{d|u} = \infty. \tag{3.28}$$

In particular, under assumption (3.28), one has

$$\limsup_{t \rightarrow \infty} \frac{S_t}{t} = 1 \quad \text{and} \quad \liminf_{t \rightarrow \infty} \frac{S_t}{t} = -1 \quad \text{a.s.} \tag{3.29}$$

The persistent random walk S is transient to ∞ (resp. transient to $-\infty$) if and only if

$$J_{u|d} = \infty \quad \text{and} \quad J_{d|u} < \infty \quad (\text{resp. } J_{u|d} < \infty \quad \text{and} \quad J_{d|u} = \infty). \tag{3.30}$$

Remark 3.2. *When $J_{u|d}$ and $J_{d|u}$ are both finite, the drift of the persistent random walk S belongs to $(-1, 1)$. Besides, in the symmetric case, that is for all $n \geq 1$, $\alpha_n^u = \alpha_n^d$, S is recurrent.*

Table 3.1: Recurrence/transience criteria.

	$\Theta_u < \infty$		$\Theta_u = \infty$
$\Theta_d < \infty$	recurrent $\mathbf{d}_S = 0$	drifting $+\infty$ $\mathbf{d}_S \in (0, 1)$	drifting $+\infty$ $\mathbf{d}_S = 1$
		drifting $-\infty$ $\mathbf{d}_S \in (-1, 0)$	
$\Theta_d = \infty$	drifting $-\infty$ $\mathbf{d}_S = -1$		recurrent $J_{u d} = J_{d u} = \infty$
			drifting $+\infty$ $\infty = J_{u d} > J_{d u}$
			drifting $-\infty$ $\infty = J_{d u} > J_{u d}$

This theorem ends the characterisation of the type of persistent random walks. In Table 3.1, the conditions for the recurrence and the transience are summarized.

Proof of Theorem 3.2. For a real sequence $(f(n))_{n \geq 0}$, define the finite difference $\Delta f(n)$, for $n \geq 0$, by $\Delta f(n) = f(n+1) - f(n)$. We apply the criteria of Erickson [5] to the randomly modified random walk $(M_n^{\xi})_{n \geq 0}$ whose cumulative distribution function is given by (3.26). Then, we shall deduce that the persistent random walk S is recurrent is and only if

$$J^+ := \sum_{n=1}^{\infty} \frac{n(-\Delta \mathcal{T}^u(n))}{\sum_{k=1}^n \mathcal{T}^d(k)} = \infty \quad \text{and} \quad J^- := \sum_{n=1}^{\infty} \frac{n(-\Delta \mathcal{T}^d(n))}{\sum_{k=1}^n \mathcal{T}^u(k)} = \infty, \quad (3.31)$$

and transient to ∞ (resp. to $-\infty$) if and only if

$$J^+ = \infty \quad \text{and} \quad J^- < \infty \quad (\text{resp. } J^+ < \infty \quad \text{and} \quad J^- = \infty). \quad (3.32)$$

To end the proof of the theorem, we only need to show that $J^+ = \infty$ if and only if $J_{u|d} = \infty$ — the other equivalence is being shown similarly. Summing by parts, we compute

$$\sum_{n=1}^N \frac{n(-\Delta \mathcal{T}^u(n))}{\sum_{k=1}^n \mathcal{T}^d(k)} = -\frac{(N+1)\mathcal{T}^u(N+1)}{\sum_{k=1}^{N+1} \mathcal{T}^d(k)} + \sum_{n=0}^N \left(1 - \frac{n\mathcal{T}^d(n+1)}{\sum_{k=1}^n \mathcal{T}^d(k)}\right) \frac{\mathcal{T}^u(n+1)}{\sum_{k=1}^{n+1} \mathcal{T}^d(k)}. \quad (3.33)$$

From this equality, it follows obviously that $J^+ = \infty$ implies that $J_{u|d} = \infty$. Conversely, assume that $J^+ < \infty$ and remark that

$$\frac{N}{\sum_{k=1}^N \mathcal{T}^d(k)} \quad (3.34)$$

is non decreasing to infinity. In facts, a simple computation gives

$$\Delta \left(\frac{N}{\sum_{k=1}^N \mathcal{T}^d(k)} \right) = \frac{\sum_{k=1}^N \mathcal{T}^d(k) - N\mathcal{T}^d(N+1)}{(\sum_{k=1}^{N+1} \mathcal{T}^d(k)) (\sum_{k=1}^N \mathcal{T}^d(k))}. \quad (3.35)$$

But the quantity $\sum_{k=1}^N \mathcal{T}^d(k) - N\mathcal{T}^d(N+1)$ is nothing but the truncated expectation $\mathbb{E}(\tau^d \mathbf{1}_{\{\tau^d \leq N\}})$ which is obviously non negative. Consequently, it follows that for all $m \geq n$

$$\frac{n(-\Delta \mathcal{T}^u(m))}{\sum_{k=1}^n \mathcal{T}^d(k)} \leq \frac{m(-\Delta \mathcal{T}^u(m))}{\sum_{k=1}^m \mathcal{T}^d(k)}. \quad (3.36)$$

Thus, summing over $m \geq n$, we obtain

$$\frac{n\mathcal{T}^u(n)}{\sum_{k=1}^n \mathcal{T}^d(k)} \leq \sum_{m=n}^{\infty} \frac{m(-\Delta \mathcal{T}^u(m))}{\sum_{k=1}^m \mathcal{T}^d(k)} \quad (3.37)$$

and the right-hand side tends to 0 as n goes to ∞ since J^+ is assumed to be finite.

Assuming $J_{\text{u|d}} = \infty$, the result of Kesten [8] applies as in lemma 3.3, and it follows that for all $c > 0$

$$\mathbb{P} \left[\tau_n^{\text{u}} \geq c \sum_{k=1}^n (1 - \xi_k) \tau^{\text{d}} \quad \text{i.o.} \right] = 1, \quad \text{and equivalently} \quad \mathbb{P} \left[\tau_n^{\text{u}} \geq c \sum_{k=1}^{N_n} \tau_k^{\text{d}} \quad \text{i.o.} \right] = 1. \quad (3.38)$$

The conditional Borel-Cantelli lemma and computations similar to those appearing in the proof of lemma 3.3 give rise to

$$\mathbb{P} \left[\sum_{n=1}^{\infty} \mathcal{F}^{\text{u}} \left(c \sum_{k=1}^{N_n} \tau_k^{\text{d}} \right) = \infty \right] = 1. \quad (3.39)$$

Since, $N_n \geq \frac{n}{3}$ for sufficiently large $n \geq 1$ and \mathcal{F}^{u} is non increasing, one deduces from (3.39)

$$\mathbb{P} \left[\sum_{n \geq 1} \mathcal{F}^{\text{u}} \left(c \sum_{k=1}^{\lfloor n/3 \rfloor} \tau_k^{\text{d}} \right) = \infty \right] = 1, \quad \text{or equivalently} \quad \mathbb{P} \left[\sum_{n \geq 1} \mathcal{F}^{\text{u}} \left(c \sum_{k=1}^n \tau_k^{\text{d}} \right) = \infty \right] = 1. \quad (3.40)$$

Thus, for all $c > 0$

$$\mathbb{P} \left(\tau_n^{\text{u}} \geq c \sum_{k=1}^n \tau_k^{\text{d}} \quad \text{i.o.} \right) = 1 \quad (3.41)$$

and the first statement follows. The second one is then obvious by symmetry of the problem. \blacksquare

Remark 3.3. Compare to J^+ , the quantity $J_{\text{u|d}}$ has the advantage to involve only the distribution tails and not their differences, i.e. their densities. The distribution tails are obviously more tractable in computations because of their monotonicity.

Remark 3.4. The quantity $1 - \frac{n \mathcal{F}^{\ell}(n+1)}{\sum_{k=1}^n \mathcal{F}^{\ell}(k)}$ in $J_{\ell|\ell}$ may be arbitrarily small, for instance when the distribution corresponding to \mathcal{F}^{ℓ} is slowly varying. However, when this quantity is well controlled — typically when it stays away from 0 — then the criteria can be rewritten in terms of

$$K_{\ell_1|\ell_2} = \sum_{n=1}^{\infty} \frac{\mathcal{F}^{\ell_1}(n)}{\sum_{k=1}^n \mathcal{F}^{\ell_2}(k)}. \quad (3.42)$$

Remark 3.5. Contrary to the well defined drift case for which a small perturbation on the parameters of a recurrent persistent random walk will lead in general to a transient behaviour, in the case of an undefined drift the persistent random walk may stay recurrent as long as the perturbation remains controlled asymptotically. To summarize, the criteria is global in the former case and asymptotic in the latter case.

Following this last remark, in the next section, we give some examples of perturbations which exhibit stability or unstability of the recurrent and transient properties in the context of $\Theta_{\text{u}} = \Theta_{\text{d}} = \infty$.

3.4 Perturbations

The comparison lemma 3.2 can be relaxed as shown by the following proposition.

Proposition 3.2 (Asymptotic comparison). Assume that the drift is undefined. Let $(\mathcal{F}^{\ell}(n))$ and $(\tilde{\mathcal{F}}^{\ell}(n))$ with $\ell \in \mathcal{A}$ be distribution tails such that, for $n \geq 1$ large enough,

$$\mathcal{F}^{\text{u}}(n) \leq \tilde{\mathcal{F}}^{\text{u}}(n) \quad \text{and} \quad \mathcal{F}^{\text{d}}(n) \geq \tilde{\mathcal{F}}^{\text{d}}(n). \quad (3.43)$$

Then, $J_{\text{u|d}} = \infty$ implies $\tilde{J}_{\text{u|d}} = \infty$, where J and \tilde{J} correspond to the quantity defined in (3.27) for the tails \mathcal{F} and $\tilde{\mathcal{F}}$ respectively.

Proof. Let $N \geq 1$ be such that the inequalities of (3.43) are satisfied for $n \geq N$. Then we set for $\ell \in \mathcal{A}$

$$\mathcal{F}_c^\ell(n) = \begin{cases} \mathcal{F}^\ell(n) & \text{for } n \geq N \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{\mathcal{F}}_c^\ell(n) = \begin{cases} \widetilde{\mathcal{F}}^\ell(n) & \text{for } n \geq N \\ 1 & \text{otherwise.} \end{cases} \quad (3.44)$$

Due to Lemma 3.2, there exists a coupling such that $S \leq \widetilde{S}$ a.s. Whatever the type of S , it follows from Theorem 3.2 that $J_{\text{u|d}}^c = \infty$ implies $\widetilde{J}_{\text{u|d}}^c = \infty$. To end the proof, it remains to show that $J_{\text{u|d}}^c = \infty$ if and only if $J_{\text{u|d}} = \infty$ (and similarly $\widetilde{J}_{\text{u|d}}^c = \infty$ if and only if $\widetilde{J}_{\text{u|d}} = \infty$). Since $\mathcal{F}^\ell(n)$ and $\mathcal{F}_c^\ell(n)$ only differ for $1 \leq n < N$, it comes that

$$\lim_{k \rightarrow \infty} \frac{\sum_{l=1}^k \mathcal{F}_c^\ell(l) - k \mathcal{F}_c^\ell(k) - \sum_{l=1}^k \mathcal{F}^\ell(l) + k \mathcal{F}^\ell(k)}{\sum_{l=1}^k \mathcal{F}^\ell(l) - k \mathcal{F}^\ell(k)} = \lim_{k \rightarrow \infty} \frac{\sum_{l=1}^{k-1} (1 - \mathcal{F}^\ell(l))}{\sum_{l=1}^k \mathcal{F}^\ell(l) - k \mathcal{F}^\ell(k)} = 0 \quad (3.45)$$

because the truncated expectation of τ^d goes to infinity. In addition, we have the asymptotic equivalence

$$\frac{\mathcal{F}^u(n)}{(\sum_{k=1}^n \mathcal{F}^d(k))^2} \underset{n \rightarrow \infty}{\sim} \frac{\mathcal{F}_c^u(n)}{(\sum_{k=1}^n \mathcal{F}_c^d(k))^2}. \quad (3.46)$$

Consequently, it follows

$$\left(1 - \frac{n \mathcal{F}^d(n)}{\sum_{\ell=1}^n \mathcal{F}^d(\ell)}\right) \frac{\mathcal{F}^u(n)}{\sum_{\ell=1}^n \mathcal{F}^d(\ell)} \underset{n \rightarrow \infty}{\sim} \left(1 - \frac{n \mathcal{F}_c^d(n)}{\sum_{\ell=1}^n \mathcal{F}_c^d(\ell)}\right) \frac{\mathcal{F}_c^u(n)}{\sum_{\ell=1}^n \mathcal{F}_c^d(\ell)}. \quad (3.47)$$

■

For two non negative sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$, we write $(a_n) \asymp (b_n)$ if there exist two positive constants $C_0 \leq C_1$ such that for all $n \geq 0$ sufficiently large,

$$C_0 a_n \leq b_n \leq C_1 a_n. \quad (3.48)$$

Corollary 3.1 (Comparison of tails). *Assume that the drift is undefined. Let $(\mathcal{F}^\ell(n))$ and $(\widetilde{\mathcal{F}}^\ell(n))$ be tails of distribution such that*

$$\mathcal{F}^\ell(n) \asymp \widetilde{\mathcal{F}}^\ell(n). \quad (3.49)$$

Then, the persistent random walk S and \widetilde{S} have the same recurrent or transient behaviour.

Proof. The assumption implies that there exist constants $C_u, C_d > 0$ such that for large n

$$\mathcal{F}^u(n) \leq C_u \widetilde{\mathcal{F}}^u(n) \quad \text{and} \quad \mathcal{F}^d(n) \leq C_d \widetilde{\mathcal{F}}^d(n). \quad (3.50)$$

For sufficiently large $n \geq 1$, we set

$$\widehat{\mathcal{F}}^u(n) = C_u \widetilde{\mathcal{F}}^u(n) \quad \text{and} \quad \widehat{\mathcal{F}}^d(n) = C_d \widetilde{\mathcal{F}}^d(n). \quad (3.51)$$

According to Proposition 3.2, we deduce that $J_{\text{u|d}} = \infty$ implies $\widehat{J}_{\text{u|d}} = \infty$. Furthermore a simple computation shows that $\widehat{J}_{\text{u|d}} = \frac{C_u}{C_d} \widetilde{J}_{\text{u|d}}$. By symmetry, the corollary follows. ■

Now, let us consider for any $\ell \in \mathcal{A}$ some perturbation γ^ℓ of α^ℓ satisfying for all $n \geq 1$,

$$\widetilde{\alpha}_n^\ell := \alpha_n^\ell + \gamma_n^\ell \in [0, 1], \quad n \geq 1, \quad (3.52)$$

and we denote by \widetilde{S} the associated persistent random walk.

Proposition 3.3 (Bounded perturbations). *Assume that the drift is undefined. Then S and \tilde{S} have the same recurrent or transient behaviour if for all $\ell \in \mathcal{A}$,*

$$\left\{ \sum_{k=1}^n \log \left(1 - \frac{\gamma_k^\ell}{1 - \alpha_k^\ell} \right) : n \geq 1 \right\} \text{ is bounded.} \quad (3.53)$$

In particular, S and \tilde{S} have the same recurrent or transient behaviour if for all $\ell \in \mathcal{A}$,

$$\limsup_{n \rightarrow \infty} \alpha_n^\ell < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |\gamma_n^\ell| < \infty. \quad (3.54)$$

The terminology *bounded perturbation* has to be understood in the sense of (3.53). Also, a perturbation will be said *unbounded* if the sequence in (3.53) is unbounded.

Proof. The tail distribution $\tilde{\mathcal{F}}^\ell$ associated to \tilde{S} satisfies

$$\tilde{\mathcal{F}}^\ell(n) \underset{n \rightarrow \infty}{\asymp} \mathcal{F}^\ell(n), \quad (3.55)$$

since

$$\log(\tilde{\mathcal{F}}^\ell(n)) = \sum_{k=1}^n \log(1 - \tilde{\alpha}_k^\ell) = \log(\mathcal{F}^\ell(n)) + \sum_{k=1}^n \log \left(1 - \frac{\gamma_k^\ell}{1 - \alpha_k^\ell} \right). \quad (3.56)$$

■

Even though general criteria for the type of the persistent random walk exist in the case of unbounded perturbations, these criteria are tedious to write precisely and irrelevant to have an insight into the phenomena. The examples of 3.2 should speak for themselves.

Example 3.2. *The following two persistent random walks associated for n sufficiently large to*

$$\alpha_{1,n}^u := \frac{1}{2n}, \quad \alpha_{1,n}^d := \frac{1}{2n} + \frac{c}{n \log(n)} \quad \text{and} \quad \alpha_n^u := \frac{1}{n}, \quad \alpha_n^d := \frac{1}{n} + \frac{c}{n \log(n) \log(\log(n))}, \quad (3.57)$$

are recurrent or transient according to $|c| \leq 1$ or $|c| > 1$.

As previously, the following proposition is not as general as it could be and more particularly the assumption on α^u . Still, the main ideas are contained in this proposition.

Proposition 3.4 (Lacunar perturbations). *Assume that there exists $0 < \lambda < 1$ and $L \subset \mathbb{N}$ such that for n sufficiently large*

$$\alpha_n^u \sim \frac{\lambda}{n \log(n)} \quad \text{and} \quad \alpha_n^d := \begin{cases} \alpha_n^u, & \text{if } n \in L, \\ 0 & \text{if } n \notin L. \end{cases} \quad (3.58)$$

Then S is recurrent.

Remark 3.6. *Note that the admissibility of the sequences $(\alpha_n^\ell)_{n \geq 1}$ prevent from choosing L with arbitrarily large gaps. Namely, the subset L needs to satisfy*

$$\sum_{n \in L} \alpha_n^u = \infty, \quad \text{or equivalently} \quad \sum_{n \in L} \frac{1}{n \log(n)} = \infty. \quad (3.59)$$

However, the set L can be chosen with zero density in the following sense

$$\lim_{n \rightarrow \infty} \frac{\text{card } L \cap [0, n]}{n} = 0. \quad (3.60)$$

Typically, the resulting persistent random walk is still recurrent if one choose

$$L = \{l_n : n \in \mathbb{N}\}, \quad \text{with } l_n := \lceil n \log(n) \cdots \log_k(n) \rceil \vee 1 \quad (3.61)$$

for n sufficiently large.

Proof. First, let us consider the quantity J^- . Let $(l_n)_{n \geq 1}$ be the increasing sequence defining L , one can check that

$$J^- = \sum_{n \geq 1} \frac{n \alpha_n^d}{\sum_{k=1}^n \mathcal{F}^u(k)} = \sum_{n \geq 1} \frac{l_n \alpha_{l_n}^u}{\sum_{k=1}^{l_n} \mathcal{F}^u(k)}. \quad (3.62)$$

Since

$$\sum_{k=1}^n \mathcal{F}^u(k) \sim \frac{n}{\log^\lambda(n)}, \quad (3.63)$$

it follows the right handside in (3.62) is infinite since the sequence $(\alpha_n^d)_{n \geq 0}$ is assumed to be admissible so that

$$\sum_{n \geq 1} \frac{1}{l_n \log(l_n)} = \infty. \quad (3.64)$$

Secondly, we need to consider J^+ . Observe that

$$\sum_{k=1}^N \mathcal{F}^d(k) \sim \sum_{k=1}^N \exp\left(-\sum_{l \in L \cap [0, n]} \alpha_l^u\right) \leq \sum_{k=1}^N \exp\left(-\frac{\lambda \text{card } L \cap [0, n]}{n \log(n)}\right), \quad (3.65)$$

dividing by N and remarking that the quantity in the exponential goes to zero when n goes to infinity, it follows that

$$\sum_{k=1}^N \mathcal{F}^d(k) = \mathcal{O}(N) \quad (3.66)$$

This bound suffices to prove that $J^+ = \infty$. ■

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