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# Multi-parametric solutions to the NLS equation.

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## Abstract

The structure of the solutions to the one dimensional focusing nonlinear Schrödinger equation (NLS) for the order  $N$  in terms of quasi rational functions is given here. We first give the proof that the solutions can be expressed as a ratio of two wronskians of order  $2N$  and then two determinants by an exponential depending on  $t$  with  $2N - 2$  parameters. It also is proved that for the order  $N$ , the solutions can be written as the product of an exponential depending on  $t$  by a quotient of two polynomials of degree  $N(N + 1)$  in  $x$  and  $t$ . The solutions depend on  $2N - 2$  parameters and give when all these parameters are equal to 0, the analogue of the famous Peregrine breather  $P_N$ . It is fundamental to note that in this representation at order  $N$ , all these solutions can be seen as deformations with  $2N - 2$  parameters of the famous Peregrine breather  $P_N$ . With this method, we already built Peregrine breathers until order  $N = 10$ , and their deformations depending on  $2N - 2$  parameters.

## 1 Introduction

The term of rogue wave was introduced in the scientific community by Draper in 1964 [1]. The usual criteria for rogue waves in the ocean, is that the vertical distance from trough to crest is two or more times greater than the average wave height among one third of the highest waves in a time series (10 to 30 min). The first rogue wave recorded by scientific measurement in North Sea was made on the oil platform of Draupner in 1995, located between Norway and Scotland. Rogue waves in the ocean have led to many marine catastrophes; it is one of the reasons why these rogue waves turn out to be so important for the scientific community. It becomes a challenge to get a better understanding of their mechanisms of formation.

The rogue waves phenomenon currently exceed the strict framework of the study of ocean's waves and play a significant role in other fields; in nonlinear optics [2], Bose-Einstein condensate [3], atmosphere [4] and even finance [5].

Here, we consider the one dimensional focusing nonlinear Schrödinger equation

(NLS) to describe the phenomena of rogue waves. The first results concerning the NLS equation date back the works of Zakharov and Shabat in 1968 who solved it using the inverse scattering method [6, 7]. The case of periodic and almost periodic algebro-geometric solutions to the focusing NLS equation were first constructed in 1976 by Its and Kotlyarov [8, 9]. In 1977 Kuznetsov found the first breather type solution of the NLS equation [10]; a similar result was given by Ma [11] in 1979. The first quasi rational solutions to NLS equation were constructed in 1983 by Peregrine [12]. In 1986 Akhmediev, Eleonski and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather [13]. Other analogues of Peregrine breathers of order 3 were constructed and initial data corresponding to orders 4 and 5 were described in a series of articles by Akhmediev et al., in particular in [14, 15] using Darboux transformations.

Quite recently, many works about NLS equation have been published using different methods. In 2010, rational solutions to the NLS equation were written as a quotient of two wronskians [16]. In 2011, the present author constructed in [17] another representation of the solutions to the NLS equation in terms of a ratio of two wronskians of even order  $2N$  composed of elementary functions using truncated Riemann theta functions depending on two parameters; rational solutions were obtained when some parameter tended to 0. In 2012, Guo, Ling and Liu found another representation of the solutions as a ratio of two determinants [19] using generalized Darboux transform; a new approach was proposed by Ohta and Yang in [20] using Hirota bilinear method; finally, the present author has obtained rational solutions in terms of determinants which do not involve limits in [21] depending on two parameters.

With this extended method, we present multi-parametric families of quasi rational solutions to the focusing NLS equation of order  $N$  in terms of determinants (determinants of order  $2N$ ) dependent on  $2N - 2$  real parameters. With this representation, at the same time the well-known ring structure, but also the triangular shapes also found by Ohta and Yang [20], Akhmediev et al. [25] are given.

The aim of this paper is to prove the representation of the solutions to the focusing NLS equation depending this time on  $2N - 2$  parameters; the proof presented in this paper with  $2N - 2$  parameters has been never published. This is the first task of the paper; then we deduce its particular degenerate representations in terms of a ratio of two determinants of order  $2N$ . The second task of the paper is to give the proof of the structure of the solution at the order  $N$  as the ratio of two polynomials of order  $N(N + 1)$  in  $x$  and  $t$  by an exponential depending on  $t$ . This representation makes possible to get all the possible patterns for the solutions to the NLS equation. It is important to stress that contrary to other methods, these solutions depending on  $2N - 2$  parameters give the Peregrine breather as particular case when all the parameters are equal to 0 : for this reason, these solutions will be called  $2N - 2$  parameters deformations of the Peregrine of order  $N$ .

The paper is organized as follows. First of all, we express the solutions of the NLS equation using Fredholm determinants from these expressed in terms of

truncated functions theta of Riemann first obtained by Its, Rybin and Salle [9]; the representation given in theorem 2.1 is different from those given in [9]. From that, we prove the representation of the solutions of the NLS equation in terms of wronskians depending on  $2N - 2$  parameters. We deduce a degenerate representation of solutions to the NLS equation depending a priori on  $2N - 2$  parameters at the order  $N$ .

Then we prove a theorem which states the structure of the quasi-rational solutions to the NLS equation. It was only conjectured in preceding works [17, 21]. Families depending on  $2N - 2$  parameters for the  $N$ -th order as a ratio of two polynomials of  $x$  and  $t$  multiplied by an exponential depending on  $t$  are obtained; it is proved that each of these polynomials have a degree equal to  $N(N + 1)$ .

## 2 Expression of solutions to the NLS equation in terms of wronskians

### 2.1 Solutions to the NLS equation in terms of $\theta$ functions

For  $r = 1, 3$ , we define

$$\begin{aligned} \theta_r(x, t) = & \sum_{k \in \{0;1\}^{2N}} \exp \left\{ \sum_{\mu > \nu, \mu, \nu=1}^{2N} \ln \left( \frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu} \right)^2 k_\mu k_\nu \right. \\ & \left. + \left( \sum_{\nu=1}^{2N} i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + \sum_{\mu=1, \mu \neq \nu}^{2N} \ln \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| + \pi i \epsilon_\nu + e_\nu \right) k_\nu \right\}, \end{aligned} \quad (1)$$

In this formula, the symbol  $\sum_{k \in \{0;1\}^{2N}}$  denotes summation over all  $2N$ -dimensional vectors  $k$  whose coordinates  $k_\nu$  are either 0 or 1.

The terms  $\kappa_\nu, \delta_\nu, \gamma_\nu$  and  $x_{r,\nu}$  are functions of the parameters  $\lambda_\nu, 1 \leq \nu \leq 2N$ ; they are defined by the formulas :

$$\begin{aligned} \kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}; \\ x_{r,\nu} = (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad r = 1, 3. \end{aligned} \quad (2)$$

The parameters  $-1 < \lambda_\nu < 1, \nu = 1, \dots, 2N$ , are real numbers such that

$$\begin{aligned} -1 < \lambda_{N+1} < \lambda_{N+2} < \dots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \dots < \lambda_1 < 1 \\ \lambda_{N+j} = -\lambda_j, \quad j = 1, \dots, N. \end{aligned} \quad (3)$$

The condition (3) implies that

$$\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j}, \quad j = 1, \dots, N. \quad (4)$$

Complex numbers  $e_\nu, 1 \leq \nu \leq 2N$  are defined in the following way :

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \leq j \leq N, \quad a, b \in \mathbf{R}. \quad (5)$$

$\epsilon_\nu \in \{0; 1\}, \varphi, \nu = 1 \dots 2N$  are arbitrary real numbers.

With these notations, the solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0, \quad (6)$$

can be expressed as ([9])

$$v(x, t) = \frac{\theta_3(x, t)}{\theta_1(x, t)} \exp(2it - i\varphi), \quad (7)$$

## 2.2 From $\theta$ functions to Fredholm determinants

To get Fredholm determinants, we have to express the functions  $\theta_r$  defined in (1) in terms of subsets of  $[1, \dots, 2N]$

$$\theta_r(x, t) = \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \times \exp \left( \sum_{\nu \in J} i\kappa_\nu x - 2\delta_\nu t + x_{r, \nu} + e_\nu \right). \quad (8)$$

In (8), the symbol  $\sum_{J \subset \{1, \dots, 2N\}}$  denotes summation over all subsets  $J$  of indices of the set  $\{1, \dots, 2N\}$ .

Let  $I$  be the unit matrix and  $C_r = (c_{jk})_{1 \leq j, k \leq 2N}$  the matrix defined by :

$$c_{\nu\mu} = (-1)^{\epsilon_\nu} \frac{\prod_{\eta \neq \mu} |\gamma_\nu + \gamma_\eta|}{\prod_{\eta \neq \nu} |\gamma_\nu - \gamma_\eta|} \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r, \nu} + e_\nu), \quad (9)$$

$$\epsilon_j = j \quad 1 \leq j \leq N, \quad \epsilon_j = N + j, \quad N + 1 \leq j \leq 2N. \quad (10)$$

Then  $\det(I + C_r)$  has the following form

$$\det(I + C_r) = \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r, \nu} + e_\nu). \quad (11)$$

Comparing this last expression (11) with the formula (8) at the beginning of this section, we have clearly the identity

$$\theta_r = \det(I + C_r). \quad (12)$$

We can give another representation of the solutions to NLS equation. To do this, let's consider the matrix  $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$  defined by :

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r, \nu} + e_\nu). \quad (13)$$

We have the equality  $\det(I + D_r) = \det(I + C_r)$ , and so the solution of NLS equation takes the form

$$v(x, t) = \frac{\det(I + D_3(x, t))}{\det(I + D_1(x, t))} \exp(2it - i\varphi). \quad (14)$$

**Theorem 2.1** *The function  $v$  defined by*

$$v(x, t) = \frac{\det(I + D_3(x, t))}{\det(I + D_1(x, t))} \exp(2it - i\varphi). \quad (15)$$

*is a solution of the focusing NLS equation with the matrix  $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$  defined by*

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu).$$

*where  $\kappa_\nu$ ,  $\delta_\nu$ ,  $x_{r,\nu}$ ,  $\gamma_\nu$ ,  $e_\nu$  being defined in (2), (3) and (5).*

### 2.3 From Fredholm determinants to wronskians

We want to express solutions to NLS equation in terms of wronskian determinants. For this, we need the following notations :

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1, 3, \quad (16)$$

with the arguments

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2, \quad 1 \leq \nu \leq 2N. \quad (17)$$

We denote  $W_r(y)$  the wronskian of the functions  $\phi_{r,1}, \dots, \phi_{r,2N}$  defined by

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_{r,\nu})_{\nu, \mu \in [1, \dots, 2N]}]. \quad (18)$$

We consider the matrix  $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$  defined in (13). Then we have the following statement

**Theorem 2.2**

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0), \quad (19)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_{r,\nu})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

**Proof :** We start to remove the factor  $(2i)^{-1} e^{i\Theta_{r,\nu}}$  in each row  $\nu$  in the wronskian  $W_r(y)$  for  $1 \leq \nu \leq 2N$ .

Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_{r,\nu}} (2i)^{-N} (2)^{-N} \times \tilde{W}_r, \quad (20)$$

with

$$\tilde{W}_r = \begin{vmatrix} (1 - e^{-2i\Theta_{r,1}}) & i\gamma_1(1 + e^{-2i\Theta_{r,1}}) & \dots & (i\gamma_1)^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_{r,1}}) \\ (1 - e^{-2i\Theta_{r,2}}) & i\gamma_2(1 + e^{-2i\Theta_{r,2}}) & \dots & (i\gamma_2)^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_{r,2}}) \\ \vdots & \vdots & \vdots & \vdots \\ (1 - e^{-2i\Theta_{r,2N}}) & i\gamma_{2N}(1 + e^{-2i\Theta_{r,2N}}) & \dots & (i\gamma_{2N})^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_{r,2N}}) \end{vmatrix}$$

The determinant  $\tilde{W}_r$  can be written as

$$\tilde{W}_r = \det(\alpha_{jk}e_j + \beta_{jk}),$$

where  $\alpha_{jk} = (-1)^k(i\gamma_j)^{k-1}$ ,  $e_j = e^{-2i\Theta_{r,j}}$ , and  $\beta_{jk} = (i\gamma_j)^{k-1}$ ,  $1 \leq j \leq N$ ,  $1 \leq k \leq 2N$ ,  
 $\alpha_{jk} = (-1)^{k-1}(i\gamma_j)^{k-1}$ ,  $e_j = e^{-2i\Theta_{r,j}}$ , and  $\beta_{jk} = (i\gamma_j)^{k-1}$ ,  $N+1 \leq j \leq 2N$ ,  $1 \leq k \leq 2N$ .

We want to calculate  $\tilde{W}_r$ . To do this, we use the following Lemma

**Lemma 2.1** *Let  $A = (a_{ij})_{i,j \in [1, \dots, N]}$ ,  $B = (b_{ij})_{i,j \in [1, \dots, N]}$ ,  $(H_{ij})_{i,j \in [1, \dots, N]}$ , the matrix formed by replacing in  $A$  the  $j$ th row of  $A$  by the  $i$ th row of  $B$ . Then*

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}) \quad (21)$$

**Proof :** We use the classical notations :  $\tilde{A} = (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$  the transposed matrix in cofactors of  $A$ . We have the well known formula  $A \times \tilde{A} = \det A \times I$ . So it is clear that  $\det(\tilde{A}) = (\det(A))^{N-1}$ .

The general term of the product  $(c_{ij})_{i,j \in [1, \dots, N]} = (a_{ij}x_i + b_{ij})_{i,j \in [1, \dots, N]} \times (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$  can be written as

$$\begin{aligned} c_{ij} &= \sum_{s=1}^N (a_{is}x_i + b_{is}) \times \tilde{a}_{js} \\ &= x_i \sum_{s=1}^N a_{is}\tilde{a}_{js} + \sum_{s=1}^N b_{is}\tilde{a}_{js} \\ &= \delta_{ij} \det(A)x_i + \det(H_{ij}). \end{aligned}$$

We get

$$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}).$$

$$\text{Thus } \det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}).$$

□.

We denote  $U = (\alpha_{ij})_{i,j \in [1, \dots, 2N]}$ ,  $V = (\beta_{ij})_{i,j \in [1, \dots, 2N]}$ .

By applying the previous lemma, one obtains :

$$\begin{aligned} \tilde{W}_r &= \det(\alpha_{ij}e_i + \beta_{ij}) \\ &= \det(\alpha_{ij}) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}) = \det(U) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(U)}), \end{aligned} \quad (22)$$

where  $(H_{ij})_{i,j \in [1, \dots, N]}$  is the matrix formed by replacing in  $U$  the  $j$ th row of  $U$  by the  $i$ th row of  $V$  defined previously.

The determinant of  $U$  of Vandermonde type is clearly equal to

$$\det(U) = i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m). \quad (23)$$

To calculate determinant  $\tilde{W}_r$ , we must compute now  $\det(H_{ij})$ . To do that, two cases must be studied :

1. For  $1 \leq j \leq N$ . The matrix  $H_{ij}$  is clearly of the VanderMonde type where the  $j$ -th row of  $U$  in  $U$  is replaced by the  $i$ -th row of  $V$ . Clearly, we have :

$$\det(H_{ij}) = (-1)^{N(2N+1)+N-1} (i)^{N(2N-1)} \times M, \quad (24)$$

where  $M = M(m_1, \dots, m_{2N})$  is the Vandermonde determinant defined by  $m_k = \gamma_k$  for  $k \neq j$  and  $m_j = -\gamma_i$ . Thus we have :

$$\begin{aligned} \det(H_{ij}) &= -(i)^{N(2N-1)} \times \prod_{2N \geq l > k \geq 1, l \neq j} (m_l - m_k) \\ &= -(i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l < j} (-\gamma_i - \gamma_l) \times \prod_{l > j} (\gamma_l + \gamma_i), \quad (25) \\ &= (-1)^j (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j} (\gamma_l + \gamma_i). \end{aligned}$$

To evaluate  $\tilde{W}_r$ , we must simplify the quotient  $q_{ij} := \frac{\det(H_{ij})}{\det(U)}$  :

$$\begin{aligned} q_{ij} &= \frac{(-1)^j (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j} (\gamma_l + \gamma_i)}{i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m)} \\ &= \frac{(-1)^j \prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l < j} (\gamma_j - \gamma_l) \prod_{l > j} (\gamma_l - \gamma_j)} = \frac{(-1)^j \prod_{l \neq j} (\gamma_l + \gamma_i)}{(-1)^{j-1} \prod_{l \neq j} (\gamma_l - \gamma_j)} = -\frac{\prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l \neq j} (\gamma_l - \gamma_j)}. \end{aligned} \quad (26)$$

We can replace  $q_{ij}$  by  $r_{ij}$  defined by  $-\frac{\prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l \neq i} (\gamma_l - \gamma_i)}$ , because  $\det(\delta_{ij} x_i + \frac{\det(q_{ij})}{\det(A)}) = \det(\delta_{ij} x_i + \frac{\det(r_{ij})}{\det(A)})$  (similar matrices).

We express  $r_{ij}$  in terms of absolute value; as  $j \in [1; N]$  and  $0 < \gamma_1 < \dots < \gamma_N < 1 < \gamma_{2N} < \dots < \gamma_{N+1}$ , we have :

$$\prod_{l \neq i} (\gamma_l - \gamma_i) = (-1)^{i-1} \prod_{l \neq i} |\gamma_l - \gamma_i|, \quad \prod_{l \neq j} (\gamma_l + \gamma_i) = \prod_{l \neq j} |\gamma_l + \gamma_i|. \quad (27)$$

So the term  $r_{ij}$  can be written as

$$r_{ij} = (-1)^i \frac{\prod_{l \neq j} |\gamma_l + \gamma_i|}{\prod_{l \neq i} |\gamma_l - \gamma_i|} = (-1)^{\epsilon(i)} \frac{\prod_{l \neq j} |\gamma_l + \gamma_i|}{\prod_{l \neq i} |\gamma_l - \gamma_i|} = c_{ij} e^{-2i\Theta_{r,i}(0)}, \quad (28)$$

with respect to the notations given in (10) and (13).

2. The same estimations for  $N+1 \leq j \leq 2N$  are made;  $\det H_{ij}$  is first

$$\det(H_{ij}) = (-1)^{N(2N+1)+N-1} (i)^{N(2N-1)} \times M, \quad (29)$$

with  $M = M(m_1, \dots, m_{2N})$  the Vandermonde determinant defined by  $m_k = \gamma_k$  for  $k \neq j$  and  $m_j = -\gamma_i$ . Thus we have :

$$\begin{aligned} \det(H_{ij}) &= (i)^{N(2N-1)} \times \prod_{2N \geq l > k \geq 1, l \neq j} (m_l - m_k) \\ &= (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l < j} (-\gamma_i - \gamma_l) \times \prod_{l > j} (\gamma_l + \gamma_i), \quad (30) \\ &= (-1)^{j-1} (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j} (\gamma_l + \gamma_i). \end{aligned}$$



The quotient  $q_{ij} := \frac{\det(H_{ij})}{\det(U)}$  equals :

$$\begin{aligned} q_{ij} &= \frac{(-1)^{j-1} (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq i, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j} (\gamma_l + \gamma_i)}{i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m)} \\ &= \frac{(-1)^{j-1} \prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l < j} (\gamma_j - \gamma_l) \prod_{l > j} (\gamma_l - \gamma_j)} = \frac{(-1)^{j-1} \prod_{l \neq j} (\gamma_l + \gamma_i)}{(-1)^{j-1} \prod_{l \neq j} (\gamma_l - \gamma_j)} = \frac{\prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l \neq j} (\gamma_l - \gamma_j)}. \end{aligned} \quad (31)$$

We replace  $q_{ij}$  by  $r_{ij}$  defined by  $\frac{\prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l \neq i} (\gamma_l - \gamma_j)}$ , for the same reason as previously exposed.

$r_{ij}$  is expressed in terms of absolute value; as  $j \in [N+1; 2N]$  and  $0 < \gamma_1 < \dots < \gamma_N < 1 < \gamma_{2N} < \dots < \gamma_{N+1}$ , we have :

$$\prod_{l \neq i} (\gamma_l - \gamma_i) = (-1)^{2N-i+N} \prod_{l \neq i} |\gamma_l - \gamma_i|, \quad \prod_{l \neq j} (\gamma_l + \gamma_i) = \prod_{l \neq j} |\gamma_l + \gamma_i|. \quad (32)$$

So the term  $r_{ij}$  can be written as

$$r_{ij} = (-1)^{N+i} \frac{\prod_{l \neq j} |\gamma_l + \gamma_i|}{\prod_{l \neq i} |\gamma_l - \gamma_j|} = (-1)^{\epsilon(i)} \frac{\prod_{l \neq j} |\gamma_l + \gamma_i|}{\prod_{l \neq i} |\gamma_l - \gamma_j|} = c_{ij} e^{-2i\Theta_{r,i}(0)}, \quad (33)$$

with respect to the notations given in (10) and (13).

Replacing  $e_i$  by  $e^{-2i\Theta_{r,i}}$ ,  $\det \tilde{W}_r$  can be expressed as

$$\begin{aligned} \det \tilde{W}_r &= \det(U) \times \det(\delta_{ij} e_i + \frac{\det(H_{ij})}{\det(U)}) = \det(U) \times \det(\delta_{ij} e_i + r_{ij}) \\ &= \det(U) \prod_{i=1}^{2N} e^{-2i\Theta_i} \det(\delta_{ij} + (-1)^{\epsilon(i)} \prod_{l \neq i} \left| \frac{\gamma_l + \gamma_i}{\gamma_l - \gamma_i} \right| e^{2i\Theta_{r,i}}). \end{aligned} \quad (34)$$

We estimate the two members of the last relation (34) in  $y = 0$ , and using (23) we obtain the following result

$$\begin{aligned} \det \tilde{W}_r(0) &= i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m) \prod_{i=1}^{2N} e^{-2i\Theta_{r,i}(0)} \\ &\times \det(\delta_{ij} + (-1)^{\epsilon(i)} \prod_{l \neq i} \left| \frac{\gamma_l + \gamma_i}{\gamma_l - \gamma_i} \right| e^{2i\Theta_{r,i}(0)}) \\ &= i^{N(2N-1)} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-2i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(\delta_{ij} + c_{ij}) \\ &= i^{N(2N-1)} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-2i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(I + C_r) \\ &= i^{N(2N-1)} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-2i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(I + D_r). \end{aligned} \quad (35)$$

Therefore, the wronskian  $W_r$  given by (20) can be written as

$$\begin{aligned} W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0) &= \prod_{j=1}^{2N} e^{i\Theta_{r,j}(0)} (2)^{-2N} (i)^{-N} \times \tilde{W}_r \\ &= \prod_{j=1}^{2N} e^{i\Theta_{r,j}(0)} (2)^{-2N} (i)^{-N} i^{N(2N-1)} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-2i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(I + D_r) \\ &= (2)^{-2N} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(I + D_r). \end{aligned} \quad (36)$$

As a consequence

$$\det(I + D_r) = k_r(0) W_r(\phi_1, \dots, \phi_{2N})(0). \quad (37)$$

□

## 2.4 Wronskian representation of solutions to the NLS equation

From the initial formulation (15) we have

$$v(x, t) = \frac{\det(I + D_3(x, t))}{\det(I + D_1(x, t))} \exp(2it - i\varphi).$$

Using (19), the following relation between Fredholm determinants and wronskians is obtained

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \dots, \phi_{r,2N})(0)$$

and

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \dots, \phi_{r,2N})(0).$$

As  $\Theta_{3,j}(0)$  contains  $N$  terms  $x_{3,j}$   $1 \leq j \leq N$  and  $N$  terms  $-x_{3,j}$   $1 \leq j \leq N$ , we have the equality  $k_3(0) = k_1(0)$ , and we get the following result :

**Theorem 2.3** *The function  $v$  defined by*

$$v(x, t) = \frac{W_3(\phi_{3,1}, \dots, \phi_{3,2N})(0)}{W_1(\phi_{1,1}, \dots, \phi_{1,2N})(0)} \exp(2it - i\varphi).$$

*is a solution of the focusing NLS equation depending on two real parameters  $a$  and  $b$  with  $\phi_\nu^r$  defined in (16)*

$$\begin{aligned} \phi_{r,\nu} &= \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), & 1 \leq \nu \leq N, \\ \phi_{r,\nu} &= \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), & N+1 \leq \nu \leq 2N, \quad r = 1, 3, \end{aligned}$$

$\kappa_\nu, \delta_\nu, x_{r,\nu}, \gamma_\nu, e_\nu$  being defined in(2), (3) and (5).

## 3 Families of multi-parametric solutions to the NLS equation in terms of a ratio of two determinants

Solutions to the NLS equation as a quotient of two determinants are constructed. Similar functions defined in a preceding work [21] are used, but modified as explained in the following. The following notations are needed :

$$\begin{aligned} X_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ Y_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2, \end{aligned}$$

for  $1 \leq \nu \leq 2N$ , with  $\kappa_\nu, \delta_\nu, x_{r,\nu}$  defined in (2).

Parameters  $e_\nu$  are defined by (5).

Here, is the crucial point : we choose the parameters  $a_j$  and  $b_j$  in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a}_{k,j} \epsilon^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b}_{k,j} \epsilon^{2k+1}, \quad 1 \leq j \leq N. \quad (38)$$

Below the following functions are used :

$$\begin{aligned}\varphi_{4j+1,k} &= \gamma_k^{4j-1} \sin X_k, & \varphi_{4j+2,k} &= \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} &= -\gamma_k^{4j+1} \sin X_k, & \varphi_{4j+4,k} &= -\gamma_k^{4j+2} \cos X_k,\end{aligned}\quad (39)$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned}\varphi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin X_{N+k},\end{aligned}\quad (40)$$

for  $1 \leq k \leq N$ .

We define the functions  $\psi_{j,k}$  for  $1 \leq j \leq 2N$ ,  $1 \leq k \leq 2N$  in the same way, the term  $X_k$  is only replaced by  $Y_k$ .

$$\begin{aligned}\psi_{4j+1,k} &= \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} &= \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} &= -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} &= -\gamma_k^{4j+2} \cos Y_k,\end{aligned}\quad (41)$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned}\psi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin Y_{N+k},\end{aligned}\quad (42)$$

for  $1 \leq k \leq N$ .

Then it is clear that

$$q(x, t) := \frac{W_3(0)}{W_1(0)}$$

can be written as

$$q(x, t) = \frac{\Delta_3}{\Delta_1} = \frac{\det(\varphi_{j,k})_{j,k \in [1,2N]}}{\det(\psi_{j,k})_{j,k \in [1,2N]}}. \quad (43)$$

We recall that  $\lambda_j = 1 - 2j\epsilon^2$ . All the functions  $\varphi_{j,k}$  and  $\psi_{j,k}$  and their derivatives depend on  $\epsilon$  and can all be prolonged by continuity when  $\epsilon = 0$ .

Then the following expansions are used

$$\varphi_{j,k}(x, t, \epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \varphi_{j,1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,1}[l] = \frac{\partial^{2l} \varphi_{j,1}}{\partial \epsilon^{2l}}(x, t, 0),$$

$$\varphi_{j,1}[0] = \varphi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\varphi_{j,N+k}(x, t, \epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \varphi_{j,N+1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \varphi_{j,N+1}[l] = \frac{\partial^{2l} \varphi_{j,N+1}}{\partial \epsilon^{2l}}(x, t, 0),$$

$$\varphi_{j,N+1}[0] = \varphi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1.$$

We have the same expansions for the functions  $\psi_{j,k}$ .

$$\psi_{j,k}(x, t, \epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \psi_{j,1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,1}[l] = \frac{\partial^{2l} \psi_{j,1}}{\partial \epsilon^{2l}}(x, t, 0),$$

$$\psi_{j,1}[0] = \psi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad 1 \leq l \leq N-1,$$

$$\psi_{j,N+k}(x, t, \epsilon) = \sum_{l=0}^{N-1} \frac{1}{(2l)!} \psi_{j,N+1}[l] k^{2l} \epsilon^{2l} + O(\epsilon^{2N}), \quad \psi_{j,N+1}[l] = \frac{\partial^{2l} \psi_{j,N+1}}{\partial \epsilon^{2l}}(x, t, 0),$$

$$\psi_{j,N+1}[0] = \psi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N, \quad 1 \leq k \leq N, \quad N+1 \leq k \leq 2N..$$

Then we get the following result :

**Theorem 3.1** *The function  $v$  defined by*

$$v(x, t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} \quad (44)$$

*is a quasi-rational solution of the NLS equation (6)*

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

where

$$\begin{aligned} n_{j1} &= \varphi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N & n_{jk} &= \frac{\partial^{2k-2} \varphi_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ n_{jN+1} &= \varphi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N & n_{jN+k} &= \frac{\partial^{2k-2} \varphi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{j1} &= \psi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N & d_{jk} &= \frac{\partial^{2k-2} \psi_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{jN+1} &= \psi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N & d_{jN+k} &= \frac{\partial^{2k-2} \psi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ & & & 2 \leq k \leq N, \quad 1 \leq j \leq 2N \end{aligned}$$

*The functions  $\varphi$  and  $\psi$  are defined in (39), (40), (41), (42).*

**Proof :** The columns of the determinants appearing in  $q(x, t)$  are combined successively to eliminate in each column  $k$  (and  $N+k$ ) of them the powers of  $\epsilon$  strictly inferior to  $2(k-1)$ ; then each common term in numerator and denominator is factorized and simplified; finally we take the limit when  $\epsilon$  goes to 0.

Precisely, first of all, the components  $j$  of the columns 1 and  $N+1$  are respectively equal by definition to  $\varphi_{j1}[0] + 0(\epsilon)$  for  $C_1$ ,  $\varphi_{jN+1}[0] + 0(\epsilon)$  for  $C_{N+1}$  of  $\Delta_3$ , and  $\psi_{j1}[0] + 0(\epsilon)$  for  $C'_1$ ,  $\psi_{jN+1}[0] + 0(\epsilon)$  for  $C'_{N+1}$  of  $\Delta_1$ .

At the first step of the reduction, we replace the columns  $C_k$  by  $C_k - C_1$  and  $C_{N+k}$  by  $C_{N+k} - C_{N+1}$  for  $2 \leq k \leq N$ , for  $\Delta_3$ ; the same changes for  $\Delta_1$  are done. Each component  $j$  of the column  $C_k$  of  $\Delta_3$  can be rewritten as  $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \varphi_{j,1}[l] (k^{2l} - 1) \epsilon^{2l}$  and the column  $C_{N+k}$  replaced by  $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \varphi_{j,N+1}[l] (k^{2l} - 1) \epsilon^{2l}$  for  $2 \leq k \leq N$ . For  $\Delta_1$ , we have the same reductions, each component  $j$  of the column  $C'_k$  can be rewritten as  $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \psi_{j,1}[l] (k^{2l} - 1) \epsilon^{2l}$  and the column  $C'_{N+k}$  replaced by  $\sum_{l=1}^{N-1} \frac{1}{(2l)!} \psi_{j,N+1}[l] (k^{2l} - 1) \epsilon^{2l}$  for  $2 \leq k \leq N$ .

The term  $\frac{k^2-1}{2} \epsilon^2$  for  $2 \leq k \leq N$  can factorized in  $\Delta_3$  and  $\Delta_1$  in each column  $k$  and  $N+k$ , and so these common terms can be simplified in numerator and denominator.

If we restrict the developments at order 1 in columns 2 and  $N + 2$ , we get respectively  $\varphi_{j1}[1] + 0(\epsilon)$  for component  $j$  of  $C_2$ ,  $\varphi_{jN+1}[1] + 0(\epsilon)$  for component  $j$  of  $C_{N+2}$  of  $\Delta_3$ , and  $\psi_{j1}[1] + 0(\epsilon)$  for component  $j$  of  $C'_2$ ,  $\psi_{jN+1}[1] + 0(\epsilon)$  for component  $j$  of  $C'_{N+2}$  of  $\Delta_1$ . This algorithm can be continued up to the columns  $C_N$ ,  $C_{2N}$  of  $\Delta_3$  and  $C'_N$ ,  $C'_{2N}$  of  $\Delta_1$ .

Then taking the limit when  $\epsilon$  tends to 0,  $q(x, t)$  can be replaced by  $Q(x, t)$  defined by :

$$Q(x, t) := \begin{array}{c} \left| \begin{array}{cccccc} \varphi_{1,1}[0] & \dots & \varphi_{1,1}[N-1] & \varphi_{1,N+1}[0] & \dots & \varphi_{1,N+1}[N-1] \\ \varphi_{2,1}[0] & \dots & \varphi_{2,1}[N-1] & \varphi_{2,N+1}[0] & \dots & \varphi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{2N,1}[0] & \dots & \varphi_{2N,1}[N-1] & \varphi_{2N,N+1}[0] & \dots & \varphi_{2N,N+1}[N-1] \end{array} \right| \\ \hline \left| \begin{array}{cccccc} \psi_{1,1}[0] & \dots & \psi_{1,1}[N-1] & \psi_{1,N+1}[0] & \dots & \psi_{1,N+1}[N-1] \\ \psi_{2,1}[0] & \dots & \psi_{2,1}[N-1] & \psi_{2,N+1}[0] & \dots & \psi_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{2N,1}[0] & \dots & \psi_{2N,1}[N-1] & \psi_{2N,N+1}[0] & \dots & \psi_{2N,N+1}[N-1] \end{array} \right| \end{array} \quad (45)$$

So the solution of the NLS equation takes the form :

$$v(x, t) = \exp(2it - i\varphi) \times Q(x, t)$$

So we get the result given in (44).  $\square$

## 4 Families of quasi-rational solutions of order $N$ depending on $2N - 2$ parameters

Here a theorem which states the structure of the quasi-rational solutions to the NLS equation is given. It was only conjectured in preceding works [17, 21]. Moreover we obtain here families depending on  $2N - 2$  parameters for the  $N$ th-order Peregrine breather including families with 2 parameters constructed in preceding works and so we get other symmetries in these deformations than those were expected.

In this section we use the notations defined in the previous sections. The functions  $\varphi$  and  $\psi$  are defined in (39), (40), (41), (42).

**Theorem 4.1** *The function  $v$  defined by*

$$v(x, t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} \quad (46)$$

*is a quasi-rational solution of the NLS equation (6) quotient of two polynomials  $N(x, t)$  and  $D(x, t)$  depending on  $2N - 2$  real parameters  $\tilde{a}_j$  and  $\tilde{b}_j$ ,  $1 \leq j \leq N - 1$ .*

*$N$  and  $D$  are polynomials of degrees  $N(N + 1)$  in  $x$  and  $t$ .*

**Proof :** From the previous result (45), we need to analyze functions  $\varphi_{k,1}$ ,  $\psi_{k,1}$  and  $\varphi_{k,N+1}$ ,  $\psi_{k,N+1}$ . Functions  $\varphi_{k,j}$  and  $\psi_{k,j}$  differ only by the term of the argument  $x_{3,k}$ , so only the study of functions  $\varphi_{k,j}$  will be carried out. Then the study of functions  $\psi_{k,j}$  can be easily deduced from the analysis of  $\varphi_{k,j}$ . The expansions of these functions in  $\epsilon$  are studied. We denote  $(l_{kj})_{k,j \in [1,2N]}$  the matrix defined by

$$l_{kj} = \frac{\partial^{2j-2}}{\partial \epsilon^{2j-2}} \varphi_{k1}, \quad l_{k,j+N} = \frac{\partial^{2j-2}}{\partial \epsilon^{2j-2}} \varphi_{k,1+N}, \quad 1 \leq j \leq N, 1 \leq k \leq 2N,$$

$\frac{\partial^0}{\partial x^0} \varphi$  meaning  $\varphi$ . Each coefficient of the matrix  $(l_{kj})_{k,j \in [1,2N]}$  must be evaluated, the power of  $x$  and  $t$  in the coefficient of  $\epsilon^{2(m-1)}$  for the column  $m \in [1, 2N]$ . We remark that with these notations, the matrix  $(l_{kj})_{k,j \in [1,2N]}$  evaluated in  $\epsilon = 0$  is exactly  $(n_{kj})_{k,j \in [1,2N]}$  defined in (45). Four cases must be studied depending on the parity of  $k$ .

1. We study  $l_{k1}$  for  $k$  odd,  $k = 2s + 1$ .

$$\begin{aligned} l_{k1} &= (-1)^s \sin(2\epsilon(1 - \epsilon^2)^{\frac{1}{2}}x + 4i\epsilon(1 - \epsilon^2)^{\frac{1}{2}}(1 - 2\epsilon^2)t \\ &\quad - i \ln \frac{1 + i\epsilon(1 - \epsilon^2)^{-\frac{1}{2}}}{1 - i\epsilon(1 - \epsilon^2)^{-\frac{1}{2}}} - e_1) \times \epsilon^{k-2}(1 - \epsilon^2)^{-\frac{k-2}{2}} \\ &= (-1)^s \sin \epsilon \left( \sum_{l=0}^p c_{2l} \epsilon^{2l} x + 2i \sum_{l=0}^p c_{2l} \epsilon^{2l} (1 - 2\epsilon^2)t + 2 \sum_{l=0}^p (-1)^l \epsilon^{2l} \frac{(1 - \epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)} \right. \\ &\quad \left. - \sum_{l=1}^{N-1} \tilde{a}_l \epsilon^{2l} + i \sum_{l=1}^{N-1} \tilde{b}_l \epsilon^{2l} + O(\epsilon^{p+1}) \right) \times \epsilon^{k-2} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\ &= (-1)^s \sin \epsilon \left( \sum_{l=0}^p (c_{2l}x + d_{2l}t + f_{2l} + O(\epsilon^{p+1})) \epsilon^{2l} \right) \times \epsilon^{k-2} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\ &= \sum_{l=0}^q \frac{(-1)^{l+s} \epsilon^{2l}}{(2l+1)!} \left( \sum_{n=0}^p (c_{2n}x + d_{2n}t + f_{2n} + O(\epsilon^{p+1})) \epsilon^{2n} \right)^{2l+1} \times \epsilon^{k-1} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\ &= \sum_{l=0}^q \frac{(-1)^{l+s} \epsilon^{2l}}{(2l+1)!} \left( \sum_{n=0}^p P_n(x, t) \epsilon^{2n} \right)^{2l+1} \times \epsilon^{k-1} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t) \end{aligned}$$

where  $P_n(x, t)$  is a polynomial of order 1 in  $x$  and  $t$ .

$$\begin{aligned} l_{k,1} &= \sum_{l=0}^q \epsilon^{2l} \sum_{\alpha_0 + \dots + \alpha_p = 2l+1} \beta_{\alpha_0, \dots, \alpha_p} P_0(x, t)^{\alpha_0} \dots P_p(x, t)^{\alpha_p} \epsilon^{2(\alpha_1 + 2\alpha_2 + p\alpha_p)} \times \epsilon^{2s} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t) \\ &= \sum_{l=0}^q \epsilon^{2l} \sum_{\alpha_0 + \dots + \alpha_p = 2l+1} Q_{\alpha_0, \dots, \alpha_p}(x, t) \epsilon^{2(\alpha_1 + 2\alpha_2 + p\alpha_p)} \times \epsilon^{2s} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t), \end{aligned}$$

where  $Q_{\alpha_0, \dots, \alpha_p}(x, t)$  is a polynomial of order  $2l + 1$  in  $x$  and  $t$ .

The terms in  $\epsilon^0$  are obtained for  $l = 0$  in the two summations with  $\alpha_0 = 1$ .

For column  $m$ , we search the terms in  $\epsilon^{2m-2}$  with the maximal power in  $x$  and  $t$ . It is obtained for  $2l + k - 1 = 2m - 2$ , which gives  $l = m - s - 1$ . The notations given in (44) are used. We get the following result

**Proposition 4.1**

$$\deg(n_{2s+1,m}) = 2(m - s) - 1 \text{ for } s \leq m - 1, \quad n_{2s+1,m} = 0 \text{ for } s \geq m. \quad (47)$$

2. We study  $l_{k1}$  for  $k$  even,  $k = 2s$ .

$$\begin{aligned} l_{k1} &= (-1)^{s+1} \cos(2\epsilon(1 - \epsilon^2)^{\frac{1}{2}}x + 4i\epsilon(1 - \epsilon^2)^{\frac{1}{2}}(1 - 2\epsilon^2)t \\ &\quad - i \ln \frac{1 + i\epsilon(1 - \epsilon^2)^{-\frac{1}{2}}}{1 - i\epsilon(1 - \epsilon^2)^{-\frac{1}{2}}} - e_1) \times \epsilon^{k-2}(1 - \epsilon^2)^{-\frac{k-2}{2}} \\ &= (-1)^{s+1} \cos \epsilon \left( \sum_{l=0}^p c_{2l} \epsilon^{2l} x + 2i \sum_{l=0}^p c_{2l} \epsilon^{2l} (1 - 2\epsilon^2)t + 2 \sum_{l=0}^p (-1)^l \epsilon^{2l} \frac{(1 - \epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)} \right. \\ &\quad \left. - \sum_{l=1}^{N-1} \tilde{a}_l \epsilon^{2l} + i \sum_{l=1}^{N-1} \tilde{b}_l \epsilon^{2l} + O(\epsilon^{p+1}) \right) \times \epsilon^{k-2} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\ &= (-1)^{s+1} \cos \epsilon \left( \sum_{l=0}^p (c_{2l}x + d_{2l}t + f_{2l} + O(\epsilon^{p+1})) \epsilon^{2l} \right) \times \epsilon^{k-2} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\ &= \sum_{l=0}^q \frac{(-1)^{l+d+1} \epsilon^{2l}}{(2l)!} \left( \sum_{n=0}^p (c_{2n}x + d_{2n}t + f_{2n} + O(\epsilon^{p+1})) \epsilon^{2n} \right)^{2l} \times \epsilon^{k-2} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\ &= \sum_{l=0}^q \frac{(-1)^{l+s+1} \epsilon^{2l}}{(2l)!} \left( \sum_{n=0}^p P_n(x, t) \epsilon^{2n} \right)^{2l} \times \epsilon^{2s-2} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t) \end{aligned}$$

where  $P_n(x, t)$  is a polynomial of order 1 in  $x$  and  $t$ .

$$\begin{aligned} l_{k,1} &= \sum_{l=0}^q \epsilon^{2l} \sum_{\alpha_0 + \dots + \alpha_p = 2l} \beta_{\alpha_0, \dots, \alpha_p} P_0(x, t)^{\alpha_0} \dots P_p(x, t)^{\alpha_p} \epsilon^{2(\alpha_1 + 2\alpha_2 + p\alpha_p)} \times \epsilon^{2s-2} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t) \\ &= \sum_{l=0}^q \epsilon^{2l} \sum_{\alpha_0 + \dots + \alpha_p = 2l} Q_{\alpha_0, \dots, \alpha_p}(x, t) \epsilon^{2(\alpha_1 + 2\alpha_2 + p\alpha_p)} \times \epsilon^{2s-2} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t), \end{aligned}$$

where  $Q_{\alpha_0, \dots, \alpha_p}(x, t)$  is a polynomial of order  $2l$  in  $x$  and  $t$ .

The terms in  $\epsilon^0$  are obtained for  $l = 0$  in the two summations with  $\alpha_0 = 1$ .

For column  $m$ , we search the terms in  $\epsilon^{2m-2}$  with the maximal power in  $x$  and  $t$ . It is obtained for  $2l + k - 2 = 2m - 2$ , which gives  $l = m - s$ .

With the notations given in (44), we have

**Proposition 4.2**

$$\deg(n_{2s,m}) = 2(m - s) \text{ for } s \leq m, \quad n_{2s,m} = 0 \text{ for } s > m. \quad (48)$$

3. We study  $l_{k, \frac{M}{2}+1}$  for  $k$  odd,  $k = 2s + 1$ .

$$\begin{aligned}
l_{k, \frac{M}{2}+1} &= (-1)^s \cos(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x - 4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)t + i \ln \frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e^{\frac{M}{2}+1}) \\
&\quad \times \epsilon^{M-k-1}(1-\epsilon^2)^{-\frac{M-k-1}{2}} \\
&= (-1)^s (\cos \epsilon (\sum_{l=0}^p c_{2l} \epsilon^{2l} x - 2i \sum_{l=0}^p c_{2l} \epsilon^{2l} (1-2\epsilon^2)t - 2 \sum_{l=0}^p (-1)^l \epsilon^{2l} \frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)} \\
&\quad - \sum_{l=1}^{N-1} \tilde{a}_l \epsilon^{2l} + i \sum_{l=1}^{N-1} \tilde{b}_l \epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{M-k-1} (\sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1})) \\
&= (-1)^s (\cos \epsilon (\sum_{l=0}^p (c_{2l}x + d_{2l}t + f_{2l}) \epsilon^{2l} + O(\epsilon^{p+1})) \times \epsilon^{M-k-1} (\sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1})) \\
&= \sum_{l=0}^q \frac{(-1)^{l+s} \epsilon^{2l}}{(2l)!} (\sum_{n=0}^p (c_{2n}x + d_{2n}t + f_{2n} + O(\epsilon^{p+1})) \epsilon^{2n})^{2l} \times \epsilon^{M-k-1} (\sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1})) \\
&= \sum_{l=0}^q \frac{(-1)^{l+s} \epsilon^{2l}}{(2l)!} (\sum_{n=0}^p P_n(x, t) \epsilon^{2n} + O(\epsilon^{p+1}))^{2l} \times \epsilon^{M-2s-2} (\sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}))
\end{aligned}$$

where  $P_n(x, t)$  is a polynomial of order 1 in  $x$  and  $t$ .

$$\begin{aligned}
l_{k, \frac{M}{2}+1} &= \sum_{l=0}^q \epsilon^{2l} \sum_{\alpha_0 + \dots + \alpha_p = 2l} \beta_{\alpha_0, \dots, \alpha_p} P_0(x, t)^{\alpha_0} \\
&\quad \dots P_p(x, t)^{\alpha_p} \epsilon^{2(\alpha_1 + 2\alpha_2 + p\alpha_p)} \times \epsilon^{M-2s-2} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t) \\
&= \sum_{l=0}^q \epsilon^{2l} \sum_{\alpha_0 + \dots + \alpha_p = 2l} Q_{\alpha_0, \dots, \alpha_p}(x, t) \epsilon^{2(\alpha_1 + 2\alpha_2 + p\alpha_p)} \times \epsilon^{M-2s-2} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t),
\end{aligned}$$

where  $Q_{\alpha_0, \dots, \alpha_p}(x, t)$  is a polynomial of order  $2l$  in  $x$  and  $t$ .

The terms in  $\epsilon^0$  (column  $\frac{M}{2} + 1$ ) are obtained for  $l = 0$  in the two summations with  $\alpha_0 = 1$ .

For column  $\frac{M}{2} + m$ , we search the terms in  $\epsilon^{2m-2}$  with the maximal power in  $x$  and  $t$ . It is obtained for  $2l + 2(N - s - 1) = 2m - 2$ , which gives  $l = m + s - N$ . Then we get the following result

**Proposition 4.3**

$$\deg(n_{2s+1, m + \frac{M}{2}}) = 2m + 2s - M \text{ for } s \geq \frac{M}{2} - m, \quad n_{2s+1, m} = 0 \text{ for } s < \frac{M}{2} - m. \quad (49)$$



4. We study  $l_{k,1+\frac{M}{2}}$  for  $k$  even,  $k = 2s$ .

$$\begin{aligned}
l_{k\frac{M}{2}+1} &= (-1)^s \sin(2\epsilon(1-\epsilon^2)^{\frac{1}{2}}x - 4i\epsilon(1-\epsilon^2)^{\frac{1}{2}}(1-2\epsilon^2)t + i \ln \frac{1+i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}}{1-i\epsilon(1-\epsilon^2)^{-\frac{1}{2}}} - e^{\frac{M}{2}+1}) \\
&\quad \times \epsilon^{M-k-1}(1-\epsilon^2)^{-\frac{M-k-1}{2}} \\
&= (-1)^s \sin \epsilon \left( \sum_{l=0}^p c_{2l} \epsilon^{2l} x - 2i \sum_{l=0}^p c_{2l} \epsilon^{2l} (1-2\epsilon^2)t - 2 \sum_{l=0}^p (-1)^l \epsilon^{2l} \frac{(1-\epsilon^2)^{-\frac{2l+1}{2}}}{(2l+1)} \right. \\
&\quad \left. - \sum_{l=1}^{N-1} \tilde{a}_l \epsilon^{2l} + i \sum_{l=1}^{N-1} \tilde{b}_l \epsilon^{2l} + O(\epsilon^{p+1}) \right) \times \epsilon^{M-k-1} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\
&= (-1)^s \sin \epsilon \left( \sum_{l=0}^p (c_{2l}x + d_{2l}t + f_{2l}) \epsilon^{2l} + O(\epsilon^{p+1}) \right) \times \epsilon^{M-k-1} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\
&= \sum_{l=0}^q \frac{(-1)^{l+s} \epsilon^{2l}}{(2l+1)!} \left( \sum_{n=0}^p (c_{2n}x + d_{2n}t + f_{2n} + O(\epsilon^{p+1})) \epsilon^{2n} \right)^{2l+1} \times \epsilon^{M-k} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right) \\
&= \sum_{l=0}^q \frac{(-1)^{l+s} \epsilon^{2l}}{(2l+1)!} \left( \sum_{n=0}^p P_n(x, t) \epsilon^{2n} + O(\epsilon^{p+1}) \right)^{2l+1} \times \epsilon^{M-2s} \left( \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^{r+1}) \right)
\end{aligned}$$

where  $P_n(x, t)$  is a polynomial of order 1 in  $x$  and  $t$ .

$$\begin{aligned}
l_{k,1} &= \sum_{l=0}^q \epsilon^{2l} \sum_{\alpha_0+\dots+\alpha_p=2l+1} \beta_{\alpha_0,\dots,\alpha_p} P_0(x, t)^{\alpha_0} \\
&\quad \dots P_p(x, t)^{\alpha_p} \epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{M-2s} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t) \\
&= \sum_{l=0}^q \epsilon^{2l} \sum_{\alpha_0+\dots+\alpha_p=2l+1} Q_{\alpha_0,\dots,\alpha_p}(x, t) \epsilon^{2(\alpha_1+2\alpha_2+p\alpha_p)} \times \epsilon^{M-2s} \sum_{l=1}^r g_{2l} \epsilon^{2l} + O(\epsilon^t),
\end{aligned}$$

where  $Q_{\alpha_0,\dots,\alpha_p}(x, t)$  is a polynomial of order  $2l+1$  in  $x$  and  $t$ .

The terms in  $\epsilon^0$  are obtained for  $l=0$  in the two summations with  $\alpha_0=1$ .

For column  $\frac{M}{2}+m$ , we search the terms in  $\epsilon^{2m-2}$  with the maximal power in  $x$  and  $t$ . It is obtained for  $2l+M-k=2m-2$ , which gives  $l=m+s-N-1$ .

Using the notations given in (44), we get the following result

**Proposition 4.4**

$$\begin{aligned}
\deg(n_{2s,m+\frac{M}{2}}) &= 2m+2s-M-1 \text{ for } s \geq \frac{M}{2}+1-M, \\
n_{2s,m+\frac{M}{2}} &= 0 \text{ for } s < \frac{M}{2}+1-m.
\end{aligned} \tag{50}$$

These results can be rewritten in the following way

**Proposition 4.5**

$$\begin{aligned}
\deg(n_{j,k}) &= 2k - j \text{ for } j \leq 2k, \\
n_{j,k} &= 0 \text{ for } j > 2k, \\
\deg(n_{j,k}) &= 2k + j - 2M - 1 \text{ for } j \geq 2M + 1 - 2k, \\
n_{j,k} &= 0 \text{ for } j < 2M + 1 - 2k.
\end{aligned} \tag{51}$$

The degree of the determinant of the matrix  $(n_{kj})_{k,j \in [1, 2N]}$  can now be evaluated.

From the previous analysis, we see that  $x$  and  $t$  have necessarily the same power in each  $n_{kj}$ . The maximal power in  $x$  and  $t$ , is successively taken in each column. It is realized by the following product

$$\prod_{j=1}^N n_{j,j} \prod_{j=1}^N n_{N+j, 2N+1-j}.$$

Applying the result given in (51) we get

$$\begin{aligned}
\deg(\det(n_{kj})_{k,j \in [1, 2N]}) &= \sum_{j=1}^N \deg(n_{j,j}) + \sum_{j=1}^N \deg(n_{N+j, 2N+1-j}) \\
&= \sum_{j=1}^N 2j - j + \sum_{j=1}^N 2(M + 1 - j) - 2M - 1 + \frac{M}{2} + j \\
&= \sum_{j=1}^N j + \sum_{j=1}^N N + 1 - j = N(N + 1).
\end{aligned}$$

It is the same for determinant  $\det(d_{kj})_{k,j \in [1, 2N]}$ , we have  $\deg(\det(d_{kj})_{k,j \in [1, 2N]}) = N(N + 1)$ .

Thus the quotient

$$\frac{\det((n_{kj})_{j,k \in [1, 2N]})}{\det((d_{kj})_{j,k \in [1, 2N]})}$$

defines a quotient of two polynomials, each of them of degree  $N(N + 1)$ , and this proves the result.

Parameters  $a_1 = \sum_{k=1}^{N-1} \tilde{a}_k \epsilon_k$  and  $a_1 = \sum_{k=1}^{N-1} \tilde{a}_k \epsilon_k$  must be chosen in the following way.

The term  $\epsilon_k$  must be a power of  $\epsilon$  to get a nontrivial solution;  $\epsilon_k$  must be a strictly positive number  $a$  in order to have a finite limit when  $\epsilon$  goes to 0. If the power of  $\epsilon$  is superior to  $2N - 2$ , the derivations going up to  $2N - 2$ , then this coefficient becomes 0 when the limit is taken when  $\epsilon$  goes to 0 and so has no relevance in the expression of the limit.

□

## 5 Conclusion

Here we proved the structure of quasi-rational solutions to the one dimensional focusing NLS equation at order  $N$ . They can be expressed as a product of an exponential depending on  $t$  by a ratio of two polynomials of degree  $N(N+1)$  in  $x$  and  $t$ . If we choose  $\tilde{a}_i = \tilde{b}_i = 0$  for  $1 \leq i \leq N-1$ , we obtain the classical (analogue) Peregrine breather. Thus these solutions appear as  $2N-2$ -parameters deformations of the Peregrine breather of order  $N$ .

The solutions for orders 3 and 4 first found by Matveev have also been explicitly found by the present author [26, 27]. We have also explicitly found the solutions at order 5 with 8 parameters [28]: these expressions are too extensive to be presented : it takes 14049 pages! For other orders 6, 7, 8, the solutions are also explicitly found but are too long to be published in any review. In the relative works [29, 30, 32, 33, 34] only the analysis has been done and figures of deformations of the Peregrine breathers has been realized. The solutions for order 9 with 16 parameters [33] and respectively for order 10 with 18 parameters are also completely found [34].

We still insist on the fact that quasi rational solutions of NLS equation can be expressed as a quotient of two polynomials of degree  $N(N+1)$  in  $x$  and  $t$  dependent on  $2N-2$  real parameters by an exponential depending on time. Among these aforementioned solutions of order  $N$ , there is one which has the largest module : it is the solution obtained in this representation when all the parameters are equal to 0; one obtains the Peregrine breather order  $N$ . His importance is due to the fact that among the solutions of order  $N$ , its module is largest, equal to  $2N+1$ . This result first formulated by Akhmediev has just been proved recently [35].

In the recent studies proposed by the author, the solutions of order  $N$  can be represented by their module in the plane  $(x; t)$ . With the representation given in this article, one obtains at order  $N$ , the configurations containing  $N(N+1)/2$  peaks, except the special case of Peregrine breather. These configurations can be classified according to the values of the parameters  $a_i$  or  $b_i$  for  $i$  varying between 1 and  $N-1$ . It is important to note that the role played by  $a_i$  or  $b_i$  for a given index  $i$  is the same one, in obtaining the configurations. The study refers to the analysis of the solutions when only one of the parameters is not equal to 0. Among these solutions, one distinguishes two types of configurations; for  $a_1$  or  $b_1$  not equal to 0, one observes triangular configurations with  $N(N+1)/2$  peaks. For  $a_i$  or  $b_i$  not equal to 0 and  $2 \leq i \leq N-1$ , one observes concentric rings. The simplest structure is obtained for  $a_{N-1}$  or  $b_{N-1}$  not equal to 0 : one obtains only one ring of  $2N-1$  peaks with in his center Peregrine breather of order  $N-2$ ; this fact was also first formulated by Akhmediev. The detailed study of the other structures is being analyzed. We hope to be able to give results soon.

We can conclude that the method described in the present paper provides a very efficient and powerful tool to get explicit solutions to the NLS equation and to understand the behavior of rogue waves.

There are currently many applications in different fields as recent works by

Akhmediev et al. [36] or Kibler et al. [37] attest it in particular. This study leads to a better understanding of the phenomenon of rogue waves, and it would be relevant to go on with higher orders.

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